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CHARACTERIZATIONS OF SEMICONTINUITY OF GENERALIZED APPROXIMATE KY FAN'S POINTS*

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Abstract: In this paper, we mainly concern with the stability for generalized approximate Ky Fan inequality problems. Firstly, we give a sufficient condition of completeness to the space P of generalized approximate Ky Fan inequality problems. Then, under new assumptions, which are different from the proper quasi-convexity and even weaker than the continuity, we provide sufficient and necessary conditions of upper semicontinuity (and usco) of solution mappings for generalized approximate Ky Fan inequality problems under perturbations. Our results extend and improve some known results in the literature.

Key words: *characterization, generalized approximate Ky Fan's efficient point, upper semicontinuity, solution mapping, perturbation*

Mathematics Subject Classification: *45M10, 58C07, 90C31*

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1 Introduction

It is well known that the Ky Fan Inequality is a very general mathematical format, which embraces the formats of several disciplines, as those for equilibrium problems of Mathematical Physics, those from Game Theory, those from Optimization and Variational Inequalities, and so on (see [5,8,9]). Since Ky Fan Inequality was introduced in [8], it has been extended and generalized to vector-valued mappings. In general, the Ky Fan Inequality with vector-valued mappings is known as the generalized Ky Fan Inequality (see [9,13,17]).

The stability analysis of solution mappings for (generalized) Ky Fan Inequality is one of continuously interesting topics in optimization theory and applications. In general, it is concerned with the study of the behavior of the solutions set of the problem when it's data is subject to change. And it may be understood as the solution set having some topological properties, such as upper/lower semicontinuity, Hölder continuity, calmness, Aubin property and metric regularity and and so on. The main goal of this kind is to provide qualitative or quantitative informations on the problem itself. In last years, many authors have intensively studied the stability of solution mappings for variational inequalities or Ky Fan inequalities when objective functions are perturbed by parameters (or functional sequences); see [2,6,11-16,19-21]. Huang et al. [12] discussed the upper semicontinuity and lower semicontinuity of

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the solution mapping to a parametric implicit Ky Fan Inequality. By virtue of a density result and scalarization technique, Gong and Yao [11] first discussed the lower semicontinuity of the solution mapping to a generalized Ky Fan Inequality. Li et al. [16] obtained sufficient conditions for the lower semicontinuity of solution mappings to a generalized Ky Fan Inequality. Peng and Yang [19] discussed the lower semicontinuity of solution mappings to two classes of parametric weak generalized Ky Fan Inequalities. Chen and Huang [6] studied the continuity of the solution mapping to a parametric weak generalized Ky Fan Inequality. Recently, Anh et al. [2] discuss the Hölder continuity of solution mappings of parametric primal and dual Ky Fan inequalities. Peng et al. [21] establish Hölder continuity of the approximate solution mapping to generalized parametric Ky Fan Inequalities by using scalarization technique.

On the other hand, essential component is also an interesting and important topic in the stability analysis of Ky Fan's inequalities or variational inequalities under perturbations. There have been some papers concerning with essential components of the solutions set to various nonlinear problems, such as optimization problems, Ky Fan inequalities and equilibrium problems, see [18, 22, 24, 25] and the references therein. Based on results of essential components, they established some sufficient conditions of the upper semicontinuity and/or the generic lower semicontinuity of (weakly) efficient solution mappings. In 1995, Tan et al. [22] established stability results for Ky Fan inequality problems in compact setting and non-compact setting, respectively. Recently, Chen and Gong [7] proved a generic stability theorem and given an existence theorem for essentially connected components of the set of solutions to symmetric generalized Ky Fan's inequalities. Very recently, by virtue of the main idea of [22, 24, 25], Li et al. [17] obtained the sufficient and necessary conditions of upper semicontinuity of the efficient solution mapping for a class of generalized Fan Ky inequality.

But, up to now, on the subject of sufficient and necessary conditions of upper/lower semicontinuity, few results for the Ky Fan inequality with mapping sequences of converging are available (it appears that only two relevant papers are [22] and [17]). Moreover, we also point out that there are two actualities as follows: (i) In [17], the continuity and proper quasiconvexity on objective functions play an important role in characterizing the semicontinuity of solution mappings of generalized Fan Ky inequalities, which are strong assumptions actually; (ii) to the best of our knowledge, no paper deals with sufficient and necessary conditions of semicontinuity of solution mappings to generalized approximate Ky Fan inequalities. Thus, a natural question is 'How to get a sufficient and necessary condition of upper or lower semicontinuity of the solution mapping for generalized approximate Ky Fan inequalities without imposing continuity and proper quasiconvexity?' The purpose of this paper is to establish sufficient and necessary conditions of the upper semicontinuity of solution mappings to generalized approximate Ky Fan inequalities by removing the assumptions of continuity and proper quasiconvexity, under perturbations of both on the objective functions and constraints set sequences. The obtained results are new and different form the corresponding results in [17, 22].

The rest of the paper is organized as follows. In Section 2, we recall some basic definitions and preliminaries from nonlinear analysis, set-valued analysis and vector optimization, which will be used in the sequel. In Section 3, we give a sufficient condition of completeness to the space P of perturbed generalized approximate Ky Fan inequality problems. Based on the result, in Section 4, we obtain sufficient and necessary conditions of upper semicontinuity of solution mappings for generalized approximate Ky Fan inequality problems under perturbations without using continuity and proper quasiconvexity of objective functions, and give examples to illustrate that our main result extends the corresponding ones in the literature.

2 Preliminaries

Throughout this paper, unless otherwise specified, let X be a real Banach space and K be a nonempty compact convex subset of X. Let $f : X \times X \longrightarrow \mathrm{R}$ be a real-valued function. Fan [8] introduced the following inequality (called Ky Fan inequality) of finding $x \in K$ such that

$$f(x, y) \geq 0, \quad \forall y \in K,$$

which plays a very important role in nonlinear analysis and was intensively studied in past years. In some cases, Ky Fan inequalities are also called equilibrium problems or systems.

Let Y be a real Banach space and Y^* be the dual space of Y. Let $C \subset Y$ be a point closed convex cone with $\mathrm{int}C \neq \varnothing$, where $\mathrm{int}C$ denotes the interior of C. In the sequel, we shall use the following ordering relations: for any $x, y \in Y$,

$$y \leqq_C x \;\Leftrightarrow\; x - y \in C; \qquad y \nleqq_C x \;\Leftrightarrow\; x - y \notin C;$$

$$y \leq_C x \;\Leftrightarrow\; x - y \in C \setminus \{0_Y\}; \qquad y \nleq_C x \;\Leftrightarrow\; x - y \notin C \setminus \{0_Y\};$$

$$y <_C x \;\Leftrightarrow\; x - y \in \mathrm{int}C; \qquad y \not<_C x \;\Leftrightarrow\; x - y \notin \mathrm{int}C.$$

Let

$$C^* := \{\xi \in Y^* : \xi(y) \geq 0, \forall y \in C\}$$

be the dual cone of C. Denote the quasi-interior of C^* by $C^\natural$, i.e.,

$$C^\natural := \{f \in Y^* : f(y) > 0,\ \forall y \in C \setminus \{0_Y\}\}.$$

It is easy to see that $C^\natural \neq \emptyset$ if and only if C has a base. Letting $e \in \mathrm{int}C$ be given, we denote the set $C^\sharp$ as

$$C^\sharp := \{\xi \in C^\natural : \xi(e) = 1\}.$$

In the sequel, we always assume that $C^\sharp \neq \varnothing$. Let K be a nonempty compact convex subset of X and $f : X \times X \longrightarrow Y$ be a vector-valued mapping. We consider the following generalized approximate Ky Fan's inequality problems of

$$\text{finding } x \in K \text{ such that } f(x, y) + \varepsilon e \nleq_C 0, \quad \forall y \in K \tag{$*_1$}$$

and

$$\text{finding } x \in K \text{ such that } f(x, y) + \varepsilon e \not<_C 0, \quad \forall y \in K \tag{$*_2$}$$

where $e \in \mathrm{int}C$, $\varepsilon \geq 0$ is any nonnegative real number. We call that a vector $x \in K$ is a generalized approximate Ky Fan's efficient (resp. weakly efficient) point of f in K if x satisfies $(*_1)$ (resp. $(*_2)$).

Special case

(i) When $\varepsilon = 0$, the generalized approximate Ky Fan inequality reduces to the generalized Ky Fan inequality in [17].

(ii) Let $\varepsilon = 0$, and $f : X \times X \to \mathrm{R}$ be real-valued, then the generalized approximate Ky Fan inequality reduces to the wellknown Ky Fan inequality in [9, 22].

Now, we recall some notions and results which will be used in the sequel.

Definition 2.1. Let E be a nonempty convex subset of a vector space X. Let f be a mapping from X to Y. We say that

(i) f is properly quasi C-convex on E, iff for every $x_1, x_2 \in E$, and any $\lambda \in [0,1]$, either

$$f(\lambda x_1 + (1-\lambda)x_2) \leqq_C f(x_1) \text{ or } f(\lambda x_1 + (1-\lambda)x_2) \leqq_C f(x_2);$$

(ii) f is C-convex on E, iff for every $x_1, x_2 \in E$, and any $\lambda \in [0,1]$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leqq_C \lambda f(x_1) + (1-\lambda) f(x_2).$$

Remark 2.2. From Definition 2.1, we can find that a C-convex function is not necessary a properly quasi C-convex function, meanwhile, a properly quasi C-convex function is also not necessary a C-convex function in general.

The following example shows that a C-convex function is not necessary a properly quasi C-convex function.

Example 2.3. Let $X = E = \mathbb{R}^2, Y = \mathbb{R}^2, C = \mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 | x \geq 0, y \geq 0\}$. Define $f : E \to Y$ by

$$f(x) = \Big(\frac{1}{2}x_1^2, 2x_2^2\Big).$$

For every $x, y \in E$, and any $\lambda \in [0,1]$, one has

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= \Big(\frac{1}{2}(\lambda x_1 + (1-\lambda)y_1\Big)^2, 2(\lambda x_2 + (1-\lambda)y_2)^2) \\ &\leqq_C \Big(\frac{1}{2}(\lambda x_1^2 + (1-\lambda)y_1^2\Big), 2(\lambda x_2^2 + (1-\lambda)y_2^2)) \\ &= \lambda f(x) + (1-\lambda) f(y), \end{aligned}$$

which means f is a C-convex function.

However, choosing $x = (2,0),\ y = (0,1),\ \lambda = \frac{1}{2}$, one get that

$$f(\lambda x + (1-\lambda)y) = \Big(\frac{1}{2}, \frac{1}{2}\Big) \nleqq_C (2,0) = f(x),$$

and

$$f(\lambda x + (1-\lambda)y) = \Big(\frac{1}{2}, \frac{1}{2}\Big) \nleqq_C (0,2) = f(y).$$

f is not a properly quasi C-convex function on E.

Definition 2.4. Let $e \in \text{int}C$ be any given point and any $\varepsilon \geq 0$. A vector-valued mapping $f : X \times X \to Y$ is approximate C-quasimonotone on X if and only if for any $x, y \in X : x \neq y$,

$$f(x,y) + \varepsilon e \nleqq_C 0 \Longrightarrow f(y,x) + \varepsilon e \leqq_C 0.$$

Definition 2.5 ([1])**.** Let X and Y be topological vector spaces, $F : X \to 2^Y$ be a set-valued mapping.

(i) F is said to be upper semicontinuous (u.s.c, for short) at $x_0 \in X$, if for any open set V with $F(x_0) \subset V$, there exists a neighborhood U of x_0 in X such that $F(x) \subset V$ for all $x \in U$;

(ii) F is said to be lower semicontinuous (l.s.c, for short) at $x_0 \in X$, if for any open set V with $F(x_0) \cap V \neq \emptyset$, there exists a neighborhood U of x_0 in X such that $F(x) \cap V \neq \emptyset$ for all $x \in U$;

(iii) F is said to be continuous at $x_0 \in X$, if it is both l.s.c and u.s.c at $x_0 \in X$. F is said to be l.s.c (resp. u.s.c) on X, iff it is l.s.c (resp. u.s.c) at each $x \in X$;

F is said to be closed, if the graph of F, i.e., $Graph(F) = \{(x, y) : x \in X, y \in F(x)\}$, is a closed set in $X \times Y$. F is said to be compact, if the closure of range $F(X)$, i.e., $\overline{F(X)}$, is compact, where $F(X) = \bigcup_{x \in X} F(x)$.

Definition 2.6. Let E be a nonempty subset of X, and let f be a mapping from E to Y. f is said to be C-upper semicontinuous (resp.$(-C)$-upper semicontinuous) at $x_0 \in E$, if for any neighborhood W of 0_Y in Y, there is a neighborhood $U(x_0)$ of x_0 such that for each $x \in U(x_0) \cap E$,

$$f(x) \in f(x_0) + W + C \text{ (resp.} f(x) \in f(x_0) + W - C).$$

f is said to be C-upper semicontinuous (resp.$(-C)$-upper semicontinuous) on E iff f is C-upper semicontinuous (resp.$(-C)$-upper semicontinuous) at every point of E.

Remark 2.7. Obviously, the C and $(-C)$-upper semicontinuity is strictly larger than the continuity of f, that is

$$\text{continuity} \implies C \text{ and } (-C)\text{-upper semicontinuity},$$

but the converse implication does not hold in general.

We give example 2.8 to illustrate that there exists function f which is C and $(-C)$-upper semicontinuous, but it is not continuous.

Example 2.8. Let $Y := \mathrm{R}^2$ and $C = \{(x, y) \in \mathrm{R}^2 : x \in \mathrm{R}, y \geq 0\}$. Clearly, C is a closed convex cone in Y. Defined $f : [-1, 1] \times [-1, 1] \to Y$ by

$$f(x, y) = \begin{cases} (0, 0), & \text{if } y = 0; \\ (-\frac{1}{2y}, -\frac{y}{3}), & \text{if } y \neq 0. \end{cases}$$

By virtue of Definition 2.6, we can verify that f is C-upper semicontinuous and $(-C)$-upper semicontinuous at $(0, 0)$, but f is not continuous at $(0, 0)$.

$\boxed{3}$ Completeness of the Space P

In this section, we establish a metric space P with perturbations on the objective functions and set sequences as follows. The space P of problems $(*_1)$ and $(*_2)$ is defined by

$$P := \Big\{ \pi = (f, K, \varepsilon) : f, K, \varepsilon \text{ satisfies the following conditions:}$$

(i) $f : X \times X \to Y$ is C and $(-C)$-upper semicontinous on $X \times X$;

$$\left.\begin{array}{l}\text{(ii) } \forall x \in X, f(x,\cdot) \text{ is } C\text{-convex on } X;\\ \text{(iii) } f \text{ is approximate } C\text{-quasimonotone on } X \times X;\\ \text{(iv) } \forall x \in X, f(x,x) \geqq_C 0 \text{ and } sup_{(x,y)\in X\times X}\|f(x,y)\| < +\infty;\\ \text{(v) } \varepsilon \in \mathrm{R}^+ \text{ and } K \text{ is nonempty compact convex subset of } X\end{array}\right\}.$$

For any $\pi_1 = (f_1, K_1, \varepsilon_1), \pi_2 = (f_2, K_2, \varepsilon_2) \in P$, define the metric as

$$\rho := \rho(\pi_1, \pi_2) = sup_{(x,y)\in X\times X}\|f_1(x,y) - f_2(x,y)\| + H(K_1, K_2) + |\varepsilon_1 - \varepsilon_2|,$$

where H is the Hausdorff metric on X, i.e., $H(K_1, K_2) = \max\{e(K_1, K_2), e(K_2, K_1)\}$ and $e(K_1, K_2) = \sup\{d(a, K_2) : a \in K_1\}$.

In this section, under weaker assumptions, we obtain the completeness of the space P with perturbed generalized approximate Ky Fan inequality problems. In the rest of paper, for convenience, for any $\delta \in \mathrm{R}, \delta > 0$, we denote $B(\delta) := \{y \in Y : \|y\| \leq \delta\}$ and $B^0(\delta) := \{y \in Y : \|y\| < \delta\}$.

Proposition 3.1. *(P, ρ) is a complete metric space.*

Proof. Obviously, (P, ρ) is a metric space as ρ is a metric. Then, we verity that (P, ρ) is complete. Let $\{\pi_n\}_{n=1}^{+\infty}$ be a Cauchy sequence of P, where $\pi_n := (f_n, K_n, \varepsilon_n)$. Then, for any $\delta > 0$, there exists a positive integer $N(\delta)$ such that

$$\rho(\pi_n, \pi_m) \leq \delta, \text{ for all } m, n \geq N(\delta).$$

This implies that for all $x, y \in X$,

$$\|f_n(x,y) - f_m(x,y)\| \leq \delta,\ H(K_n, K_m) \leq \delta \text{ and } |\varepsilon_n - \varepsilon_m| \leq \delta. \tag{3.1}$$

Then, for fixed $x, y \in X$, $\{f_n(x,y)\}_{n=1}^{+\infty}$ is a Cauchy sequence in $(Y, \|\cdot\|)$, $\{K_n\}_{n=1}^{+\infty}$ is a Cauchy sequence of $L(X)$, where $L(X)$ is all nonempty compact convex subsets of X and $\{\varepsilon_n\}_{n=1}^{+\infty}$ is a Cauchy sequence in R. From the completeness of R, Y and $(L(X), H)$, there exist $\varepsilon \in \mathrm{R}, f(x,y) \in Y$ and $K \in L(X)$ such that

$$f_n(x,y) \xrightarrow{\|\cdot\|} f(x,y),\ K_n \xrightarrow{H} K \text{ and } \varepsilon_n \to \varepsilon. \tag{3.2}$$

For fixed $n \geq N(\delta)$ and any $(x,y) \in X \times X$, let $m \to +\infty$, it follows from (3.1) and (3.2) that

$$f_n(x,y) \in f(x,y) + B(\delta) \text{ and } f(x,y) \in f_n(x,y) + B(\delta). \tag{3.3}$$

Now we need to show $\pi := (f, K, \varepsilon) \in M$, and we divide the proof into four steps.

(i) For each $x \in X$, $f(x,\cdot)$ is C-convex on X. For any positive integer n, any $\lambda \in [0,1]$ and for every $y_1, y_2 \in X$, by the C-convexity of $f_n(x,\cdot)$,

$$f_n(x, \lambda y_1 + (1-\lambda)y_2) \leqq_C \lambda f_n(x, y_1) + (1-\lambda) f_n(x, y_2),$$

which together with (3.2) and the closedness of C yields that

$$f(x, \lambda y_1 + (1-\lambda)y_2) \leqq_C \lambda f(x, y_1) + (1-\lambda) f(x, y_2),\ \forall \lambda \in [0,1].$$

Then, $f(x, \cdot)$ is C-convex on X.

(ii) f is approximate C-quasimonotone on $X \times X$. Indeed, for any $x, y \in X : x \neq y$, $f(x, y) + \varepsilon e \nleqq_C 0$, i.e., $f(x, y) + \varepsilon e \notin -C$. By the openness of $Y \setminus -C$ and $f_n \xrightarrow{\|\cdot\|} f$, we obtain that when n is large enough, $f_n(x, y) + \varepsilon e \in Y \setminus -C$, that is,

$$f_n(x, y) + \varepsilon e \nleqq_C 0.$$

From the approximate C-quasimonotonicity of f_n, one has

$$f_n(y, x) + \varepsilon e \in -C.$$

Therefore, together with $f_n \xrightarrow{\|\cdot\|} f$ and the closedness of C yields that

$$f(y, x) + \varepsilon e \in -C,$$

which implies that $f(y, x) + \varepsilon e \leqq_C 0$.

(iii) f is C and $(-C)$-upper semicontinous on $X \times X$. Since f_n is C-upper semicontinous on $X \times X$, for any fixed $(x_0, y_0) \in X \times X$ and for any $\delta \geq 0$, there exists a neighborhood U of (x_0, y_0) in $X \times X$ such that

$$f_n(x, y) \in f_n(x_0, y_0) + B^0(\delta) + C, \forall (x, y) \in U. \tag{3.4}$$

Combining (3.3) and (3.4), when $n \to \infty$ we have

$$\begin{aligned} f(x, y) &\in f_n(x, y) + B(\delta) \\ &\subset f_n(x_0, y_0) + B^0(\delta) + C + B(\delta) \\ &\subset f(x_0, y_0) + B(\delta) + B^0(\delta) + C + B(\delta) \\ &\subset f(x_0, y_0) + B^0(3\delta) + C, \forall (x, y) \in U. \end{aligned}$$

By the arbitrariness of δ, we get f is C-upper semicontinous. Similarly, using the same method, we can get f is $(-C)$-upper semicontinous on $X \times X$.

(iv) Finally, we can easily verify $sup_{(x,y)\in X\times X}\|f(x, y)\| < +\infty$ and $f(x, x) \geqq_C 0$ for every $x \in X$. Therefore, there exists $\pi := (f, K, \varepsilon) \in P$. By (3.2) and (3.3), we also get that $\pi_n \xrightarrow{\rho} \pi$. Thus, (P, ρ) is a complete metric space. This completes the proof. □

Remark 3.2. (i) For each $\pi := (f, K, \varepsilon) \in P$, we denote by $S_\varepsilon(\pi)$ and $S^w_\varepsilon(\pi)$ the set of generalized approximate Ky Fan's efficient points and generalized approximate Ky Fan's weak efficient points, respectively. When $\varepsilon = 0$, $S_\varepsilon(\pi)$ and $S^w_\varepsilon(\pi)$ collapse to $S(\pi)$ and $S^w(\pi)$, respectively.

(ii) In the paper, we mainly discuss the characterization of u.s.c of approximate solution mappings, so we always assume that $S_\varepsilon(\pi)$ is nonempty for each $\pi = (f, K, \varepsilon) \in P$. Then, $S^w_\varepsilon(\pi) \neq \varnothing$ since $S_\varepsilon(\pi) \subset S^w_\varepsilon(\pi)$ by definition.

Remark 3.3. By using the assumptions of C-convexity and C and $(-C)$-upper semicontinuity, which are differ from the properly quasi C-convexity and even weaker than the continuity of f, respectively (see Remark 2.2 and Remark 2.6), we obtain the the completeness of the space P. The result is different from the corresponding ones in the literature.

4 Characterization of u.s.c of Approximate Solution Mappings

In this section, we mainly characterize the upper semicontinuity of the solution mapping for the generalized approximate Ky Fan inequality.

Some facts about u.s.c appear as follows.

Lemma 4.1 ([1,3]). *Let X and Y be topological vector spaces, $F : X \to 2^Y$ be a set-valued mapping. If F has compact values (i.e., $F(x)$ is a compact set for each $x \in X$), then F is u.s.c at $x_0 \in X$ if and only if for any net (x_α) in X with $x_\alpha \to x_0$ and for any $y_\alpha \in F(x_\alpha)$, there exist $y_0 \in F(x_0)$ and a subnet (y_β) of (y_α) such that $y_\beta \to y_0$.*

Lemma 4.2 ([1]). *Let X and Y be two locally convex Hausdorff spaces, $F : X \to 2^Y$ be a set-valued mapping.*

(i) *If Y is compact and F is closed, then F is u.s.c.*

(ii) *If F is u.s.c with closed values, then F is a closed.*

Lemma 4.3 ([25]). *Let K and K_n $(n = 1, 2, ...)$ all be nonempty compact subsets of Hausdorff topological space X with $K_n \xrightarrow{H} K$. Then the following statements holds:*

(i) *$\cup_{n=1}^{+\infty} k_n \cup K$ is also a nonempty compact subset of X;*

(ii) *If $x_n \in K_n$ converging to x, then $x \in K$.*

Lemma 4.4 ([3]). *Let Y be a topological vector space. For each neighborhood U of zero in Y, there exist two neighborhoods U_1, U_2 of zero in Y such that $U_1 + U_2 \subset U$.*

Theorem 4.5 (Characterization of u.s.c of approximate solution mappings). *Let $\pi := (f, K, \epsilon) \in P$. Then, the approximate efficient solution mapping $S_\varepsilon : P \rightrightarrows X$ is u.s.c at π if and only if $S_\varepsilon(\pi) = S^w_\varepsilon(\pi)$.*

Proof. (i) (Necessity) We proof $S_\varepsilon(\pi) = S^w_\varepsilon(\pi)$. Suppose to the contrary that $S_\varepsilon(\pi) \neq S^w_\varepsilon(\pi)$. Then there exists $x' \in S^w_\varepsilon(\pi)$ such that $x' \notin S_\varepsilon(\pi)$ as $S_\varepsilon(\pi) \subset S^w_\varepsilon(\pi)$. Now, we may structure a sequence:

$$\pi_n := (f_n, K_n, \varepsilon_n) \ \text{ such that } \ \pi_n \in P \ \text{ and } \ \pi_n \xrightarrow{\rho} \pi.$$

For each $x, y \in X$, let $\eta : X \times X \to \mathrm{R}$ be defined by

$$\eta(x, y) := \Big(\frac{\|y - x'\|}{3}\Big) - \frac{\|x - x'\|}{3}\Big) \vee \Big(-\frac{1}{2}\Big) \wedge \Big(\frac{1}{2}\Big).$$

Then, $\eta(\cdot, \cdot)$ is continuous on $X \times X$; for every $x \in X, \eta(x, x) = 0$ and $\eta(x, \cdot)$ is convex on X.

Let $e \in \mathrm{int}C$. For each positive integer n, define $\pi_n := (f_n, K_n, \varepsilon_n)$ as follows:

$$f_n := f(x, y) + \frac{\eta(x, y)}{2n} e, \quad \forall x, y \in X,$$

$$K_n := \Big(1 - \frac{1}{2n + 1}\Big)K + \frac{1}{2n + 1}x' \text{ and } \varepsilon_n := \varepsilon + \frac{\eta(x, y)}{2n}.$$

We can verify that $\pi_n \in P$, $\pi_n := (f_n, K_n, \varepsilon_n) \xrightarrow{\rho} \pi := (f, K, \varepsilon)$ and $x' \in K_n \subset K$ easily.

Now, we can conclude that $x' \in S_\varepsilon(\pi_n)$. Otherwise, there exists $y' \in K_n$ with $y' \neq x'$ such that

$$f_n(x', y') + \varepsilon e = f(x', y') + \Big(\frac{\eta(x', y')}{2n} + \varepsilon\Big) e \in -C \setminus \{0\}.$$

This implies that there exists $\nu \in C \setminus \{0\}$ such that

$$f_n(x', y') + \varepsilon e = f(x', y') + \frac{\eta(x', y')}{2n} e + \varepsilon e = -\nu. \tag{4.1}$$

By (4.1), $\nu \in C \setminus \{0\}$ and $\frac{\eta(x',y')}{n} > 0$, we get $f(x', y') + \varepsilon e = -\frac{\eta(x',y')}{2n} e - \nu \in -\text{int}C$, which contradicts the fact $x' \in S^w_\varepsilon(\pi)$. Then, we have

$$x' \in S_\varepsilon(\pi_n). \tag{4.2}$$

Since $x' \notin S_\varepsilon(\pi)$, we can find V which is a open neighborhood of $S_\varepsilon(\pi)$ in X, such that

$$x' \notin V. \tag{4.3}$$

It follows from (4.2)-(4.3), $\pi_n \xrightarrow{\rho} \pi$ and the upper semicontinuity of S_ε that

$$S_\varepsilon(\pi_n) \subset V \text{ and } x' \in V,$$

which contradicts (4.3). Thus, we obtain $S_\varepsilon(\pi) = S^w_\varepsilon(\pi)$.

(ii) (Sufficiency) We prove the approximate weak efficient solution mapping $S^w_\varepsilon : P \rightrightarrows X$ is u.s.c at π when $S_\varepsilon(\pi) = S^w_\varepsilon(\pi)$. On the contrary, assume that S^w_ε is not upper semicontinuous at π. Then, there exist an open set U with $S^w_\varepsilon(\pi) \subset U$, a sequence $\{\pi_n := (f_n, K_n, \varepsilon_n)\} \subset P$ converging to $\pi := (f, K, \epsilon)$ with $x_n \in S^w_\varepsilon(\pi_n)$, s.t.

$$x_n \notin U, \forall n \in \mathbb{N}.$$

By Lemma 4.3, $\cup_{n=1}^{+\infty} K_n \cup K$ is compact due to the compactness of K and K_n for any n. Then, $\{x_n\}_{n=1}^{+\infty} \subset \cup_{n=1}^{+\infty} K_n \cup K$ has a convergent subsequence. Without loss of generality, assume that $\{x_n\}_{n=1}^{+\infty}$ is convergent. By virtue of Lemma 4.3, we have

$$x_0 := lim_{n \to +\infty} x_n \in K.$$

Now, we proceed to show $x_0 \in S^W_\varepsilon(\pi)$. If not, $x_0 \notin S^w_\varepsilon(\pi)$, there exists $z_0 \in K$ such that

$$f(x_0, z_0) + \varepsilon e <_C 0, \ i.e., \ f(x_0, z_0) + \varepsilon e \in -\text{int}C. \tag{4.4}$$

For (4.4), there exists a neighborhood W of zero in Y, such that

$$f(x_0, z_0) + \varepsilon e + W \in -\text{int}C. \tag{4.5}$$

By virtue of Lemma 4.4, there exist two neighborhoods W_1, W_2 of zero in Y such that

$$W_1 + W_2 \subset W. \tag{4.6}$$

It follows from $\pi_n \xrightarrow{\rho} \pi$ that $f_n \xrightarrow{\|\cdot\|} f$, i.e., for above $W_1 > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$f_n(x_0, z_0) \in f(x_0, z_0) + W_1, \forall n \geq n_1. \tag{4.7}$$

By virtue of $(-C)$-upper semicontinuity of f_n, for W_2, there exists $n_2 \in \mathbb{N}$, when $n \geq n_2$, we have

$$f_n(x_n, z_n) \in f_n(x_0, z_0) + W_2 - C, \tag{4.8}$$

where $(x_n, z_n) \in U(x_0, z_0)$ ($U(x_0, z_0)$ is a neighborhood of (x_0, z_0)). Letting $\gamma_0 := max\{\gamma_1, \gamma_2\}$, from (4.5)-(4.8), when $n \geq \max\{n_1, n_2\}$, we have

$$\begin{aligned} f_n(x_n, z_n) - f(x_0, z_0) &= f_n(x_n, z_n) - f_n(x_0, z_0) + f_n(x_0, z_0) - f(x_0, z_0) \\ &\in W_2 - C + W_1 \\ &\subset W - C. \end{aligned}$$

By $\phi \in C^\sharp$, one has

$$\phi(f_n(x_n, z_n)) \leq \phi(f(x_0, z_0)). \tag{4.9}$$

Combining (4.4) and (4.9), for all n large enough, we have

$$\phi(f_n(x_n, z_n)) + \varepsilon \leq \phi(f(x_0, z_0)) + \varepsilon <_C 0,$$

which contradicts the fact $x_n \in S^w_\varepsilon(\pi_n)$. Thus,

$$x_0 \in S^w_\varepsilon(\pi) \subset U. \tag{4.10}$$

On the other hand, since $x_n \notin U$ and $x_n \to x_0$, then from the openness of U, one has

$$x_0 \notin U,$$

which contradicts (4.10). Hence, we conclude that S_ε is u.s.c. at π. This completes the proof. □

Remark 4.6. Theorem 4.5 generalizes and improves the main result of [17] (see, [17, Theorem 4.2]) in the following three aspects:

(i) the key assumption of the continuity for object function in [17] is removed by the C and $(-C)$ -upper semicontinuity, which is weaker than the continuity;

(ii) the properly quasi C-convexity of object function is replaced by the C-convexity, which is differ from the former;

(iii) the model is extended from generalized Ky Fan's inequality to generalized approximate Ky Fan's inequality.

Moreover, the proof method for u.s.c is different from the one in [17], and Theorem 4.5 also generalizes and improves the corresponding results of Tan et al. [22].

Remark 4.7. In fact, when the assumption $f(x, x) \geqq_C 0 (\forall x \in X)$ is relaxed to $f(x, x) + \varepsilon e \geqq_C 0 (\forall x \in X)$ in Theorem 4.5, we can also get the result. The following example is given to illustrate the case (the space of problems can be marked as P' in this case).

Example 4.8. Let $X := [-5, 5] \subset \mathrm{R}, Y := \mathrm{R}^2, C := \mathrm{R}^2_+ = \{x = (x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$, let $\varepsilon = \frac{1}{8}, \varepsilon_n = \frac{1}{8} + \frac{1}{n^2}, e = (\frac{1}{2}, \frac{1}{2})$, and let $K := [0, 1]$, and $K_n := [\frac{1}{n}, 1 + \frac{1}{n}]$. Define the mappings $f, f_n : X \times X \longrightarrow \mathrm{R}^2$ by

$$f(x, y) = \Big(-\frac{1}{16} + x, \frac{1}{3}x^2(x - y) - \frac{1}{16}\Big), \text{ for all } x, y \in K$$

and

$$f_n(x,y) = \Big(-\frac{1}{16} + x - \frac{1}{2n^2}, \frac{1}{3}x^2(1+\frac{2}{3n^2})(x-y) - \frac{1}{16} - \frac{1}{2n^2}\Big), \text{ for all } x, y \in K.$$

Then we can verity that $f(x,x) \ngeqq_C 0$, but $f(x,x) + \varepsilon e \geqq_C 0$, for any $x \in X$ in the (new) system P', $\pi_n := (f_n, A_n, \varepsilon_n) \in P'$ and $\pi_n := (f_n, K_n, \varepsilon_n) \xrightarrow{\rho} \pi := (f, K, \varepsilon) \in P'$ when $n \to +\infty$.

It follows from direct computation shows that

$$S_\varepsilon(\pi) = S_\varepsilon^w(\pi) = [0,1] \text{ and } S_\varepsilon(\pi_n) = S_\varepsilon^w(\pi_n) = \Big[\frac{1}{n}, 1 + \frac{1}{n}\Big].$$

From Definition 2.5, we can check that $S_\varepsilon : P' \rightrightarrows X$ is u.s.c at π easily.

By virtue of Theorem 4.5 and the definition of usco (u.s.c and compact valued, see also [4, Chapter 8.2, p.190]), we can obtain Theorem 4.9 as follows.

Theorem 4.9 (Characterization of usco for approximate solution mappings)**.** *Let $\pi := (f, K, \epsilon) \in P$. Then, the approximate efficient solution mapping $S_\varepsilon : P \rightrightarrows X$ is usco at π if and only if $S_\varepsilon(\pi) = S_\varepsilon^w(\pi)$.*

Proof. (i) Obviously, the necessity can be obtained directly by Theorem 4.5.

(ii) For sufficiency, from Theorem 4.5, we can first obtain that S_ε is u.s.c. at π. Then by virtue of Berge's Theorem 3 in [3, Chapter 6.1], $S_\varepsilon(\pi)$ is compact. Therefore, we conclude that S_ε is usco at π. This completes the proof. ☐

When (i) $\varepsilon = 0$ and (ii) the C and $(-C)$ -upper semicontinuity collapses to the continuity, we can get the space of problems P^* as

$$\begin{aligned} P^* := \Big\{ & \pi = (f, K) : f, K \text{ satisfies the following conditions:} \\ & \text{(i) } f : X \times X \to Y \text{is continous on } X \times X; \\ & \text{(ii) } \forall x \in X, f(x, \cdot) \text{ is } C\text{-convex on } X; \\ & \text{(iii) } f \text{ is } C\text{-quasimonotone on } X \times X; \\ & \text{(iv) } \forall x \in X, f(x,x) \geqq_C 0 \text{ and } sup_{(x,y)\in X\times X} \|f(x,y)\| < +\infty; \\ & \text{(v) } K \text{ is nonempty compact convex subset of } X \Big\}. \end{aligned}$$

Then, we can get the following Corollary.

Corollary 4.10. *Assume $\pi := (f, K) \in P^*$ and $S(\pi) \neq \emptyset$. Then the efficient solution mapping $S : P^* \rightrightarrows X$ is usco at π if and only if $S(\pi) = S^w(\pi)$.*

Finally, we give Example 4.11 to illustrate Corollary 4.10.

Example 4.11. Let $X := [-7, 7] \subset \mathrm{R}, Y := \mathrm{R}^2, C := \mathrm{R}^2_+ = \{x = (x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$, and let $K = [0,1], K_n = [\frac{1}{n}, 1 - \frac{1}{4n}]$ $(n \geq 2)$. Define the mapping $f : X \times X \longrightarrow \mathbb{R}^2$ by

$$f(x,y) := \Big(\frac{1}{2}y - \frac{x}{2} + \frac{5}{4}, \frac{1}{2}y - \frac{1}{2}x\Big),$$

and define $f_n : X \times X \longrightarrow \mathbb{R}^2$ by

$$f_n(x,y) := \Big(\frac{1}{2}y - \frac{x}{2} + \frac{5}{4} - \frac{1}{3n^2},\ \frac{1}{2}y - \frac{1}{2}x + \frac{1}{n} - \frac{1}{n^2}\Big).$$

Therefore, $\pi_n := (f_n, K_n) \in P^*$ and $\pi_n := (f_n, K_n) \xrightarrow{\rho} \pi := (f, K) \in P^*$ when $n \to +\infty$, and we can verity that all assumptions in Corollary 4.10 are satisfied.

It follows from direct computation shows that

$$S(\pi) = S^w(\pi) = [0,1] \text{ and } S(\pi_n) = S^w(\pi_n) = \Big[\frac{1}{n}, 1 - \frac{1}{4n}\Big].$$

Therefor, $S : P^* \rightrightarrows X$ is usco (u.s.c and compact valued) at π and Corollary 4.10 is applicable.

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