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ON ROBUST APPROXIMATE OPTIMAL SOLUTIONS FOR UNCERTAIN CONVEX OPTIMIZATION AND APPLICATIONS TO MULTI-OBJECTIVE OPTIMIZATION

Xiang-Kai Sun*, Xiao-Bing Li†, Xian-Jun Long‡ and Zai-Yun Peng§

Abstract: In this paper, we consider robust approximate optimal solutions for a convex optimization problem in the face of data uncertainty. By employing conjugate analysis, we first introduce a robust type closed convex constraint qualification, and then we establish necessary and sufficient optimality conditions for robust approximate optimal solutions of this uncertain convex optimization problem by using robust optimization approach (worst-case approach). In addition, we also introduce types of Wolfe and Mond-Weir robust dual problems and investigate robust approximate weak and strong duality relations. Moreover, as applications, using a scalarization method, we establish optimality conditions and duality theorems for weakly robust approximate efficient solutions of an uncertain multi-objective optimization problem.

Key words: *approximate optimal solutions, uncertain optimization, Wolfe and Mond-Weir type robust duality*

Mathematics Subject Classification: *90C25, 90C31, 90C46*

1 Introduction

It is well known that characterizing approximate solutions of an optimization problem is essential since, numerically, it is much more advantageous to obtain an approximate solution. In the early 1980s, several researchers have been interested in ε-optimal solutions (approximate solutions) for nonlinear programming problems from a computational point of view, see Refs. [20, 24, 28]. Subsequently, much attention has been paid to characterize approximate solutions of an optimization problem. And some results on various types of approximate optimality conditions, approximate multiplier rules, approximate saddle-point theorems, and approximate duality theorems for ε-solutions of several scalar and vector optimization problems have been obtained, see, for example, [12–15, 23, 26, 34, 37].

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Recently, optimization problems in the face of data uncertainty have received a great deal of attention due to the reality of uncertainty in many real-world optimization problems. As we know, robust optimization approach (worst-case approach), has emerged as a powerful deterministic approach for studying optimization problems with data uncertainty [3], associates an uncertain optimization problem with its robust counterpart [2]. And many researchers have obtained different kinds of characterizations of the optimal solutions for scalar and vector optimization problems under uncertainty data by using robust optimization methodology, see, for example, [9, 19, 29–31, 35] and the references therein. However, to the best of our knowledge, there have been few papers on approximate solutions of an optimization problem in the face of data uncertainty, see Ref. [22]. So, the aim of this paper is to establish necessary and sufficient optimality conditions for ε-optimal solutions with a constraint qualification hypothesis and to provide some corresponding robust approximate duality results for an uncertain optimization problem. To do this, let X and Y be locally convex vector spaces, $C \subseteq X$ be a nonempty closed convex set, $K \subseteq Y$ be a nonempty closed convex cone, $f : X \to \mathbb{R}$ be a convex function, and $g : X \to Y$ be a K-convex function. Consider the following convex optimization problem

$$(\mathrm{P}) \qquad \begin{cases} \inf \ f(x) \\ \text{s.t. } \ g(x) \in -K, \\ \qquad x \in C. \end{cases}$$

This problem (P) in the face of data uncertainty can be captured by the following optimization problem

$$(\mathrm{UP}) \qquad \begin{cases} \inf \ f(x) \\ \text{s.t. } \ g(x,v) \in -K, \\ \qquad x \in C. \end{cases}$$

Here, Z is a locally convex vector space, v is an uncertain parameter and belongs to a convex and compact uncertainty set $\mathcal{V} \subseteq Z$.

Following [2], in this paper, we consider robust approximate optimal solutions for (UP) by examining its robust (worst-case) counterpart

$$(\mathrm{RUP}) \qquad \begin{cases} \inf \ f(x) \\ \text{s.t. } \ g(x,v) \in -K, \forall v \in \mathcal{V}, \\ \qquad x \in C, \end{cases}$$

where the uncertain constraint is enforced for every parameter v in the prescribed uncertainty set $\mathcal{V}$. It is worth noting that the robust counterpart, which is termed as the robust optimization problem, aims at finding a worst-case solution that is immunized against the data uncertainty.

As was mentioned above, the aim of this paper is to generalize the theory of approximate optimality conditions and approximate duality theorems for certain optimization problems to the uncertain cases. We make some contributions to the study of approximate theory for uncertain optimization problems. More precisely, by using a robust type closed convex constraint qualification, we first obtain necessary and sufficient optimality conditions for robust approximate optimal solutions of an uncertain convex optimization problem which provides a new generalization of the celebrated necessary and sufficient optimality conditions for approximate optimal solutions of certain convex optimization problems to uncertain cases. Then, along with approximate optimality conditions, we propose types of Wolfe and Mond-Weir robust dual problems to the primal one, and examine robust approximate weak and strong duality, respectively. As a consequence, by using a scalarization method (see also [10, 11, 21]), we establish necessary and sufficient optimality conditions for weakly robust approximate efficient solutions and to provide some corresponding approximate robust duality results for a multi-objective optimization problem with uncertainty data. This approach seems to be new in the literature, and we hope it will provide a useful opportunity to learn about approximate theory for an uncertain multi-objective optimization problem from the related scalar optimization problem.

The paper is organized as follows. In Section 2, we recall some notions and give some preliminary results. In Section 3, we first introduce a robust type closed convex constraint qualification, and then obtain necessary and sufficient optimality conditions for robust approximate optimal solutions of (UP). We also show that our results encompass as special cases some optimization problems considered in the recent literature. In Section 4, we first introduce a Wolfe type and a Mond-Weir type robust dual problems for (UP) by virtue of robust optimization approach, and then discuss the robust approximate weak duality and strong duality properties. In Section 5, we establish optimality conditions and duality theorems for weakly robust approximate efficient solutions of an uncertain multi-objective optimization problem.

2 Preliminaries

In this paper, we use the standard notation, please see [27, 36]. Unless otherwise specified, let X be a locally convex vector space with its topological dual space X^*, endowed with the weak* topology $w(X^*, X)$. For $x^* \in X^*$ and $x \in X$, $\langle x^*, x\rangle$ denotes the value of x^* at x, that is $\langle x^*, x\rangle = x^*(x)$. Given a set $D \subseteq X^*$ or $D \subseteq X^* \times \mathbb{R}$, the weak* closure (resp. convex hull) of D is denoted by $\mathrm{cl}^{w^*} D$ (resp. co D). Moreover, for a nonempty set $C \subseteq X$, the indicator function δ_C of C is defined as $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ if $x \notin C$.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real valued function. The effective domain, the epigraph and the conjugate function of f are defined by

$$\mathrm{dom}\ f := \{x \in X : f(x) < +\infty\},$$

$$\mathrm{epi} f := \{(x, r) \in X \times \mathbb{R} : f(x) \le r\},$$

and

$$f^*(x^*) := \sup_{x\in X} \{\langle x^*, x\rangle - f(x)\},$$

respectively. Let $\bar{x} \in \mathrm{dom}\ f$. Then, following [16], we have

$$\mathrm{epi}\ f^* = \bigcup_{\varepsilon\ge 0} \Big\{\big(x^*, \langle x^*, \bar{x}\rangle + \varepsilon - f(\bar{x})\big) : x^* \in \partial_\varepsilon f(\bar{x})\Big\}, \tag{2.1}$$

where, for a given $\varepsilon \ge 0$, the ε-subdifferential of f at $\bar{x} \in \mathrm{dom}\ f$ is defined by

$$\partial_\varepsilon f(\bar{x}) := \{x^* \in X^* : f(x) \ge f(\bar{x}) + \langle x^*, x - \bar{x}\rangle - \varepsilon, \forall x \in X\},$$

and $\partial_\varepsilon f(\bar{x}) = \emptyset$, for any $\bar{x} \notin \mathrm{dom}\ f$. If $\varepsilon = 0$, the set $\partial f(\bar{x}) := \partial_0 f(\bar{x})$ is the classical subdifferential of convex analysis, that is,

$$\partial f(\bar{x}) = \{x^* \in X^* : f(x) \ge f(\bar{x}) + \langle x^*, x - \bar{x}\rangle, \forall x \in X\}.$$

Let Y be another locally convex vector space with its topological dual space Y^*, endowed with the weak* topology $w(Y^*, Y)$. Let $K \subseteq Y$ be a nonempty closed convex cone. Its (positive) dual cone is defined by

$$K^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0, \forall y \in K\}.$$

We say that vector valued function $h : X \to Y$ is K-convex, iff for any $x, y \in X$ and $\alpha \in [0, 1]$,

$$h(\alpha x + (1-\alpha)y) - \alpha h(x) - (1-\alpha)h(y) \in -K.$$

Moreover, we say that h is K-concave iff $-h$ is K-convex. For convenience, for any $\lambda \in Y^*$, the composition of mapping $\lambda \circ h$, will be denoted by λh. Obviously, h is K-convex if and only if λh is a convex function, for each $\lambda \in K^*$. Similarly, h is K-concave if and only if λh is a concave function, for each $\lambda \in K^*$.

In this paper, we endow $X^* \times \mathbb{R}$ with the product topology of $w(X^*, X)$ and the usual Euclidean topology. Now, we give the following important results which will be used in the following sections

Lemma 2.1 ([4])**.** Let $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$ be proper convex functions such that $\mathrm{dom} f_1 \cap \mathrm{dom} f_2 \neq \emptyset$.

(i) If f_1 and f_2 are lower semicontinuous, then,

$$\text{epi } (f_1 + f_2)^* = \text{cl } (\text{epi } f_1^* + \text{epi } f_2^*).$$

(ii) If one of f_1 and f_2 is continuous at some $\bar{x} \in \mathrm{dom}\ f_1 \cap \mathrm{dom}\ f_2$, then,

$$\text{epi } (f_1 + f_2)^* = \text{epi } f_1^* + \text{epi } f_2^*.$$

We conclude this section with a remark that an element $p \in X^*$ can be regarded as a function on X in such a way that $p(x) := \langle p, x\rangle$, for any $x \in X$. Thus, for any $\alpha \in \mathbb{R}$ and any function $f : X \to \mathbb{R} \cup \{+\infty\}$, we have

$$(f + p + \alpha)^*(x^*) = f^*(x^* - p) - \alpha, \text{ for each } x^* \in X^*, \tag{2.2}$$

and

$$\text{epi } (f + p + \alpha)^* = \text{epi } f^* + (p, -\alpha). \tag{2.3}$$

3 Robust Approximate Optimality Conditions

In this section, we will give optimality conditions for robust approximate optimal solutions of (UP) by examining (RUP). So, we first recall some important concepts which will be used in this paper.

Definition 3.1. The robust feasible set of (UP) is defined by

$$\mathcal{F} := \Big\{x \in C : g(x, v) \in -K, \forall v \in \mathcal{V}\Big\}.$$

Remark 3.2. In the special case that $\mathcal{V}$ is a singleton, the robust feasible set $\mathcal{F}$ becomes the feasible set of (P), that is,

$$\mathcal{F}_0 := \Big\{x \in C : g(x) \in -K\Big\}.$$

Definition 3.3. Let $\varepsilon \geq 0$ and $\bar{x} \in \mathcal{F}$.

(i) We say that $\bar{x}$ is a robust ε-optimal solution of (UP), iff $\bar{x}$ is an ε-optimal solution of (RUP), i.e.,

$$f(x) \geq f(\bar{x}) - \varepsilon, \forall x \in \mathcal{F}.$$

(ii) We say that $\bar{x}$ is a robust optimal solution of (UP), iff $\bar{x}$ is an optimal solution of (RUP), i.e.,

$$f(x) \geq f(\bar{x}), \forall x \in \mathcal{F}.$$

Now, we will obtain a multiplier characterization for robust approximate optimal solutions of (UP). To begin with, we first introduce a robust type closed convex constraint qualification.

Definition 3.4. We say that robust type closed convex constraint qualification (RCCCQ) holds, iff

$$\bigcup_{v\in\mathcal{V},\lambda\in K^*} \text{epi } \left((\lambda g)(\cdot, v)\right)^* + \text{ epi } \delta_C^* \text{ is a weak}^* \text{ closed convex set.}$$

Remark 3.5. (i) A special case of (RCCCQ), where $C = X$, can be found in [29]. In this case, some completely characterizations of this constraint qualification have been obtained. For more details, please see [29, Propositions 3.1, 3.2 and 3.3].

(ii) In the special case when $\mathcal{V}$ is a singleton, (RCCCQ) becomes the following constraint qualification

$$\text{(CCQ)} \qquad \bigcup_{\lambda\in K^*} \text{epi } (\lambda g_0)^* + \text{ epi } \delta_C^* \text{ is a weak}^* \text{ closed set.}$$

It is also important to note that (CCQ) was introduced in [17] to study duality results and optimality conditions for convex optimization problems. And various sufficient conditions, including generalized interior-type conditions and metric regularity conditions, for (CCQ) have been obtained (see [4, 7, 8] for details).

Now, following [29], we give some characterizations of $\bigcup_{v\in\mathcal{V},\lambda\in K^*} \text{epi } \left((\lambda g)(\cdot, v)\right)^* +$ epi δ_C^*.

Proposition 3.6. *Let $g : X \times Z \to Y$ be a continuous function such that for any $\lambda \in S^*, v \in \mathcal{V} \subseteq Z$, $(\lambda, v) \mapsto (\lambda g)(x, v)$ is a concave function for any $x \in X$ and let $\mathcal{V}$ be a compact convex set. Then,*

$$\bigcup_{v\in\mathcal{V},\lambda\in K^*} \text{epi } \left((\lambda g)(\cdot, v)\right)^* + \text{ epi } \delta_C^*$$

is a convex set.

Proof. By Proposition 3.2 in [29], we know that $\bigcup_{v\in\mathcal{V},\lambda\in K^*} \text{epi } \left((\lambda g)(\cdot, v)\right)^*$ is a convex set. Moreover, it is easy to see that epi $\delta_C^* = C^* \times \mathbb{R}_+$ is a convex set. Thus, $\bigcup_{v\in\mathcal{V},\lambda\in K^*} \text{epi } \left((\lambda g)(\cdot, v)\right)^* +$ epi δ_C^* is a convex set and the proof is complete. □

By using the similar method of [29, Proposition 3.3], we can easily get the following result.

Proposition 3.7. *Let $g : X \times Z \to Y$ be a continuous function such that for any $v \in Z$, $g(\cdot, v)$ is a K-convex function and let $\mathcal{V} \subseteq Z$ be a compact set. Suppose that* int $S \neq \emptyset$ *and there exists $x_0 \in X$ such that $g(x_0, v) \in -\text{int } S$, for any $v \in \mathcal{V}$. Then,*

$$\bigcup_{v\in\mathcal{V},\lambda\in K^*} \text{epi } \left((\lambda g)(\cdot, v)\right)^* + \text{ epi } \delta_C^*$$

is a weak* closed set.

Slightly extending Theorem 3.1 in [29] to a robust convex optimization problem with a geometric constraint, we can obtain the robust version of Farkas Lemma for convex functions.

Lemma 3.8. *Let $g : X \times Z \to Y$ be a continuous function such that for any $v \in Z$, $g(\cdot, v)$ is a K-convex function. Then, the following statements are equivalent:*

(i) $\Big\{x \in C : g(x, v) \in -K, v \in \mathcal{V}\Big\} \subseteq \{x \in X : f(x) \geq 0\}$.

(ii) $(0, 0) \in \text{ epi } f^* + \text{cl}^{w^*}\text{co}\left(\bigcup_{v\in\mathcal{V},\lambda\in K^*} \text{ epi } \left((\lambda g)(\cdot, v)\right)^* + \text{ epi } \delta_C^*\right)$.

We are now in a position to present necessary and sufficient optimality conditions for robust approximate optimal solutions of (UP) using (RCCCQ).

Theorem 3.9. *For the problem* (UP), *let $\varepsilon \geq 0$, $\bar{x} \in \mathcal{F}$ and let $g : X \times Z \to Y$ be a continuous function such that for any $v \in Z$, $g(\cdot, v)$ is a K-convex function. Suppose that* (RCCCQ) *holds. Then, $\bar{x}$ is a robust ε-optimal solution of* (UP) *if and only if there exist $\bar{v} \in \mathcal{V}$, $\bar{\lambda} \in K^*$, and $\varepsilon_i \geq 0$, $i = 1, 2, 3$, such that*

$$0 \in \partial_{\varepsilon_1} f(\bar{x}) + \partial_{\varepsilon_2}\left(\left(\bar{\lambda} g\right)(\cdot, \bar{v})\right)(\bar{x}) + \partial_{\varepsilon_3}\delta_C(\bar{x}), \tag{3.1}$$

and

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon = \left(\bar{\lambda} g\right)(\bar{x}, \bar{v}). \tag{3.2}$$

Proof. Let $\bar{x} \in \mathcal{F}$ be a robust ε-optimal solution of (UP). Then, for any $x \in \mathcal{F}$,

$$f(x) \geq f(\bar{x}) - \varepsilon.$$

For any $x \in \mathcal{F}$, set

$$\phi(x) := f(x) - f(\bar{x}) + \varepsilon.$$

Then,

$$g(x, v) \in -K, v \in \mathcal{V}, x \in C \Longrightarrow \phi(x) \geq 0.$$

By Lemma 3.8, we get

$$(0, 0) \in \text{epi } \phi^* + \text{cl}^{w^*}\text{co}\left(\bigcup_{v\in\mathcal{V},\lambda\in K^*} \text{epi } \left((\lambda g)(\cdot, v)\right)^* + \text{ epi } \delta_C^*\right). \tag{3.3}$$

By (2.3), we know that

$$\text{epi } \phi^* = \text{epi } f^* + (0, f(\bar{x}) - \varepsilon). \tag{3.4}$$

On the other hand, since (RCCCQ) holds, it follows from (3.3) and (3.4) that

$$(0, \varepsilon - f(\bar{x})) \in \text{ epi } f^* + \bigcup_{v \in \mathcal{V}, \lambda \in K^*} \text{epi } ((\lambda g)(\cdot, v))^* + \text{ epi } \delta_C^*.$$

Thus, there exist $\bar{v} \in \mathcal{V}$, and $\bar{\lambda} \in K^*$ such that

$$(0, \varepsilon - f(\bar{x})) \in \text{ epi } f^* + \text{ epi } \left((\bar{\lambda} g)(\cdot, \bar{v})\right)^* + \text{ epi } \delta_C^*.$$

This follows that there exist $(u^*, r) \in$ epi f^*, $(v^*, s) \in$ epi $\left((\bar{\lambda} g)(\cdot, \bar{v})\right)^*$, and $(w^*, t) \in$ epi δ_C^* such that

$$u^* + v^* + w^* = 0 \tag{3.5}$$

and

$$r + s + t = \varepsilon - f(\bar{x}). \tag{3.6}$$

Moreover, by (2.1), there exist $\varepsilon_i \geq 0$, $i = 1, 2, 3$, such that

$$u^* \in \partial_{\varepsilon_1} f(\bar{x}), \text{ and } r = \langle u^*, \bar{x} \rangle + \varepsilon_1 - f(\bar{x}),$$

$$v^* \in \partial_{\varepsilon_2} \left(\left(\bar{\lambda} g\right)(\cdot, \bar{v})\right)(\bar{x}), \text{ and } s = \langle v^*, \bar{x} \rangle + \varepsilon_2 - \left(\bar{\lambda} g\right)(\bar{x}, \bar{v}),$$

$$w^* \in \partial_{\varepsilon_3} \delta_C(\bar{x}), \text{ and } t = \langle w^*, \bar{x} \rangle + \varepsilon_3.$$

Together with (3.5) and (3.6), we know that

$$0 \in \partial_{\varepsilon_1} f(\bar{x}) + \partial_{\varepsilon_2} \left(\left(\bar{\lambda} g\right)(\cdot, \bar{v})\right)(\bar{x}) + \partial_{\varepsilon_3} \delta_C(\bar{x}),$$

and

$$\begin{aligned} \varepsilon - f(\bar{x}) &= r + s + t \\ &= \langle u^* + v^* + w^*, \bar{x} \rangle + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - f(\bar{x}) - \left(\bar{\lambda} g\right)(\bar{x}, \bar{v}) \\ &= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - f(\bar{x}) - \left(\bar{\lambda} g\right)(\bar{x}, \bar{v}). \end{aligned}$$

This follows that

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \left(\bar{\lambda} g\right)(\bar{x}, \bar{v}) = \varepsilon.$$

Thus, (3.1) and (3.2) hold.

Conversely, assume that there exist $\bar{v} \in \mathcal{V}$, $\bar{\lambda} \in K^*$, and $\varepsilon_i \geq 0$, $i = 1, 2, 3$, such that (3.1) and (3.2) hold. By (3.1), there exist $u^* \in \partial_{\varepsilon_1} f(\bar{x})$, $v^* \in \partial_{\varepsilon_2} \left(\left(\bar{\lambda} g\right)(\cdot, \bar{v})\right)(\bar{x})$, and $w^* \in \partial_{\varepsilon_3} \delta_C(\bar{x})$, such that

$$u^* + v^* + w^* = 0. \tag{3.7}$$

Since $u^* \in \partial_{\varepsilon_1} f(\bar{x})$, $v^* \in \partial_{\varepsilon_2} \left(\left(\bar{\lambda} g\right)(\cdot, \bar{v})\right)(\bar{x})$, and $w^* \in \partial_{\varepsilon_3} \delta_C(\bar{x})$, we obtain that, for any $x \in \mathcal{F}$,

$$f(x) - f(\bar{x}) \geq \langle u^*, x - \bar{x} \rangle - \varepsilon_1,$$

$$\left(\bar{\lambda} g\right)(x, \bar{v}) - \left(\bar{\lambda} g\right)(\bar{x}, \bar{v}) \geq \langle v^*, x - \bar{x} \rangle - \varepsilon_2,$$

and

$$\delta_C(x) - \delta_C(\bar{x}) \geq \langle w^*, x - \bar{x} \rangle - \varepsilon_3.$$

Then, adding these inequalities yield,

$$\begin{aligned} & f(x)-f(\bar{x})+\left(\bar{\lambda} g\right)(x, \bar{v})-\left(\bar{\lambda} g\right)(\bar{x}, \bar{v}) \\ \geq\ & \langle u^*+v^*+w^*, x-\bar{x}\rangle-\varepsilon_1-\varepsilon_2-\varepsilon_3, \text{ for any } x \in \mathcal{F}. \end{aligned}$$

Moreover, together with $\left(\bar{\lambda} g\right)(x, \bar{v}) \leq 0$ and (3.7), we get,

$$f(x)-f(\bar{x})-\left(\bar{\lambda} g\right)(\bar{x}, \bar{v}) \geq-\varepsilon_1-\varepsilon_2-\varepsilon_3, \text{ for any } x \in \mathcal{F}.$$

Then, it follows from (3.2) that

$$f(x) \geq f(\bar{x})-\varepsilon, \text{ for any } x \in \mathcal{F}.$$

Thus, $\bar{x}$ is a robust ε-optimal solution of (UP). This completes the proof. □

Remark 3.10. Let us mention that the optimality condition given in Theorem 3.9 was established in [22] by assuming that $X=\mathbb{R}^n$, $Z=\mathbb{R}^q$, $K=\mathbb{R}_+^m$, $g(x, v)=(g_1(x, v), \ldots, g_m(x, v))$, and g_i is a real valued function, $i=1, .., m$. So, the results obtained in [22] can be regarded as a special case of Theorem 3.9.

Now, we give an example to explain Theorem 3.9.

Example 3.11. Let $X=\mathbb{R}^2$, $Y=Z=\mathbb{R}$, $K=\mathbb{R}_+$, $C=\mathbb{R}_+^2$ and $\mathcal{V}=[-1,1]$. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ and $g: \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$ be defined respectively by

$$f(x)=2 x_1+x_2^2, \quad g(x, v)=x_1^2-v x_1,$$

for any $x:=(x_1, x_2) \in \mathbb{R}^2$ and $v \in[-1,1]$.

Obviously, for the problem (UP), $\mathcal{F}=\{0\} \times[0,+\infty)$, and we can easily get

$$\bigcup_{v \in[-1,1], \lambda \in \mathbb{R}_+} \operatorname{epi}\ ((\lambda g)(\cdot, v))^*+\ \operatorname{epi}\ \delta_C^*=\mathbb{R} \times(-\mathbb{R}_+) \times \mathbb{R}_+.$$

So, the conditions of Theorem 3.9 are satisfied. Therefore, we can characterize the robust ε-optimal solution of (UP).

For example, let $\bar{x}:=(0,0) \in \mathcal{F}$ and $\varepsilon=1$. Obviously, $(0,0)$ is a robust ε-optimal solution of (UP). On the other hand, for instance, when $\varepsilon_1=\frac{1}{4}$, $\varepsilon_2=\varepsilon_3=\frac{1}{2}$, $v=1$ and $\lambda=\frac{1}{2}$, we can easily check that $\partial_{\varepsilon_1} f(\bar{x})=\{2\} \times[-1,1]$, $\partial_{\varepsilon_2}\left(\left(\bar{\lambda} g\right)(\cdot, \bar{v})\right)(\bar{x})=\left[-\frac{3}{2}, \frac{1}{2}\right] \times\{0\}$, $\partial_{\varepsilon_3} \delta_C(\bar{x})=-\mathbb{R}_+^2$, and then

$$\partial_{\varepsilon_1} f(\bar{x})+\partial_{\varepsilon_2}\left(\left(\bar{\lambda} g\right)(\cdot, \bar{v})\right)(\bar{x})+\partial_{\varepsilon_3} \delta_C(\bar{x})=\left(-\infty, \frac{5}{2}\right] \times(-\infty, 1].$$

Thus, (3.1) and (3.2) hold.

In the special case when $\mathcal{V}$ is a singleton, we can easily obtain the following result which has been investigated in [1].

Corollary 3.12. *For the problem* (P)*, let* $\varepsilon \geq 0$*,* $\bar{x} \in \mathcal{F}_0$ *and let* $g: X \rightarrow Y$ *be a continuous* K*-convex function. Suppose that* (CCQ) *holds. Then,* $\bar{x}$ *is an* ε*-optimal solution of* (P) *if and only if there exist* $\bar{\lambda} \in K^*$*, and* $\varepsilon_i \geq 0$*,* $i=1,2,3$*, such that*

$$0 \in \partial_{\varepsilon_1} f(\bar{x})+\partial_{\varepsilon_2}\left(\bar{\lambda} g\right)(\bar{x})+\partial_{\varepsilon_3} \delta_C(\bar{x}),$$

and

$$\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon=\left(\bar{\lambda} g\right)(\bar{x}).$$

Letting $\varepsilon = 0$ in Theorem 3.9, we can easily get the following necessary and sufficient optimality conditions for robust optimal solutions of (UP).

Theorem 3.13. *For the problem* (UP)*, let $g : X \times Z \to Y$ be a continuous function such that for any $v \in Z$, $g(\cdot, v)$ is a K-convex function. Suppose that* (RCCCQ) *holds and $\bar{x} \in \mathcal{F}$. Then, $\bar{x}$ is a robust optimal solution of* (UP) *if and only if there exist $\bar{v} \in \mathcal{V}$, and $\bar{\lambda} \in K^*$, such that*

$$0 \in \partial f(\bar{x}) + \partial\left(\left(\bar{\lambda} g\right)(\cdot, \bar{v})\right)(\bar{x}) + \partial \delta_C(\bar{x}),$$

and

$$\left(\bar{\lambda} g\right)(\bar{x}, \bar{v}) = 0.$$

Remark 3.14. In [19], the similar result of Theorem 3.13 has been obtained in finite dimensional spaces by using a Slater type constraint qualification. However, by virtue of a closed convex constraint qualification and a different proof approach, we investigate (UP) in locally convex Hausdorff topological vector spaces. Moreover, as was mentioned in [18], the closed convex constraint qualification is weaker than Slater type constraint qualifications. So, the result obtained in Theorem 3.13 is different from the result obtained in [19].

In the special case when $\varepsilon = 0$ and $\mathcal{V}$ is a singleton, we can easily obtain the following result. Related results can be found in [1, 4, 8, 17].

Corollary 3.15. *For the problem* (P)*, let $g : X \to Y$ be a continuous K-convex function. Suppose that* (CCQ) *holds and $\bar{x} \in \mathcal{F}_0$. Then, $\bar{x}$ is an optimal solution of* (P) *if and only if there exists $\bar{\lambda} \in K^*$, such that*

$$0 \in \partial f(\bar{x}) + \partial\left(\bar{\lambda} g\right)(\bar{x}) + \partial \delta_C(\bar{x}),$$

and

$$\left(\bar{\lambda} g\right)(\bar{x}) = 0.$$

4 Robust Approximate Duality Theorems

In this section, by virtue of robust optimization approach, we first introduce a Wolfe [32] type robust dual problem ($\mathrm{RD_W}$) and a Mond-Weir [25] type robust dual problem ($\mathrm{RD_{MW}}$), respectively, for (UP), and then discuss the robust approximate weak and strong duality properties between the corresponding problems.

Let $y \in C$, $\varepsilon \geq 0$ and $\lambda \in K^*$. To (UP), we attach the *Wolfe type robust dual* problem

$$(\mathrm{RD_W}) \qquad \begin{cases} \max_{(y,\lambda,v)} \left\{ f(y) + (\lambda g)(y, v) \right\} \\ \text{s.t.} \quad 0 \in \partial_{\varepsilon_1} f(y) + \partial_{\varepsilon_2}\left((\lambda g)(\cdot, v)\right)(y) + \partial_{\varepsilon_3} \delta_C(y), \\ \qquad \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \leq \varepsilon, \\ \qquad \lambda \in K^*, y \in C, v \in \mathcal{V}, \end{cases}$$

and the *Mond-Weir type robust dual* problem

$$(\mathrm{RD_{MW}}) \qquad \begin{cases} \max_{(y,\lambda,v)} \ f(y) \\ \text{s.t.} \quad 0 \in \partial_{\varepsilon_1} f(y) + \partial_{\varepsilon_2}\left((\lambda g)(\cdot, v)\right)(y) + \partial_{\varepsilon_3} \delta_C(y), \\ \qquad \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - (\lambda g)(y, v) \leq \varepsilon, \\ \qquad \lambda \in K^*, y \in C, v \in \mathcal{V}. \end{cases}$$

Remark 4.1. In the special case that $\varepsilon = 0$, and there is no uncertainty in the constraint function, (UP) becomes (P), ($\mathrm{RD_W}$) and ($\mathrm{RD_{MW}}$) collapse to the classical *Wolfe type dual* problem

$$(\mathrm{D_W}) \qquad \begin{cases} \max_{(y,\lambda)} \ \Big\{f(y) + (\lambda g)(y)\Big\} \\ \text{s.t.} \quad 0 \in \partial f(y) + \partial\left(\lambda g\right)(y) + \partial \delta_C(y), \\ \qquad\quad \lambda \in K^*, y \in C, \end{cases}$$

and the classical *Mond-Weir type dual* problem

$$(\mathrm{D_{MW}}) \qquad \begin{cases} \max_{(y,\lambda)} \ f(y) \\ \text{s.t.} \quad 0 \in \partial f(y) + \partial\left(\lambda g\right)(y) + \partial \delta_C(y), \\ \qquad\quad (\lambda g)(y) \geq 0, \\ \qquad\quad \lambda \in K^*, y \in C, \end{cases}$$

respectively. For more details on classical Wolfe type and Mond-Weir type dual problems, please see [5] and the references therein.

Now, we only prove Wolfe type robust ε-weak and ε-strong duality properties, since Mond-Weir type robust ε-weak and ε-strong duality properties can be proved similarly.

Theorem 4.2 (Wolfe type robust ε-weak duality)**.** *Let $\varepsilon \geq 0$. For any feasible x of* (RUP) *and any feasible (y, λ, v) of* ($\mathrm{RD_W}$)*, we have*

$$f(x) \geq f(y) + (\lambda g)(y, v) - \varepsilon.$$

Proof. Let x be a feasible solution of (RUP) and let (y, λ, v) be a feasible solution of ($\mathrm{RD_W}$). Then, $y \in C$, $\lambda \in K^*$, $v \in \mathcal{V}$, and there exist $\varepsilon_i \geq 0$, $i = 1, 2, 3$, such that

$$0 \in \partial_{\varepsilon_1} f(y) + \partial_{\varepsilon_2}\left(\left(\lambda g\right)(\cdot, v)\right)(y) + \partial_{\varepsilon_3} \delta_C(y), \tag{4.1}$$

and

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \leq \varepsilon. \tag{4.2}$$

By (4.1), there exist $u^* \in \partial_{\varepsilon_1} f(y)$, $v^* \in \partial_{\varepsilon_2}\left(\left(\lambda g\right)(\cdot, v)\right)(y)$, and $w^* \in \partial_{\varepsilon_3} \delta_C(y)$, such that

$$u^* + v^* + w^* = 0. \tag{4.3}$$

By $u^* \in \partial_{\varepsilon_1} f(y)$,

$$f(x) - f(y) \geq \langle u^*, x - y\rangle - \varepsilon_1.$$

Thus, by $v^* \in \partial_{\varepsilon_2}\left(\left(\lambda g\right)(\cdot, v)\right)(y)$, $w^* \in \partial_{\varepsilon_3} \delta_C(y)$, (4.2) and (4.3), we have

$$\begin{aligned} & f(x) - \Big(f(y) + \left(\lambda g\right)(y, v)\Big) \\ \geq \ & -\langle v^* + w^*, x - y\rangle - \varepsilon_1 - \left(\lambda g\right)(y, v) \\ = \ & -\langle v^*, x - y\rangle - \langle w^*, x - y\rangle - \varepsilon_1 - \left(\lambda g\right)(y, v) \\ \geq \ & -\left(\lambda g\right)(x, v) + \left(\lambda g\right)(y, v) - \varepsilon_2 - \delta_C(x) + \delta_C(y) - \varepsilon_3 - \varepsilon_1 - \left(\lambda g\right)(y, v) \\ = \ & -\left(\lambda g\right)(x, v) - \varepsilon_1 - \varepsilon_2 - \varepsilon_3. \\ \geq \ & -\left(\lambda g\right)(x, v) - \varepsilon. \end{aligned}$$

Moreover, since x is a feasible solution of (RUP), we get $g(x,v) \in -K$. Then, from $\lambda \in K^*$, we obtain that $(\lambda g)(x,v) \leq 0$. Thus,

$$f(x) - \Big(f(y) + (\lambda g)(y,v)\Big) \geq -\varepsilon.$$

This completes the proof. □

Theorem 4.3 (Wolfe type robust ε-strong duality)**.** *Let $\varepsilon \geq 0$, $\bar{x} \in \mathcal{F}$ and let $g : X \times Z \to Y$ be a continuous function such that for any $v \in Z$, $g(\cdot, v)$ is a K-convex function. Suppose that* (RCCCQ) *holds. If $\bar{x}$ is a robust ε-optimal solution of* (UP)*, then, there exist $\bar{v} \in \mathcal{V}$, and $\bar{\lambda} \in K^*$, such that $\left(\bar{x}, \bar{\lambda}, \bar{v}\right)$ is a robust 2ε-optimal solution of* ($\mathrm{RD_W}$)*.*

Proof. Let $\bar{x} \in \mathcal{F}$ be a robust ε-optimal solution of (UP). By Theorem 3.9, there exist $\bar{v} \in \mathcal{V}$, $\bar{\lambda} \in K^*$, and $\varepsilon_i \geq 0$, $i = 1, 2, 3$, such that

$$0 \in \partial_{\varepsilon_1} f(\bar{x}) + \partial_{\varepsilon_2}\left(\left(\bar{\lambda} g\right)(\cdot, \bar{v})\right)(\bar{x}) + \partial_{\varepsilon_3}\delta_C(\bar{x}), \tag{4.4}$$

and

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon = \left(\bar{\lambda} g\right)(\bar{x}, \bar{v}). \tag{4.5}$$

Moreover, by (4.5) and $\left(\bar{\lambda} g\right)(\bar{x}, \bar{v}) \leq 0$, we obtain that

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \leq \varepsilon. \tag{4.6}$$

So, it follows from (4.4) and (4.6) that $\left(\bar{x}, \bar{\lambda}, \bar{v}\right)$ is a feasible solution of ($\mathrm{RD_W}$). Then, for any feasible solution (y, λ, v) of ($\mathrm{RD_W}$),

$$\begin{aligned} f(\bar{x}) + \left(\bar{\lambda} g\right)(\bar{x}, \bar{v}) - \Big(f(y) + (\lambda g)(y,v)\Big) &= f(\bar{x}) - \Big(f(y) + (\lambda g)(y,v)\Big) + \left(\bar{\lambda} g\right)(\bar{x}, \bar{v}) \\ &\geq -\varepsilon + \left(\bar{\lambda} g\right)(\bar{x}, \bar{v}) \\ &= -\varepsilon + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon \\ &\geq -2\varepsilon. \end{aligned}$$

where the first inequality is from the Wolfe type robust ε-weak duality. Thus, $\left(\bar{x}, \bar{\lambda}, \bar{v}\right)$ is a robust 2ε-optimal solution of ($\mathrm{RD_W}$). □

In the special case when $\varepsilon = 0$, and $\mathcal{V}$ is a singleton, we can obtain the following characterization of classical Wolfe type duality which has been established in [5, 6, 22, 29] under different kinds of constraint qualifications.

Corollary 4.4. *Let $g : X \to Y$ be a continuous K-convex function. Suppose that* (CCQ) *holds and $\bar{x} \in \mathcal{F}_0$. If $\bar{x}$ is an optimal solution of* (P)*, then, there exists $\bar{\lambda} \in K^*$, such that $\left(\bar{x}, \bar{\lambda}\right)$ is an optimal solution of* ($\mathrm{D_W}$)*, and the objective values of* (P) *and* ($\mathrm{D_W}$) *are equal.*

Similarly, we can obtain the following Mond-Weir type robust ε-weak and ε-strong duality properties.

Theorem 4.5 (Mond-Weir type robust ε-weak duality)**.** *Let $\varepsilon \geq 0$. For any feasible x of* (RUP) *and any feasible (y, λ, v) of* ($\mathrm{RD_{MW}}$)*, Then,*

$$f(x) \geq f(y) - \varepsilon.$$

Theorem 4.6 (Mond-Weir type robust ε-strong duality)**.** *Let $\varepsilon \geq 0$, $\bar{x} \in \mathcal{F}$ and let $g : X \times Z \to Y$ be a continuous function such that for any $v \in Z$, $g(\cdot, v)$ is a K-convex function. Suppose that* (RCCCQ) *holds. If $\bar{x}$ is a robust ε-optimal solution of* (UP)*, then, there exist $\bar{v} \in \mathcal{V}$, and $\bar{\lambda} \in K^*$, such that $(\bar{x}, \bar{\lambda}, \bar{v})$ is a robust 2ε-optimal solution of* (RD_{MW})*.*

In the special case when $\varepsilon = 0$, and $\mathcal{V}$ is a singleton, we can obtain the following characterization for classical Mond-Weir type duality. For related results, please see [5, 6].

Corollary 4.7. *Let $g : X \to Y$ be a continuous K-convex function. Suppose that* (CCQ) *holds and $\bar{x} \in \mathcal{F}_0$. If $\bar{x}$ is an optimal solution of* (P)*, then, there exists $\bar{\lambda} \in K^*$, such that $(\bar{x}, \bar{\lambda})$ is an optimal solution of* (D_{MW})*. and the objective values of* (P) *and* (D_{MW}) *are equal.*

5 Applications to Multi-Objective Optimization

In this section, as an application of the general results of the previous sections, we examine a classical multi-objective optimization problem

$$(\text{MP}) \qquad \begin{cases} \min & \Big(f_1(x), f_2(x), \ldots, f_n(x)\Big) \\ \text{s.t.} & g(x) \in -K, \\ & x \in C. \end{cases}$$

where $C \subseteq X$ is a nonempty convex set, $f_i : X \to \mathbb{R}$ is a continuous convex function for any $i = 1, \ldots, n$. Moreover, the feasible set of (MP) is also defined by

$$\mathcal{F}_0^{\text{M}} := \Big\{x \in C : g(x) \in -K\Big\}.$$

This problem (MP) in the face of data uncertainty can be captured by the following multi-objective optimization problem

$$(\text{UMP}) \qquad \begin{cases} \min & \Big(f_1(x), f_2(x), \ldots, f_n(x)\Big) \\ \text{s.t.} & g(x, v) \in -K, \\ & x \in C. \end{cases}$$

Similarly, we will obtain some completely characterizations of weakly robust approximate efficient solutions of (UMP) by examining its robust (worst-case) counterpart (RUMP)

$$(\text{RUMP}) \qquad \begin{cases} \min & \Big(f_1(x), f_2(x), \ldots, f_n(x)\Big) \\ \text{s.t.} & g(x, v) \in -K, \forall v \in \mathcal{V}, \\ & x \in C. \end{cases}$$

The robust feasible set of (UMP) is also defined by

$$\mathcal{F}^{\text{M}} := \Big\{x \in C : g(x, v) \in -K, \forall v \in \mathcal{V}\Big\}.$$

In this paper, we only deal with weakly robust approximate efficient solutions for (UMP), since one can undertake other kinds of robust approximate efficient solutions for (UMP) in the same manner.

Definition 5.1. Let $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{R}^n_+$ and $\bar{x} \in \mathcal{F}^{\text{M}}$.

(i) $\bar{x}$ is said to be a weakly robust ε-efficient solution of (UMP), iff there does not exist $x \in \mathcal{F}^{\mathrm{M}}$, such that

$$f_i(x) < f_i(\bar{x}) - \varepsilon_i, \text{ for all } i = 1, \dots, n.$$

(ii) $\bar{x}$ is said to be a weakly robust efficient solution of (UMP), iff there does not exist $x \in \mathcal{F}^{\mathrm{M}}$, such that

$$f_i(x) < f_i(\bar{x}), \text{ for all } i = 1, \dots, n.$$

Remark 5.2. In the case that when $\mathcal{V}$ is a singleton, the notions of weakly robust ε-efficient solution and weakly robust efficient solution stated in Definition 5.1 become the notions of weakly ε-efficient solution and weakly efficient solution of (MP), respectively. See [5, 10, 12, 21] for details.

The following result gives a simple characterization of weakly robust ε-efficient solutions of (UMP), which plays an important role in the study of this kind of solutions. This result can be easily obtained using separation theorem, see also [21, Proposition 8.2].

Proposition 5.3. Let $\bar{x} \in \mathcal{F}^{\mathrm{M}}$ and $\varepsilon := (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{R}^n_+$. Then, $\bar{x}$ is a weakly robust ε-efficient solution of (UVP) if and only if there exist $\mu_i \geq 0$, $i = 1, \dots, n$, $\sum_{i=1}^n \mu_i = 1$, such that

$$\sum_{i=1}^n \mu_i f_i(x) \geq \sum_{i=1}^n \mu_i f_i(\bar{x}) - \sum_{i=1}^n \mu_i \varepsilon_i, \text{ for any } x \in \mathcal{F}^{\mathrm{M}}.$$

Now, by using the similar methods of Section 3, we characterize the corresponding weakly robust approximate efficient solutions of (UMP).

Theorem 5.4. *For the problem* (UMP), *let* $\bar{x} \in \mathcal{F}^{\mathrm{M}}$, $\varepsilon := (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{R}^n_+$ *and let* $g : X \times Z \to Y$ *be a continuous function such that for any* $v \in Z$, $g(\cdot, v)$ *is a* K*-convex function. Suppose that* (RCCCQ) *holds. Then,* $\bar{x}$ *is a weakly robust* ε*-efficient solution of* (UMP) *if and only if there exist* $\bar{\mu}_i \geq 0$, $i = 1, \dots, n$, $\sum_{i=1}^n \bar{\mu}_i = 1$, $\bar{\lambda} \in K^*$, $\bar{v} \in \mathcal{V}$, $\alpha_i \geq 0$, $i = 1, \dots, n$, $\beta \geq 0$, *and* $\gamma \geq 0$, *such that*

$$0 \in \sum_{i=1}^n \partial_{\alpha_i} \left(\bar{\mu}_i f_i\right)(\bar{x}) + \partial_\beta \left(\left(\bar{\lambda} g\right)(\cdot, \bar{v})\right)(\bar{x}) + \partial_\gamma \delta_C(\bar{x}), \tag{5.1}$$

and

$$\sum_{i=1}^n \alpha_i + \beta + \gamma - \left(\bar{\lambda} g\right)(\bar{x}, \bar{v}) = \sum_{i=1}^n \bar{\mu}_i \varepsilon_i. \tag{5.2}$$

Proof. Let $\bar{x} \in \mathcal{F}$ be a weakly robust ε-efficient solution of (UMP). By Proposition 5.3, there exist $\bar{\mu}_i \geq 0$, $i = 1, \dots, n$, $\sum_{i=1}^n \bar{\mu}_i = 1$, such that

$$\sum_{i=1}^n \bar{\mu}_i f_i(x) \geq \sum_{i=1}^n \bar{\mu}_i f_i(\bar{x}) - \sum_{i=1}^n \bar{\mu}_i \varepsilon_i, \text{ for any } x \in \mathcal{F}^{\mathrm{M}}.$$

For any $x \in \mathcal{F}^{\mathrm{M}}$, set

$$\psi(x) := \sum_{i=1}^n \bar{\mu}_i f_i(x) - \sum_{i=1}^n \bar{\mu}_i f_i(\bar{x}) + \sum_{i=1}^n \bar{\mu}_i \varepsilon_i.$$

Then,

$$g(x,v) \in -K, v \in \mathcal{V}, x \in C \Longrightarrow \psi(x) \geq 0.$$

Moreover, we know that

$$\text{epi } \psi^* = \sum_{i=1}^{n} \text{epi } (\bar{\mu}_i f_i)^* + \left(0, \sum_{i=1}^{n} \bar{\mu}_i f_i(\bar{x}) - \sum_{i=1}^{n} \bar{\mu}_i \varepsilon_i\right).$$

So, using the same methods of Theorem 3.9, we can easily get

$$\left(0, \sum_{i=1}^{n} \bar{\mu}_i \varepsilon_i - \sum_{i=1}^{n} \bar{\mu}_i f_i(\bar{x})\right) \in \sum_{i=1}^{n} \text{epi } (\bar{\mu}_i f_i)^* + \bigcup_{v \in \mathcal{V}, \lambda \in K^*} \text{epi } \left((\lambda g)(\cdot, v)\right)^* + \text{ epi } \delta_C^*.$$

Thus, there exist $\bar{v} \in \mathcal{V}$, and $\bar{\lambda} \in K^*$ such that

$$\left(0, \sum_{i=1}^{n} \bar{\mu}_i \varepsilon_i - \sum_{i=1}^{n} \bar{\mu}_i f_i(\bar{x})\right) \in \sum_{i=1}^{n} \text{epi } (\bar{\mu}_i f_i)^* + \text{ epi } \left((\bar{\lambda} g)(\cdot, \bar{v})\right)^* + \text{ epi } \delta_C^*.$$

This follows that there exist $(u_i^*, r_i) \in$ epi $(\bar{\mu}_i f_i)^*$, $i = 1, \ldots, n$, $(v^*, s) \in$ epi $\left((\bar{\lambda} g)(\cdot, \bar{v})\right)^*$, and $(w^*, t) \in$ epi δ_C^*, such that

$$\sum_{i=1}^{n} u_i^* + v^* + w^* = 0, \tag{5.3}$$

and

$$\sum_{i=1}^{n} r_i + s + t = \sum_{i=1}^{n} \bar{\mu}_i \varepsilon_i - \sum_{i=1}^{n} \bar{\mu}_i f_i(\bar{x}). \tag{5.4}$$

Moreover, by (2.1), there exist $\alpha_i \geq 0$, $i = 1, \ldots, n$, $\beta \geq 0$, and $\gamma \geq 0$, such that

$$u_i^* \in \partial_{\alpha_i} (\bar{\mu}_i f_i)(\bar{x}), \text{ and } r_i = \langle u_i^*, \bar{x} \rangle + \alpha_i - \bar{\mu}_i f_i(\bar{x}), \quad i = 1, 2, \ldots, n,$$

$$v^* \in \partial_\beta \left(\left(\bar{\lambda} g\right)(\cdot, \bar{v})\right)(\bar{x}), \text{ and } s = \langle v^*, \bar{x} \rangle + \beta - \left(\bar{\lambda} g\right)(\bar{x}, \bar{v}),$$

$$w^* \in \partial_\gamma \delta_C(\bar{x}), \text{ and } t = \langle w^*, \bar{x} \rangle + \gamma.$$

Together with (5.3) and (5.4), we know that

$$0 \in \sum_{i=1}^{n} \partial_{\alpha_i} \left(\bar{\mu}_i f_i\right)(\bar{x}) + \partial_\beta \left(\left(\bar{\lambda} g\right)(\cdot, \bar{v})\right)(\bar{x}) + \partial_\gamma \delta_C(\bar{x}),$$

and

$$\begin{aligned} & \sum_{i=1}^{n} \bar{\mu}_i \varepsilon_i - \sum_{i=1}^{n} \bar{\mu}_i f_i(\bar{x}) \\ = \;& \sum_{i=1}^{n} r_i + s + t \\ = \;& \left\langle \sum_{i=1}^{n} u_i^* + v^* + w^*, \bar{x} \right\rangle + \sum_{i=1}^{n} \alpha_i + \beta + \gamma - \sum_{i=1}^{n} \bar{\mu}_i f_i(\bar{x}) - \left(\bar{\lambda} g\right)(\bar{x}, \bar{v}) \end{aligned}$$

$$= \sum_{i=1}^{n}\alpha_i+\beta+\gamma-\sum_{i=1}^{n}\bar{\mu}_i f_i(\bar{x})-\left(\bar{\lambda}g\right)(\bar{x},\bar{v}).$$

This follows that

$$\sum_{i=1}^{n}\alpha_i+\beta+\gamma-\left(\bar{\lambda}g\right)(\bar{x},\bar{v})=\sum_{i=1}^{n}\bar{\mu}_i\varepsilon_i.$$

Thus, (5.1) and (5.2) hold.

Conversely, assume that there exist $\bar{\mu}_i \geq 0$, $i=1,\ldots,n$, $\sum_{i=1}^{n}\bar{\mu}_i=1$, $\bar{\lambda}\in K^*$, $\bar{v}\in\mathcal{V}$, $\alpha_i\geq 0$, $i=1,\ldots,n$, $\beta\geq 0$, and $\gamma\geq 0$, such that (5.1) and (5.2) hold. By (5.1), there exist $u_i^*\in\partial_{\alpha_i}(\bar{\mu}_i f_i)(\bar{x})$, $i=1,\ldots,n$, $v^*\in\partial_\beta\left(\left(\bar{\lambda}g\right)(\cdot,\bar{v})\right)(\bar{x})$, and $w^*\in\partial_\gamma\delta_C(\bar{x})$, such that

$$\sum_{i=1}^{n}u_i^*+v^*+w^*=0. \tag{5.5}$$

Since $u_i^*\in\partial_{\alpha_i}(\bar{\mu}_i f_i)(\bar{x})$, $i=1,\ldots,n$, $v^*\in\partial_\beta\left(\left(\bar{\lambda}g\right)(\cdot,\bar{v})\right)(\bar{x})$, and $w^*\in\partial_\gamma\delta_C(\bar{x})$, we obtain that, for any$x\in\mathcal{F}^{\mathrm{M}}$,

$$\bar{\mu}_i f_i(x)-\bar{\mu}_i f_i(\bar{x})\geq\langle u_i^*,x-\bar{x}\rangle-\alpha_i,$$

$$\left(\bar{\lambda}g\right)(x,\bar{v})-\left(\bar{\lambda}g\right)(\bar{x},\bar{v})\geq\langle v^*,x-\bar{x}\rangle-\beta,$$

and

$$\delta_C(x)-\delta_C(\bar{x})\geq\langle w^*,x-\bar{x}\rangle-\gamma.$$

Then, adding these inequalities yields, for any $x\in\mathcal{F}^{\mathrm{M}}$,

$$\begin{aligned}&\sum_{i=1}^{n}\bar{\mu}_i f_i(x)-\sum_{i=1}^{n}\bar{\mu}_i f_i(\bar{x})+\left(\bar{\lambda}g\right)(x,\bar{v})-\left(\bar{\lambda}g\right)(\bar{x},\bar{v})\\ \geq\ &\left\langle\sum_{i=1}^{n}u_i^*+v^*+w^*,x-\bar{x}\right\rangle-\sum_{i=1}^{n}\alpha_i-\beta-\gamma.\end{aligned}$$

Moreover, together with $\left(\bar{\lambda}g\right)(x,\bar{v})\leq 0$ and (5.5), we get,

$$\sum_{i=1}^{n}\bar{\mu}_i f_i(x)-\sum_{i=1}^{n}\bar{\mu}_i f_i(\bar{x})-\left(\bar{\lambda}g\right)(\bar{x},\bar{v})\geq-\sum_{i=1}^{n}\alpha_i-\beta-\gamma,\ \text{for any } x\in\mathcal{F}^{\mathrm{M}}.$$

Then, it follows from (5.2) that

$$\sum_{i=1}^{n}\bar{\mu}_i f_i(x)\geq\sum_{i=1}^{n}\bar{\mu}_i f_i(\bar{x})-\sum_{i=1}^{n}\bar{\mu}_i\varepsilon_i,\ \text{for any } x\in\mathcal{F}^{\mathrm{M}}.$$

Thus, $\bar{x}$ is a weakly robust ε-efficient solution of (UMP). This completes the proof. □

In the special case when $\mathcal{V}$ is a singleton, we can easily obtain the following result.

Corollary 5.5. *For the problem* (MP)*, let* $\bar{x}\in\mathcal{F}_0^{\mathrm{M}}$*,* $\varepsilon:=(\varepsilon_1,\ldots,\varepsilon_n)\in\mathbb{R}_+^n$ *and let* $g: X\to Y$ *be a continuous* K*-convex function. Suppose that* (CCQ) *holds. Then,* $\bar{x}$ *is a weakly*

ε-efficient solution of (MP) *if and only if there exist* $\bar{\mu}_i \geq 0$, $i = 1, \dots, n$, $\sum_{i=1}^{n} \bar{\mu}_i = 1$, $\bar{\lambda} \in K^*$, $\alpha_i \geq 0$, $i = 1, \dots, n$, $\beta \geq 0$, *and* $\gamma \geq 0$, *such that*

$$0 \in \sum_{i=1}^{n} \partial_{\alpha_i} (\bar{\mu}_i f_i)(\bar{x}) + \partial_\beta (\bar{\lambda} g)(\bar{x}) + \partial_\gamma \delta_C(\bar{x}),$$

and

$$\sum_{i=1}^{n} \alpha_i + \beta + \gamma - (\bar{\lambda} g)(\bar{x}) = \sum_{i=1}^{n} \bar{\mu}_i \varepsilon_i.$$

Similarly, taking $\varepsilon_i = 0$, $i = 1, \dots, n$, in Theorem 5.4, we can easily get the following necessary and sufficient conditions for weakly robust efficient solutions of (UMP).

Theorem 5.6. *For the problem* (UMP), *let* $g : X \times Z \to Y$ *be a continuous function such that for any* $v \in Z$, $g(\cdot, v)$ *is a* K*-convex function. Suppose that* (RCCCQ) *holds and* $\bar{x} \in \mathcal{F}^{\mathrm{M}}$. *Then,* $\bar{x} \in \mathcal{F}^{\mathrm{M}}$ *is a weakly robust efficient solution of* (UMP) *if and only if there exist* $\bar{\mu}_i \geq 0$, $i = 1, \dots, n$, $\sum_{i=1}^{n} \mu_i = 1$, $\bar{\lambda} \in K^*$, *and* $\bar{v} \in \mathcal{V}$, *such that*

$$0 \in \sum_{i=1}^{n} \partial (\bar{\mu}_i f_i)(\bar{x}) + \partial \left((\bar{\lambda} g)(\cdot, \bar{v})\right)(\bar{x}) + \partial \delta_C(\bar{x}),$$

and

$$(\bar{\lambda} g)(\bar{x}, \bar{v}) = 0.$$

In the special case when $\mathcal{V}$ is a singleton and $\varepsilon_i = 0$, $i = 1, \dots, n,$, we can easily obtain the following result.

Corollary 5.7. *For the problem* (MP), *let* $g : X \to Y$ *be a continuous* K*-convex function. Suppose that* (CCQ) *holds and* $\bar{x} \in \mathcal{F}_0^{\mathrm{M}}$. *Then,* $\bar{x}$ *is a weakly efficient solution of* (MP) *if and only if there exist* $\bar{\mu}_i \geq 0$, $i = 1, \dots, n$, $\sum_{i=1}^{n} \bar{\mu}_i = 1$, *and* $\bar{\lambda} \in K^*$, *such that*

$$0 \in \sum_{i=1}^{n} \partial (\bar{\mu}_i f_i)(\bar{x}) + \partial (\bar{\lambda} g)(\bar{x}) + \partial \delta_C(\bar{x}),$$

and

$$(\bar{\lambda} g)(\bar{x}) = 0.$$

Finally, in this section, we introduce a Wolfe type ($\mathrm{RMD_W}$) and a Mond-Weir type robust multi-objective dual problem ($\mathrm{RMD_{MW}}$), respectively, for (UMP), and discuss the robust approximate weak and strong multi-objective duality properties between the corresponding problems. As was mentioned above, we also only deal with weakly robust ε-efficient solutions for the corresponding problems.

Now, let $y \in X$, $\varepsilon := (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{R}_+^n$, $\mu := (\mu_1, \dots, \mu_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^{n} \mu_i = 1$, and $\lambda \in K^*$. To (UMP), we attach the *Wolfe type robust multi-objective dual problem* with respect to weakly efficient solutions

$$(\mathrm{RMD_W}) \quad \begin{cases} \max_{(y,\mu,\lambda,v)} \ \Big(f_1(y) + (\lambda g)(y, v), \dots, f_n(y) + (\lambda g)(y, v)\Big) \\ \text{s.t.} \quad 0 \in \sum_{i=1}^{n} \partial_{\alpha_i} (\mu_i f_i)(y) + \partial_\beta \Big((\lambda g)(\cdot, v)\Big)(y) + \partial_\gamma \delta_C(y), \\ \qquad \sum_{i=1}^{n} \alpha_i + \beta + \gamma \leq \sum_{i=1}^{n} \mu_i \varepsilon_i, \\ \qquad \mu_i \geq 0, i = 1, \dots, n, \sum_{i=1}^{n} \mu_i = 1, \\ \qquad \lambda \in K^*, y \in C, v \in \mathcal{V}, \end{cases}$$

and the *Mond-Weir type robust multi-objective dual problem* with respect to weakly efficient solutions

$$(\mathrm{RMD_{MW}})\qquad \begin{cases} \max_{(y,\mu,\lambda,v)} \Big(f_1(y),\dots,f_n(y)\Big) \\ \text{s.t.}\quad 0\in\sum_{i=1}^n \partial_{\alpha_i}(\mu_i f_i)(y)+\partial_\beta((\lambda g)(\cdot,v))(y)+\partial_\gamma\delta_C(y), \\ \qquad \sum_{i=1}^n \alpha_i+\beta+\gamma-(\lambda g)(y,v)\le\sum_{i=1}^n \mu_i\varepsilon_i, \\ \qquad \mu_i\ge 0, i=1,\dots,n, \sum_{i=1}^n \mu_i=1, \\ \qquad \lambda\in K^*, y\in C, v\in\mathcal{V}, \end{cases}$$

where the maximization is also over all the parameter $v\in\mathcal{V}$.

Remark 5.8. In the special case that $\varepsilon_i=0$, $i=1,\dots,n$ and $\mathcal{V}$ is a singleton, (UMP) becomes (MP), ($\mathrm{RMD_W}$) and ($\mathrm{RMD_{MW}}$) collapse to the *Wolfe type multi-objective dual* problem with respect to weakly efficient solutions

$$(\mathrm{MD_W})\qquad \begin{cases} \max_{(y,\mu,\lambda)} \Big(f_1(y)+(\lambda g)(y),\dots,f_n(y)+(\lambda g)(y)\Big) \\ \text{s.t.}\quad 0\in\sum_{i=1}^n \partial(\mu_i f_i)(y)+\partial(\lambda g)(y)+\partial\delta_C(y), \\ \qquad \mu_i\ge 0, i=1,\dots,n, \sum_{i=1}^n \mu_i=1, \\ \qquad \lambda\in K^*, y\in C, \end{cases}$$

and the *Mond-Weir type multi-objective dual problem* with respect to weakly efficient solutions

$$(\mathrm{MD_{MW}})\qquad \begin{cases} \max_{(y,\mu,\lambda)} \Big(f_1(y),\dots,f_n(y)\Big) \\ \text{s.t.}\quad 0\in\sum_{i=1}^n \partial(\mu_i f_i)(y)+\partial(\lambda g)(y)+\partial\delta_C(y), \\ \qquad (\lambda g)(y)\ge 0, \\ \qquad \mu_i\ge 0, i=1,\dots,n, \sum_{i=1}^n \mu_i=1, \\ \qquad \lambda\in K^*, y\in C, \end{cases}$$

respectively. For more details on Wolfe type and Mond-Weir type multi-objective dual problems with respect to other kinds of efficient solutions, see also [5,9–12] and the references therein.

In this section, a weakly robust (approximate) efficient solution and a weakly (approximate) efficient solution of a "max" multi-objective optimization problem like the robust multi-objective dual problem ($\mathrm{RMD_W}$) and ($\mathrm{RMD_{MW}}$) is similarly defined as in Definition 5.1. In what follows, we use the following notation for convenience:

$$u\prec v\Leftrightarrow v-u\in\ \mathrm{int}\ \mathbb{R}^m_+,\quad u\not\prec v\Leftrightarrow v-u\notin\ \mathrm{int}\ \mathbb{R}^m_+.$$

Similarly, we first give the following Wolfe type robust ε-weak and ε-strong multi-objective duality properties.

Theorem 5.9 (Wolfe type robust ε-weak multi-objective duality)**.** *Let* $\varepsilon:=(\varepsilon_1,\dots,\varepsilon_n)\in\mathbb{R}^n_+$. *For any feasible* x *of* (RUMP) *and any feasible* (y,μ,λ,v) *of* ($\mathrm{RMD_W}$), *we have*

$$\Big(f_1(x),\dots,f_n(x)\Big)\not\prec\Big(f_1(y)+(\lambda g)(y,v)-\varepsilon_1,\dots,f_n(y)+(\lambda g)(y,v)-\varepsilon_n\Big).$$

Proof. Let x be a feasible solution of (RUMP) and (y,μ,λ,v) be a feasible solution of ($\mathrm{RMD_W}$). Then, $y\in C$, $\mu_i\ge 0$, $i=1,\dots,n$, $\sum_{i=1}^n\mu_i=1$, $\lambda\in K^*$, $v\in\mathcal{V}$, and there exist $\alpha_i\ge 0$, $i=1,\dots,n$, $\beta\ge 0$, and $\gamma\ge 0$, such that

$$0\in\sum_{i=1}^n \partial_{\alpha_i}(\mu_i f_i)(y)+\partial_\beta((\lambda g)(\cdot,v))(y)+\partial_\gamma\delta_C(y), \tag{5.6}$$

and

$$\sum_{i=1}^{n} \alpha_i + \beta + \gamma \leq \sum_{i=1}^{n} \mu_i \varepsilon_i.$$

Assume to the contrary that

$$\Big(f_1(x), \ldots, f_n(x)\Big) \prec \Big(f_1(y) + (\lambda g)(y, v) - \varepsilon_1, \ldots, f_n(y) + (\lambda g)(y, v) - \varepsilon_n\Big).$$

Hence,

$$\Big\langle \mu, \Big(f_1(x) - f_1(y) - (\lambda g)(y, v) + \varepsilon_1, \ldots, f_n(x) - f_n(y) - (\lambda g)(y, v) + \varepsilon_n\Big)\Big\rangle < 0,$$

due to $\mu \in \mathbb{R}^n_+$. This inequality is equivalent to

$$\sum_{i=1}^{n} \mu_i f_i(x) < \sum_{i=1}^{n} \mu_i f_i(y) + (\lambda g)(y, v) - \sum_{i=1}^{n} \mu_i \varepsilon_i. \tag{5.7}$$

On the other hand, by (5.6), there exist $u_i^* \in \partial_{\alpha_i}(\mu_i f_i)(y)$, $v^* \in \partial_\beta \left((\lambda g)(\cdot, v)\right)(y)$, and $w^* \in \partial_\gamma \delta_C(y)$, such that

$$\sum_{i=1}^{n} u_i^* + v^* + w^* = 0.$$

By $u_i^* \in \partial_{\alpha_i}(\mu_i f_i)(y)$,

$$\mu_i f_i(x) - \mu_i f_i(y) \geq \langle u_i^*, x - y\rangle - \alpha_i, i = 1, \ldots, n.$$

This follows that

$$\sum_{i=1}^{n} \mu_i f_i(x) - \sum_{i=1}^{n} \mu_i f_i(y) \geq \left\langle \sum_{i=1}^{n} u_i^*, x - y \right\rangle - \sum_{i=1}^{n} \alpha_i.$$

Then, following the lines in the proof of Theorem 4.2, we can justify that

$$\sum_{i=1}^{n} \mu_i f_i(x) - \left(\sum_{i=1}^{n} \mu_i f_i(y) + (\lambda g)(y, v)\right) \geq -\sum_{i=1}^{n} \mu_i \varepsilon_i.$$

This contradicts (5.7), and so the proof is complete. □

Theorem 5.10 (Wolfe type robust ε-strong multi-objective duality)**.** *Let $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{R}^n_+$, $\bar{x} \in \mathcal{F}^{\mathrm{M}}$, and let $g : X \times Z \to Y$ be a continuous function such that for any $v \in Z$, $g(\cdot, v)$ is a K-convex function. Suppose that* (RCCCQ) *holds. If $\bar{x}$ is a weakly robust ε-efficient solution of* (UMP)*, then, there exist $\bar{\mu}_i \geq 0$, $i = 1, \ldots, n$, $\sum_{i=1}^{n} \bar{\mu}_i = 1$, $\bar{v} \in \mathcal{V}$, and $\bar{\lambda} \in K^*$, such that $\left(\bar{x}, \bar{\mu}, \bar{\lambda}, \bar{v}\right)$ is a weakly robust 2ε-efficient solution of* ($\mathrm{RMD_W}$)*.*

Proof. Let $\bar{x}$ be a weakly robust ε-efficient solution of (UMP). By Theorem 5.4, there exist $\bar{\mu}_i \geq 0$, $i = 1, \ldots, n$, $\sum_{i=1}^{n} \bar{\mu}_i = 1$, $\bar{\lambda} \in K^*$, $\bar{v} \in \mathcal{V}$, $\alpha_i \geq 0$, $i = 1, \ldots, n$, $\beta \geq 0$, and $\gamma \geq 0$, such that

$$0 \in \sum_{i=1}^{n} \partial_{\alpha_i}\left(\bar{\mu}_i f_i\right)(\bar{x}) + \partial_\beta \left(\left(\bar{\lambda} g\right)(\cdot, \bar{v})\right)(\bar{x}) + \partial_\gamma \delta_C(\bar{x}),$$

and

$$\sum_{i=1}^{n} \alpha_i + \beta + \gamma - \left(\bar{\lambda} g\right)(\bar{x}, \bar{v}) = \sum_{i=1}^{n} \bar{\mu}_i \varepsilon_i. \tag{5.8}$$

Moreover, by (5.8) and $\left(\bar{\lambda} g\right)(\bar{x}, \bar{v}) \leq 0$, we obtain that

$$\sum_{i=1}^{n} \alpha_i + \beta + \gamma \leq \sum_{i=1}^{n} \bar{\mu}_i \varepsilon_i.$$

Then, $\left(\bar{x}, \bar{\mu}, \bar{\lambda}, \bar{v}\right)$ is a feasible solution of $(\mathrm{RMD_W})$. Moreover, for any feasible solution (y, μ, λ, v) of $(\mathrm{RMD_W})$ and $\bar{\mu}_i \geq 0$, $i = 1, \ldots, n$, $\sum_{i=1}^{n} \bar{\mu}_i = 1$. By using the similar methods of Theorem 4.3, we can easily get

$$\sum_{i=1}^{n} \bar{\mu}_i f_i(\bar{x}) + \left(\bar{\lambda} g\right)(\bar{x}, \bar{v}) - \left(\sum_{i=1}^{n} \bar{\mu}_i f_i(y) + (\lambda g)(y, v)\right) \geq -2 \sum_{i=1}^{n} \bar{\mu}_i \varepsilon_i.$$

By Proposition 5.3, $\left(\bar{x}, \bar{\mu}, \bar{\lambda}, \bar{v}\right)$ is a weakly robust 2ε-efficient solution of $(\mathrm{RMD_W})$. □

Similarly, in the special case when $\varepsilon = 0$, and $\mathcal{V}$ is a singleton, we obtain the following Wolfe type multi-objective duality. Related results can be found in [5, 10, 11] under various conditions imposed on the objective function and the constraint conditions, or under different kinds of constraint qualifications.

Corollary 5.11. *Let $g : X \to Y$ be a continuous K-convex function. Suppose that* (CCQ) *holds and $\bar{x} \in \mathcal{F}$. If $\bar{x}$ is a weakly efficient solution of* (MP)*, then, there exist $\bar{\mu}_i \geq 0$, $i = 1, \ldots, n$, $\sum_{i=1}^{n} \bar{\mu}_i = 1$, and $\bar{\lambda} \in K^*$, such that $\left(\bar{x}, \bar{\mu}, \bar{\lambda}\right)$ is a weakly efficient solution of* $(\mathrm{MD_W})$.

Similarly, we obtain the following Mond-Weir type robust ε-weak and ε-strong multi-objective duality properties.

Theorem 5.12 (Mond-Weir type robust ε-weak multi-objective duality)**.** *Let $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{R}^n_+$. For any feasible x of* (RUMP) *and any feasible (y, μ, λ, v) of* $(\mathrm{RMD_{MW}})$*, we have*

$$\left(f_1(x), \ldots, f_n(x)\right) \not< \left(f_1(y) - \varepsilon_1, \ldots, f_n(y) - \varepsilon_n\right).$$

Theorem 5.13 (Mond-Weir type robust ε-strong multi-objective duality)**.** *Let $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{R}^n_+$, $\bar{x} \in \mathcal{F}$, and let $g : X \times Z \to Y$ be a continuous function such that for any $v \in Z$, $g(\cdot, v)$ is a K-convex function. Suppose that* (RCCCQ) *holds. If $\bar{x}$ is a weakly robust ε-efficient solution of* (UMP)*, then, there exist $\bar{\mu}_i \geq 0$, $i = 1, \ldots, n$, $\sum_{i=1}^{n} \bar{\mu}_i = 1$, $\bar{v} \in \mathcal{V}$, and $\bar{\lambda} \in K^*$, such that $\left(\bar{x}, \bar{\mu}, \bar{\lambda}, \bar{v}\right)$ is a weakly robust 2ε-efficient solution of* $(\mathrm{RMD_{MW}})$.

Similarly, in the special case when $\varepsilon = 0$, and $\mathcal{V}$ is a singleton, we can obtain the following Mond-Weir type multi-objective duality. Related results can be found in [4, 10, 33].

Corollary 5.14. *Let $g : X \to Y$ be a continuous K-convex function. Suppose that* (CCQ) *holds and $\bar{x} \in \mathcal{F}$. If $\bar{x}$ is a weakly efficient solution of* (MP)*, then, there exist $\bar{\mu}_i \geq 0$, $i = 1, \ldots, n$, $\sum_{i=1}^{n} \bar{\mu}_i = 1$, and $\bar{\lambda} \in K^*$, such that $\left(\bar{x}, \bar{\mu}, \bar{\lambda}\right)$ is a weakly efficient solution of* $(\mathrm{MD_{MW}})$.

6 Conclusions

In this paper, we consider robust approximate optimal solutions for a convex optimization problem under uncertainty in the constraint function. By using the framework of robust optimization approach (the worst-case approach), we obtain optimality theorems and duality theorems for robust approximate optimal solutions of the uncertain convex optimization problem. We also show that our results encompass as special cases some optimization problems considered in the recent literature. Moreover, we apply the proposed approach to investigate weakly robust approximate efficient solutions for a multi-objective optimization problem in the face of data uncertainty in the constraint function.

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References

[1] F.S. Bai, Z.Y. Wu and D.L. Zhu, Sequential Lagrange multiplier condition for ε-optimal solution in convex programming, *Optimization* 57 (2008) 669–680.

[2] A. Ben-Tal, L.E. Ghaoui and A. Nemirovski, *Robust Optimization*, Princeton Series in Applied Mathematics, 2009.

[3] A. Ben-Tal and A. Nemirovski, Robust optimization-methodology and applications, *Math. Program.* Ser. B 92 (2002) 453-480.

[4] R.I. Boţ, *Conjugate Duality in Convex Optimization*, Springer-Verlag, Berlin, 2010.

[5] R.I. Boţ and S.M. Grad and G. Wanka, *Duality in Vector Optimization*, Springer-Verlag, Berlin, 2009.

[6] R.I. Boţ and S.M. Grad, Wolfe duality and Mond-Weir duality via perturbations, *Nonlinear Anal.* 73 (2010), 374–384.

[7] R.I. Bot and G. Wanka, An alternative formulation for a new closed cone constraint qualification, *Nonlinear Anal.* 64 (2006) 1367–1381.

[8] R.S. Burachik and V. Jeyakumar, A simple closure condition for the normal cone intersection formula, *Proc. Amer. Math. Soc.* 133 (2004) 1741–1748.

[9] T. D. Chuong, Optimality and duality for robust multiobjective optimization problems, *Nonlinear Anal.* 134 (2016) 127–143.

[10] T.D. Chuong and D.S. Kim, Optimality conditions and duality in nonsmooth multiobjective optimization problems, *Ann. Oper. Res.* 217 (2014) 117–136.

[11] T.D. Chuong and D.S. Kim, D.S.: Nondifferentiable minimax programming problems with applications, *Ann. Oper. Res.* 251 (2017) 73–87.

[12] T.D. Chuong and D.S Kim, Approximate solutions of multiobjective optimization problems, *Positivity* 20 (2016) 187–207.

[13] J. Dutta, Necessary optimality conditions and saddle points for approximate optimization in Banach spaces, *TOP* 13 (2005) 127–143.

[14] Y. Gao, S.H. Hou and X.M. Yang, Existence and optimality conditions for approximate solutions to vector optimization problems, *J. Optim. Theory Appl.* 152 (2012), 97–120.

[15] A. Hamel, An ε-Lagrange multiplier rule for a mathematical programming problem on Banach spaces, *Optimization* 49 (2001) 137–149.

[16] V. Jeyakumar, Asymptotic dual conditions characterizing optimality for convex programs, *J. Optim. Theory Appl.* 93 (1997) 153–165.

[17] V. Jeyakumar, G.M. Lee and N. Dinh, A closed cone constraint qualification for convex optimization, Applied Mathematics Research Report AMR04/8, University of New South Wales, Sydney, Australia, 2004.

[18] V. Jeyakumar and G.Y. Li, Strong duality in robust convex programming: complete characterizations, *SIAM J. Optim.* 20 (2010) 3384–3407.

[19] V. Jeyakumar, G.M. Lee and G.Y. Li, Characterizing robust solution sets of convex programs under data uncertainty, *J. Optim. Theory Appl.* 164 (2015) 407–435.

[20] S.S. Kutateladze, Convex ε-programming, *Soviet Math. Doklady* 20 (1979) 391–393.

[21] G.M. Lee, G.S. Kim and N. Dinh, Optimality conditions for approximate solutions of convex semiinfinite vector optimization problems,in: *Recent Developments in Vector Optimization, Vector Optimization*, Q.H. Ansar and J.-C. Yao (eds.), vol. 1, Springer, Berlin, 2012, pp. 275–295.

[22] J.H. Lee and G.M. Lee, On ε-solutions for convex optimization problems with uncertainty data, *Positivity* 16 (2012) 509–526.

[23] J.C. Liu, ε-Duality theorem of nondifferentiable nonconvex multiobjective programming, *J. Optim. Theory Appl.* 69 (1991) 153–167.

[24] P. Loridan, Necessary conditions for ε-optimality, *Math. Program.* 19 (1982) 140–152.

[25] B. Mond and T. Weir, Generalized concavity and duality, in: Generalized Concavity in Optimization and Economics, S. Schaible, W.T. Ziemba (eds.), Academic Press, New York, 1981, pp. 263–279.

[26] Q.S. Qiu and X.M. Yang, Some properties of approximate solutions for vector optimization problem with set-valued functions, *J. Global Optim.* 47 (2010) 1–12.

[27] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[28] J.J. Strodiot, V.H. Nguyen and N. Heukemes, ε-Optimal solutions in nondifferentiable convex programming and some related questions, *Math. Program.* 25 (1983) 307–328.

[29] X.K. Sun and Y. Chai, On robust duality for fractional programming with uncertainty data, *Positivity* 18 (2014) 9–28.

[30] X.K. Sun, X.J. Long, H.Y. Fu and X.B. Li, Some characterizations of robust optimal solutions for uncertain fractional optimization and applications, *J. Ind. Manag. Optim.* 13 (2017) 803–824.

[31] X.K. Sun, Z.Y. Peng and X.L. Guo, Some characterizations of robust optimal solutions for uncertain convex optimization problems, *Optim. Lett.* 10 (2016) 1463–1478.

[32] P. Wolfe, A duality theorem for nonlinear programming, *Q. Appl. Math.* 19 (1961) 239–244.

[33] X.M. Yang, K.L. Teo and X.Q. Yang, Duality for a class of nondifferentiable multiobjective programming problems, J. Math. Anal. Appl. 252 (2000) 999–1005.

[34] K. Yokoyama, ε-Optimality criteria for convex programming problems via exact penalty functions, *Math. Program.* 56 (1992) 233–243.

[35] H. Yu and H.M. Liu, Robust multiple objective game theory, *J. Optim. Theory Appl.* 159 (2013) 272–280.

[36] C. Zalinescu, *Convex Analysis in General Vector Spaces*, World Scientific, London, 2002.

[37] Z.A. Zhou, X.M. Yang and J.W. Peng, ε-optimality conditions of vector optimization problems with set-valued maps based on the algebraic interior in real linear spaces, *Optim. Lett.* 8 (2014) 1047–1061.

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Xiang-Kai Sun
College of Mathematics and Statistics
Chongqing Technology and Business University
Chongqing 400067, P. R. China
E-mail address: sxkcqu@163.com

Xiao-Bing Li
College of Mathematics and Statistics
Chongqing JiaoTong University University
Chongqing 400074, China
E-mail address: xiaobinglicq@126.com

Xian-Jun Long
College of Mathematics and Statistics
Chongqing Technology and Business University
Chongqing 400067, P. R. China
E-mail address: xianjunlong@hotmail.com

Zai-Yun Peng
College of Mathematics and Statistics
Chongqing JiaoTong University University
Chongqing 400074, China
E-mail address: pengzaiyun@126.com