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TROPICAL PRODUCTION TECHNOLOGIES

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Abstract: This paper proposes to model production technologies with an upper semilattice structure endowed with translation homothetic properties. In some recent papers, idempotent convex structures were proposed to measure performance of firms. These approaches involved a divisibility assumption and it was shown that one can propose a framework to model technologies. Along this line, performance of firms can be evaluated by computing Farrell efficiency measures in closed form. However, there exist other distance functions based upon an additive structure. The paper proposes a framework allowing to take into account both semilattice structure and translation homothetic properties in productivity measurement. To do that, the concept of Max-Plus convexity is considered. Max-Plus convex technologies combine both an upper semilattice structure and an additivity assumption. All this analysis relies on the field of tropical idempotent mathematics.

Key words: *technology, upper semilattice, translation homotheticity,* max-*plus convex sets, nonparametric models, DEA*

Mathematics Subject Classification: *06A06, 06A12, 52A01, 91B38*

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1 Introduction

In microeconomic production theory, a technology is the set of all technically feasible combinations of output and inputs.

Convexity is a key assumption in production economics. For example, the DEA model introduced in [12] allows to compute the efficiency score of firms using linear programming methods and yields a procedure to rank them. One can loosely say that these DEA technologies are basically constructed from the convex hull of all the observed production vectors representing each firm, respectively.

The convexity assumption of the technology assumes away some economically important technological features such as indivisible production activities, economies of scale (increasing returns to scale), and economies of specialization, which all result from concavities in production functions. Estimation of efficiency and scale economies of firms against convex production technologies appears problematic. Recently, Briec and Liang [11] developed a $\mathbb{B}$-convex framework to measure performance of firms. This approach involves a multiplicative idempotent algebraic structure. It was shown in [11] that one can propose a DEA framework to model the technology. Along this line performance of firms can be evaluated by computing Farrell efficiency measures in closed form. There exist, however, several other approaches relaxing convexity. Among others, Deprins, Simar and Tulkens [17] and

Tulkens [41] introduced the so called FDH (Free Disposal Hull) method relaxing convexity and assuming only a strong disposal assumption of the technology.

$\mathbb{B}$-convexity involves a multiplicative structure of the technology and distance functions. However, there exist many other distance functions based upon translation properties and additive structures. For example, the Blackorby-Donaldson's translation function [4] is a translation homothetic distance function that can be used to measure efficiency. Similar distance functions were proposed in consumer and production theory by Luenberger [28,29] and Chambers, Chung and Färe [13]. A recent overview on the axomatic of efficiency measurement can be found in the works by Russell and Schworm [37,38]. The paper proposes a framework allowing to take into account translation homothetic structures in productivity measurement. This we do by modelling production technologies with an upper semilattice structure satisfying a translation homothetic property. Hence, the concept of max-plus convexity making the formal substitution $+ \mapsto \max$ and $\cdot \mapsto +$ is considered. In finite dimensional space, Max-Plus convex and $\mathbb{B}$-convex sets are isomorphic and, consequently, a proposition that is true in the framework of $\mathbb{B}$-convexity holds, with obvious lexical modifications, in Max-Plus convexity. A general overview on Max-Plus algebra can be found in [21]. We call technologies combining both an upper semilattice structure and an additivity assumption, Max-Plus production technologies. In the following, tropical technologies will be a generic term to describe both Max-Plus and $\mathbb{B}$-convex technologies. This terminology comes from the fact that all this analysis relies on the field of tropical mathematics. In general, the term "tropical" stands for algebraic structures involving an idempotent semi-ring.

Max-Plus convex sets belong to a special class of path-connected and contractible semilattices defined over the nonnegative Euclidean orthant. An upper (lower) semilattice is a partially ordered set in which each pair of elements has a least upper (greatest lower) bound. Max-Plus convexity has two basic implications on the nature of a production technology and the way outputs are produced from inputs. First, since a Max-Plus convex set is an upper semilattice, the least upper bound of a pair of input vectors can produce the upper bound of the outputs they can individually produce. It has been shown in [9] that these technologies belong to the class of the Kohli input price (KI) nonjoint technologies (see for instance [27]). Notice that this assumption is no less intuitive than the one of convexity. Namely, convex technologies imply that the convex combination of a pair of input vectors can produce the convex combination of the output vectors they can individually produce. The second implication is additivity. Max-Plus convex sets, by virtue of their nature, have a path-connected structure which results from the fact that inputs and outputs can be additively contracted. Therefore, a Max-Plus convexity assumption of the technology, implicitly, assumes that inputs and output are translatable. The non-convex structure of tropical production technologies can be related to the presence of indivisibilities in all multistage production process. Tone and Sahoo [40] analyzed in details the potential for reorganization of inputs, which can emerge due to indivisibility of specific inputs. They showed that the presence of indivisibilities makes the technology structure non-convex.

Paralleling Charnes, Cooper and Rhodes, a nonparametric production model can be derived from the weakly monotonic Max-Plus convex hull of a data set. From a computational standpoint, one can show that the translation distance function can be calculated in closed form for each unit and requires a smaller number of arithmetic operations than the one involved by DEA.

The paper unfolds as follows. Section 2 presents the background. In section 3, Max-Plus convexity is introduced in a production context. Section 5 proposes and develops a Max-Plus convex DEA model. Dual properties are analyzed in section 4. Finally, it is shown that the translation distance function can be computed in closed form.

2 The Production Model

The following subsections are devoted to present basic concepts of production theory as well as traditional methods for estimating the production frontier in a nonparametric context.

2.1 Definitions and Concepts

Let us define the notation used in the paper. For $z, w \in \mathbb{R}^d$ we denote $z \leq w \iff z_i \leq w_i$ $\forall i \in \{1, ..., d\}$. Let $\mathbb{R}^d_+ = \{z \in \mathbb{R}^d : 0 \leq z\}$ be the non negative Euclidean d-orthant.

Now let $m, n \in \mathbb{N}$ be two positive natural numbers such that $d = m + n$. A production technology transforms inputs $x = (x_1, ..., x_m)$ into outputs $y = (y_1, ..., y_n)$. The set $T \subset \mathbb{R}^{m+n}_+$ of all input-output vectors that are feasible is called the production technology and is defined as follows:

$$T := \left\{(x, y) \in \mathbb{R}^{m+n}_+ : x \text{ can produce } y\right\}. \tag{2.1}$$

T is also characterized by an input correspondence $L : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ and an output correspondence $P : \mathbb{R}^m_+ \rightrightarrows \mathbb{R}^n_+$, where:

$$L(y) := \left\{x \in \mathbb{R}^m_+ : (x, y) \in T\right\} \tag{2.2}$$

is the set of all input vectors that yield at least y and

$$P(x) := \left\{y \in \mathbb{R}^n_+ : (x, y) \in T\right\} \tag{2.3}$$

is the set of all the output vectors obtainable from x. Observe that T just defines a binary relation $T \subset \mathbb{R}^m_+ \times \mathbb{R}^n_+$ and L, P are its "lower" and "upper" level sets. In the remainder of the paper we denote each input-output vector as

$$z = (x, y) \in T. \tag{2.4}$$

Now, let

$$K = \mathbb{R}^m_+ \times (-\mathbb{R}^n_+) \tag{2.5}$$

denotes the strong disposal cone. There are some assumptions that can be made on the production technology (see Shephard [39] for a taxonomy):

TC : *For any* $z \in T$, $(z - K) \cap T$ *is bounded and* T *is a closed set.*
TS : T *is strongly disposable, i.e.* $T = (T + K) \cap \mathbb{R}^d_+$.

TC is a standard mathematical requirement. Moreover, it postulates an infinite output cannot be produced from a finite input. TS defines a technology with strongly disposable inputs and outputs i.e., fewer outputs can always be produced with more inputs, and inversely. Notice that, actually, we do not assume that $0 \in T$, i.e., no outputs without inputs.

In the following, these axioms are restricted to input and output sets. For all $y \in \mathbb{R}^n_+$, we consider the following axioms.

LC : $L(y)$ *is a closed set.*
LS : $L(y)$ *is strongly disposable, i.e.* $L(y) = L(y) + \mathbb{R}^m_+$.

LS imposes that an output can always be produced with more inputs. For all $x \in \mathbb{R}^m_+$, one can impose assumptions on the output set.

PC : $P(x)$ is a closed and bounded set.
PS : $P(x)$ is strongly disposable, i.e. $P(x) = (P(x) - \mathbb{R}^n_+) \cap \mathbb{R}^n_+$.

Notice that PC means that for $x \in \mathbb{R}^m_+$, $P(x)$ is a compact set. This assumption imposes that an infinite output cannot be produced from a finite input. PS imposes that less output can always be produced with the same input.

Technical efficiency can be measured by introducing the concept of an input distance function. One can loosely say that this distance function selects the closest point to any observed firms on the boundary of the production set. Along this line, the problem of measuring technical efficiency can be readily solved by linear programming. Let us define $R_x = \{\mu x : \mu > 0\}$. The Farrell input technical efficiency measure introduced by Farrel [18] is essentially the inverse of the Shephard's distance function [39]. It is the map $E_{\text{in}} : \mathbb{R}^{m+n}_+ \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined as:

$$E_{\text{in}}(x, y) := \inf \{\lambda \geq 0 : (\lambda x, y) \in T\} \tag{2.6}$$

if $R_x \cap L(y) \neq \emptyset$ and takes the value $+\infty$ otherwise.

In the input oriented case, this measure indicates the minimum contraction of an input vector by a scalar λ still remaining in the technology. Equivalently, it measures the maximal amount an input vector can be shrunk along a ray until it reaches the isoquant of the input set $L(y)$.

In the following, we introduce the input Blackorby-Donaldson translation function [4]. Let $\mathbb{1}_m$ be the m-dimensional vector whose all the components are one. The input Blackorby-Donaldson translation function is the map

$$\mathbb{D}_{\text{in}} : \mathbb{R}^{m+n}_+ \times \mathbb{R}^{m+n}_+ \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\} \tag{2.7}$$

defined by:

$$\mathbb{D}_{\text{in}}(x, y) := \sup\{\delta \in \mathbb{R} : (x - \delta \mathbb{1}_m, y) \in T\} \tag{2.8}$$

if $(x - \delta \mathbb{1}_m, y) \in T$ for some $\delta \in \mathbb{R}$ and taking the value $-\infty$ otherwise. One can equivalently write

$$\mathbb{D}_{\text{in}}(x, y) = \sup\{\delta \in \mathbb{R} : x - \delta \mathbb{1}_m \in L(y)\}. \tag{2.9}$$

This definition implies that if $(x, y) \notin T$ then $\mathbb{D}_{\text{in}}(x, y) = -\infty$. The vector $\delta \mathbb{1}_m$ determines the direction in which $\mathbb{D}_{\text{in}}(x, y)$ is defined. Thus, this function is defined by contracting inputs in a preassigned direction. This translation function is a special case of the directional distance function defined by Chambers, Chung and Färe [13] taking the direction $g = (\mathbb{1}_m, 0)$. In their paper they prove that for all $\alpha \in \mathbb{R}$, if $x + \alpha \mathbb{1} \in \mathbb{R}^m_+$ then $\mathbb{D}_{\text{in}}(x + \alpha \mathbb{1}, y) = \mathbb{D}_{\text{in}}(x, y) + \alpha$. In the case where the input set is strongly disposable, the input translation distance function is weakly monotonic, i.e. $u \geq x \Longrightarrow \mathbb{D}_{\text{in}}(u, y) \geq \mathbb{D}_{\text{in}}(x, y)$. The translation distance function can be viewed as a restricted case of the **topical functions** introduced in [23] (see [35] for related topics). A function $f : \mathbb{R}^m \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is called topical if this function is weakly monotonic with respect to the usual partial order defined over $\mathbb{R}^m$ and satisfies translation homotheticity ($f(x + \alpha \mathbb{1}) = f(x) + \alpha$ for all $x \in \mathbb{R}^m$ and all $\alpha \in \mathbb{R}$). It follows that for all $y \in \mathbb{R}^n_+$ the map $x \mapsto \mathbb{D}_{\text{in}}(x, y)$ satisfies the translation property of topical functions when $x + \alpha \mathbb{1}_m \in \mathbb{R}^m_+$. The input oriented translation function is also

Symmetrically, one can define an output Farrell measure $E_{\text{out}} : \mathbb{R}^{m+n}_+ \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$ defined by: $E_{\text{out}}(x, y) := \sup \{\theta > 0 : (x, \theta y) \in T\}$ if $R_y \cap P(x) \neq \emptyset$ and takes the value $-\infty$ otherwise. This searches for the maximum expansion of an output vector by a scalar θ to the production frontier. These input and output distance functions can be interpreted in term of co-gauge and gauge respectively.

related to the **nonlinear scalarization function** defined in [22] and [33] (see also [25] and Definition 1.40, p.13). Making obvious changes of variables, it is also related to the nonlinear functional introduced in [24] (see Definition 2.23, p.39).

The output-oriented Blackorby-Donaldson Translation function is the map $\mathbb{D}_{\text{out}} : \mathbb{R}_+^{m+n} \times \mathbb{R}_+^{m+n} \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ defined by: $\mathbb{D}_{\text{out}}(x, y) := \sup\{\delta : y + \delta \mathbb{1}_n \in P(x)\}$. In such a case the outputs are expanded in the direction determined by the vector $\mathbb{1}_n$. As in the input case, this output translation function can be related to topical functions and nonlinear scalarization functions (see [35], [25], [24]). Finally, one can introduce a graph oriented translation function $\mathbb{D} : \mathbb{R}_+^{m+n} \times \mathbb{R}_+^{m+n} \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$\mathbb{D}(x, y) := \sup\{\delta : (x - \delta \mathbb{1}_m, y + \delta \mathbb{1}_n) \in T\} \tag{2.10}$$

if $(x - \delta \mathbb{1}_m, y + \delta \mathbb{1}_n) \in T$ for some $\delta \in \mathbb{R}$ and taking the value $-\infty$ otherwise. This function is defined by simultaneously contracting inputs and expanding outputs.

The following subsection presents a classical approach for estimating a production technology.

A production technology T satisfies a constant returns to scale assumption if and only if for all $(x, y) \in T$ and all $\theta \geq 0$, $(\theta x, \theta y) \in T$. In such a case we have $E_{\text{in}}(x, y) = [E_{\text{out}}(x, y)]^{-1}$. Following Chambers [14] a technology is graph translation homothetic if and only if for all $(x, y) \in T$ and all $\delta \in \mathbb{R}$

$$(x + \delta \mathbb{1}_m, y + \delta \mathbb{1}_n) \geq 0 \text{ implies } (x + \delta \mathbb{1}_m, y + \delta \mathbb{1}_n) \in T. \tag{2.11}$$

In the remainder, we shall say that a map $f : \mathbb{R}^d \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is translation homothetic if and only if for all $\alpha \in \mathbb{R}$ and all $z \in \mathbb{R}^d$ we have $f(z + \alpha \mathbb{1}_d) = \alpha + f(z)$.

2.2 Nonparametric Technologies

Following Farrell [18], Charnes, Cooper and Rhodes [12] introduced the DEA model. In their approach the technology is derived from the smallest convex cone containing all the observed firms. This representation involves a constant return to scale assumption of the technology. Such a production model is said to be nonparametric because it does not require some functional specification of the production frontier. Suppose that $A = \{z^1, ..., z^l\}$ represents a set of l observed firms $z^k = (x^k, y^k)$ $(k = 1, ..., l)$ operating in specific sector of the economy. Under a constant returns to scale assumption, this nonparametric technology is defined by

$$T_c^+ := \Big\{(x, y) \in \mathbb{R}_+^d : x \geq \sum_{k=1}^{l} \mu_k \, x^k, y \leq \sum_{k=1}^{l} \mu_k \, y^k, \mu \geq 0\Big\}. \tag{2.12}$$

Equivalently, this subset is the weakly monotonic conical convex hull of A that is $T_c^+ = (Cc(A) + K) \cap \mathbb{R}_+^d$. The convex hull and the conical convex hull of a finite set A are denoted $Co(A)$ and $Cc(A)$ respectively. In our notations, the superscript "+" is justified by the fact that traditional convexity involves additivity.

Following Banker, Charnes, Cooper [2], the production technology can also be defined as the weakly monotonic convex hull of the observations. The production set is then modelled adding the constraint $\sum_{k=1}^{l} \mu_k = 1$ in equation (2.12). This specification implies a variable returns to scale assumption. The production technology is then defined by:

$$T_v^+ := \Big\{(x, y) \in \mathbb{R}_+^d : x \geq \sum_{k=1}^{l} \mu_k \, x^k, y \leq \sum_{k=1}^{l} \mu_k \, y^k, \mu \geq 0, \sum_{k=1}^{l} \mu_k = 1\Big\}. \tag{2.13}$$

We have equivalently $T_v^+ = (Co(A) + K) \cap \mathbb{R}_+^d$. The above approach summarizes the so called DEA method (Data Envelopment Analysis) which leads to an operational definition of the production set. Technical efficiency can then be measured using linear programming.

Notice that there exist some models in which convexity is dropped. A classical example is the FDH approach introduced and developed in [17, 41] (FDH stands for "Free Disposal Hull"). The technology is then the smallest set containing the data and satisfying a strong disposal assumption.

More recently $\mathbb{B}$-convex nonparametric models were introduced in [9]. A subset C of $\mathbb{R}_+^n$ is $\mathbb{B}$-convex if and only if for all $z, u \in C$ and all $t \in [0,1]$, $z \vee tu \in C$. The subset of $\mathbb{R}_+^{m+n}$ defined by

$$T_{\max} := \Big\{(x,y) \in \mathbb{R}_+^{m+n} : x \geq \bigvee_{k=1}^{l} t_k x^k, y \leq \bigvee_{k=1}^{l} t_k y^k, \max_{k=1,\ldots,l} t_k = 1, t \geq 0\Big\} \tag{2.14}$$

is called a **$\mathbb{B}$-convex nonparametric estimation** of the production technology.
It has been proved in [9] that $T_{\max}$ is a closed $\mathbb{B}$-convex set. Consequently, it also has an upper semilattice structure.

The next section presents Max-Plus convexity and analyzes its connections to the production model.

3 Max-Plus Convexity and Tropical Technologies

This section introduces **Max-Plus convexity**. In finite dimensional space $\mathbb{B}$-convex sets are homeomorphic to Max-Plus convex sets.

3.1 Max-Plus Background

The so called Max-Plus algebra denoted by $\mathbb{R}_{\max} = \big(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes\big)$ is the semiring composed of the set $\mathbb{R} \cup \{-\infty\}$ which is defined by the maximization operation as addition: $s \oplus t := \max(s,t)$ and the usual addition operation as multiplication: $s \otimes t := s + t$. The Max-Plus algebra is an example of a Maslov's semimodule structure. In the following a subset M of $\mathbb{R} \cup \{-\infty\}$ is Max-Plus-convex if and only if for all $z, u \in M$ and all $t \in [-\infty, 0]$, $z \oplus tu \in M$.

We only wish here to point the fact that Max-Plus convex sets are isomorphic to $\mathbb{B}$-convex sets and consequently, all that has been proved in the framework of $\mathbb{B}$-convexity holds with obvious lexical modifications, in Max-Plus convexity.
For z and z' in $\mathbb{R}_{\max}^d$ let $d_{\mathrm{M}}(z,z') :=\| \mathbf{e}^z - \mathbf{e}^{z'} \|_\infty$ where $\mathbf{e}^z := (e^{z_1}, ..., e^{z_d})$, with the convention $e^{-\infty} = 0$, and, for $y \in \mathbb{R}_+^d$, $\| y \|_\infty = \max_{i=1,\ldots,d} y_i$. The map $z \mapsto \mathbf{e}^z$ is a homeomorphism from $\mathbb{R}_{\max}^d$ with the metric d_{M} to $\mathbb{R}_+^d$ endowed with the metric induced by the norm $\| \cdot \|_\infty$; its inverse is the map $\mathbf{ln}(y) := (\ln(y_1), ..., \ln(y_d))$ from $\mathbb{R}_+^d$ to $\mathbb{R}_{\max}^d$, with the convention $\ln(0) = -\infty$.
The following two assertions hold and are equivalent:

A referee asked for the mnemonic value of $\mathbb{B}$ in $\mathbb{B}$-convexity. It was mentioned in [7] that the name of one of the authors originates from Britany.

Where for all $z^1, z^2, ..., z^l \in \mathbb{R}^d$

$$\bigvee_{k=1}^{l} z_k := \big(\max\{z_1^1, ..., z_1^l\}, \ldots, \max\{z_d^1, ..., z_d^l\}\big) \text{ and } \bigwedge_{k=1}^{l} z_k := \big(\min\{z_1^1, ..., z_1^l\}, \ldots, \min\{z_d^1, ..., z_d^l\}\big).$$

1. A subset C of $\mathbb{R}^d_{\max}$ is Max-Plus convex if and only if the set $\{\mathbf{e}^x : x \in C\}$ is a $\mathbb{B}$-convex subset of $\mathbb{R}^d_+$.
2. A subset C of $\mathbb{R}^d_+$ is $\mathbb{B}$-convex convex if and only if the set $\{\mathbf{ln}(x) : x \in C\}$ is a Max-Plus convex subset of $\mathbb{R}^d_{\max}$.

One can loosely say that Max-Plus convexity is obtained from usual convexity making the formal substitution $+ \mapsto \oplus$ and $\cdot \mapsto \otimes$. Semilattices play a key role in such a context context. From the notation above, it follows that a subset $L \subset \mathbb{R}^d$ is an upper-semilattice if and only if $\forall z, t \in L$ then $z \oplus t \in L$, and it is a **lower-semilattice** if and only if $\forall z, t \in L$ then $z \ominus t \in L$.

Paralleling the usual multiplication of vectors by scalar numbers, for all $s \in \mathbb{R}_{\max}$ and all $z \in \mathbb{R}^d_{\max}$ the Max-Plus multiplication by a scalar number is defined by:

$$s \otimes z := (s \otimes z_1, ..., s \otimes z_d) = (s + z_1, ..., s + z_d) = z + s\mathbb{1}_d. \tag{3.1}$$

Clearly, the input translation function can then be defined as

$$\mathbb{D}_{\text{in}}(x, y) = \sup\{\delta \in \mathbb{R} : (-\delta) \otimes x \in L(y)\}. \tag{3.2}$$

On the output side, we have

$$\mathbb{D}_{\text{out}}(x, y) = \sup\{\delta \in \mathbb{R} : \delta \otimes y \in P(y)\}. \tag{3.3}$$

Graph translation homotheticity can be redefined as

$$(\delta \otimes x, \delta \otimes y) \in T \tag{3.4}$$

for all $\delta \in \mathbb{R}$ such that $(\delta \otimes x, \delta \otimes y) \geq 0$.

3.2 Max-Plus Convex Sets

Definition 3.1. A subset M of $\mathbb{R}^d_{\max}$ is Max-Plus convex, if and only if for all $z, u \in M$, and all $s \in [-\infty, 0]$ we have:

$$z \oplus (s \otimes u) \in M.$$

Paralleling this definition a set C such that for all $z, u \in C$, and $s \in \mathbb{R}_{\max}$ implies $z \oplus (s \otimes u) \in C$ is called a **Max-Plus convex cone**. A Max-Plus convex set M satisfies the following properties:

(i) M is an upper-semilattice;

(ii) M is a path-connected set with respect to the topology induced by the distance d_M;

(iii) If $\{z^1, ..., z^l\} \subset M$, with $s_k \leq 0$ for all $k = 1, ..., l$ and $\bigoplus_{k=1}^l s_k = 0$ then we have $\bigoplus_{k=1}^l (s_k \otimes z^k) \in M$;

In the remainder, we denote for all $z^1, ..., z^l \in \mathbb{R}^d$

$$\bigoplus_{k=1}^{l} z^k := \bigvee_{k=1}^{l} z^k = \left(\max\{z^1_1, ..., z^l_1\}, \ldots, \max\{z^1_d, ..., z^l_d\}\right)$$

and

$$\bigominus_{k=1}^{l} z^k := \bigwedge_{k=1}^{l} z^k = \left(\min\{z^1_1, ..., z^l_1\}, \ldots, \min\{z^1_d, ..., z^l_d\}\right).$$

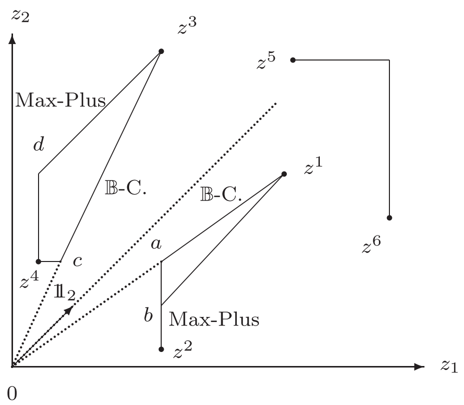

Figure 3.1 Max-Plus and 𝔹-convex segment lines.

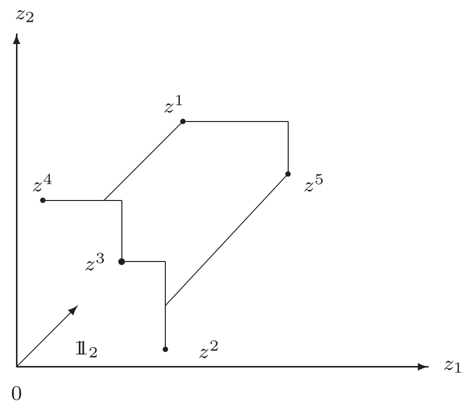

Figure 3.2 Max-Plus convex hull of 5 points.

(iv) If $N \subset \mathbb{R}^d$ is Max-Plus convex set, then $M \cap N$ is a Max-Plus convex set.

Remark 3.2. If L is a strongly disposable subset of $\mathbb{R}^d_{\max}$ then it is a Max-Plus convex set, since $\forall z, u \in L$ and $s \in [-\infty, 0]$, we have $(s \otimes z) \oplus u \geq u$ and L is a strongly disposable set implies $(s \otimes z) \oplus u \in L$.

One can then introduce the notion of Max-Plus convex hull.

Definition 3.3. Let $A = \{z^1, ..., z^l\}$ be a finite part of $\mathbb{R}^d_{\max}$. The set

$$\mathbb{M}(A) := \Big\{ \bigoplus_{k=1}^{l} (s_k \otimes z^k) : s \in [-\infty, 0]^l, \bigoplus_{k=1}^{l} s_k = 0 \Big\}$$

is called the Max-Plus convex hull of A.

The geometrical representation of the Max-Plus convex hull of two or several points in each case is given in the following two figures.

In Figure 3.1, the difference between the convex lines joining two points in 𝔹-convexity and Max-Plus convexity are analyzed.

In Max-Plus algebra, the convex hull of $\{z^1, z^2\}$ is the broken line $[z^1, b, z^2]$. Notice that the segment line $[z^1, a]$ belongs to the affine line spanned from z^1 in the direction of the vector $\mathbb{1}_2$. The 𝔹-convex hull (𝔹-C) is the broken line $[z^1, a, z^2]$ and point a belongs to the ray spanned from the origin to z^1. The convex lines joining z^3 and z^4 differ more significantly. The Max-Plus convex hull of $\{z^3, z^4\}$ is the broken line $[z^3, d, z^4]$ and the segment line $[z^3, z^d]$ belong to the affine line spanned from z^3 in the direction of the vector $\mathbb{1}_2$. The 𝔹-convex hull is the broken line $[z^3, d, z^4]$. In the last case, z^5 and z^6 are not ordered and the convex hull is identical in both cases. Along this line, Figure 3.2 depicts the smallest Max-Plus convex sets which contains 5 points. These examples show that 𝔹-convexity is based upon a multiplicative structure and Max-Plus convexity upon an additive one. Along this line, Max-Plus structures of the production technology will be useful to compute translation distance function.

3.3 Structure of Max-Plus Technologies

In this section the implications of Max-Plus convexity to the technology from an economic viewpoint are analyzed. Max-Plus convexity means that the technology obeys two basic properties. First it is endowed with a upper semilattice structure: the least upper bound of two input vectors allows to produce the least upper bound of the output vectors they can individually produce. This upper semilattice structure stands in place of the additive structure inherited from the traditional convexity assumption. Moreover, Max-Plus convexity implies that the production vectors satisfy a translation homotheticity property as usual convexity. We have mentioned above that a tropical technology is a path-connected upper semilattice. From an economical viewpoint, connexity is important because it allows the possibility of continuously transforming a production technique.

The following result establishes a relationship between graph translation homotheticity and Max-Plus convexity. We say that a technology T is: (i) lower graph translation homothetic if and only if $\forall \delta \leq 0$ and all $(x, y) \in T$, such that $(\delta \otimes x, \delta \otimes y) = (x + \delta \mathbb{1}_m, y + \delta \mathbb{1}_n) \geq 0$ we have $(\delta \otimes x, \delta \otimes y) \in T$; (ii) upper graph translation homothetic if and only if $\forall \delta \geq 0$ and all $(x, y) \in T$, $(\delta \otimes x, \delta \otimes y) \in T$.

Lemma 3.4. *Suppose that T is an upper semilattice.*

(a) *If $0 \in T$ and T is Max-Plus convex then T satisfies the lower graph translation homothetic assumption.*

(b) *If T is lower graph translation homothetic then it is Max-Plus convex.*

Proof. (a) Suppose that $0 \in T$ and T is Max-Plus convex. Therefore, for all $z \in T$ and all $\delta \leq 0$, we have $0 \oplus (\delta \otimes z) \in T$. Moreover, for all $\delta \leq 0$ if $\delta \otimes z \in T$, then $\delta \otimes z \geq 0$. It follows that

$$0 \oplus (\delta \otimes z) = (\delta \otimes z) \in T,$$

which proves the first part of the statement. (b) If T is lower graph translation homothetic, for all $\delta \leq 0$ and all $z \in T$, if $\delta \otimes z \geq 0$ then $\delta \otimes z \in T$. Since T is an upper semilatice for all $z, w \in T$, $w \oplus (\delta \otimes z) \in T$. Therefore T is Max-Plus convex, which ends the proof. □

This result shows that technologies having an upper semilattice structure and satisfying lower graph translation homotheticity are Max-Plus convex. Consequently, under reasonable assumptions, an upper semilattice technology may be Max-Plus convex.

Notice that if a technology is Max-Plus convex then the input and output sets are Max-Plus convex.
The next statement establishes that, whenever the input set satisfies a strong disposal assumption, it is Max-Plus convex.

Lemma 3.5. *For all $y \in \mathbb{R}^n_+$, if the input set $L(y)$ satisfies (LS) then it is a Max-Plus convex set.*

Proof. Suppose $x \in L(y)$. If the strong disposal assumption holds, then $L(y) = L(y) + \mathbb{R}^m_+$. For all $u \in L(y)$ and all $\lambda \leq 0$, $x \oplus (\lambda \otimes u) \geq x$. Consequently $x \oplus (\lambda \otimes u) \in x + \mathbb{R}^m_+ \subset L(y) + \mathbb{R}^m_+ = L(y)$, which ends the proof. □

This result shows that there exists a large class of production technologies whose input correspondences satisfy a Max-Plus convexity property. Max-Plus convexity is more restrictive in an output context. The next statement proves that the output sets have a supremum with respect to the partial order $\leq$. One can then prove that, under a strong disposal assumption (PS), this supremum entirely characterizes the output set.

Proposition 3.6. *Suppose that the output set $P(x)$ satisfies (PC) and is Max-Plus convex. Then:*
(a) *There exists a supremum $\overline{y}_x \in P(x)$ such that $\forall y \in P(x)$, $y \leq \overline{y}_x$.*
(b) *Moreover*

$$P(x) = \Big\{ y \in \mathbb{R}^n_+ : \bigoplus_{j=1}^{n} y_j \otimes (-\overline{y}_{x,j}) \leq 0 \Big\}.$$

Proof. (a) Since PC holds the result can be deduced from Briec, Horvath and Rubinov [8] who proved that every compact Max-Plus convex set has a supremum. (b) We have established above that, if $P(x) \neq 0$ is a closed and bounded set satisfying a strong disposal assumption, then there exists a supremum element $\overline{y}_x$ such that $P(x) = \big(\overline{y}_x - \mathbb{R}^n_+\big) \cap \mathbb{R}^n_+$. Equivalently, one can write

$$P(x) = \{y \in \mathbb{R}^n : 0 \leq y \leq \overline{y}_x\}.$$

Consequently, $P(x) = \{y \in \mathbb{R}^n_+ : \bigoplus_{j=1}^{n} y_j \otimes (-\overline{y}_{x,j}) \leq 0\}$. □

It was established in [9] that technologies whose output set is a compact upper semilattice having a maximal element belong to the class of the Kohli technologies. A technology $T \subset \mathbb{R}^m_+ \times \mathbb{R}^n_+$ is a **Kohli input price (KI) nonjoint technology** if there exist n single output technologies $T^j \subset \mathbb{R}^m_+ \times \mathbb{R}_+$ such that, for all input vectors $x \in \mathbb{R}^m_+$, $T(x) = T^1(x) \times \cdots \times T^n(x)$. This is a generalization of the fixed-coefficient Leontief transformation. More details can be found in [27](see p 215). One can give an immediate characterization of the technology using the fact that the output set has a maximal element. The next results ends this section proving that a multi-output Max-Plus convex technology satisfying a strong disposal assumption has a functional representation.

Proposition 3.7. *Suppose that T is Max-Plus convex and satisfies TC. Then there exists a vector function $F : \mathbb{R}^m_+ \longrightarrow \mathbb{R}^n_+$ such that*

$$T = \big\{(x, y) \in \mathbb{R}^{m+n}_+ : y \leq F(x)\big\}.$$

Proof. The proof is obtained by defining a map $F : \mathbb{R}^m_+ \longrightarrow \mathbb{R}^n_+$ such that $F(x) = \bar{y}_x$, where for all $x \in \mathbb{R}^m$ $\bar{y}_x$ is the supremum of $P(x)$.□

Notice that this statement is generally wrong for convex multi-output technologies. This functional representation comes from the fact that a Max-Plus convex technology satisfying a strong disposal assumption involves a cubic multi-output technology (see Proposition 3.6.)

Notice that using some functional characterization of half-space in Max-Plus convexity (see for instance [10,15,26]), on can give the following example of a multi-output production technology having a functional characterization:

$$T = \left\{ (x, y) \in \mathbb{R}^{m+n}_+ : \bigoplus_{i=1}^{m} (a_i \otimes x_i) \oplus c \leq \bigoplus_{j=1}^{n} (b_j \otimes y_j) \oplus d \right\} \tag{3.5}$$

where $a \in \mathbb{R}^m_+$, $b \in \mathbb{R}^n_+$ and $c, d \geq 0$. If $c = d = 0$, it is easy to check that the technology is graph translation homothetic.

4 Some Duality Results for Max-Plus Convex Technologies

In this section, some aspects of the economic meaning of Max-Plus convexity are analyzed. Among other things, we focus on the dual characterization of input and output sets for technologies satisfying a Max-Plus convexity assumption. The key discrepancy, by comparison with vector spaces, is that one needs to characterize the half-spaces in Max-Plus convexity (see for instance [10, 26, 42]). To develop a dual framework based upon gauge functions one uses specific functional forms to separate a point from a subspace, or more generally, from a convex set. It follows that a large class of input and output sets can be expressed as the intersection of a collection of suitable half-spaces.

4.1 Duality and Input Translation Distance Function

Let $C_{\max} : \mathbb{R}^m_+ \times \mathbb{R}^n_+ \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ be the function defined by:

$$C_{\max}(w, y) := \inf_{x \in L(y)} \max_{i=1,\ldots,m} w_i x_i. \tag{4.1}$$

This function maps each input price $w \in \mathbb{R}^m_+$ and each output vector $y \in \mathbb{R}^n_+$ to the minimum of the maximal individual cost of each input. Thus, it is called the **max-cost function**. Basically, this function has some formal analogy with the cost function making the formal substitution $+ \to \max$. By the way, $C_{max}(w, y)$ can be seen as some kind of a max-plus support function. Not surprisingly, its economic interpretations is different. For example, let us consider a firm that selects some input by minimizing this max-cost function. The map $x \mapsto \max_{i=1,\ldots,m} w_i x_i$ can be interpreted as the maximum of the individual costs of each factor. By definition, $C_{\max}(w, y)$ gives the minimum amount of the maximal cost required to produce a production vector y. It has been established in [9, 11] that Farrell technical efficiency measure is interpretable as a maximum ratio between the max-cost function and the max-cost of the observed input vector.

This duality property only requires a strong disposal assumption. However, it has been shown in [9] that $\mathbb{B}$-convexity is sufficient. Under strong disposal assumption of input sets (in such a case they are Max-Plus convex), the following result establishes a dual characterization of the input sets involving the function $C_{\max}(w, y)$.

Proposition 4.1. *Let $y \in \mathbb{R}^n_+$ and suppose that $L(y)$ satisfies (LC) and (LS). Then, we have:*

$$L(y) = \Big\{x \in \mathbb{R}^m_+ : \max_{i=1,\ldots,m} w_i x_i \geq C_{\max}(w, y),\ \forall w \geq 0\Big\}.$$

This property was established in [9]. It can also be deduced, for example, from [15, 26]. Using equivalent notations we have:

$$L(y) = \Big\{x \in \mathbb{R}^m_+ : \bigoplus_{i=1}^{m} w_i x_i \geq C_{\max}(w, y),\ \forall w \geq 0\Big\}. \tag{4.2}$$

Notice that this duality result cannot be obtained making the formal substitution $+ \mapsto \oplus$ and deriving a duality result from [9,11]. Indeed it combines both the multiplicative structure of the max-cost function and the additive nature of the translation function. For all $w \in \mathbb{R}^m_+$, let $I(w) := \{i \in \{1, ..., m\} : w_i > 0\}$, be the support of w.

Proposition 4.2. *If $L(y)$ satisfies (LC) and (LS), then for all input vectors x:*

$$\mathbb{D}_{\text{in}}(x, y) = \inf_{w \geq 0} \max_{i \in I(w)} \Big\{x_i - \frac{C_{\max}(w, y)}{w_i}\Big\}$$

$$= \inf_{w\geq 0} \bigoplus_{i\in I(w)} \Big\{x_i \otimes (-\frac{C_{\max}(w,y)}{w_i})\Big\}$$

and

$$\begin{aligned} C_{\max}(w,y) &= \inf_{x\geq 0} \Big\{ \max_{i=1,\dots,m} \big\{w_i(x_i - \mathbb{D}_{\text{in}}(x,y))\big\}\Big\} \\ &= \inf_{x\geq 0} \Big\{ \bigoplus_{i=1}^{m} w_i(x_i - \mathbb{D}_{\text{in}}(x,y))\Big\}. \end{aligned}$$

Proof. First, one equivalently has

$$\mathbb{D}_{\text{in}}(x,y) = \inf\{\delta \in \mathbb{R} : x - \delta \mathbb{1}_m \notin L(y)\}.$$

Since $L(y) = \bigcap_{w\in\mathbb{R}^m_+}\{x \in \mathbb{R}^m_+ : \max_{i\in I(w)} w_i x_i \geq C_{\max}(w,y)\}$, it follows that

$$\begin{aligned} \mathbb{R}^m_+ \backslash L(y) &= \bigcup_{w\in\mathbb{R}^m_+} \mathbb{R}^m_+ \backslash \{x \in \mathbb{R}^m_+ : \max_{i\in I(w)} w_i x_i \geq C_{\max}(w,y)\} \\ &= \bigcup_{w\in\mathbb{R}^m_+} \{x \in \mathbb{R}^m_+ : \min_{i\in I(w)} w_i x_i < C_{\max}(w,y)\}. \end{aligned}$$

Consequently,

$$\begin{aligned} \mathbb{D}_{\text{in}}(x,y) &= \inf\{\delta \in \mathbb{R} : x - \delta\mathbb{1}_m \in \bigcup_{w\in\mathbb{R}^m_+} \{x \in \mathbb{R}^m_+ : \max_{i\in I(w)} w_i x_i < C_{\max}(w,y)\}\} \\ &= \inf_{w\in\mathbb{R}^m_+} \inf\{\delta \in \mathbb{R} : x - \delta\mathbb{1}_m \in \{x \in \mathbb{R}^m_+ : \max_{i\in I(w)} w_i x_i < C_{\max}(w,y)\}\} \\ &= \inf_{w\in\mathbb{R}^m_+} \inf\{\delta \in \mathbb{R} : \max_{i\in I(w)} w_i(x_i - \delta) < C_{\max}(w,y)\} \\ &= \inf_{w\geq 0} \max_{i\in I(w)} \Big\{x_i - \frac{C_{\max}(w,y)}{w_i}\Big\}. \end{aligned}$$

To prove the second part of the statement, we note that the map $x \mapsto \max_{i=1,\dots,m} w_i x_i$ is nondecreasing over $\mathbb{R}^m_+$. Hence, $C_{\max}(w,y) = \inf\{\max_{i=1,\dots,m} w_i x_i : x \in L(x)\}$ and

$$\max_{i\in I(w)} w_i\big(x - \mathbb{D}_{\text{in}}(x,y)\mathbb{1}_m\big)_i \leq \max_{i\in I(w)} w_i x_i.$$

Moreover, for all $x \in \mathbb{R}^m_+$, $x - \mathbb{D}_{\text{in}}(x,y)\mathbb{1}_m \in L(y)$. It follows that

$$C_{\max}(w,y) \leq \max_{i=1,\dots,m} \big\{w_i(x - \mathbb{D}_{\text{in}}(x,y)\mathbb{1}_m)_i\big\} \leq \max_{i=1,\dots,m} \{w_i y_i\},$$

which ends the proof. □

4.2 Output Translation Distance Function and Duality

Following the same logic as in the case of the inputs sets, one could define some min-revenue function in the case of output sets. However, in max-Plus convex structures, the output set has but a very simple dual characterization. Therefore, we just give a simple formulation based upon the maximal element of the output set. To do that an assumption of **translation maximality** is introduced. Let M be a closed Max-Plus convex set. We say that M is translation maximal, if and only if for all $x, y \in M$ such that $x \leq y$ and $t \geq 0$ then $tx \ominus y \in M$. For all $y \in \mathbb{R}^n$, let $\Delta(y) := \{y + \delta \mathbb{1}_n : \delta \in \mathbb{R}\}$ be the affine line spanned from y in the direction of the vector of the ones.

Proposition 4.3. *Let $x \in \mathbb{R}^m_+$. Suppose that $P(x)$ satisfies (PC) and is translation maximal and Max-Plus convex. Let $\overline{y}_x$ be the supremum of $P(x)$. Suppose moreover that $\Delta(y) \cap P(x) \neq \emptyset$. We have:*

$$\mathbb{D}_{\text{out}}(x, y) = \bigoplus_{j=1}^{n} \big(\overline{y}_{x,j} \otimes (-y_j)\big).$$

Proof. By definition $y + \mathbb{D}_{\text{out}}(x, y)\mathbb{1}_n \in P(x)$. Therefore,

$$\bigoplus_{j=1}^{n} (y_j + \mathbb{D}_{\text{out}}(x, y)) \otimes (-\overline{y}_{x,j}) \leq 0$$

and $\mathbb{D}_{\text{out}}(x, y) \leq \bigoplus_{j=1}^{n} \bar{y}_j \otimes (-y_j)$. Suppose that $\mathbb{D}_{\text{out}}(x, y) < (\bigoplus_{j=1}^{n} \overline{y}_{x,j}) \otimes (-y_j)$. This implies that $y + \mathbb{D}_{\text{out}}(x, y)\mathbb{1}_n < \bar{y}$. Therefore, one can find some $\bar{\delta} > \mathbb{D}_{\text{out}}(x, y)$ such that $y + \bar{\delta}\mathbb{1}_n \leq \overline{y}_x$. However, by definition $y + \bar{\delta}\mathbb{1}_n \notin P(x)$. Since

$$y + \bar{\delta}\mathbb{1}_n = y + \mathbb{D}_{\text{out}}(x, y)\mathbb{1}_n + (\bar{\delta} - \mathbb{D}_{\text{out}}(x, y))\mathbb{1}_n,$$

this contradicts translation maximality. □

This result has an immediate corollary in the case where the output set satisfies a strong disposal assumption.

Corollary 4.4. *Let $x \in \mathbb{R}^m_+$. Suppose that $P(x)$ is a Max-Plus convex set satisfying (PC) and (PS). Let $\overline{y}_x$ be the supremum of $P(x)$. Suppose moreover that $\Delta(y) \cap P(x) \neq \emptyset$. We have:*

$$\mathbb{D}_{\text{out}}(x, y) = \bigoplus_{j=1}^{n} \big(\overline{y}_{x,j} \otimes (-y_j)\big).$$

5 Tropical Nonparametric Technologies

In line with the convex production models proposed in equations (2.12) and (2.13) it is quite natural to propose a class of Max-Plus convex nonparametric models. Specifically, this section focusses on two models postulating variable and constant returns to scale respectively.

5.1 Max-Plus Nonparametric Production Technologies

We propose a Max-Plus convex nonparametric model for estimating a technology given an observed data set A. Let $A = \{(x^k, y^k) : k = 1, ..., l\} \subset \mathbb{R}^{m+n}_+$ be a set of l observed production vectors. The subset of $\mathbb{R}^{m+n}_+$ defined by

$$T_v^{\oplus} := \Big\{(x, y) \in \mathbb{R}^d_+ : x \geq \bigoplus_{k=1}^{l} (s_k \otimes x^k),\ y \leq \bigoplus_{k=1}^{l} (s_k \otimes y^k),\ \max_{k=1,\dots,l} s_k = 0, s \in \mathbb{R}^l\Big\}, \quad (5.1)$$

is called a **Max-Plus nonparametric estimation** of the production technology. One can equivalently write $T_v^{\oplus} = (\mathbb{M}(A) + K) \cap \mathbb{R}_+^{m+n}$, where $K = \mathbb{R}_+^m \times (-\mathbb{R}_+^n)$ and $\mathbb{M}(A)$ is the Max-Plus convex hull of A defined in Definition 3.3. The following result establishes the basic properties of these technologies.

Proposition 5.1. *For all subsets $A = \{(x^k, y^k) : k = 1, ..., l\} \subset \mathbb{R}_+^{m+n}$ of l observed production vectors, the nonparametric technology $T_v^{\oplus}$ is a Max-Plus convex subset of $\mathbb{R}_+^{m+n}$ and satisfies TC and TS.*

Proof. We first prove that $T_v^{\oplus}$ is a Max-Plus convex set. Assume that $z, z' \in T_v^{\oplus}$. In this case, there exists $s_1, ..., s_l \leq 0$, with $\bigoplus_{k=1}^l s_k = 0$, such that $x \geq \bigoplus_{k=1}^l s_k \otimes x^k$ and $y \leq \bigoplus_{k=1}^l s_k \otimes y^k$.

Moreover, there exists $s'_1, ..., s'_l \leq 0$ with $\bigoplus_{k=1}^l s'_k = 0$ such that we have $x' \geq \bigoplus_{k=1}^l s'_k \otimes x^k$ and $y' \leq \bigoplus_{k=1}^l s'_k \otimes y^k$.

Now, let $t, t' \leq 0$ such that $t \oplus t' = 0$. We have

$$(t \otimes x) \oplus (t' \otimes x') \geq \Big(t \bigoplus_{k=1}^{l} (s_k \otimes x^k)\Big) \oplus \Big(t' \bigoplus_{k=1}^{l} (t'_k \otimes x^k)\Big).$$

Then, we deduce that

$$\Big(t \bigoplus_{k=1}^{l} (s_k \otimes x^k)\Big) \oplus \Big(t' \bigoplus_{k=1}^{l} (s'_k \otimes x^k)\Big) = \bigoplus_{k=1}^{l} \Big((t \otimes s_k) \oplus (t' \otimes s'_k)\Big) \otimes x^k.$$

Similarly one has $(t \otimes y) \oplus (t' \otimes y') \leq \bigoplus_{k=1}^l \Big((t \otimes s_k) \oplus (t' \otimes s'_k)\Big) \otimes y^k$. Since we have

$$\bigoplus_{k=1}^{l} \Big((t \otimes s_k) \oplus (t' \otimes s'_k)\Big) = 0$$

we deduce that $(t \otimes z) \oplus (t' \otimes z') \in T_v^{\oplus}$ and consequently $T_v^{\oplus}$ is a Max-Plus convex set.

TS is immediate from the construction of $T_v^{\oplus}$. Let us prove TC. Since $\mathbb{T}(A)$ is a closed set $T_v^{\oplus}$. For all $z \in T$, since $T_v^{\oplus} = (\mathbb{M}(A) + K) \cap \mathbb{R}_+^{m+n}$, for all $(u, v) \in T_v^{\oplus}$ we have $v \leq \bigoplus_{k=1}^l y^k$. Consequently, for all $(u, v) \in (z - K) \cap T_v^{\oplus}$ we have $v \leq \bigoplus_{k=1}^l y^k$. Hence $(z - K) \cap T_v^{\oplus}$ is a subset of the box $\Big[0, (x, \bigoplus_{k=1}^l y^k)\Big]$ which is bounded and this ends the proof. □

To make a comparison, we depict the convex and Max-Plus convex cases respectively respectively in Figures 5.1 and 5.2.

In Figure 2.1, the technology is the weakly monotonic convex hull of the data set. The addition of the cone is necessary for the definition of strong disposal hull of the data sample. A similar procedure is applied to the Max-Plus convex hull of A to construct the production set $T_{\max}$ (see Figure 2.2). Given a set of input-output vectors, the respective production frontiers are different. Remark that the slope of the frontier is locally nonincreasing between points z^1 and z^2 and is locally nondecreasing between points z^2 and z^4. This makes a significant difference with the convex model where the slope of the frontier is nonincreasing. In line with Proposition 3.4, one can see that the technology is not lower translation homothetic in Figure 5.1 because $0 \notin T_v^{\oplus}$.

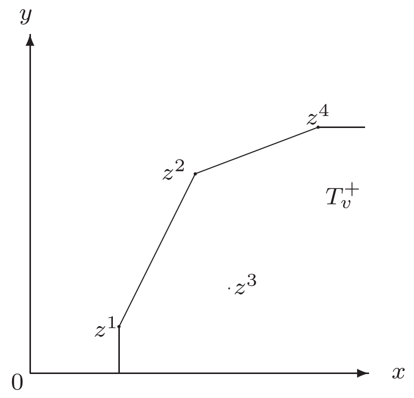

Figure 5.1 DEA nonparametric estimation

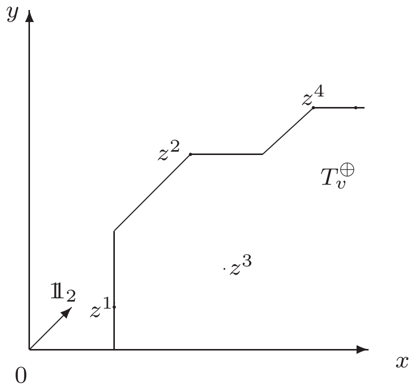

Figure 5.2 Max-Plus estimation

Figure 1: DEA nonparametric estimation

Figure 2: Max-Plus estimation

Paralleling the standard DEA model, it is quite natural to define a graph translation homothetic Max-Plus nonparametric model of the technology. This we do by dropping the last constraint in equation (5.1). The following technology is Max-Plus convex and satisfies a constant return to scale assumption.

$$T_c^{\oplus} := \Big\{(x,y) \in \mathbb{R}_+^d : x \geq \bigoplus_{k=1}^{l} s_k \otimes x^k,\ y \leq \bigoplus_{k=1}^{l} s_k \otimes y^k, s \in \mathbb{R}^l\Big\}. \tag{5.2}$$

Proposition 5.2. *For all subsets $A = \{z^1, ..., z^l\} \subset \mathbb{R}_+^d$ of l observed production vectors, the nonparametric technology $T_c^{\oplus}$ is a Max-Plus convex subset of $\mathbb{R}_+^d$ and satisfies TC and TS. Moreover, it is graph translation homothetic.*

5.2 Computing Distance Functions and Max-Plus Convex Nonparametric Technologies

In this section we provide a method for calculating the translation distance function. A system of **Max-Plus linear inequalities** is a set of linear inequalities in the same variables defined with respect to the Max-Plus algebra. The next result provides necessary and sufficient conditions for the existence of some solution. In the following, we denote ϕ_{-1} the map which associates to any positive vector the vector of the inverse components. Such a system has an equivalent one replacing the usual addition with the scalar multiplication. To do that, one can state the following result in Max-Plus that can be derived from [11]. Let us consider the maximum-inequations systems:

$$\begin{cases} \bigvee_{k=1}^{l} a_i^k x_k & \leq b_i, \quad i = 1, ..., m \\ \bigvee_{k=1}^{l} c_j^k x_k & \geq d_j, \quad j = 1, ..., n \end{cases} \tag{5.3}$$

where for $k = 1, ..., l$, $a^k \in \mathbb{R}^m_{++}$ and $b \in \mathbb{R}^m_{++}$, $c^k \in \mathbb{R}^n_{++}$ and $d \in \mathbb{R}^n_{++}$. Let $\mathcal{S}$ be the solution set of this systems. It was established in [11] that

$$\mathcal{S} \neq \emptyset \iff \bigwedge_{i=1}^{m} b_i \phi_{-1}(a_i) \in \mathcal{S}.$$

In the following, an equivalent form is provided in Max-Plus convexity. The proof is straightforward from [11].

Proposition 5.3. *Let us consider the two systems of Max-Plus linear inequalities:*

$$\begin{cases} \bigoplus_{k=1}^{l} a_i^k \otimes x_k & \leq b_i, \quad i = 1, ..., m \\ \\ \bigoplus_{k=1}^{l} c_j^k \otimes x_k & \geq d_j, \quad j = 1, ..., n \end{cases} \tag{5.4}$$

where for $k = 1, ..., l$, $a^k \in \mathbb{R}^m$, $b \in \mathbb{R}^m$, $c^k \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$. Then, this system has some solution if and only if $\bigominus_{i=1,...,m} b_i \otimes (-a_i)$ is solution.

We first recall some results obtained in the context of $\mathbb{B}$-convexity in [9] and [11] where it has been shown that Farrell input and output efficiency measure can be calculated in closed form. For all $A = \{(x^k, y^k) : k = 1, ..., l\} \subset \mathbb{R}^{m+n}_{++}$, let us denote

$$\alpha_{\bar{k},k} = \min_{i=1,...,m} \frac{x_i^{\bar{k}}}{x_i^k}.$$

Suppose moreover that $T = T_{\max}$. In [9] and [11] it has been established that for each $\bar{k} \in [l]$, the input Farrell technical efficiency measure is:

$$E_{\text{in}}(x^{\bar{k}}, y^{\bar{k}}) = \max\Big\{ \max_{j=1,...,n} \min_{\substack{k \\ y_j^{\bar{k}} \leq y_j^k}} \{\frac{y_j^{\bar{k}}}{y_k^k \alpha_{\bar{k},k}}\}, \min_k \frac{1}{\alpha_{\bar{k},k}} \Big\}. \tag{5.5}$$

Moreover the Farrell output measure is:

$$E_{\text{out}}(x^{\bar{k}}, y^{\bar{k}}) = \min_{j=1,...,n} \max_k \Big\{ \frac{y_j^k \min\{\alpha_{\bar{k},k}, 1\}}{y_j^{\bar{k}}} \Big\}. \tag{5.6}$$

Using some transformations similar to that one used to prove Proposition 5.3, it is easy to establish a closed formula for input and output translation distance functions.

Proposition 5.4. *Let $A = \{(x^k, y^k) : k = 1, ..., l\} \subset \mathbb{R}^d_+$. If $T = T_v^{\oplus}$ then for all $\bar{k} \in \{1, ..., l\}$ the input translation distance function is:*

$$\mathbb{D}_{\text{in}}(x^{\bar{k}}, y^{\bar{k}}) = \min\Big\{ \min_{j=1,...,n} \max_{\substack{k=1,...,l \\ y_j^{\bar{k}} \leq y_j^k}} \{ -y_j^{\bar{k}} + y_j^k + \beta_{\bar{k},k} \}, \max_{k=1,...,l} \beta_{\bar{k},k} \Big\}$$

where

$$\beta_{\bar{k},k} = \min_{i=1,...,m} \{x_i^{\bar{k}} - x_i^k\}.$$

Moreover, the output distance function is:

$$\mathbb{D}_{\text{out}}(x^{\bar{k}}, y^{\bar{k}}) = \min_{j=1,...,n} \max_{k=1,...,l} \left\{ y_j^k - y_j^{\bar{k}} + \min\left\{\beta_{\bar{k},k}, 0\right\} \right\}.$$

Proof. We compute the translation distance function which is obtained by computing the Debreu-Farrell input measure of technical efficiency that is the inverse of the Shephard distance function. This measure is defined as:

$$S_{\text{in}}(x,y) = \sup\{\mu > 0 : \mu^{-1}x \in L(y)\}.$$

Fix $(x^{\bar{k}}, y^{\bar{k}}) \in \mathbb{R}^d_+$. The program to solve is :

$$\begin{aligned} &\max \delta \\ &\bigoplus_{k=1}^{l} s_k \otimes x^k \leq (-\delta) \otimes x^{\bar{k}} \\ &\bigoplus_{k=1}^{l} s_k \otimes y^k \geq y^{\bar{k}} \\ &\max_{k=1,\ldots,l} s_k = 0, s \in \mathbb{R}^l \end{aligned} \tag{5.7}$$

To make the analogy with the $\mathbb{B}$-convexity, at this stage, we consider an exponential transformation of the data. The map $z \mapsto \mathbf{e}^z$ is an homeomorphism from the set $(\mathbb{R} \cup \{-\infty\})^d$ to $\mathbb{R}^d_+$. Taking the exponential function on both side and replacing the symbol $\oplus$ with $\vee$ yields:

$$\begin{aligned} &\max \delta \\ &\bigvee_{k=1}^{l} e^{s_k}\mathbf{e}^{x^k} \leq e^{-\delta}\mathbf{e}^{x^{\bar{k}}} \\ &\bigvee_{k=1}^{l} e^{s_k}\mathbf{e}^{y^k} \geq \mathbf{e}^{y^{\bar{k}}} \\ &\max_{k=1,\ldots,l} e^{s_k} = 1, \mathbf{e}^s \geq 0 \end{aligned} \tag{5.8}$$

Setting $t_k = e^{s_k}$, $\tilde{x}^k = \mathbf{e}^{x^k}$, $\tilde{y}^k = \mathbf{e}^{y^k}$ and $\mu = e^{\delta}$, we obtain the program

$$\begin{aligned} &\max \ln \mu \\ &\bigvee_{k=1}^{l} t_k\tilde{x}^k \leq \mu^{-1}\tilde{x}^{\bar{k}} \\ &\bigvee_{k=1}^{l} t_k\tilde{y}^k \geq \tilde{y}^{\bar{k}} \\ &\max_{k=1,\ldots,l} t_k = 1, t \geq 0 \end{aligned} \tag{5.9}$$

The optimal value $\mu_\star$ of this program is the Shephard distance function computed for the technology

$$\tilde{T}_{\max} = \Big\{(x,y) \in \mathbb{R}^{m+n}_+ : x \geq \bigvee_{k=1}^{l} t_k\tilde{x}_k, y \leq \bigvee_{k=1}^{l} t_k\tilde{y}_k, \max_{k=1,\ldots,l} t_k = 1, t{\geq}0\Big\}. \tag{5.10}$$

It follows that if $\lambda_\star$ is the Farrell Technical efficiency measure, we have the equality $\mathbb{D}_{\text{in}}(x^{\bar{k}}, y^{\bar{k}}) = -\ln \lambda_\star$.

For all $A = \{(\tilde{x}^k, \tilde{y}^k) : k = 1, ..., l\} \subset \mathbb{R}^{m+n}_{++}$, let us denote

$$\tilde{\alpha}_{\bar{k},k} = \min_{i=1,\ldots,m} \Big(\frac{\tilde{x_i}^{\bar{k}}}{\tilde{x_i}^{k}}\Big).$$

Using equation (5.5), the input Farrell technical efficiency measure can be computed from the technology $\tilde{T}_{\max}$, we obtain

$$\tilde{E}_{\text{in}}(\tilde{x}^{\bar{k}}, \tilde{y}^{\bar{k}}) = \max\Big\{ \max_{j=1,\ldots,n} \min_{\substack{k \\ \tilde{y}_{\bar{k},j} \leq \tilde{y}_{k,j}}} \Big\{\frac{\tilde{y}_{\bar{k},j}}{\tilde{y}_{k,j}\tilde{\alpha}_{\bar{k},k}}\Big\}, \min_k \frac{1}{\tilde{\alpha}_{\bar{k},k}}\Big\}.$$

Using the fact that $\mathbb{D}_{\text{in}}(x^{\bar{k}}, y^{\bar{k}}) = -\ln \lambda_\star$, we obtain the result. The proof of the output case is very similar using equation (5.6). □

In the next statement, we establish a formula for a graph translation function.

Proposition 5.5. *Let* $A = \{(x^k, y^k) : k = 1, ..., l\} \subset \mathbb{R}^d_+$. *(a) If* $T = T^{\oplus}_v$ *then for all* $\bar{k} \in \{1, ..., l\}$ *the graph translation distance function is:*

$$\mathbb{D}(x^{\bar{k}}, y^{\bar{k}}) = \min\Bigg\{\Big[\min_{j=1,\ldots,n} \max\Big\{\frac{1}{2} \max_{\substack{k=1,\ldots,l \\ y_j^{\bar{k}} \leq y_j^k - \beta_{\bar{k},k}}} \{-y_j^{\bar{k}} + y_j^k + \beta_{\bar{k},k}\}, \gamma_{\bar{k},j}\Big\}\Big], \max_{k=1,\ldots,l} \beta_{\bar{k},k}\Bigg\}.$$

where

$$\beta_{\bar{k},k} = \min_{i=1,\ldots,m} \{x_i^{\bar{k}} - x_i^k\} \quad \textit{and} \quad \gamma_{\bar{k},j} = \max_{\substack{k=1,\ldots,l \\ y_j^{\bar{k}} \geq y_j^k - \beta_{\bar{k},k}}} \{-y_j^{\bar{k}} + y_j^k\}.$$

(b) If $T = T^{\oplus}_c$ *then for all* $\bar{k} \in [l]$ *the graph distance function is:*

$$\mathbb{D}(x^{\bar{k}}, y^{\bar{k}}) = \frac{1}{2}\mathbb{D}_{\text{in}}(x^{\bar{k}}, y^{\bar{k}}) = \frac{1}{2}\mathbb{D}_{\text{out}}(x^{\bar{k}}, y^{\bar{k}}) = \frac{1}{2} \min_{j=1,\ldots,n} \max_{k=1,\ldots,l} \{-y_j^{\bar{k}} + y_j^k + \beta_{\bar{k},k}\}.$$

Proof. (*a*) Fix $(x^{\bar{k}}, y^{\bar{k}}) \in \mathbb{R}^d_+$. The program to solve is :

$$\max\Big\{\delta : \bigoplus_{k=1}^{l} s_k \otimes x^k \leq (-\delta) \otimes x^{\bar{k}}, \bigoplus_{k=1}^{l} s_k \otimes y^k \geq \delta \otimes y^{\bar{k}}, \max_{k=1,\ldots,l} s_k = 0\Big\} \tag{5.11}$$

We have the equivalence of inequation systems:

$$\begin{cases} \bigoplus_{k=1}^{l} s_k \otimes x^k & \leq & (-\delta) \otimes x^{\bar{k}} \\ \bigoplus_{k=1}^{l} s_k \otimes y^k & \geq & \delta \otimes y^{\bar{k}} \\ \bigoplus_{k=1}^{l} s_k & = 0, & s \in \mathbb{R} \end{cases}$$

and

$$\begin{cases} \bigotimes_{k=1}^{l} s_k \otimes x^j & \leq & (-\delta) \otimes x^{\bar{k}} & \\ \bigoplus_{k=1}^{l} s_k & \leq & 0 & (\mathcal{S}^\star) \\ \bigoplus_{k=1}^{l} s_k \otimes y^k & \geq & \delta \otimes y^{\bar{k}} & \\ \bigoplus_{k=1}^{l} s_k & \geq & 0. & s \in \mathbb{R} \end{cases}$$

Now, from Proposition 5.3 the set of solutions $\mathcal{S}^\star$ is non empty if and only if $\ominus_{i=1}^{m}[-(\delta)\otimes x_i^{\bar{k}}\otimes(-x_i)]\ominus 0$ is solution, where the x_i's denote the vectors $(x_i^1,...,x_i^l)$.

We need to solve the maximization program:

$$\begin{aligned}
&\sup\delta\\
&st.\max_{k=1,\ldots,l}\Big\{x_i^k+\min\big\{\min_{i=1,\ldots,m}\{x_i^{\bar{k}}-x_i^j\},\delta\big\}\Big\}\le x_i^{\bar{k}}\quad i=1,...,m\\
&\max_{k=1,\ldots,l}\Big\{y_j^k+\min\big\{-2\delta+\min_{i=1,\ldots,m}\{x_i^{\bar{k}}-x_i^k\},-\delta\big\}\Big\}\ge y_j^{\bar{k}}\quad j=1...n\\
&\max_{k=1,\ldots,l}\Big\{\min\big\{-\delta+\min_{i=1,\ldots,m}\{x_i^{\bar{k}}-x_i^k\},0\big\}\Big\}=0.
\end{aligned}$$

Set

$$\beta_{\bar{k},k}=\min_{i=1,\ldots,m}\{x_i^{\bar{k}}-x_i^k\}.$$

From the notations above, this problem can be rewritten with a slight rearrangement

$$\begin{aligned}
&\sup\delta\\
&st.\max_{k=1,\ldots,l}\Big\{\min\big\{x_i^k+\beta_{\bar{k},k},\delta+x_i^k\big\}\Big\}\le x_i^{\bar{k}}\ i=1,...,m\\
&\max_{k=1,\ldots,l}\Big\{\min\big\{-2\delta+y_j^k+\beta_{\bar{k},k},-\delta+y_j^k\big\}\Big\}\ge y_j^{\bar{k}}\ j=1,...,m\\
&\max_{k=1,\ldots,l}\Big\{\min\big\{-\delta+\beta_{\bar{k},k},0\big\}\Big\}=0
\end{aligned}$$

For $\mu_1,\mu_2,b>0$, we have:

(i) $\sup\Big\{\delta\in\mathbb{R}:\min\big\{\mu_1,\mu_2+\delta\big\}\le b\Big\}=\begin{cases}+\infty & b\ge\mu_1\\ b-\mu_2 & b<\mu_1\end{cases}$

(ii) $\sup\Big\{\delta\in\mathbb{R}:\min\big\{\mu_1-2\delta,\mu_2-\delta\big\}\ge b\Big\}=\begin{cases}\mu_2-b & \text{if}\quad b\ge 2\mu_2-\mu_1\\ (\mu_1-b)/2 & \text{if}\quad b\le 2\mu_2-\mu_1\end{cases}$

(iii) $\sup\Big\{\delta\in\mathbb{R}:\min\{\mu_1-\delta,\mu_2\}=b\Big\}=\begin{cases}-\infty & \text{if}\quad b>\mu_2\\ \mu_1-b & \text{if}\quad b\le\mu_2\end{cases}$

Since, by definition $x_i^{\bar{k}}\ge\beta_{\bar{k},j}+x_i^k$, it follows that only the constraints $j=1,...,n$ and the last constraints are active. Moreover, from condition (ii), $y_j^{\bar{k}}\le y_j^k-\beta_{\bar{k},k}$ implies that $\sup\big\{\delta\in\mathbb{R}:\min\big\{-2\delta+y_j^k+\beta_{\bar{k},k},-\delta+y_j^k\big\}\ge y_j^{\bar{k}}\big\}=\frac{1}{2}\max_{\substack{k=1,\ldots,l\\ y_j^{\bar{k}}\le y_j^k-\beta_{\bar{k},k}}}\big\{-y_j^{\bar{k}}+y_j^k+\beta_{\bar{k},j}\big\}$ $\forall k,j$. Conversely, $y_j^{\bar{k}}\ge y_j^k-\beta_{\bar{k},k}$ implies that

$$\sup\big\{\delta\in\mathbb{R}:\min\big\{-2\delta+y_j^k+\beta_{\bar{k},k},-\delta+y_j^k\big\}\ge y_j^{\bar{k}}\big\}=\max_{\substack{k=1,\ldots,l\\ y_j^{\bar{k}}\ge y_j^k-\beta_{\bar{k},k}}}\beta_{\bar{k},k}.$$

Moreover $\sup\big\{\delta\in\mathbb{R}:\min\big\{-\delta+\beta_{\bar{k},k},0\big\}=0\big\}=\beta_{\bar{k},k}$, which yields the result. (b) The computation of $\mathbb{D}_{\text{in}}(x^{\bar{k}},y^{\bar{k}})$ is immediate by dropping the constraint $\max_k s_k\le 0$. Since $T_c^{\oplus}$ is graph translaton homothetic, the other identities follow from [6]. □

5.3 Numerical Example

The following data sample can be found in Färe, Grosskopf and Lovell [20].

Table 1. Data Sample

Firms	Input	Output 1	Output 2
1	2	3/2	1
2	2	2	1
3	4	3	2
4	6	6	6
5	7	6	6
6	8	7	4
7	9	7	4

The values of the efficiency measures for Max-Plus and $\mathbb{B}$-convex technologies are listed in Table 2 and 3. They are also compared to the efficiency scores computed using the traditional DEA model.

Table 2. Efficiency scores under a VRS assumption.

Firms	DEA Input Farrell VRS	$\mathbb{B}$-Convex Input Farrell VRS	Max-Plus Input Translation VRS	DEA Output Farrel VRS	$\mathbb{B}$-Convex Output Farrel VRS	Max-Plus Output Translation VRS	Max-Plus Graph Translation VRS
1	1	1	0	1	0.75	0.5	0
2	1	1	0	1	1	0	0
3	0.75	0.75	1	1.333	0.75	1	0.5
4	1	1	0	1	0	0	0
5	0.857	0.857	1	1	1	0	0
6	1	1	0	1	1	0	0
7	0.888	0.888	1	1	1	0	0

Table 3. Efficiency scores under CRS and GTH assumptions.

Firms	DEA Input Farrell CRS	DEA Output Farrell CRS	$\mathbb{B}$-convex Input Farrell CRS	$\mathbb{B}$-convex Output Farrell CRS	DEA Graph Translation CRS	Max-Plus Graph Translation GTH
1	0.75	1.333	0.75	1.333	0.25	0.25
2	1	1	1	1	0	0
3	0.75	1.333	0.75	1.33	0.5	0.5
4	1	1	1	1	0	0
5	0.857	1.666	0.857	1.166	0.5	0.5
6	0.875	1.142	0.875	1.142	0.495	0.5
7	0.778	1.285	0.888	1.125	1	1

In Table 2, we assume that the technologies satisfy a variable returns to scale assumption (VRS). One can see that all the efficient points in the Max-Plus production set are efficient in the $\mathbb{B}$-convex one and conversely. Moreover, the input scores computed using the DEA and $\mathbb{B}$-convex models are identical. In general, however, this is not true. These results come from the configuration of the data set and the fact that the input dimension is 1. For example, if the efficiency measures are output oriented, then firm 1 is efficient in the DEA case and inefficient under either a Max-Plus or a $\mathbb{B}$-convex assumption. Remark that both in the Max-Plus and $\mathbb{B}$-convex cases, firms may be efficient (resp. inefficient) and inefficient (resp. efficient) respectively in the input and output dimension. This is the case for firms 1, 5 and 7. Notice that only firm 3 is inefficient in the graph oriented case. Table 3 reports the cases of constant returns to scale (CRS) and graph translation nomothetic assumptions (GTH). One can see, not surprisingly, that there are more inefficient firms since the production sets are larger than in the VRS case. The results are identical in the DEA and $\mathbb{B}$-convex cases excepted for firm 7. Moreover, computing the graph translation distance function in the DEA case, one can see that the results are identical to those obtained in the tropical case, excepted for firm 6.

Conclusion

In this paper, a framework to analyze Max-Plus production technologies was proposed. In a nonparametric context, it was established that input, output and graph distance functions can be calculated in closed form (this is not the case for DEA models). A duality result was established in the paper that holds for a large class of non-convex technologies. Deeper investigations should be done concerning the aggregation of Max-Plus technologies in view of the additive nature of the translation distance function. Along this line, there are some potential empirical applications of these models to many fields of economics and management.

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