Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 13, No. 4, pp. 571–591, October 2017

A CLASS OF DIFFERENTIAL INVERSE MIXED VARIATIONAL INEQUALITIES IN FINITE DIMENSIONAL SPACES*

Wei Li, Xing Wang, Xue-song Li and Nan-jing Huang†

Abstract: In this paper, we introduce and study a new class of differential inverse mixed variational inequality in finite dimensional Euclidean spaces, which consists of a system of ordinary differential equations and an inverse mixed variational inequality. We study linear growth properties and characteristics of the set of solutions for the inverse mixed variational inequality and prove some existence theorems of weak solutions for the differential inverse mixed variational inequality in the sense of Carathéodory by applying a result on the differential inclusion involving an upper semicontinuous set-valued mapping with nonempty closed and convex values. Moreover, we show the convergence of the Euler time-stepping method to a weak solution of the differential inverse mixed variational inequality by employing some results concerned with differential inclusions.

Key words: *differential inverse mixed variational inequality, linear growth, Carathéodory weak solution, convergence*

Mathematics Subject Classification: *49J40, 47J15*

1 Introduction

In 2008, Pang and Stewart [23] studied differential variational inequalities (DVIs) in finite dimensional Euclidean space which provides a modeling ground for many applied problems in engineering and economics such as differential Nash games, electrical circuits, robotics, earthquake engineering, and structural dynamics see [1–3, 9]. In their seminal papers the authors have already shown that the DVI unifies several mathematical problem classes that include ordinary differential equations (ODEs) with smooth and discontinuous right-hand sides, dynamic complementarity systems, differential algebraic equations (DAEs), and evolutionary variational inequalities. In 2010, Li et al. [20] introduced and investigated a class of differential mixed variational inequalities (DMVIs) in finite-dimensional Euclidean spaces which generalized the corresponding results of [23]. Stewart [27] investigated the uniqueness for a class of index-one DVIs. Recently, Wang et al. [31] studied stability for differential mixed variational inequalities (DMVIs). They proved an existence theorem of Carathéodory weak solutions for DMVI and obtained some upper semicontinuity and continuity results concerned with the Carath odory weak solution set mapping for DMVI. For more related results, we refer to [4, 5, 7, 8, 21, 22, 26, 29, 30, 32] and the references therein.

*This work was supported by the National Natural Science Foundation of China (11301359, 11471230, 11501263, 11671282, 41704132).

†Corresponding author.

On the other hand, the inverse variational inequality (IVI) was firstly proposed by He and Liu [11] in 2006. It has many applications in various areas, such as market equilibrium problems in economics and normative flow control problems appeared in transportation and telecommunication networks (see [10, 11]). In 2008, Yang [28] discussed the dynamic power price problem, in both the discrete and the evolutionary cases, and characterized the optimal price as a solution of IVI. Hu and Fang [15] investigated the well-posedness of IVI, and established some characterizations of the well-posedness of inverse variational inequalities under some suitable conditions. Recently, Hu and Fang [16] studied Levitin-Polyak well-posedness by perturbations of inverse variational inequalities. As a generalization of IVI, Li et al. [19] introduced and studied a new inverse mixed variational inequality (IMVI) in the setting of Hilbert spaces. They constructed an iterative algorithm for solving IMVI and proved the convergence of sequences generated by the algorithm. Some related works for IVIs can be found in [12, 13, 24] and the references therein.

Very recently, Li et al. [18] introduced and studied a new class of differential inverse variational inequality (DIVI) in finite dimensional Euclidean spaces, which consists of a system of ordinary differential equations and an inverse variational inequality. They showed the linear growth of the set of solutions for DIVIs and the existence theorems of Carathéodory weak solutions for DIVIs. They also gave an application to the time-dependent spatial price equilibrium control problem. We note that IMVI is not only a generalization of IVI but also provides a model to the study for traffic network equilibrium control problems [19]. Therefore, it is important and interesting to study some kinds of differential inverse mixed variational inequalities under suitable conditions. The main purpose of this paper is to introduce and study a new class of differential inverse mixed variational inequality in finite dimensional Euclidean spaces, which consists of an ordinary differential equation and an inverse mixed variational inequality.

Recall IMVI in finite-dimensional Euclidean space as follows: find $x \in R^n$ such that

$$f(x) \in K, \quad \langle x' - f(x), x\rangle + \varphi(x') - \varphi(f(x)) \geq 0, \quad \forall x' \in K,$$

where $f : R^n \to R^n$ is a mapping, $K \subset R^n$ is a closed convex subset, $\varphi : R^n \to (-\infty, +\infty]$ is a proper lower semicontinuous (l.s.c.) convex functional. Let $S(K, f, \varphi)$ denote the set of solutions for IMVI. Specially, if f^{-1} exists, setting $y = f(x)$ and $g(y) = f^{-1}(y)$, then it is easy to see that IMVI reduces to MVI: find $x \in K$, such that

$$\langle g(x), x' - x\rangle + \varphi(x') - \varphi(x) \geq 0, \quad \forall x' \in K,$$

where $g : R^n \to R^n$ is a mapping. If φ is the indicator functional on K, then IMVI reduces to the following classical IVI: find $x \in R^n$ such that

$$f(x) \in K, \quad \langle x' - f(x), x\rangle \geq 0, \quad \forall x' \in K.$$

Combining the differential equations with the inverse mixed variational inequalities, in this paper, we consider an initial-value differential inverse mixed variational inequality (DIMVI) as follows:

$$\begin{cases} \dot{x}(t) &=& a(t, x(t)) + b(t, x(t))u(t), \\ u(t) &\in& S(K, G(t, x(t)) + F(\cdot), \varphi), \quad \forall t \in [0, T], \\ x(0) &=& x^0, \end{cases} \tag{1.1}$$

where $K \subset R^n$ is a nonempty closed convex subset, $\Omega \equiv [0, T] \times R^m$, $a : \Omega \to R^m, b : \Omega \to R^{m\times n}, G : \Omega \to R^n$ are given functions, $F : R^n \to R^n$ is a linear function and

$\varphi : R^n \to (-\infty, +\infty]$ is a proper l.s.c. convex functional. We are interested in finding the time-dependent trajectories $x(t)$ and $u(t)$ such that (1.1) holds in the weak sense of Carathéodory for $t \in [0, T]$, which means that x is an absolutely continuous function on $[0, T]$ and u is an integrable function on $[0, T]$ such that the differential equation is satisfied for almost all $t \in [0, T]$ and $u(t) \in S(K, G(t, x(t)) + F(\cdot), \varphi)$ for almost all $t \in [0, T]$.

It is well known that various theoretical results, numerical algorithms and applications have been studied extensively for classical differential (inverse) variational inequalities in the literature (see, for example, [4, 5, 7, 8, 18, 20–23, 27, 29–32] and the references therein). Nevertheless, in some real situations, it is necessary to study the generalized differential variational inequality model where the state variable is described by an ordinary differential equation and the control variable is governed by an inverse mixed variational inequality, that is, DIMVI.

1.1 An example of DIMVI

In this subsection, we will give an example of DIMVI in the time-dependent spatial price equilibrium control problem.

Assume that a single commodity is produced at m supply markets, with typical supply market denoted by i and is consumed at n demand markets, with typical demand market denoted by j, during the time interval $[0, T]$ with $T > 0$. Let (i, j) denote the typical pair of producers and consumers for $i = 1, \cdots, m$ and $j = 1, \cdots, n$. Let $S_i(t)$ be the supply of the commodity produced at supply market i at time $t \in [0, T]$ and group the supplies into a column vector $S(t) \in R^m$. Let $D_j(t)$ be the demand of the commodity associated with demand market j at time $t \in [0, T]$ and group the demands into a column vector $D(t) \in R^n$. Let $x_{ij}(t)$ be the commodity shipment from supply market i to demand market j at time $t \in [0, T]$ and group the commodity shipments into a column vector $x(t) \in R^{mn}$. Assume that, for all $t \in [0, T]$, $S_i(t) = \sum_{j=1}^n x_{ij}(t)$ and $D_j(t) = \sum_{i=1}^m x_{ij}(t)$.

Now, we consider the problem from the policy-makers point of view and formulate the time-dependent optimal control equilibrium problem. Under this perspective, by adjusting taxes $u(t)$, it is possible to control the resource exploitations $S(x(t), u(t))$ at supply markets and the consumption $D(x(t), u(t))$ at demands markets. Similar to Scrimali [24], let

$$W(t, x(t), u(t)) = (S(x(t), u(t)), D(x(t), u(t))), \quad \forall t \in [0, T].$$

Assume that the function $W(t, x, u)$ can be written as $W(t, x(t), u(t)) = \widetilde{G}(t, x(t)) + \widetilde{F}(u(t))$ for all $t \in [0, T]$ such that $\widetilde{G}(t, x)$ is a Carathéodory function and $\widetilde{F}(u)$ is Lipschitz continuous. Moreover, we assume that there exists $\gamma(t) \in L^2(0, T)$ such that $\|\widetilde{G}(t, x)\| \leq \gamma(t) + \|x\|$. Then we know that W is a mapping from $[0, T] \times L^2([0, T], R^{mn}) \times L^2([0, T], R^{m+n})$ to $L^2([0, T], R^{m+n})$. Finally, the capacity constrains are assumed to be independent of x and u. Thus, we are led to the following lower and upper capacity constrains $\underline{w}(t) = (\underline{S}(t), \underline{D}(t))$ and $\overline{w}(t) = (\overline{S}(t), \overline{D}(t))$, where $\underline{S}(t), \overline{S}(t) \in L^2([0, T], R^m)$, $\underline{D}(t), \overline{D}(t) \in L^2([0, T], R^n)$, $0 \leq \underline{S}(t) < \overline{S}(t)$ for almost all $t \in [0, T]$ and $0 \leq \underline{D}(t) < \overline{D}(t)$ for almost all $t \in [0, T]$.

Let us introduce the set of feasible states as follows:

$$\mathcal{K} = \left\{ w \in L^2([0, T], R^{m+n}) : \underline{w}(t) \leq w(t) \leq \overline{w}(t) \quad \text{for almost all } \ t \in [0, T] \right\}.$$

Similar to the definition of Scrimali [24], we call that $u^*(t)$ is an optimal regulatory tax if it makes the corresponding state $W(t, x(t), u^*(t))$ satisfying the constraint $W(t, x(t), u^*(t)) \in$

$\mathcal{K}$ and for almost all $t \in [0,T]$, the following three conditions are satisfied:

$$\begin{aligned} W_r(t,x(t),u^*(t)) = \overline{w}_r(t) \quad &\Rightarrow \quad u_r^*(t) \geq 0, \qquad r = 1,2,\cdots,m+n, \\ W_r(t,x(t),u^*(t)) = \underline{w}_r(t) \quad &\Rightarrow \quad u_r^*(t) \leq 0, \qquad r = 1,2,\cdots,m+n, \\ \underline{w}_r(t) < W_r(t,x(t),u^*(t)) < \overline{w}_r(t) \quad &\Rightarrow \quad u_r^*(t) = 0, \qquad r = 1,2,\cdots,m+n. \end{aligned}$$

By employing Theorem 2 of Scrimali [24], a regulatory tax vector $u^*(t) \in L^2([0,T],R^{m+n})$ is optimal if and only if it solves the following variational inequality:

$$W(t,x(t),u^*(t)) \in \mathcal{K}, \quad \int_0^T \langle w(t) - W(t,x(t),u^*(t)), u^*(t)\rangle \, dt \leq 0, \quad \forall w(t) \in \mathcal{K}. \tag{1.2}$$

Suppose that $u(t)$ can be presented as the following form:

$$u(t) = Q(W(t,x(t),u(t))) + A(W(t,x(t),u(t))), \quad \forall t \in [0,T],$$

where $Q, A : L^2([0,T],R^{m+n}) \to L^2([0,T],R^{m+n})$ are two mappings such that Q^{-1} (the inverse of Q) exists. Under the above assumptions, we know that (1.2) can be rewritten as follows:

$$\begin{aligned} W(t,x(t),u^*(t)) \in \mathcal{K}, \quad &\int_0^T \langle w(t) - W(t,x(t),u^*(t)), Q(W(t,x(t),u^*(t)))\rangle \, dt \\ - &\int_0^T \langle w(t) - W(t,x(t),u^*(t)), A(W(t,x(t),u^*(t)))\rangle \, dt \leq 0, \quad \forall w(t) \in \mathcal{K}. \end{aligned} \tag{1.3}$$

Moreover, assume that $\widetilde{Q}(t,x(t),u(t)) = Q(W(t,x(t),u(t)))$ and let

$$\varphi(w(t)) = \Big\langle w(t), A(Q^{-1}(\widetilde{Q}(t,x(t),u(t))))\Big\rangle, \quad \forall w \in L^2([0,T],R^{m+n}).$$

Then it is easy to see that (1.3) can be formulated as follows:

$$\begin{aligned} Q^{-1}(\widetilde{Q}(t,x(t),u^*(t))) \in \mathcal{K}, \quad &\int_0^T \Big\langle w(t) - Q^{-1}(\widetilde{Q}(t,x(t),u^*(t))), \widetilde{Q}(t,x(t),u^*(t))\Big\rangle \, dt \\ - \int_0^T \varphi(w(t))dt + &\int_0^T \varphi(Q^{-1}(\widetilde{Q}(t,x(t),u^*(t))))dt \leq 0, \quad \forall w(t) \in \mathcal{K}. \end{aligned} \tag{1.4}$$

On the other hand, we know that there is a relationship between the change rate of commodity shipments $x(t)$ and regulatory taxes $u(t)$ with the commodity shipments $x(t)$. We require that

$$\dot{x}(t) = f(t,x(t)) + B(t,x(t))u(t), \qquad \text{for almost all } t \in [0,T], \tag{1.5}$$

where $f : [0,T] \times R^{mn} \to R^{mn}$ and $B : [0,T] \times R^{mn} \to R^{mn \times (m+n)}$ are two functions satisfying some suitable conditions.

Combining (1.4) and (1.5), we can show that $(x(t),u(t))$ is a Carathéodory weak solution of the following DIVI problem (see Lemma 4.1):

$$\begin{cases} \dot{x}(t) &= f(t,x(t)) + B(t,x(t))u(t), \\ u(t) &\in \mathrm{S}(-\mathcal{K}, -Q^{-1}(\widetilde{Q}(t,x(t),\cdot)), -\varphi), \\ x(0) &= x_0. \end{cases} \tag{1.6}$$

The differential mixed variational inequality (1.6) is nothing but a form DIMVI (1.1) associated to the mappings f, B, $-Q^{-1}\widetilde{Q}$, to the functional $-\varphi$, and to the constrain set $-\mathcal{K}$.

2 Preliminaries

In this section, we recall some preliminaries that shall be used in what follows.

Definition 2.1. A function $a : \Omega \to R^n$ (resp., $b : \Omega \to R^{n\times m}$) is said to be Lipschitz continuous if there exists a constant $L_a > 0$ (resp., $L_b > 0$) such that, for any (t_1, x), $(t_2, y) \in \Omega$,

$$\| a(t_1, x) - a(t_2, y) \| \leq L_a(|t_1 - t_2| + \| x - y \|)$$
$$(\text{resp.}, \| b(t_1, x) - b(t_2, y) \| \leq L_b(|t_1 - t_2| + \| x - y \|)).$$

Definition 2.2 ([23])**.** A function $f : R^n \to R^n$ is said to be monotone plus on a convex set $K \subset R^n$ if f is monotone on K, i.e.,

$$\langle f(v) - f(u), v - u\rangle \geq 0, \quad \forall v, u \in K,$$

and the following plus property holds: for any $v, u \in K$,

$$\langle f(v) - f(u), v - u\rangle = 0 \Rightarrow f(v) = f(u).$$

Definition 2.3 ([17])**.** Let A be a nonempty subset of R^n. The noncompactness measure μ of set A is defined by

$$\mu(A) = \inf\{\epsilon > 0 : A \subset \bigcup_{i=1}^{n} A_i,\ \text{diam}A_i < \epsilon,\ i = 1, 2, \cdots, n\},$$

where diam means the diameter of a set.

Lemma 2.4 ([14, Theorem 1.2.8]))**.** *Let $\varphi : R^n \to (-\infty, +\infty]$ be a proper l.s.c. convex functional. Then φ is bounded from below by an affine function, i.e., there exists a point $(y^0, \beta) \in R^n \times R$ such that*

$$\varphi(x) \geq \langle x, y^0\rangle + \beta, \quad \forall x \in R^n.$$

Lemma 2.5 ([20])**.** *Let $\varphi : R^n \to (-\infty, +\infty]$ be a proper l.s.c. convex functional. Suppose that the function $\varphi(u(\cdot))$ is integrable on $[0, T]$ for every $u \in L^2[0, T]$. Then*

$$\phi(u) = \int_0^T \varphi(u(t))dt, \quad u \in L^2[0, T]$$

is a proper l.s.c. convex functional.

In the rest of this paper, we assume that the following conditions (A) and (B) hold:

(A) a and b are Lipschitz continuous functions on Ω with Lipschitz constants $L_a > 0$ and $L_b > 0$, respectively;

(B) a and b are bounded on Ω with

$$\sigma_b \equiv \sup_{(t,x)\in\Omega} \|b(t, x)\| < \infty, \quad \sigma_a \equiv \sup_{(t,x)\in\Omega} \|a(t, x)\| < \infty.$$

Let $\mathbb{F} : \Omega \rightrightarrows R^m$ be a set-valued map defined as follows:

$$\mathbb{F}(t, x) \equiv \{a(t, x) + b(t, x)u : u \in S(K, G(t, x) + F, \varphi)\}. \tag{2.1}$$

Lemma 2.6 ([23])**.** *Let $\mathbb{F} : \Omega \rightrightarrows R^m$ be an upper semicontinuous set-valued map with nonempty closed convex values. Suppose that there exists a scalar $\rho_{\mathbb{F}} > 0$ satisfying*

$$\sup\{\| y \|: y \in \mathbb{F}(t, x)\} \leq \rho_{\mathbb{F}}(1+ \| x \|), \quad \forall (t, x) \in \Omega. \tag{2.2}$$

Then, for every $x^0 \in R^n$, the $DI : \dot{x} \in \mathbb{F}(t, x), x(0) = x^0$ has a weak solution in the sense of Carathéodory.

Lemma 2.7 ([23])**.** *Let $h : \Omega \times R^m \to R^m$ be a continuous function and $U : \Omega \rightrightarrows R^n$ a closed set-valued map such that, for some constant $\eta_U > 0$,*

$$\sup_{u \in U(t,x)} \| u \| \leq \eta_U(1+ \| x \|), \quad \forall (t, x) \in \Omega.$$

Let $v : [0, T] \to R^m$ be a measurable function and $x : [0, T] \to R^m$ a continuous function satisfying $v(t) \in h(t, x(t), U(t, x(t)))$ for almost all $t \in [0, T]$. Then there exists a measurable function $u : [0, T] \to R^n$ such that $u(t) \in U(t, x(t))$ and $v(t) = h(t, x(t), u(t))$ for almost all $t \in [0, T]$.

Definition 2.8. A sequence $\{x_n\} \subset R^n$ is called an α-approximating sequence for $IMVI(K, f, \varphi)$ iff there exists $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$f(x_n) \in K, \quad \langle f(x_n) - f', x_n\rangle + \varphi(f(x_n)) - \varphi(f') \leq \frac{\alpha}{2}\|f(x_n) - f'\|^2 + \varepsilon_n, \quad \forall f' \in K, \forall n \in N,$$

where α is a nonnegative number.

Remark 2.9. When φ is the indicator functional on K, α-approximating sequence for $IMVI(K, f, \varphi)$ reduces to α-approximating sequence for $IVI(K, f)$ in [15].

Lemma 2.10. *Let $\alpha \geq 0$ and $f : R^n \to R^n$ be a function and $K \subset R^n$ be a nonempty convex set. Assume that $x^* \in R^n$ satisfying $f(x^*) \in K$. Then*

$$\langle f(x^*) - f', x^*\rangle + \varphi(f(x^*)) - \varphi(f') \leq 0, \quad \forall f' \in K$$

if and only if

$$\langle f(x^*) - f', x^*\rangle + \varphi(f(x^*)) - \varphi(f') \leq \frac{\alpha}{2}\|f(x^*) - f'\|^2, \quad \forall f' \in K.$$

Proof. It is easy to see that the necessity holds. Next we prove the sufficiency. Since K is convex and $f(x^*) \in K$, for any given $g \in K$ and $t \in [0, 1]$, we know that $f(x^*) + t(g - f(x^*)) \in K$. Applying the convexity of φ, it follows that

$$\begin{aligned} & t\langle f(x^*) - g, x^*\rangle + t\varphi(f(x^*)) - t\varphi(g) \\ = \; & \langle tf(x^*) - tg, x^*\rangle + \varphi(f(x^*)) + (t - 1)\varphi(f(x^*)) - t\varphi(g) \\ \leq \; & \langle f(x^*) - [f(x^*) + t(g - f(x^*))], x^*\rangle + \varphi(f(x^*)) - \varphi(tg + (1 - t)f(x^*)) \\ \leq \; & \frac{\alpha}{2}\|f(x^*) - (f(x^*) + t(g - f(x^*))\|^2 \\ = \; & \frac{\alpha t^2}{2}\|f(x^*) - g\|^2. \end{aligned} \tag{2.3}$$

Now (2.3) shows that

$$\langle f(x^*) - g, x^*\rangle + \varphi(f(x^*)) - \varphi(g) \leq \frac{\alpha t}{2}\|f(x^*) - g\|^2.$$

Letting $t \to 0$, we have

$$\langle f(x^*) - g, x^* \rangle + \varphi(f(x^*)) - \varphi(g) \leq 0, \qquad \forall g \in K.$$

This completes the proof. □

Remark 2.11. If φ is the indicator functional on a nonempty convex set $K \subset R^n$, then $IMVI(K, f, \varphi)$ reduces to $IVI(K, f)$. Thus, Lemma 2.10 extends Lemma 1.1 in [15].

3 Existence of Solutions for DIMVI (1.1)

Lemma 3.1. *Let K be a nonempty closed convex set of R^n and (a, b, G) satisfy conditions (A) and (B) above. Let $F : R^n \to R^n$ be continuous and $\varphi : R^n \to (-\infty, +\infty]$ be proper l.s.c. convex. Suppose that there exists a constant $\rho > 0$ such that, for any $q \in G(\Omega)$,*

$$\sup\{\| u \|: u \in S(K, q + F, \varphi)\} \leq \rho(1 + \| q \|). \tag{3.1}$$

Then there exists a constant $\rho^{\mathbb{F}} > 0$ such that (2.2) holds for the map $\mathbb{F} > 0$ defined by (2.1). Hence $\mathbb{F}$ is upper semicontinuous and closed-valued on Ω.

Proof. Since a and G are Lipschitz continuous on Ω, there exist $\rho_a > 0$ and $\rho_G > 0$ such that, for all $(t, x) \in \Omega$,

$$\|a(t, x)\| \leq \rho_a(1 + \|x\|), \quad \|G(t, x)\| \leq \rho_G(1 + \|x\|).$$

It is easy to see that there exists $\rho^{\mathbb{F}} > 0$ such that (2.2) holds. Thus the set-valued map $\mathbb{F}$ has linear growth.

Next we prove $\mathbb{F}$ is upper semicontinuity on Ω. Since $\mathbb{F}$ has linear growth, the upper semicontinuous of $\mathbb{F}$ holds if $\mathbb{F}$ is closed (see [23]). Assume that the sequence $\{(t_n, x_n)\} \subset \Omega$ converges to some vector $(t_0, x_0) \in \Omega$ and $\{a(t_n, x_n) + b(t_n, x_n)u_n\}$ converges to some vector $z_0 \in R^m$ as $n \to \infty$, where $u_n \in S(K, G(t_n, x_n) + F, \varphi)$ for every n, which means that

$$G(t_n, x_n) + F(u_n) \in K \tag{3.2}$$

and

$$\langle F' - G(t_n, x_n) - F(u_n), u_n \rangle + \varphi(F') - \varphi(G(t_n, x_n) + F(u_n)) \geq 0, \quad \forall F' \in K. \tag{3.3}$$

It follows from (3.1) that the sequence $\{u_n\}$ is bounded and so $\{u_n\}$ has a convergent subsequence (denoted it by $\{u_n\}$ again) with a limit $u_0 \in R^n$. Since F is continuous and K is closed, by (3.2), one has

$$G(t_n, x_n) + F(u_n) \to G(t_0, x_0) + F(u_0) \in K.$$

Moreover, the lower semicontinuity of φ implies that

$$\varphi(G(t_0, x_0) + F(u_0)) \leq \liminf_{n \to \infty} \varphi(G(t_n, x_n) + F(u_n)).$$

Thus, for any $F' \in K$,

$$\begin{aligned} & \langle F' - G(t_0, x_0) - F(u_0), u_0 \rangle + \varphi(F') - \varphi(G(t_0, x_0) + F(u_0)) \\ \geq\ & \liminf_{n \to \infty} \{ \langle F' - G(t_n, x_n) - F(u_n), u_n \rangle + \varphi(F') - \varphi(G(t_n, x_n) + F(u_n))\} \\ \geq\ & 0, \end{aligned} \tag{3.4}$$

which means that $u_0 \in S(K, G(t_0, x_0) + F, \varphi)$ and

$$a(t_n, x_n) + b(t_n, x_n)u_n \to z_0 = a(t_0, x_0) + b(t_0, x_0)u_0 \in \mathbb{F}(t_0, x_0).$$

Therefore, $\mathbb{F}$ is closed. This completes the proof. □

Remark 3.2. We note that Lemma 3.1 extends Lemma 2.5 in [18]. Moreover, if $(q+F)^{-1}$ exists and continuous, then Lemma 3.1 is also an extension of Lemma 2.3 in [20].

Lemma 3.3. *Let (a, b, G) satisfy conditions (A) and (B) mentioned above and $K \subset R^n$ be nonempty, closed and convex. Let $\varphi : R^n \to (-\infty, +\infty]$ be a proper l.s.c. convex functional and $F : R^n \to R^n$ be continuous and monotone plus function on R^n. Suppose that $S(K, q + F, \varphi) \neq \emptyset$ for all $q \in G(\Omega)$. Then $S(K, q+F, \varphi)$ is closed and convex for all $q \in G(\Omega)$.*

Proof. Let $\{u_n\} \subset S(K, q + F, \varphi)$ with $u_n \to u_0$. Then $q + F(u_n) \in K$ and for any $F' \in K$,

$$\langle F' - q - F(u_n), u_n\rangle + \varphi(F') - \varphi(q + F(u_n)) \geq 0.$$

Since K is closed, F is continuous on R^n and φ is lower semicontinuous, we know that $q + F(u_0) \in K$ and for any $F' \in K$,

$$\begin{aligned} & \langle F' - q - F(u_0), u_0\rangle + \varphi(F') - \varphi(q + F(u_0)) \\ \geq\ & \liminf_{n\to\infty}\{\ \langle F' - q - F(u_n), u_n\rangle + \varphi(F') - \varphi(q + F(u_n))\} \\ \geq\ & 0. \end{aligned} \tag{3.5}$$

This means that $u_0 \in S(K, q + F, \varphi)$ and so $S(K, q + F, \varphi)$ is closed for all $q \in G(\Omega)$. Next we prove that $S(K, q + F, \varphi)$ is convex for all $q \in G(\Omega)$. Let $u_1, u_2 \in S(K, q + F, \varphi)$. Then one has

$$q + F(u_1) \in K, \quad q + F(u_2) \in K, \tag{3.6}$$

and for any $F' \in K$,

$$\langle F' - q - F(u_1), u_1\rangle + \varphi(F') - \varphi(q + F(u_1)) \geq 0 \tag{3.7}$$

and

$$\langle F' - q - F(u_2), u_2\rangle + \varphi(F') - \varphi(q + F(u_2)) \geq 0. \tag{3.8}$$

It follows from (3.6) that, for $\lambda \in (0, 1)$,

$$\lambda(q + F(u_1)) + (1 - \lambda)(q + F(u_2)) = q + \lambda F(u_1) + (1 - \lambda)F(u_2) = q + F(\tilde{u}) \in K,$$

where $\tilde{u} = \lambda u_1 + (1 - \lambda)u_2$. Letting $F' = q + F(u_2)$ in (3.7), we have

$$\langle F(u_2) - F(u_1), u_1\rangle + \varphi(q + F(u_2)) - \varphi(q + F(u_1)) \geq 0. \tag{3.9}$$

Letting $F' = q + F(u_1)$ in (3.8), one has

$$\langle F(u_1) - F(u_2), u_2\rangle + \varphi(q + F(u_1)) - \varphi(q + F(u_2)) \geq 0. \tag{3.10}$$

Adding (3.9) and (3.10), we obtain

$$\langle F(u_2) - F(u_1), u_1 - u_2\rangle \geq 0.$$

Since F is monotone plus, we know that $F(u_2) = F(u_1)$. It follows from (3.7) and (3.8) that

$$\langle F' - q - F(u_1), \lambda u_1 + (1 - \lambda)u_2\rangle + \varphi(F') - \lambda\varphi(q + F(u_1)) - (1 - \lambda)\varphi(q + F(u_2)) \geq 0.$$

Since φ is convex, we get

$$\begin{aligned}
& \langle F' - q - F(\lambda u_1 + (1-\lambda)u_2), \lambda u_1 + (1-\lambda)u_2\rangle + \varphi(F') - \varphi(q + F(\lambda u_1 + (1-\lambda)u_2)) \\
\geq\ & \langle F' - q - \lambda F(u_1) - (1-\lambda)F(u_2), \lambda u_1 + (1-\lambda)u_2\rangle \\
& + \varphi(F') - \lambda\varphi(q + F(u_1)) - (1-\lambda)\varphi(q + F(u_2)) \\
=\ & \langle F' - q - F(u_1), \lambda u_1 + (1-\lambda)u_2\rangle + \varphi(F') - \lambda\varphi(q + F(u_1)) - (1-\lambda)\varphi(q + F(u_2)) \\
\geq\ & 0.
\end{aligned}$$

It follows that $\lambda u_1 + (1-\lambda)u_2 \in S(K, q + F(\cdot), \varphi)$ and so $S(K, q + F(\cdot), \varphi)$ is convex for all $q \in G(\Omega)$. This completes the proof. □

Remark 3.4. It is worth to mention that Lemma 3.3 extends Lemma 2.6 in [18].

Lemma 3.5. *Let (a, b, G) satisfy conditions (A) and (B) above, $K \subset R^n$ be a nonempty, closed and convex set, $\varphi : R^n \to (-\infty, +\infty]$ be a proper l.s.c. convex functional and $F : R^n \to R^n$ be continuous and monotone plus function on R^n. Suppose that $S(K, q+F, \varphi) \neq \emptyset$ for all $q \in G(\Omega)$ and a constant $\rho > 0$ exists such that (3.1) holds for all $q \in G(\Omega)$. Then DIMVI(1.1) has weak solutions in the sense of Carathéodory.*

Proof. Similar the proof of Proposition 6.1 of [23], applying Lemmas 2.6, 2.7, 3.1 and 3.3, we can obtain that DIMVI (1.1) has weak solutions in the sense of Carathéodory. This completes the proof. □

Remark 3.6. We would like to point out that Lemma 3.5 extends Lemma 2.7 in [18].

By Lemma 3.5, we know that DIMVI (1.1) has solutions in the weak sense of Carathéodory if (3.1) holds. Next, in Theorems 3.1-3.5, we shall give some conditions to guarantee that (3.1) holds.

Let α, F, K be defined as in the previous sections. Consider the α-approximating solution set $T_\alpha(\varepsilon)$ of IMVI$(K, q + F, \varphi)$:

$$\begin{aligned}
T_\alpha(\varepsilon) = \{x \in R^n : q + F(x) \in K, \langle q + F(x) - F', x\rangle + \varphi(q + F(x)) - \varphi(F') \\
\leq \frac{\alpha}{2}\|q + F(x) - F'\|^2 + \varepsilon, \quad \forall F' \in K, \forall \varepsilon \geq 0\}.
\end{aligned}$$

Theorem 3.7. *Let $K \subset R^n$ be a nonempty, closed and convex set, $F : R^n \to R^n$ be a continuous function and $\varphi : R^n \to (-\infty, +\infty]$ be a proper l.s.c. convex functional. If $T_\alpha(\varepsilon) \neq \emptyset$ for any $\varepsilon > 0$ and*

$$diam\, T_\alpha(\varepsilon) \to 0 \quad as \quad \varepsilon \to 0,$$

then IMVI$(K, q + F, \varphi)$ has only one solution and there exists $\rho > 0$ such that (3.1) holds for all $q \in R^n$.

Proof. : Let $\{u_n\} \subset R^n$ be α-approximating sequences for IMVI$(K, q + F, \varphi)$. Then there exists $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that $q + F(u_n) \in K$ and

$$\langle q+F(u_n)-F', u_n\rangle + \varphi(q+F(u_n)) - \varphi(F') \leq \frac{\alpha}{2}\|q+F(u_n)-F'\|^2 + \varepsilon_n, \quad \forall F' \in K, \forall n \in N, \tag{3.11}$$

which implies that $u_n \in T_\alpha(\varepsilon)$. Since $diam\, T_\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$, we deduce that $\{u_n\}$ is a Cauchy sequence. Let $u_n \to \bar{u} \in R^n$. By the continuity of F and the closedness of K, we have

$$q + F(u_n) \to q + F(\bar{u}) \in K$$

Now the lower semicontinuity of φ implies that

$$\liminf_{n\to\infty} \varphi(q+F(u_n)) \geq \varphi(q+F(\bar{u})).$$

It follows from (3.11) that, for all $F' \in K$

$$\begin{aligned} & \langle q+F(\bar{u})-F', \bar{u}\rangle + \varphi(q+F(\bar{u})) - \varphi(F') \\ \leq\ & \liminf_{n\to\infty}\{\langle q+F(u_n)-F', u_n\rangle + \varphi(q+F(u_n)) - \varphi(F')\} \\ \leq\ & \liminf_{n\to\infty}(\frac{\alpha}{2}\|q+F(u_n)-F'\|^2 + \varepsilon_n) \\ =\ & \frac{\alpha}{2}\|q+F(\bar{u})-F'\|^2. \end{aligned}$$

By Lemma 2.10, we get

$$\langle q+F(\bar{u})-F', \bar{u}\rangle + \varphi(q+F(\bar{u})) - \varphi(F') \leq 0, \quad \forall F' \in K.$$

This shows that $\bar{u}$ is a solution of IMVI$(K, q+F, \varphi)$.

Next we prove the uniqueness of the solution of IMVI$(K, q+F, \varphi)$. Let $\bar{u}_1$ and $\bar{u}_2$ be two solutions of IMVI$(K, q+F, \varphi)$. Then $\bar{u}_1, \bar{u}_2 \in T_\alpha(\varepsilon)$ for all $\varepsilon > 0$. It follows that

$$\|\bar{u}_1 - \bar{u}_2\| \leq diam T_\alpha(\varepsilon) \to 0 \ (\varepsilon \to 0),$$

which implies that $\bar{u}_1 = \bar{u}_2$ and so there exists $\rho > 0$ such that (3.1) holds for all $q \in R^n$. This completes the proof. □

Theorem 3.8. *Let $K \subset R^n$ be a nonempty, closed and convex set, $F : R^n \to R^n$ be a continuous and monotone plus function, $\varphi : R^n \to (-\infty, +\infty)$ be a proper l.s.c. convex functional, $\mu(T_\alpha(\varepsilon))$ be noncompactness measure of set $T_\alpha(\varepsilon)$. Suppose the following conditions hold:*

(i) *$T_\alpha(\varepsilon) \neq \emptyset$ for all $\varepsilon > 0$ and $\mu(T_\alpha(\varepsilon)) \to 0$ as $\varepsilon \to 0$;*

(ii) *there exists a vector $F_0 \in K$ such that*

$$\limsup_{\|u\|\to\infty} \frac{\langle F^0 - F(u), u\rangle - \langle F(u), y^0\rangle}{\|u\|^2} < 0, \tag{3.12}$$

where y^0 is defined in Lemma 2.4.

Then $S(K, q+F, \varphi)$ is a nonempty closed convex set for all $q \in R^n$, and there exists $\rho > 0$ such that (3.1) holds for all $q \in R^n$.

Proof. Similar to the proof of Theorem 2.2 in [15], we can easily obtain that $S(K, q+F, \varphi)$ is nonempty for all $q \in R^n$ by virtue of condition (i). It follows from Lemma 3.3 that $S(K, q+F, \varphi)$ is nonempty closed and convex for all $q \in R^n$. Next we prove the second assertion. If the assertion is not true, then there exist sequences $\{q_k\} \subset R^n$ and $\{u_k\} \subset R^n$ such that, for any $F' \in K$, $q_k + F(u_k) \in K$ and

$$\langle F' - q_k - F(u_k), u_k\rangle + \varphi(F') - \varphi(q+F(u_k)) \geq 0 \tag{3.13}$$

with $\|u_k\| > k(1+\|q_k\|)$. It is obvious that $\{u_k\}$ is unbounded and $\lim_{k\to\infty} \frac{\|q_k\|}{\|u_k\|} = 0$. Taking $F' = F^0$ in (3.13), we have

$$\langle F^0 - q_k - F(u_k), u_k\rangle + \varphi(F^0) - \varphi(q_k + F(u_k)) \geq 0$$

and so

$$\langle F^0 - F(u_k), u_k\rangle - \varphi(q_k + F(u_k)) \geq \langle q_k, u_k\rangle - \varphi(F^0).$$

Since φ is a proper l.s.c. convex functional, it follows from Lemma 2.4 that

$$\langle F^0 - F(u_k), u_k\rangle - \langle q_k + F(u_k), y^0\rangle - \beta \geq \langle q_k, u_k\rangle - \varphi(F^0)$$

This shows that

$$\langle F^0 - F(u_k), u_k\rangle - \langle F(u_k), y^0\rangle \geq \langle q_k, u_k\rangle - \varphi(F^0) + \beta + \langle q_k, y^0\rangle$$

and so

$$\limsup_{k\to+\infty} \frac{\langle F^0 - F(u_k), u_k\rangle - \langle F(u_k), y^0\rangle}{\|u_k\|^2} \geq 0,$$

which contradicts with (3.12). This completes the proof. □

Theorem 3.9. *Let $K \subset R^n$ be a nonempty, closed and convex set, $F : R^n \to R^n$ be a continuous and monotone plus function, $\varphi : R^n \to (-\infty, +\infty)$ be a proper l.s.c. convex functional, $\mu(T_\alpha(\varepsilon))$ be noncompactness measure of set $T_\alpha(\varepsilon)$. Suppose the following conditions hold:*

(i) $T_\alpha(\varepsilon) \neq \emptyset$ for all $\varepsilon > 0$ and $\mu(T_\alpha(\varepsilon)) \to 0$ as $\varepsilon \to 0$;

(ii) there exists a vector $F_0 \in K$ such that

$$\frac{\langle F^0 - F(u), u\rangle - \langle F(u), y^0\rangle}{\|u\|} \to -\infty \quad as \quad \|u\| \to +\infty, \tag{3.14}$$

where y^0 is defined as in Lemma 2.4.

Then $S(K, q + F, \varphi)$ is a nonempty closed convex set for all $q \in R^n$, and there exists $\rho > 0$ such that (3.1) holds for all $q \in M$, where $M \subset R^n$ is a bounded subset.

Proof. By Theorem 3.8, we know that $S(K, q + F, \varphi)$ is nonempty closed and convex for all $q \in R^n$. Next we prove the second assertion. Suppose to the contrary that there exist $\{q_k\} \subset M$ and $\{u_k\} \subset R^n$ such that, for any $F' \in K$,

$$\langle F' - q_k - F(u_k), u_k\rangle + \varphi(F') - \varphi(q_k + F(u_k)) \geq 0$$

and $\|u_k\| > k(1 + \|q_k\|)$. It is easy to see that $\{u_k\}$ is unbounded and $\lim_{k\to\infty} \frac{\|q_k\|}{\|u_k\|} = 0$. Similar to the proof of Theorem 3.8, we know that

$$\langle F^0 - F(u_k), u_k\rangle - \langle F(u_k), y^0\rangle \geq \langle q_k, u_k\rangle - \varphi(F^0) + \beta + \langle q_k, y^0\rangle.$$

Since $\{q_k\}$ and $\varphi(F^0)$ are bounded, there exists $N > 0$ and $D < 0$ such that, for all $k > N$,

$$\frac{\langle F^0 - F(u_k), u_k\rangle - \langle F(u_k), y^0\rangle}{\|u_k\|} \geq D,$$

which contradicts with (3.14). This completes the proof. □

Theorem 3.10. *Let $K = R^n$, $F : R^n \to R^n$ be a continuous and monotone plus function and $\varphi : R^n \to (-\infty, +\infty)$ be a proper l.s.c. convex functional. Suppose that $(q + F)^{-1}$ is single-valued continuous on R^n, and a vector $F_0 \in R^n$ exists such that* (3.12) *holds. Then $S(K, q + F, \varphi)$ is a nonempty closed convex set for all $q \in R^n$, and there exists a $\rho > 0$ such that (3.1) holds for all $q \in R^n$.*

Proof. For any $u \in R^n$, let $S(u) = (q + F)^{-1}(u)$. Since F is monotone, we have

$$\langle S(u_1) - S(u_2), u_1 - u_2\rangle = \langle y_1 - y_2, q + F(y_1) - q - F(y_2)\rangle = \langle y_1 - y_2, F(y_1) - F(y_2)\rangle \geq 0,$$

where $S(u_1) = y_1$ and $S(u_2) = y_2$. This shows that S is monotone on R^n. By Theorem 3.1 of [6], we know that $SOL(R^n, S, \varphi)$ is nonempty, here $SOL(R^n, S, \varphi)$ denotes the solution set of VI(R^n, S, φ). Thus, there exists $v_0 \in R^n$ such that

$$\langle S(v_0), v - v_0\rangle + \varphi(v) - \varphi(v_0) \geq 0, \quad \forall v \in R^n.$$

Letting $S(v_0) = u_0$, one has

$$\langle u_0, v - q - F(u_0)\rangle + \varphi(v) - \varphi(q + F(u_0)) \geq 0, \quad \forall v \in R^n,$$

which means that $u_0 \in S(R^n, q + F, \varphi)$. By Lemma 3.3 we know that $S(K, q + F, \varphi)$ is nonempty closed and convex for all $q \in R^n$. It follows from Theorem 3.8 that there exists $\rho > 0$ such that (3.1) holds for all $q \in R^n$. This completes the proof. □

Corollary 3.11. *Under the conditions of Theorem 3.10 we can obtain Theorem 3.3 in [20]. Moreover, if for any $v \in R^n$, $\varphi(v) = 0$, then we can obtain Proposition 6.2 in [23].*

Proof. For any $u \in R^n$, let $S(u) = (q + F)^{-1}(u)$. From Theorem 3.10 we know that S is monotone continuous. Letting $v = q + F(u)$, it follows from Lemma 2.4 that

$$\begin{aligned} \frac{\langle S(v), v - F_0\rangle + \varphi(v)}{\|v\|^2} &= \frac{\langle u, q + F(u) - F_0\rangle + \varphi(q + F(u))}{\|v\|^2} \\ &\geq \frac{\langle u, F(u) - F_0\rangle + \langle u, q\rangle + \langle q + F(u), y^0\rangle + \beta}{\|q + F\|^2\|u\|^2}. \end{aligned}$$

By (3.12), we have

$$\begin{aligned} &\liminf_{\|v\|\to+\infty} \frac{\langle S(v), v - F_0\rangle + \varphi(v)}{\|v\|^2} \\ \geq\ &\liminf_{\|u\|\to+\infty} \frac{\langle u, F(u) - F_0\rangle + \langle u, q\rangle + \langle q + F(u), y^0\rangle + \beta}{\|q + F\|^2\|u\|^2} \\ \geq\ &-\limsup_{\|u\|\to+\infty} \frac{\langle u, F_0 - F(u)\rangle - \langle u, q\rangle - \langle F(u), y^0\rangle - \langle q, y^0\rangle - \beta}{\|q + F\|^2\|u\|^2} \\ \geq\ &-\limsup_{\|u\|\to+\infty} \frac{\langle u, F_0 - F(u)\rangle - \langle F(u), y^0\rangle}{\|q + F\|^2\|u\|^2} > 0. \end{aligned}$$

Thus, we know that Theorem 3.3 in [20] holds. Moreover, when $\varphi(v) = 0$, it is easy to see that Proposition 6.2 in [23] holds. This completes the proof. □

Theorem 3.12. *Let $K = R^n$ and $F : R^n \to R^n$ be a continuous and monotone plus function. Let $\varphi : R^n \to (-\infty, +\infty)$ be a proper l.s.c. convex functional. Suppose that $(q+F)^{-1}$ is single-valued continuous on R^n, and there exists a vector $F_0 \in R^n$ such that (3.14) holds. Then $S(K, q+F, \varphi)$ is a nonempty closed convex set for all $q \in R^n$, and there exists $\rho > 0$ such that (3.1) holds for all $q \in M$, where $M \subset R^n$ is a bounded set.*

Proof. By Theorem 3.10, we know that $S(K, q+F, \varphi)$ is a nonempty closed convex set for all $q \in R^n$. It follows from Theorem 3.9 that there exists $\rho > 0$ such that (3.1) holds for all $q \in M$. This completes the proof. □

Corollary 3.13. *Under the conditions in Theorem 3.12, we can obtain Theorem 3.2 in [20].*

Proof. By Theorem 3.10, we know that $S(K, q+F, \varphi)$ is a nonempty closed convex set for all $q \in R^n$. For any $q \in M$,

$$\frac{\langle S(v), v - F_0\rangle + \varphi(v)}{\|v\|} \geq \frac{\langle u, F(u) - F_0\rangle + \langle u, q\rangle + \langle q + F(u), y^0\rangle + \beta}{\|q+F\|\|u\|}.$$

It follows from (3.14) that

$$\frac{\langle S(v), v - F_0\rangle + \varphi(v)}{\|v\|} \to +\infty, \quad as \quad \|v\| \to \infty.$$

Thus, it is clear that Theorem 3.2 in [20] holds. This completes the proof. □

Theorem 3.14. *Let $F : R^n \rightrightarrows R^n$ be a continuous and monotone plus function and (a, b, G) satisfy conditions (A) and (B). Let $K \subset R^n$ be a nonempty, closed and convex subset, and $\varphi : R^n \to (-\infty, +\infty)$ be a proper l.s.c. convex functional. Then DIMVI(1.1) has weak solutions in the sense of Carathéodory if any one of the following conditions holds:*

(a) $T_\alpha(\varepsilon) \neq \emptyset$ *for any* $\varepsilon > 0$ *and* $\mathrm{diam} T_\alpha(\varepsilon) \to 0$ *as* $\varepsilon \to 0$;

(b) (i) *for any* $\varepsilon > 0$, $T_\alpha(\varepsilon) \neq \emptyset$ *and* $\mu(T_\alpha(\varepsilon)) \to 0$ *as* $\varepsilon \to 0$;
   (ii) *there exists a vector* $F_0 \in K$ *such that (3.12) holds;*

(c) $G(\Omega)$ *is bounded and*

   (i) *for any* $\varepsilon > 0$, $T_\alpha(\varepsilon) \neq \emptyset$ *and* $\mu(T_\alpha(\varepsilon)) \to 0$ *as* $\varepsilon \to 0$;
   (ii) *there exists a vector* $F_0 \in K$ *such that* (3.14) *holds;*

(d) $K = R^n$, $(q+F)^{-1}$ *is single-valued continuous on* R^n *and there exists a vector* $F_0 \in R^n$ *such that* (3.12) *holds;*

(e) $G(\Omega)$ *is bounded,* $K = R^n$, $(q+F)^{-1}$ *is single-valued continuous on* R^n *and there exists a vector* $F_0 \in R^n$ *such that* (3.14) *holds.*

Proof. It follows from Theorems 3.7-3.12 that $S(K, q+F, \varphi)$ is a nonempty closed convex set satisfying (3.1). By Lemma 3.5, we know that DIMVI (1.1) has weak solutions in the sense of Carathéodory. This completes the proof. □

Remark 3.15. It is worth mentioning that Theorem 3.14 extends some corresponding results of Theorem 3.8 in [18].

4 Computational Methods for DIMVI (1.1)

Lemma 4.1. *Let $G : \Omega \times R^n \to R^n$ be Lipschitz continuous and $F : L^2[0,T] \to L^2[0,T]$ be continuous. Assume that $G(t,x(t)) + F(u(t)) \in K$ for all $x \in C([0,T];R^n)$ and $u \in L^2[0,T]$ with $t \in [0,T]$. Suppose that $\varphi(v(t))$ is integrable on $[0,T]$ for all $v \in L^2[0,T]$. Then $((x(t),u(t))$ is a weak solutions in the sense of Carathéodory for DIMVI (1.1) if and only if x is an absolutely continuous function on $[0,T]$ and u is an integrable function on $[0,T]$ such that the differential equation in (1.1) holds for almost all $t \in [0,T]$ and for any continuous function $P : [0,T] \to K$,*

$$\int_0^T \{\langle P(t) - G(t,x(t)) - F(u(t)), u(t)\rangle + \varphi(P(t)) - \varphi[G(t,x(t)) + F(u(t))]\}\, dt \geq 0. \quad (4.1)$$

Proof. $\Rightarrow$. Suppose that $((x(t),u(t))$ is a weak solutions in the sense of Carathéodory for DIMVI (1.1), then x is an absolutely continuous function on $[0,T]$ and u is an integrable function on $[0,T]$ such that the differential equation in (1.1) holds for almost all $t \in [0,T]$ and $u(t) \in S(K, G(t,x(t)) + F, \varphi)$ for almost all $t \in [0,T]$. Thus, it is easy to see that, for any continuous function $P : [0,T] \to K$, inequality (4.1) holds.

$\Leftarrow$. We assume that the contrary holds. Then there exists a subset $E \subset [0,T]$ with $\hat{m}(E) > 0$ (where $\hat{m}(E)$ denotes the Lebesgue measure of E) such that, for any $t \in E$,

$$u(t) \notin S(K, G(t,x(t)) + F, \varphi).$$

By Lusin theorem, we know that there exists a closed subset E_1 of E with $m(E_1) > 0$ such that $u(t)$ is continuous on E_1. Thus, there exists a closed subset E_2 of E_1 with $m(E_2) > 0$ such that $\varphi[G(t,x(t)) + F(u(t)]$ is continuous on E_2 and so there exists a $P_0 \in K$ such that, for almost all $t \in E_2$,

$$\int_{E_2} \{\langle P_0 - G(t,x(t)) - F(u(t), u(t)\rangle + \varphi(P_0) - \varphi[G(t,x(t)) + F(u(t))]\}\, dt < 0.$$

Let

$$F_0(t) = \begin{cases} P_0, & t \in E_2, \\ G(t,x(t)) + F(u(t)), & t \in [0,T]\backslash E_2. \end{cases} \quad (4.2)$$

Then it is clear that $F_0(t)$ is an integrable function on $[0,T]$. Since the space of continuous functions $C([0,T];R^n)$ is dense in $L^1([0,T];R^n)$, we can approximate $F_0(t) \in L^1([0,T];R^n)$ by continuous function. Thus, there exists a continuous function $\bar{F}(t) : [0,T] \to K$ such that

$$\int_0^T \left\{\langle \bar{F}(t) - G(t,x(t)) - F(u(t)), u(t)\rangle + \varphi(P_0) - \varphi[G(t,x(t)) + F(u(t))]\right\} dt < 0,$$

which contradicts with (4.1). This completes the proof. □

Now we discuss the convergence for a weak solution of the initial-value DIMVI(1.1). Let us choose an equidistant grid $0 = t_0 < t_1 < \cdots t_N = T$, with stepsize $h = \frac{T}{N}$. Setting $x^{h,0} = x^0$, we compute iterates

$$\{x^{h,1}, x^{h,2}, \cdots, x^{h,N}\} \subset R^m, \quad \{u^{h,1}, u^{h,2}, \cdots, u^{h,N}\} \subset R^n \quad (4.3)$$

as follows: for $i = 0, 1, \cdots, N_h$,

$$\begin{cases} x^{h,i+1} &= x^{h,i} + h\left[a(t_{h,i+1}, \theta x^{h,i} + (1-\theta)x^{h,i+1}) + b(t_{h,i}, x^{h,i})u^{h,i+1}\right], \\ u^{h,i+1} &\in S(K, G(t_{h,i+1}, x^{h,i+1}) + F, \varphi), \end{cases} \quad (4.4)$$

where $N_h = \frac{T}{h} - 1$. Let $\hat{x}^h(\cdot)$ be the continuous piecewise linear interpolant of the family $\{x^{h,i+1}\}$ and $\hat{u}^h(\cdot)$ be the constant piecewise interpolant of the family $\{u^{h,i+1}\}$, i.e.,

$$\begin{cases} \hat{x}^h(t) &= x^{h,i} + \frac{t-t_i}{h}(x^{h,i+1} - x^{h,i}), \quad \forall t \in [t_{h,i}, t_{h,i+1}], \\ \hat{u}^h(t) &= u^{h,i+1}, \quad \forall t \in (t_i, t_{i+1}] \end{cases} \tag{4.5}$$

for $i = 0, 1, \cdots, N_h$. We denote, by $L^2[0,T]$, the set of all measurable functions $u : [0,T] \to R^n$ satisfying $\int_0^T \|u(t)\|^2 dt < \infty$, in which the inner product is defined as

$$\langle u, v \rangle = \int_0^T \langle u(t), v(t) \rangle dt, \quad \forall u, v \in L^2[0,T].$$

Lemma 4.2. *Let a, b, G satisfy (A)and (B). Suppose that $S(K, q + F, \varphi)$ satisfies (3.1). Then there exist $h_1 > 0, \rho_u$ and ψ_x such that for all $h \in (0, h_1]$ and all nonnegative integers i with $(i+1)h \leq T$,*

$$\begin{cases} \|u^{h,i+1}\| &\leq \rho_u(1 + 2\|x^{h,i}\|), \\ \|x^{h,i+1} - x^{h,i}\| &\leq h\psi_x(1 + \|x^{h,i}\|). \end{cases} \tag{4.6}$$

Proof. Throughout the proof below, the scalar $h > 0$ is sufficiently small. Applying Lemma 7.1 of [23], we have

$$\|x^{h,i+1} - x^{h,i}\| \leq h \frac{\rho_a(1 + \|x^{h,i}\|) + \sigma_b\|u^{h,i+1}\|}{1 - h(1-\theta)\rho_a}.$$

Let

$$\rho_x = \frac{\rho_a + \sigma_b}{1 - h(1-\theta)\rho_a}.$$

Then

$$\|x^{h,i+1} - x^{h,i}\| \leq \rho_x h(1 + \|x^{h,i}\| + \|u^{h,i+1}\|). \tag{4.7}$$

By the linear growth of solutions to IMVI, one has

$$\begin{aligned} \|u^{h,i+1}\| &\leq \rho(1 + \|G(t_{h,i+1}, x^{h,i+1})\| \\ &\leq \rho(1 + \rho_G\|1 + \|x^{h,i+1}\|) \\ &\leq \rho(1 + \rho_G(1 + \|x^{h,i}\| + \rho_x h(1 + \|x^{h,i}\| + \|u^{h,i+1}\|) \\ &\leq \rho + \rho\rho_G + \rho_x h\rho\rho_G + (\rho\rho_G + \rho_x h\rho\rho_G)\|x^{h,i}\| + \rho_x h\rho\rho_G\|u^{h,i+1}\|. \end{aligned}$$

Letting $M = \rho + \rho\rho_G + \rho_x h\rho\rho_G$ and $N = \rho_x\rho\rho_G$, we have

$$\|u^{h,i+1}\| \leq M + M\|x^{h,i}\| + Nh\|u^{h,i+1}\|$$

which implies that

$$(1 - hN)\|u^{h,i+1}\| \leq M(1 + \|x^{h,i}\|).$$

Choosing $0 < h < \frac{1}{N}$, we get

$$\|u^{h,i+1}\| \leq \frac{M}{1 - hN}(1 + \|x^{h,i}\|)$$

and so there exists a $\rho_u > 0$ such that

$$\|u^{h,i+1}\| \leq \rho_u(1 + 2\|x^{h,i}\|). \tag{4.8}$$

Taking $h_1 = \frac{1}{2N}$ and $\psi_x = \rho_x + 2\rho_x\rho_u$, it follows from (4.7) and (4.8) that, for $h \in (0, h_1]$,

$$\begin{aligned} \|x^{h,i+1} - x^{h,i}\| &\leq h\rho_x(1 + \|x^{h,i}\| + \rho_u(1 + 2\|x^{h,i}\|)) \\ &\leq h(\rho_x + \rho_x\rho_u + (\rho_x + 2\rho_x\rho_u)\|x^{h,i}\|)) \\ &\leq h(\rho_x + 2\rho_x\rho_u)(1 + \|x^{h,i}\|) \\ &\leq h\psi_x(1 + \|x^{h,i}\|). \end{aligned}$$

This completes the proof. □

Similar to the proof of Proposition 7.1 of [23], we can obtain the following result.

Proposition 4.3. *Let (a, b, G) satisfy conditions (A) and (B) above and, $K \subset R^n$ be a nonempty, closed and convex set and $F : R^n \rightrightarrows R^n$ be a continuous and monotone plus function and $\varphi : R^n \to (-\infty, +\infty)$ be a proper l.s.c. convex functional. Suppose that there exists a constant $\rho > 0$ such that (3.1) holds for all $q \in G(\Omega)$. Then there exists a scalar $h_R > 0$ such that, for any $h \in (0, h_R]$, $\theta \in [0, 1]$ and $x^0 \in R^n$, there exists a pair $(x^{h,i+1}, u^{h,i+1})$ satisfying (4.4) for every $i = 0, 1, \cdots, N_h$.*

Theorem 4.4. *Let (a, b, G) satisfy conditions (A) and (B) above, $K \subset R^n$ be a nonempty closed and convex set. Let the mapping $F : R^n \to R^n$ be continuous and monotone plus, $\varphi : R^n \to (-\infty, +\infty)$ be a proper l.s.c. convex and integrable functional on R^n. Suppose that $S(K, G(t, x) + F(\cdot), \varphi) \neq \emptyset$, there exists a constant $M > 0$ such that for any $y \in R^n$, $\varphi(y) \leq M$, moreover there exists a constant $\rho > 0$ such that (3.1) holds. Then every sequence pair $\{(\hat{x}^h, \hat{u}^h)\}$ defined by (4.5) has a subsequence pair $\{(\hat{x}^{h_v}, \hat{u}^{h_v})\}$ such that $\hat{x}^{h_v} \to \tilde{x}$ uniformly in [0,T] and $\hat{u}^{h_v} \to \tilde{u}$ weakly in $L^2([0, T])$, as $v \to \infty$. Furthermore, suppose that $F(u) = \psi(Eu)$, where $E \in R^{n\times n}$ and $\psi : R^n \to R^n$ is Lipschitz continuous, and a constant $C > 0$ exists such that for all h sufficiently small,*

$$\|Eu^{h,i+1} - Eu^{h,i}\| \leq hC, \tag{4.9}$$

Then, $(\tilde{x}, \tilde{u})$ is a Carathéodory weak solution of the initial-value DIMVI(1.1).

Proof. By Lemma 4.2, we have

$$\|u^{h,i+1}\| \leq \rho_u(1 + 2\|x^{h,i}\|)$$

and

$$\|x^{h,i+1} - x^{h,i}\| \leq h\psi_x(1 + \|x^{h,i}\|).$$

It follows from Lemma 7.2 of [23] that there exist constants $c_{0,x}, c_{1,x}, c_{1,u}$ such that, for any $h \in (0, h_1]$ and any $i = 0, 1, \cdots, N_h$,

$$\begin{cases} \|x^{h,i+1}\| \leq c_{0,x} + c_{1,x}\|x^0\|, \\ \|u^{h,i+1}\| \leq c_{0,u} + c_{1,u}\|x^0\|. \end{cases} \tag{4.10}$$

By (4.10) and (4.7), we deduce that, for $h > 0$ sufficiently small, there exists $L_{x_0} > 0$, which is independent of h, such that

$$\|x^{h,i+1} - x^{h,i}\| \leq L_{x_0}h, \quad i = 0, 1, \ldots, N_h. \tag{4.11}$$

It follows from (4.5) that $\hat{x}^h$ is Lipschitz continuous on $[0, T]$, and the Lipschitz constant is independent of h. Thus, there exists an $h_0 > 0$ such that the family of functions $\{\hat{x}^h\}(h \in (0, h_0])$ is an equicontinuous family of functions. Letting

$$\|\hat{x}^h\|_{L^\infty} = \sup_{t\in[0,T]} \|\hat{x}^h(t)\|,$$

it follows from (4.5) and (4.10) that $\{\hat{x}^h\}$ is uniformly bounded. By using the Arzelá-Ascoli theorem, there exists a sequence $\{h_v\} \downarrow 0$ such that $\{\hat{x}^{h_v}\}$ converges uniformly to $\tilde{x}$ on $[0, T]$. Thus, from (3.1) and (4.10), we know that the iterates $\{u^{h,i+1}\}$ is uniformly bounded in the L^∞ norm on $[0, T]$ and so $\{\hat{u}^h\}$ is uniformly bounded in the L^∞ norm on $[0, T]$. Since $L^2[0, T]$ is a reflective Banach space, it is easy to know that there exists a sequence $\{h_v\} \downarrow 0$ such that $\hat{u}^{h_v} \to \tilde{u}$ weakly in $L^2[0, T]$.

Now we show that $(\tilde{x}, \tilde{u})$ is a Carathéodory weak solution of DIMVI(1.1).

(I) We first prove that $\tilde{x}(0) = x^0$. In fact, since $\hat{x}^h(0) = x^0$ for all $h > 0$ sufficiently small and $\hat{x}^{h_v} \to \tilde{x}$ uniformly as $v \to \infty$, we know that $\tilde{x}(0) = x^0$.

(II) We next show that, for almost all $t \in [0, T]$,

$$\tilde{u}(t) \in S(K, G(t, \tilde{x}(t)) + F(\cdot), \varphi).$$

In fact, by Theorem 7.1 in [23], we know that $\{G(t, \hat{x}^{h_v}) + F(\hat{u}^{h_v})\}$ converges to $G(t, \tilde{x}) + F(\tilde{u})$. It follows from Lemma 2.5 and Corollary 3.35 [25] that ϕ is an l.s.c. functional in the weak topology, where ϕ is defined in Lemma 2.5. Consequently, for any continuous function $\bar{F} : [0, T] \to K$, one has

$$\begin{aligned}\limsup_{v\to\infty} &\int_0^T \left\{\varphi(\bar{F}(t)) - \varphi[G(t, \hat{x}^{h_v}(t)) + \psi(E\hat{u}^{h_v}(t))]\right\} dt \\ &\le \int_0^T \left\{\varphi(\bar{F}(t)) - \varphi[G(t, \tilde{x}(t)) + \psi(E\tilde{u}(t))]\right\} dt\end{aligned}$$

and so

$$\begin{aligned}\limsup_{v\to\infty} &\int_0^T \{\langle \bar{F}(t) - G(t, \hat{x}^{h_v}(t)) - \psi(E\tilde{u}^{h_v}), \hat{u}^{h_v}(t)\rangle \\ &\qquad + \varphi(\bar{F}(t)) - \varphi[G(t, \tilde{x}^{h_v}(t)) + \psi(E\tilde{u}^{h_v})]\}dt \\ &\le \int_0^T \{\langle \bar{F}(t) - G(t, \tilde{x}(t)) - \psi(E\tilde{u}), \tilde{u}(t)\rangle \\ &\qquad + \varphi(\bar{F}(t)) - \varphi[G(t, \tilde{x}(t)) + \psi(E\tilde{u})]\}dt.\end{aligned}$$

On the other hand, since

$$\hat{x}^h(t) = x^{h,i} + \frac{t - t_i}{h}(x^{h,i+1} - x^{h,i}), \quad \forall t \in [t_{h,i}, t_{h,i+1}],$$

we have

$$\|\hat{x}^h - x^{h,i+1}\| = \left\|\frac{t - t_i - h}{h}(x^{h,i+1} - x^{h,i})\right\|.$$

Applying (4.11), we get

$$\|\tilde{x}^h - x^{h,i+1}\| \le L_{x^0} h$$

and so

$$\begin{aligned}&\Bigg\|\sum_{i=0}^{N_h} \int_{t_{h,i}}^{t_{h,i+1}} \left\{\langle G(t_{h,i+1}, x^{h,i+1}) - G(t, \hat{x}^h), u^{h,i+1}\rangle\right. \\ &\quad \left. +\ \varphi(G(t_{h,i+1}, x^{h,i+1}) + \psi(Eu^{h,i+1})) - \varphi[G(t, \hat{x}^h) + \psi(Eu^{h,i+1})]\right\} dt\Bigg\| \\ &\le \ Nh\{(L_G h + L_G L_{x^0} h)\|u^{h,i+1}\|_{L^\infty} + 2M\}. \qquad (4.12)\end{aligned}$$

It follows from (4.12) that

$$\begin{aligned}
&\int_0^T \{\langle \bar{F} - G(t,\hat{x}^h) - \psi(E\hat{u}^h), \hat{u}^h\rangle + \varphi(\bar{F}) - \varphi(G(t,\hat{x}^h) + \psi(E\hat{u}^h))\}dt \\
= \;& \sum_{i=0}^{N_h} \int_{t_{h,i}}^{t_{h,i+1}} \{\langle \bar{F}(t) - G(t,\hat{x}^h(t)) - \psi(Eu^{h,i+1}), u^{h,i+1}\rangle \\
& + \varphi(\bar{F}(t)) - \varphi[G(t,\hat{x}^h(t)) + \psi(Eu^{h,i+1})]\}dt \\
= \;& \sum_{i=0}^{N_h} \int_{t_{h,i}}^{t_{h,i+1}} \{\langle \bar{F}(t) - G(t_{h,i+1},x^{h,i+1}) - \psi(Eu^{h,i+1}), u^{h,i+1}\rangle \\
& + \varphi(\bar{F}(t)) - \varphi[G(t_{h,i+1},x^{h,i+1}) + \psi(Eu^{h,i+1})]\}dt \\
& + \sum_{i=0}^{N_h} \int_{t_{h,i}}^{t_{h,i+1}} \{\langle G(t_{h,i+1},x^{h,i+1}) - G(t,\hat{x}^h), u^{h,i+1}\rangle \\
& + \varphi[G(t_{h,i+1},x^{h,i+1}) + \psi(Eu^{h,i+1})] - \varphi[G(t,\hat{x}^h) + \psi(Eu^{h,i+1})]\}dt \\
\geq \;& h\sum_{i=0}^{N_h} \frac{1}{h} \int_{t_{h,i}}^{t_{h,i+1}} \langle \bar{F}(t) - G(t_{h,i+1},x^{h,i+1}) - \psi(Eu^{h,i+1}), u^{h,i+1}\rangle \\
& + \varphi(\bar{F}(t)) - \varphi[G(t_{h,i+1},x^{h,i+1}) + \psi(Eu^{h,i+1})]dt \\
& - Nh[L_G h(1+L_{x^0})\|\hat{u}^h\|_{L^\infty} + 2M]. \qquad (4.13)
\end{aligned}$$

Now the convexity of K shows that

$$\frac{1}{h}\int_{t_{h,i}}^{t_{h,i+1}} \bar{F}(t)dt \in K.$$

Since $u^{h,i+1} \in S(K, G(t_{h,i+1}, x^{h,i+1}) + F, \varphi)$, one has

$$\limsup_{h\to 0} \int_0^T \{\langle \bar{F}(t) - G(t,\hat{x}^h(t)) - \psi(E\hat{u}^h(t)), \hat{u}^h(t)\rangle + \varphi(\bar{F}(t)) - \varphi[G(t,\hat{x}^h(t)) + \psi(E\hat{u}^h(t))]\}dt \geq 0.$$

It follows from (4.13) that, for all continuous functions $\bar{F} : [0,T] \to K$,

$$\int_0^T \{\langle \bar{F}(t) - G(t,\tilde{x}(t)) - \psi(E\tilde{u}(t)), \tilde{u}(t)\rangle + \varphi(\bar{F}(t)) - \varphi[G(t,\tilde{x}(t)) + \psi(E\tilde{u}(t))]\}dt \geq 0.$$

From Lemma 4.1, it is easy to see that, for almost all $t \in [0,T]$,

$$\tilde{u}(t) \in S(K, G(t,x(t)) + F(\cdot), \varphi).$$

(III) Similar to the proof of Theorem 7.1 in [23], we can show that, for any $0 \leq s \leq t \leq T$,

$$\tilde{x}(t) - \tilde{x}(s) = \int_s^t [a(\tau,\tilde{x}(\tau)) + b(\tau,\tilde{x}(\tau))\tilde{u}(\tau)]d\tau.$$

From (I)-(III) discussed above, we know that $(\tilde{x}, \tilde{u})$ is a Carathéodory weak solution of DIMVI(1.1). This completes the proof. □

Acknowledgements

The authors are grateful to the editor and the referees for their valuable comments and suggestions.

References

[1] B. Brogliato, Some perspectives on analysis and control of complementry systems, *IEEE Trans. Automat. Control* 48 (2003) 918–935.

[2] M.K. Camlibel, W.P.M.H. Heemels and J.M. Schumacher, Consistency of a time-stepping method for a class of pieceise linear networks, *IEEE Trans. Circuit Syst.* 49 (2002) 349–357.

[3] M.K. Camlibel, J.S. Pang and J. Shen, Lyapunov stability of complementarity and extended systems, *SIAM J. Optim.* 17 (2006), 1056–1101.

[4] X.J. Chen and Z.Y. Wang, Convergence of regularized time-stepping methods for differential variational inequalities, *SIAM J. Optim.* 23 (2013) 1647–1671.

[5] X.J. Chen and Z.Y. Wang, Differential variational inequalitiy approach to dynamic games with shared constraints, *Math. Program. Ser. A* 146 (2014) 379–408.

[6] I. Ekeland and R. Témam, *Convex Analysis and Variational Problems*, Society for Industrial and Applied Mathematics, Philadephia, 1999.

[7] J. Gwinner, On a new class of differential variational inequalities and a stability result, *Math. Program. Ser. B* 139 (2013) 205–221.

[8] L.S. Han and J.S. Pang,: Non-zenoness of a class of differential quasi-variational inequalities. *Math. Program. Ser. A* 121 (2010) 171–199.

[9] L.S. Han, A. Tiwari, M.K. Camlibel and J.S. Pang, Convergence of time-stepping schemes for passive and extended linear complementrity systems, *SIAM J. Numer. Anal.* **47** (2009) 3768–3796.

[10] B.S. He, X.Z. He and H.X. Liu, Sloving a class of constrained 'blak-box' inverse variational inequalities, *Eur. J. Oper. Res.* 204 (2010) 391–401.

[11] B.S. He and H.X. Liu, *Inverse variational inequalities in economics-applications and algorithms*, Sciencepaper Online, 2006.

[12] X.Z. He and H.X. Liu, Inverse variational inequalities with projection-based solution methods, *Eur. J. Oper. Res.* 208 (2011) 12–18.

[13] B.S. He, H.X. Liu, M. Li and X.Z. He, *PPA-based methods for monotone inverse variational inequalities*, Sciencepaper Online, 2006.

[14] V.M. Hokkanen and G. Moroşanu, *Functional Methods in Differential Equations*, Chapman Hall/CRC, 2002.

[15] R. Hu and Y.P. Fang, Well-posedness of inverse variational inequalities, *J. Convex Anal.* 15 (2008) 427–437.

[16] R. Hu and Y.P. Fang, Levitin-Polyak well-posedness by perturbations of inverse variational inequalities, *Optim. Lett.* 7 (2013) 343–359.

[17] K. Kuratowski, *Topology, Vols 1 and 2*, Academic Press, New York, NY, 1968.

[18] W. Li, X. Wang and N.J. Huang, Differential inverse variational inequalities in finite dimensional spaces, *Acta Math. Sci.* 35B (2015) 407–422.

[19] X. Li, X.S. Li, N.J. Huang, A generalized f-projection algorithm for inverse mixed variational inequalities, *Optim. Lett.* 8 (2014) 1063–1076.

[20] X.S. Li, N.J. Huang and D. O'Regan, Differential mixed variational inqualities in finite dimensional spaces, *Nonlinear Anal.* 72 (2010) 3875–3886.

[21] M. Mazade and L. Thibault, Regularization of differential variational inequalities with locally prox-regular sets, *Math. Program. Ser. B* 139 (2013) 243–269.

[22] J.S. Pang and J. Shen, Strongly regular differential variational systems, *IEEE Trans. Automat. Control* 52 (2007) 242–255.

[23] J.S. Pang and D. Stewart, Differential variational inqualities, *Math. Program. Ser. A* 113 (2008) 345–424.

[24] L. Scrimali, An inverse variational inequality approach to the evolutionary spatial price equilibrium problem, *Optim. Eng.* 13 (2012) 375–387.

[25] S. Serfaty, *Functional Analysis Notes Fall*, Yevgeny Vilensky, New York, 2004.

[26] D.E. Stewart, Convergence of a time-stepping scheme for rigid body dynamics and resolution of painlevé's problem, *Arch. Ration. Mech. Anal.* 145(1998) 215–260.

[27] D.E. Stewart, Uniqueness for index-one differential variational inequalities, *Nonlinear Anal. Hybrid Syst.* 2 (2008) 812–818.

[28] J. Yang, Dynamic power price problem: an inverse variational inequality approach, *J. Ind. Manag. Optim.* 4 (2008) 673–684.

[29] X. Wang and N.J. Huang, Differential vector variational inequalities in finite-dimensional spaces, *J. Optim. Theory Appl.* 158 (2013) 109–129.

[30] X. Wang and N.J. Huang, A class of differential vector variational inequalities in finite-dimensional spaces, *J. Optim. Theory Appl.* 162 (2014), 633–648.

[31] X. Wang, W. Li, X.S. Li and N.J. Huang, Stability for differential mixed variational inequalities, *Optim. Lett.* 8 (2014), 1873–1887.

[32] X. Wang, G.J. Tang, X.S. Li and N.J. Huang, Differential quasi-variational inequalities in finite dimensional spaces, *Optimization* 64 (2015), 895–907.

Manuscript received 4 January 2016
revised 5 May 2016
accepted for publication 18 May 2017

Wei Li
Geomathematics Key Laboratory of Sichuan Province
Chengdu University of Technology
Chengdu, Sichuan, 610059, P. R. China
and
State Key Laboratory of Geohazard Prevention and Geoenvironment Protection
Chengdu, 610059, P.R. China
E-mail address: lovelylw@126.com

Xing Wang
School of Information Technology
Jiangxi University of Finance and Economics
Nanchang, 330013, P.R. China
E-mail address: wangxing0793@163.com

Xue-song Li
Department of Mathematics, Sichuan University
Chengdu, 610064, P.R. China
E-mail address: xuesongli78@hotmail.com

Nan-jing Huang
Department of Mathematics, Sichuan University
Chengdu, 610064, P.R. China
E-mail address: nanjinghuang@hotmail.com