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GENERIC STABILITY OF THE SOLUTION MAPPING FOR SEMI-INFINITE VECTOR OPTIMIZATION PROBLEMS IN BANACH SPACES*

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Abstract: In this paper, we consider the generic stability of the weakly efficient solution mapping for semi-infinite vector optimization problems under perturbations of both objective and constraint functions. We show that the weakly efficient solution mapping is upper semicontinuous and, in the sense of Baire category, most of semi-infinite vector optimization problems are essential. As application of our results, we obtain the density of the set of all problems whose weakly efficient solution mappings is continuous. Our results improve the corresponding results in the literatures.

Key words: *generic stability, semi-infinite vector optimization problem, weakly efficient solution mapping, feasible mapping, semicontinuity*

Mathematics Subject Classification: *90C29; 90C34; 49K40*

1 Introduction

In recent years, semi-infinite optimization problems became an active research topic in mathematical programming due to its extensive applications in many fields such as reverse Chebyshev approximate, robust optimization, minimax problems, design centering and disjunctive programming. A large number of results have appeared in the literature; see [12,16,20,23,24] and the references therein.

It is well-known that stability is very interesting and important in optimization theory and applications. It may be understood as the solution set having some topological properties, such as semicontinuity, well-posedness, essential stability and so on. Chuong and Yao [7] established sufficient conditions for the pseudo-Lipschitz property of the Pareto solution mapping for parametric convex semi-infinite vector optimization problems with convex perturbation of the objective function and continuous perturbation of the constraint. Chuong et al. [8] obtained sufficient conditions for the lower and upper semicontinuity of the Pareto solution mapping for semi-infinite vector optimization problems with the continuous

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perturbation of both objective and constraint functions. Chuong [4] obtained the lower semicontinuity of the Pareto solution mapping for quasiconvex semi-infinite vector optimization problems. Fan et al. [9] obtained the upper semicontinuity of the weakly efficient solution mapping for semi-infinite vector optimization problems with the continuous perturbation of both objective and constraint functions. Recently, Gong [13] obtained sufficient conditions of lower semicontinuity of efficient solution mappings for semi-infinite vector optimization problems by scalarization methods. Very recently, Peng et al. [19] derived a sufficient condition of the lower semicontinuity of the Pareto solution mapping for nonconvex semi-infinite vector optimization problems with the continuous perturbation of the objective function and cone-lower semicontinuous perturbation of the constraint function. For the sensitivity of semi-infinite vector optimization problems, we refer readers to [5, 6, 26].

On the other hand, essential stability was firstly introduced by Fort [11] for the study of fixed points of a continuous mapping. Since then, essentiality was applied in many nonlinear problems such as KKM points, vector equilibrium problems and Nash equilibrium problems: see [3, 14, 18, 22, 27, 29, 30]. Recently, Xiang and Zhou [28] obtained the essential stability of efficient solution sets for continuous vector optimization problems. Song et al. [21] generalized the results obtained by Xiang and Zhou [28] to a set-valued case. They obtained the essential stability of efficient solution sets for set-valued optimization problems with the only perturbation of the objective function in compact metric spaces. Long et al. [15] obtained the essential stability of the weakly efficient solution mapping for set-valued optimization problems with the perturbation of both the objective function and the constraint set in noncompact Banach spaces.

Recently, Fan et al. [10] obtained the essential stability for convex semi-infinite vector optimization problems with continuous perturbation of the objective and constraint functions. Since convexity and continuity do not satisfy in many cases. A natural question is 'How to weaken the convexity and continuity to get the essential stability for semi-infinite vector optimization problems?' This paper is an effort in this section.

The rest of the paper is organized as follows. In Section 2 we recall some basic definitions and some known results. Section 3 obtains some properties of the feasible set mapping and an existence theorem of the weakly efficient solution for semi-infinite vector optimization problems. Section 4 contains the main results of this paper. We obtain a sufficient condition of the upper semicontinuity of the weakly efficient solution mapping for semi-infinite vector optimization problems. Moreover, we derive that, in the sense of Baire category, most of semi-infinite vector optimization problems are essential. Our results obtained in this paper do not required the continuity of the objective and constraint functions. Our results generalize the corresponding results in [10].

2 Preliminaries

Throughout this paper, unless specified otherwise, we assume that X is a normed linear space, Y and Z are Banach spaces with norms denoted by $\|\cdot\|$. Let $C \subseteq Y$ be a closed convex cone with nonempty interior int$C$, which induces an order in Y, i.e., for any $x, y \in Y$, $x \leq_C y$ if and only if $y - x \in C$. The corresponding ordered vector space is denoted by (Y, C). Let A be a nonempty compact convex subset of X.

In this paper, we consider the following semi-infinite vector optimization problem:

$$\begin{aligned}\text{(SIVP)} \quad & \text{Min}_C\, f(x),\\ & \text{s.t.} \quad g(x,t) \leq_K b(t),\ t \in T,\\ & \qquad\ \ x \in A,\end{aligned}$$

where T is a nonempty compact subset of a Hausdorff topological space, $f : A \to Y$ is a vector-valued function, $C \subseteq Y$ is a closed convex cone, $g : A \times T \to Z$ and $b : T \to Z$ are two vector-valued functions, $K \subseteq Z$ is a closed convex cone.

We will use the following definitions of continuity for a set-valued mapping.

Definition 2.1. Let X and Y be topological vector spaces and $F : X \to 2^Y$ be a set-valued mapping. T is said to be

(i) upper semicontinuous at $x_0 \in X$ if, for any open set V containing $F(x_0)$, there exists a neighborhood U of x_0 such that $F(x) \subset V$ for all $x \in U$; F is said to be upper semicontinuous on X if it is upper semicontinuous at each $x \in X$.

(ii) lower semicontinuous at $x_0 \in X$ if, for any open set V with $F(x_0) \cap V \neq \emptyset$, there exists a neighborhood U of x_0 such that $F(x) \cap V \neq \emptyset$ for all $x \in U$; F is said to be lower semicontinuous on X if it is lower semicontinuous at each $x \in X$.

(iii) continuous on X if it is both upper semicontinuous and lower semicontinuous on X.

(iv) closed if $\mathrm{Graph}(F) := \{(x, y) : x \in X,\ y \in F(x)\}$ is a closed set in $X \times Y$.

Lemma 2.2 ([1])**.** *Let X and Y be metric spaces and $F : X \to 2^Y$ be a set-valued mapping.*

(i) *If F is closed and Y is compact, then F is upper semicontinuous.*

(ii) *If F is upper semicontinuous and for any $x \in X$, $F(x)$ is a closed set, then F is closed.*

(iii) *F is lower semicontinuous at $x_0 \in X$ if and only if for any $y \in F(x_0)$ and any sequence $\{x_\alpha\}$ with $x_\alpha \to x_0$, there exists a sequence $\{y_\alpha\}$ such that $y_\alpha \in F(x_\alpha)$ and $y_\alpha \to y$.*

Definition 2.3 ([17])**.** Let X and Y be topological vector spaces, E be a nonempty subset of X. A mapping $h : E \to Y$ is said to be C-lower semicontinuous at $x_0 \in E$ if, for any open neighborhood V of 0 in Y, there exists an open neighborhood U of x_0 such that

$$h(x) \in h(x_0) + V + C \text{ (or equivalently}, h(x_0) \in h(x) + V - C), \ \forall\, x \in U \cap E.$$

h is said to be C-lower semicontinuous on E iff h is C-lower semicontinuous at every point of E; and h is said to be C-upper semicontinuous on E iff $-h$ is C-lower semicontinuous on E.

Remark 2.4. It is easy to see that a continuous function is C-lower semicontinuous and C-upper semicontinuous, but the converse is not true as demonstrated by the following example.

Example 2.5. Let $Y = \mathbb{R}^2$ and $C = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R},\ y \geq 0\}$. Clearly, C is a closed convex cone in Y. Define $h : [-1, 1] \times [-1, 1] \to Y$ by

$$h(x, y) = \begin{cases} (0, 0), & \text{if } y = 0, \\ (1/y, 0), & \text{if } y \neq 0. \end{cases}$$

It is easy to see that h is C-lower semicontinuous and C-upper semicontinuous at $(0, 0)$, but not continuous at $(0, 0)$.

Remark 2.6. For any $\varepsilon > 0$, we denote $B(\varepsilon) := \{y \in Y : \|y\| \leq \varepsilon\}$ and $B^\circ(\varepsilon) := \{y \in Y : \|y\| < \varepsilon\}$. Then, the open neighborhood V can be replaced by $B^\circ(\varepsilon)$ in Definition 2.3 when Y is a normed space.

Definition 2.7. Let E be a nonempty convex subset of X. A mapping $h : E \to Y$ is said to be

(1) C-convex on E if, for any x_1, $x_2 \in E$ and $\lambda \in [0, 1]$, one has

$$\lambda h(x_1) + (1 - \lambda)h(x_2) \in h(\lambda x_1 + (1 - \lambda)x_2) + C.$$

(2) natural quasi-C-convex on E if, for any x_1, $x_2 \in E$ and $\lambda \in [0, 1]$, there exists $\lambda' \in [0, 1]$, such that

$$\lambda' h(x_1) + (1 - \lambda')h(x_2) \in h(\lambda x_1 + (1 - \lambda)x_2) + C.$$

Remark 2.8. Every C-convex function is a natural quasi-C-convex function, but the converse is not true illustrated by the following example.

Example 2.9 ([25])**.** Let $X = \mathbb{R}$, $E = [0, 1]$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+ = \{(x, y) : x \geq 0, y \geq 0\}$. Define $h : E \to Y$ by

$$h(x) = (x^2, 1 - x^2).$$

Then h is natural quasi-C-convex on E, but it is not C-convex.

A topological space X is said to be a Baire space if the following condition holds: given any countable collection $\{A_n\}_{n=1}^{+\infty}$ of the closed subsets of X each of which has empty interior in X, their union $\cup A_n$ also has empty interior in X. A G_δ set in a topological space X is a set that equals a countable intersection of open sets of X. We now recall the following lemmas which will be used in the sequel.

Lemma 2.10 ([2, Baire category theorem])**.** *If X is a compact Hausdorff space or a complete metric space, then X is a Baire space.*

Lemma 2.11 ([11, Theorem 2])**.** *Let Y be a metric space, X be a Baire space and $h : X \to 2^Y$ be upper semicontinuous with compact values. Then there exists a dense G_δ subset Q of X such that h is lower semicontinuous at each $x \in Q$.*

3 Existence of Weakly Efficient Solutions

In this section, we recall some results about the feasible set mapping of semi-infinite vector optimization problems. We also obtain an existence result of weakly efficient solutions for the semi-infinite vector optimization problem.

$$\begin{aligned}\Theta = \{\sigma := (g, b) : \ & g : A \times T \to Z \text{ satisfies: for any } t \in T,\\ & g(\cdot, t) \text{ is } K\text{-lower semicontinuous and naturally quasi } K\text{-convex on A},\\ & b : T \to Z \text{ is continuous}\}.\end{aligned}$$

For any pair $\sigma_1 = (g_1, b_1)$, $\sigma_2 = (g_2, b_2) \in \Theta$, we define

$$\rho'(\sigma_1, \sigma_2) := \sup_{(x,t) \in A \times T} \|g_1(x, t) - g_2(x, t)\| + \sup_{t \in T} \|b_1(t) - b_2(t)\|.$$

Assume that $\sup_{(x,t)\in A\times T} \|g(x,t)\| < +\infty$ and $\sup_{t\in T} \|b(t)\| < +\infty$ for all $\sigma \in \Theta$. Obviously, (Θ, ρ') is a metric space.

For $\sigma \in \Theta$, we consider the stability of the feasible set mapping $F : \Theta \to 2^A$ such that

$$F(\sigma) = \{x \in A : g(x,t) \leq_K b(t),\ \forall\, t \in T\}.$$

In the rest of this section, we assume that for any $\sigma \in \Theta$, $F(\sigma) \neq \emptyset$.

Proposition 3.1. *Let $\sigma \in \Theta$. Then $F(\sigma)$ is a convex set.*

Proof. Let $x_1, x_2 \in F(\sigma)$ and $x_\lambda := \lambda x_1 + (1-\lambda)x_2$ for any $\lambda \in [0,1]$. Then $x_1, x_2 \in A$. Since A is a convex set, $x_\lambda \in A$. Now we prove that $x_\lambda \in F(\sigma)$ for any $\lambda \in [0,1]$. As $x_1, x_2 \in F(\sigma)$, for any $\lambda \in [0,1]$, we have

$$\lambda(g(x_1,t) - b(t)) \in -K,\ \forall\, t \in T, \tag{3.1}$$

$$(1-\lambda)(g(x_2,t) - b(t)) \in -K,\ \forall\, t \in T. \tag{3.2}$$

By the naturally quasi K-convexity of $g(\cdot,t)$ on A for any $t \in T$, there exists $\theta \in [0,1]$ such that

$$g(x_\lambda,t) \in \theta g(x_1,t) + (1-\theta)g(x_2,t) - K.$$

This fact together with (3.1) and (3.2) yields, for any $t \in T$,

$$\begin{aligned} g(x_\lambda,t) - b(t) &\in \theta(g(x_1,t) - b(t)) + (1-\theta)(g(x_2,t) - b(t)) - K \\ &\subset -K - K - K \subset -K. \end{aligned}$$

It follows that $x_\lambda \in F(\sigma)$. Therefore, $F(\sigma)$ is convex. This completes the proof. □

Lemma 3.2 ([19])**.** *Let $\sigma \in \Theta$. Then the feasible set mapping F is closed at σ.*

Since A is compact and F is closed at σ, by Lemma 2.2 (i), we obtain that F is upper semicontinuous at σ.

Corollary 3.3. *Let $\sigma \in \Theta$. Then the feasible set mapping F is upper semicontinuous at σ.*

The following lemma was given in [19]. For the sake of convenience, we give a short proof here.

Lemma 3.4. *Let $\sigma \in \Theta$. Then, the feasible set mapping F is lower semicontinuous at σ.*

Proof. Let us consider the following two cases.

Case 1: $F(\sigma)$ is a singleton set. Denote the unique point by x_0. Suppose by contradiction that F is not lower semicontinuous at σ. Then there exist a sequence $\{\sigma_n\} \subseteq \Theta$ with $\sigma_n \to \sigma$ and an open neighborhood U of x_0 such that

$$F(\sigma_n) \cap U = \emptyset \quad \text{for all } n = 1, 2, \cdots. \tag{3.3}$$

Since $F(\sigma_n) \neq \emptyset$, we can take $x_n \in F(\sigma_n)$. Note that A is compact, without loss of generality we can assume that $x_n \to x'$. By Lemma 3.2, F is closed at σ. It follows that $x' \in F(\sigma)$. And so $x_0 = x'$ since $F(\sigma)$ is a singleton set. This contradicts (3.3) by $x_n \to x'$. Hence, F is lower semicontinuous at σ.

Case 2: $F(\sigma)$ is not a singleton set. Let V be an open convex set such that $V \cap F(\sigma) \neq \emptyset$. Let $x_0 \in V \cap F(\sigma)$. Then

$$g(x_0,t) - b(t) \leq_K 0,\ \forall\, t \in T. \tag{3.4}$$

Since $F(\sigma)$ is not a singleton set, we can choose $x' \in F(\sigma)$ with $x' \neq x_0$. Let $r \in (0,1)$ be such that

$$x_r := x_0 + r(x' - x_0) \in V.$$

From (3.4) and $x' \in F(\sigma)$, for any $t \in T$ and $\lambda \in [0,1]$, we have

$$\lambda(g(x_0,t) - b(t)) + (1-\lambda)(g(x',t) - b(t)) \leq_K 0. \tag{3.5}$$

Since $g(\cdot,t)$, $t \in T$ is naturally quasi K-convex on A, there exists $r' \in [0,1]$ such that

$$g(x_r,t) \leq_K r'g(x_0,t) + (1-r')g(x',t),\ \forall\, t \in T.$$

This together with (3.5) yields

$$g(x_r,t) - b(t) \leq_K 0,\ \forall\, t \in T. \tag{3.6}$$

Let $\sigma_n = (g_n, b_n) \in \Theta$, $n = 1,2,\cdots$ such that $\sigma_n \to \sigma$. We now claim that there exists $n_0 \in \mathbb{N}$ such that $x_r \in F(\sigma_n)$ for all $n \geq n_0$. In fact, if there does not exist $n_0 \in \mathbb{N}$ such that $x_r \in F(\sigma_n)$ for all $n \geq n_0$; or equivalently, if for any $n \in \mathbb{N}$, $x_r \notin F(\sigma_n)$, then there exists $t' \in T$ such that

$$g_n(x_r,t') - b_n(t') \not\leq_K 0,$$

or equivalently,

$$g_n(x_r,t') - b_n(t') \in Y \backslash -K.$$

Note that $Y \backslash -K$ is an open set. Then there exists a neighborhood U of 0 in Y such that

$$g_n(x_r,t') - b_n(t') + U \subset Y \backslash -K.$$

For above U, there exists $\varepsilon > 0$ such that $U_{\frac{\varepsilon}{2}} + U_{\frac{\varepsilon}{2}} \subset U$, where $U_{\frac{\varepsilon}{2}} := \{y \in Y : \|y\| \leq \frac{\varepsilon}{2}\}$. As $\sigma_n \to \sigma$, for above $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that for $n \geq n_1$,

$$g_n(x_r,t') - g(x_r,t') \in U_{\frac{\varepsilon}{2}}.$$

Similarly, there exists $n_2 \in \mathbb{N}$ such that for $n \geq n_2$,

$$b_n(t') - b(t') \in U_{\frac{\varepsilon}{2}}.$$

It follows that for $n \geq \max\{n_1, n_2\}$,

$$\begin{aligned} g(x_r,t') - b(t') &= g(x_r,t') - g_n(x_r,t') + g_n(x_r,t') - b_n(t') + b_n(t') - b(t') \\ &\in U_{\frac{\varepsilon}{2}} + U_{\frac{\varepsilon}{2}} + g_n(x_r,t') - b_n(t') \\ &\subset g_n(x_r,t') - b_n(t') + U \\ &\subset Y \backslash -K, \end{aligned}$$

which contradicts (3.6). Thus, there exists $n_0 \in \mathbb{N}$ such that $x_r \in F(\sigma_n)$ for all $n \geq n_0$ and so $V \cap F(\sigma_n) \neq \emptyset$ for all $n \geq n_0$. Therefore, F is lower semicontinuous at σ. The proof is complete. □

Now, we give an existence result for problem (SVIP).

Proposition 3.5. *If $f : A \to Y$ is C-lower semicontinuous on A and $\sigma \in \Theta$, then the weakly efficient solution set $S_w(f,\sigma) := \{x \in F(\sigma) : f(y) - f(x) \notin -intC,\ \forall\, y \in F(\sigma)\}$ is nonempty.*

Proof. By Lemma 3.2, F is closed at σ. Then, $F(\sigma)$ is a closed set. Since A is compact, $F(\sigma)$ is compact. By the C-lower semicontinuity of f on A and Corollary 5.10 in [17, p. 60], we get the efficient solution set $S(f,\sigma) := \{x \in F(\sigma) : f(y) - f(x) \notin -C\backslash\{0\},\ \forall\, y \in F(\sigma)\}$ is nonempty. Therefore, $S_w(f,\sigma)$ is nonempty since $S(f,\sigma) \subset S_w(f,\sigma)$. □

4 Generic Stability of the Solution Set Mapping

In this section, we discuss the generic stability of the weakly efficient solution set mapping for semi-infinite vector optimization problems.

$$M = \{(f,\sigma):\ f: A \to Y \text{ is } C\text{-lower semicontinuous and } C\text{-upper semicontinuous on A}, \\ \sigma \in \Theta \text{ and } F(\sigma) \neq \emptyset\}.$$

For any pair $\pi_1 = (f_1, \sigma_1)$, $\pi_2 = (f_2, \sigma_2) \in M$, we define

$$\rho(\pi_1, \pi_2) := \sup_{x \in A} \|f_1(x) - f_2(x)\| + \rho'(\sigma_1, \sigma_2).$$

Assume that $\sup_{x\in A} \|f(x)\| < +\infty$. Clearly, (M, ρ) is a metric space. For any $\pi = (f, \sigma) \in M$, Proposition 3.5 implies problem (SIVP) must have at least one weakly efficient solution. Denote by $\Gamma(\pi)$ the weak efficient solution set of problem (SIVP) respect to π, i.e., $\Gamma(\pi) := \{x \in F(\sigma) : f(y) - f(x) \notin -\text{int}C,\ \forall\, y \in F(\sigma)\}$. Then Γ defines a set-valued map from M to A and $\Gamma(\pi) \neq \emptyset$ for each $\pi \in M$.

We first prove the following lemma.

Lemma 4.1. (M, ρ) *is a complete metric space.*

Proof. Let $\{\pi_n\}$ be any Cauchy sequence in M, where $\pi_n = (f_n, \sigma_n) = (f_n, (g_n, b_n))$, $n = 1, 2, \cdots$. Then, for any $\varepsilon > 0$, there exists a positive integer N such that

$$\rho(\pi_n, \pi_m) < \varepsilon,\ \forall\, n, m > N.$$

This implies that for any $x \in A$ and $t \in T$,

$$\|f_n(x) - f_m(x)\| < \varepsilon, \quad \|g_n(x,t) - g_m(x,t)\| < \varepsilon \quad \text{and} \quad \|b_n(t) - b_m(t)\| < \varepsilon. \tag{4.1}$$

Thus, for any fixed $x \in A$ and $t \in T$, $\{f_n(x)\}$ is a Cauchy sequence in $(Y, \|\cdot\|)$ and $\{g_n(x,t)\}$ and $\{b_n(t)\}$ are Cauchy sequences in $(Z, \|\cdot\|)$. Since Y and Z are Banach spaces, there exist $f(x) \in Y$, $g(x,t) \in Z$ and $b(t) \in Z$ such that

$$f_n(x) \xrightarrow{\|\cdot\|} f(x), \quad g_n(x,t) \xrightarrow{\|\cdot\|} g(x,t) \quad \text{and} \quad b_n(t) \xrightarrow{\|\cdot\|} b(t).$$

It follows that $f : A \to Y$, $g : A \times T \to Z$ and $b : T \to Z$. Since $\|\cdot\|$ is continuous, by (4.1), for any fixed $n > N$ and any $x \in A$ and $t \in T$, let $m \to +\infty$, we have

$$\|f_n(x) - f(x)\| \le \varepsilon, \tag{4.2}$$

$$\|g_n(x,t) - g(x,t)\| \le \varepsilon \tag{4.3}$$

and

$$\|b_n(t) - b(t)\| \le \varepsilon. \tag{4.4}$$

We now prove $\pi := (f, \sigma) \in M$, where $(f, \sigma) = (f, (g, b))$. We divide the proof into four steps.

(i) b is continuous on T. In fact, for any fixed $t_0 \in T$, since b_n is continuous on T, for any $\varepsilon > 0$, there exists a neighborhood $N(t_0)$ of t_0 in T such that

$$\|b_n(t) - b_n(t_0)\| < \varepsilon,\ \forall\, t \in N(t_0). \tag{4.5}$$

From (4.4) and (4.5), we have that, for any $t \in N(t_0)$,

$$\begin{aligned}\|b(t) - b(t_0)\| &\le \|b(t) - b_n(t)\| + \|b_n(t) - b_n(t_0)\| + \|b_n(t_0) - b(t_0)\| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon,\end{aligned}$$

which implies that b is continuous on $t_0 \in T$. By the arbitrariness of t_0, b is continuous on T.

(ii) $g(\cdot, t)$ is naturally quasi K-convex on A for any $t \in T$. Indeed, for any $t \in T$, for any n, $x_1, x_2 \in A$ and $\lambda \in [0, 1]$, by the naturally quasi K-convexity of $g_n(\cdot, t)$ on A, there exists $\lambda' \in [0, 1]$ such that

$$g_n(\lambda x_1 + (1-\lambda)x_2, t) - \lambda' g_n(x_1, t) - (1-\lambda')g_n(x_2, t) \in -K.$$

Since $g_n \xrightarrow{\|\cdot\|} g$ and K is closed,

$$g(\lambda x_1 + (1-\lambda)x_2, t) - \lambda' g(x_1, t) - (1-\lambda')g(x_2, t) \in -K.$$

It follows that $g(\cdot, t)$ is naturally quasi K-convex on A for any $t \in T$.

(iii) f is C-lower semicontinuous on A. Since f_n is C-lower semicontinuous on A, for any fixed $x_0 \in A$, there exists an open neighborhood U of x_0 such that

$$f_n(x) \in f_n(x_0) + B^\circ(\varepsilon) + C, \ \forall\, x \in U \cap A. \tag{4.6}$$

Combining (4.2) and (4.6) yields

$$\begin{aligned} f(x) &\in f_n(x) + B(\varepsilon) \\ &\subset f_n(x_0) + B^\circ(\varepsilon) + C + B(\varepsilon) \\ &\subset f(x_0) + B^\circ(\varepsilon) + B(\varepsilon) + C + B(\varepsilon) \\ &\subset f(x_0) + B^\circ(3\varepsilon) + C. \end{aligned}$$

This implies that f is C-lower semicontinuous on A. Similarly, we can prove that f is C-upper semicontinuous on A and $g(\cdot, t)$ is K-lower semicontinuous on A for any $t \in T$.

(iv) $F(\sigma) \neq \emptyset$. Since $F(\sigma_n) \neq \emptyset$ for all n, there exists $x_n \in F(\sigma_n)$ for all n. Without loss of generality, we can assume that $x_n \to x_0 \in A$ as A is compact. By Lemma 3.2, F is closed at σ. Note that $\sigma_n \to \sigma$ and $x_n \to x_0$. This implies $x_0 \in F(\sigma)$ and so $F(\sigma) \neq \emptyset$.

Hence, $\pi = (f, \sigma) \in M$ and $\lim_{n\to+\infty} \rho(\pi_n, \pi) = 0$. Therefore, (M, ρ) is a complete metric space. The proof is complete. □

Theorem 4.2. *The weakly efficient solution mapping $\Gamma : M \to 2^A$ is upper semicontinuous with compact values on M.*

Proof. Let $\pi = (f, \sigma) \in M$. First we show that Γ is closed at π. Let $\pi_n = (f_n, \sigma_n) \in M$ and $x_n \in \Gamma(\pi_n)$ be sequences such that $\pi_n \xrightarrow{\rho} \pi$ and $x_n \to x_0$ as $n \to +\infty$. We shall prove that $x_0 \in \Gamma(\pi)$.

Suppose by contradiction that $x_0 \notin \Gamma(\pi)$. Then there exists $y_0 \in F(\sigma)$ such that $f(x_0) - f(y_0) \in \text{int}C$. It follows that there exists $\varepsilon_0 > 0$ such that

$$f(x_0) - f(y_0) + B^\circ(\varepsilon_0) \subset \text{int}C. \tag{4.7}$$

For $\sigma_n \to \sigma$ and $y_0 \in F(\sigma)$, by the lower semicontinuity of F at σ, there exists $y_n \in F(\sigma_n)$ such that $y_n \to y_0$. Since f is C-lower semicontinuous, for $x_n \to x_0$ and for above ε_0, there exists positive integer N_1 such that

$$f(x_n) \in f(x_0) + B^\circ(\frac{1}{4}\varepsilon_0) + C, \ \forall\, n > N_1, \tag{4.8}$$

Also by the C-upper semicontinuity of f, for $y_n \to y_0$ and for above ε_0, there exists positive integer N_2 such that

$$f(y_n) \in f(y_0) + B^\circ(\frac{1}{4}\varepsilon_0) - C, \ \forall\, n > N_2. \tag{4.9}$$

Note that $f_n \to f$. Then there exists a positive integer N_3 such that

$$f_n(x) \in f(x) + B(\frac{1}{4}\varepsilon_0) \text{ (or } f(x) \in f_n(x) + \frac{1}{4}B(\varepsilon_0)), \ \forall\, x \in A, \ \forall\, n > N_3, \tag{4.10}$$

Let $N = \max\{N_1, N_2, N_3\}$. From (4.7)-(4.10), for all $n > N$, we have

$$\begin{aligned} f_n(x_n) - f_n(y_n) &= f(x_0) - f(y_0) + f_n(x_n) - f(x_n) \\ &\quad + f(y_n) - f_n(y_n) + f(x_n) - f(x_0) + f(y_0) - f(y_n) \\ &\subset f(x_0) - f(y_0) + B(\frac{1}{4}\varepsilon_0) + B(\frac{1}{4}\varepsilon_0) + B^\circ(\frac{1}{4}\varepsilon_0) + C + B^\circ(\frac{1}{4}\varepsilon_0) + C \\ &\subset f(x_0) - f(y_0) + B^\circ(\varepsilon_0) + C \\ &\subset C + \text{int}C \subset \text{int}C, \end{aligned}$$

which contradicts the fact that $x_n \in \Gamma(\pi_n)$. It follows that $x_0 \in \Gamma(\pi)$ and so Γ is closed at π.

Since A is compact and Γ is closed at π, by Lemma 2.2 (i), Γ is upper semicontinuous at π. Moreover, Γ is closed at π implies that $\Gamma(\pi)$ is a closed set. By the compactness of A, $\Gamma(\pi)$ is compact. Therefore, by the arbitrariness of π, Γ is upper semicontinuous with compact values on M. This completes the proof. □

Definition 4.3. Let $\pi \in M$. The weakly efficient solution set $\Gamma(\pi)$ is called stable if the set-valued mapping Γ is continuous at π.

Remark 4.4. The following example shows that there exists $\pi \in M$ such that $\Gamma(\pi)$ is not stable.

Example 4.5 ([10])**.** Let $A = [-1, 2] \times [-1, 2] \subset \mathbb{R}^2$, $K = \mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$ and $T = [1, 2] \cup [3, 4] \cup [5, 6] \cup [7, 8]$. Let $f(x) = x$,

$$g(x,t) = \begin{cases} tx_1 - t, & \text{if } t \in [1,2]; \\ -tx_1, & \text{if } t \in [3,4]; \\ tx_2 - t, & \text{if } t \in [5,6]; \\ -tx_2, & \text{if } t \in [7,8]; \end{cases} \quad g_k(x,t) = \begin{cases} tx_1 - t, & \text{if } t \in [1,2]; \\ -tx_1, & \text{if } t \in [3,4]; \\ tx_2 - t, & \text{if } t \in [5,6]; \\ \frac{tx_1}{k} - tx_2, & \text{if } t \in [7,8] \end{cases}$$

for every $x = (x_1, x_2) \in A$ and $b(t) = 0$ for any $t \in T$. Let $\pi = (f, \sigma)$ and $\pi_k = (f, \sigma_k)$, where $\sigma = (g, b)$ and $\sigma_k = (g_k, b)$. It follows that $\pi \to \pi_k$, $F(\sigma) = \{(x_1, x_2) : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$, $F(\sigma_k) = \{(x_1, x_2) : \frac{x_1}{k} \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$, $\Gamma(\pi) = \{(x_1, x_2) : x_1 = 0, 0 \leq x_2 \leq 1\} \cup \{(x_1, x_2) : x_2 = 0, 0 \leq x_1 \leq 1\}$ and $\Gamma(\pi_k) = \{(x_1, x_2) : x_1 = 0, 0 \leq x_2 \leq 1\}$. It is easy to see that Γ is not lower semicontinuous at σ.

Definition 4.6. Let $\pi \in M$. A point $x \in \Gamma(\pi)$ is said to be essential if, for any open neighborhood U of x in A, there exists an open neighborhood V of π in M such that $\Gamma(\pi') \cap U \neq \emptyset$ for all $\pi' \in V$. π is said to be essential if every $x \in \Gamma(\pi)$ is essential.

Remark 4.7. From Definition 4.6, it is easy to see that the set-valued mapping Γ is lower semicontinuous at $\pi \in M$ if and only if π is essential.

The following theorem is devoted to the generic stability for semi-infinite vector optimization problems.

Theorem 4.8. *There exists a dense G_δ subset of M such that π is essential for every $\pi \in G_\delta$.*

Proof. By Lemmas 4.1 and 2.10, M is a Baire space. By Theorem 4.2, $\Gamma : M \to 2^A$ is upper semicontinuous with compact values on M. From Lemma 2.11, there exists a dense G_δ subset of M such that Γ is lower semicontinuous at each $\pi \in G_\delta$. By the Definition 4.3, the result follows. □

From Lemma 2.11 and the proof of Theorem 4.8, we obtain the density of the set of all problems whose weakly efficient solution set mapping is continuous.

Corollary 4.9. *Let $M_1 := \{\pi \in M : \Gamma$ is continuous at $\pi\}$. Then M_1 contains a dense G_δ subset of M.*

Remark 4.10. Corollary 4.9 generalizes and improves Theorem 3.6 of Fan et al. [10], one of main results of [10], from the following five aspects:

(1) the setting of the finite dimensional Euclidean spaces is generalized to Banach spaces;

(2) the convexity of f has been removed;

(3) the continuity of f is relaxed to the C-lower semicontinuity and the C-upper semicontinuity;

(4) the C-convexity of g with respect to the first argument is extended to the natural quasi-C-convexity;

(5) the continuity of g is relaxed to the K-lower semicontinuity.

Remark 4.11. It is noted that Corollary 4.9 shows that M_1 not only is a dense subset but also contains a dense G_δ subset of M. Moreover, every semi-infinite vector optimization problem can be approximated by stable semi-infinite vector optimization problem (at which the solution set mapping is continuous).

Remark 4.12. Example 4.5 shows that there exists $\pi \in M$ such that π is not essential.

The following theorem gives a sufficient condition that $\pi \in M$ is essential.

Theorem 4.13. *If $\pi \in M$ and $\Gamma(\pi)$ is a singleton set, then π is essential.*

Proof. Assume that $\Gamma(\pi) = \{x_0\}$. Let U be any open set in X such that $\Gamma(\pi) \cap U \neq \emptyset$. It follows that $x_0 \in U$ and $\Gamma(\pi) \subset U$. By Theorem 4.2, Γ is upper semicontinuous at $\pi \in M$. Then there exists an open neighborhood V of π in M such that $\Gamma(\pi') \subset U$ for each $\pi' \in V$. This implies that $\Gamma(\pi') \cap U \neq \emptyset$ for each $\pi' \in V$. Therefore, Γ is lower semicontinuous at π. By Remark 4.7, π is essential. □

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