Pacific Journal of Optimization
Vol. 13, No. 4, pp. 707–730, October 2017

# DECOMPOSITION-BASED APPROACHES FOR MULTIOBJECTIVE COMPOSITE SYSTEMS*

Francisca Miguel† and Trinidad Gómez

**Abstract:** Decision making in a composite system usually involves considering all of the components from which it is formed. In some cases, these components or subsystems may determine a part of the feasible set or they may impact the objectives of the whole system. This paper studies the relationship between the efficient set of the multiobjective problem associated with the composite system and the efficient sets of the multiobjective problems corresponding to the subsystems. In addition to simple examples appearing through this paper, the redundancy allocation problem is considered to illustrate the applicability of the results obtained.

**Key words:** *vector optimization, global optimization, large scale problem, mathematical programming*

**Mathematics Subject Classification:** *90C06, 90C26, 90C29, 90C30*

---

## 1 Introduction

Decision-making processes in public and private organizations often require the explicit consideration of different components which have some decision capabilities, and that are affected by the problem under study. In this context, we use the term "composite system" to refer to a system consisting of several quasi autonomous components. These units can be independent or can interact with each other.

When the decision-making process is modeled as an optimization problem, different elements (decision variables, goals, constraints, etc.) appear to come from the different units implied in that process. There are usually multiple conflicting objectives and consequently, the optimization problem has to be treated in a multiobjective programming framework.

The underlying optimization problem (overall problem) is often complex to solve, and an effective strategy is to reduce such complexity by applying decomposition techniques. The idea is to split the overall system according to the nature of the problem so that the generated subproblems are easier to solve. In some cases, those subproblems have to be coordinated in order to obtain a solution for the original problem ( [20,30]).

Different decomposition approaches can be found in the literature. Some of them divide the global problem by leaning on the hierarchy that underlies many of the real organizations ( [8,12,19,24]). In such cases, the decomposition is carried out through the introduction of a

*This research has been partially funded by the Junta de Andaluca (PAI groups SEJ-417 and SEJ-532) and by the Ministerio de Economa y Competitividad (Research Project MTM2010-14992 and ECO2013-47129-C4-1-R).

†Corresponding author.

set of auxiliary parameters (coordination variables) which allows that the subproblems (lower level) are solved separately, but coordinated by a master problem (top level). An overview of these schemes can be found in [9,15,17] and [3], among others. Other approaches, principally in the engineering field, divide the global problem according to the different disciplines or knowledge areas involved in it. Different methods in Multidisciplinary Design Optimization (MDO), such as Concurrent Subspace Optimization (CSSO), Collaborative Optimization (CO), Bilevel Integrated System Synthesis (BLISS), among others, have proved to be useful tools for engineering design problems ( [7, 21, 26]). Basically, these schemes differ in how the solution of the decoupled optimization problems are coordinated and how consistency between subsystems is managed. See [1, 27] and [18] for details of these methods.

In a multiobjective context, some applied decomposition approaches are heuristic, without rigorous mathematical validation or use some specific resolution technique (weighting method, epsilon-restriction method) that requires certain assumptions in the problem such as differentiability, convexity, etc. ( [7, 20] ).

We analyze multiobjective optimization problems where various multiobjective optimization subproblems appear and we offer the theorist framework which allows replacing the resolution of the original problem by the resolution of one or more optimization subproblems. This could provide a saving in the computational effort. In this paper, we follow a previous work ( [6]), but consider new situations which may arise in the natural structure of the overall problem, for example, decision variables which can be expressed as linear combinations of a finite number of variables, objective functions which depend on decision variables and intermediate functions, and complex systems that extend the previous cases with local and global variables.

As pointed out previously, the overall problem is considered as a system with multiple components, and its general formulation can be stated as:

$$\min f(x) \text{ subject to } x \in X \subseteq \mathbb{R}^n, \tag{1.1}$$

where $X \subseteq \mathbb{R}^n$, $X \neq \emptyset$ is the feasible set, $x = (x_1, x_2, \ldots, x_n) \in X$ is a vector of the decision variables; the objective function vector to be minimized $f$ is defined in a subset of $\mathbb{R}^n$ containing $X$ and it takes values in a subset of $\mathbb{R}^p$. Here $\mathbb{R}^n$ (resp. $\mathbb{R}^p$) is $n$-dimensional (resp. $p$-dimensional) Euclidean space.

Generally, in multiobjective problems (MOPs), a unique optimal solution does not exist, due to the conflicting objectives; however a set of solutions which are called efficient solutions is produced. The set of efficient solutions for the overall problem is denoted by $\mathcal{E}(X, f, \mathbb{R}^p_{\geqq})$, where $\mathbb{R}^p_{\geqq}$ is the domination cone given by $\mathbb{R}^p_{\geqq} = \{u \in \mathbb{R}^p : u_i \geqq 0 \text{ for } i = 1, 2, \ldots, p\}$, i.e. the non-negative orthant of $\mathbb{R}^p$. In the literature, this domination cone is known as the Pareto cone. As in [22, p. 28] we consider the set $\mathcal{E}(X, f, \mathbb{R}^p_{\geqq})$ defined in the following way.

$$\mathcal{E}(X, f, \mathbb{R}^p_{\geqq}) = \{x \in X : \text{ there is no } \bar{x} \in X \text{ such that } f(\bar{x}) \leq f(x)\}. \tag{1.2}$$

where $f(\bar{x}) \leq f(x)$ denotes $f_i(\bar{x}) \leq f_i(x)$ for $i = 1, 2, \ldots, p$ and $f_j(\bar{x}) < f_j(x)$ for some $j \in \{1, 2, \ldots, p\}$.

A feasible solution is efficient if there is no other feasible solution that can improve any one objective without degrading at least one other (see [23]). If $x$ is efficient, the outcome $f(x)$ is called a non-dominated point. The set of all efficient solutions is called the Pareto set, $\mathcal{E}(X, f, \mathbb{R}^p_{\geqq})$, and the set of all non-dominated points is called the Pareto front $\mathcal{E}(f(X), \mathbb{R}^p_{\geqq}) = f(\mathcal{E}(X, f, \mathbb{R}^p_{\geqq}))$. Throughout this work we will suppose that all the efficient sets are non-empty, which can be guaranteed under relatively weak assumptions on $X$ and $f$ (see [22, p. 49-51] for details of these assumptions).

The objective of the present paper is to study the relationship between the efficient set of the overall problem and the efficient sets of the subproblems, considering a variety of configurations based on the functional dependencies between decision variables and objective functions included in the problem. Special emphasis is put here on the way that the efficient sets of the subproblems partially or completely determine the efficient set of the overall problem, without additional consideration on the later problem of choosing one of the elements of this set as the final solution for the multiobjective composite system. Extensive literature is devoted to the study of this topic. An overview of those using decomposition and coordination methods is given in [3]. Furthermore, two different interactive decision-making procedures were introduced in [4].

The remainder of this paper is organized as follow: basic composite systems are presented in Section 2. The goal of this section is to reduce the multiobjective problem to a finite set of subproblems that are more easily treated due to the simplicity of the corresponding feasible sets and / or their objective functions. Furthermore, we analyze the relationships of efficiencies between the original problem and the subproblems considered. In Subsection 2.1, we consider that the feasible set can be given as a sum, a homothety or a linear combination of other sets. In subsection 2.2 the problem is linked to another problem, generally easier that the given, where the feasible sets of both are the same, but the objective functions are related by some given function. Section 3 extends some previous results and presents some new results for the composite systems that are combinations of two or more basic cases. Here, we study systems that are composed of $N$ subsystems and depend on two types of vector variables; one of them (local variables) captures information from the subsystems, while the other (global variables) gathers global aspects of the system. We consider two different cases according to whether the subsystems depend on the global variable (Subsection 3.1 and 3.2) or not. Furthermore, through out this paper the theoretical results are illustrated with simple examples in order to be clearer. In Section 4, we illustrate the way some of the methods and results of the previous sections shed new light on the solution of the multiobjective redundancy allocation problem. Finally, a summary of conclusions is given in Section 5.

## 2 Basic Composite Systems

We start our presentation with systems which are call basic composite systems. We can use these systems as building blocks to develop various complex composite systems. For each type of system we formulate the specific global problem for that system i.e. feasible set, and the objective function. We then build the pertinent efficient sets (1.2) and present any necessary auxiliary concepts. Finally, we state and prove at least one proposition defining a relationship between the efficient sets of the subproblems and the efficient set of the system.

### 2.1 Algebraic Operations on Feasible Set

We start by studying the case of one system with one decision variable, which is the result of some basic algebraic operations, i.e., addition, multiplication or linear combination.

#### 2.1.1 The Minkowski Sum

The first system we consider is the general MOP (1.1) but where the feasible decision $x$ is equal to the sum of two decision variables. More explicitly, we consider that the global feasible set is the Minkowski sum (also called the sumset) of two subsets, $X_1$ and $X_2$, of the set $\mathbb{R}^n$. Minkowski sums are used in many applications, for example, in economics to optimize the production potential of a group of companies.

The global formulation for this specific case is written as

$$\min f(x) \text{ subject to } x \in X = X_1 + X_2. \tag{2.1}$$

Due to the structure of the feasible set, the subsystems for this case are given by

$$\min f(x_i) \text{ subject to } x_i \in X_i,\ i = 1, 2. \tag{2.2}$$

It is interesting to regard the efficient set of this global problem in terms of $\mathcal{E}(X_i, f, \mathbb{R}^p_{\geqq})$ for $i = 1, 2$. For this the objective function $f$ is required to show good behavior relative to the Minkowski sum. So if $X, X_1, X_2 \subseteq A$ and $f : A \subseteq \mathbb{R}^n \to B \subseteq \mathbb{R}^p$ then $f$ is said to be an *additive function* with respect to the decomposition $X = X_1 + X_2$ if, whenever $x_1 \in X_1$ and $x_2 \in X_2$, then $f(x_1) + f(x_2) \in B$ and $f(x_1 + x_2) = f(x_1) + f(x_2)$.

A relationship between the non-dominated points for Problem (2.1) and Problems (2.2) was first obtained by Sawaragi et al. [22, p. 35]. In the subsequent proposition we reformulate this connection in a similar context for the efficient points.

**Proposition 2.1.** *If $f$ is an additive function with respect to decomposition $X = X_1 + X_2$, then*

$$\mathcal{E}(X, f, \mathbb{R}^p_{\geqq}) \subseteq \mathcal{E}(X_1, f, \mathbb{R}^p_{\geqq}) + \mathcal{E}(X_2, f, \mathbb{R}^p_{\geqq}).$$

In general converse inclusion of Proposition 2.1 is false, as we show in the following example.

**Example 2.2.** Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where $f(x, y) = [x - y, -x]$ and

$$\begin{aligned} X_1 &= \{(x_1, y_1) : \quad 0 \leqq x_1 \leqq 1, \quad 0 \leqq y_1 \leqq 2 - 2x_1\} \subseteq \mathbb{R}^2, \\ X_2 &= \{(x_2, y_2) : \quad 0 \leqq x_2 \leqq 1, \quad 2x_2 - 2 \leqq y_2 \leqq 0\} \subseteq \mathbb{R}^2. \end{aligned}$$

Figure 1(a) shows the feasible sets of the subproblems, and Figure 1(b) shows the feasible set of the global problem. Then $(1, 0) \in \mathcal{E}(X_1, f, \mathbb{R}^2_{\geqq})$ and $(0, 0) \in \mathcal{E}(X_2, f, \mathbb{R}^2_{\geqq})$. However, $(1, 0)$ is not efficient in $X$ for $f$ (for example,$(1, 2) \in X_1 + X_2 = X$ and $f(1, 2) < f(1, 0)$).

In Proposition 2.5 we improve the assertion of Proposition 2.1 in the case that $X_1$ and $X_2$ are compact sets and $f$ is a continuous map, which is additive with respect to decomposition $X = X_1+X_2$. We previously prove two lemmas. The first gives an elementary property, which will also be used in several sections of this paper and can be proved by direct verification. The second is a well known result, called *domination property*. It can be obtained as an immediate consequence of Theorem 4.1 of Hartley [10].

**Lemma 2.3.** *If $\mathcal{E}(X, f, \mathbb{R}^p_{\geqq}) \subseteq Y \subseteq X$, then $\mathcal{E}(X, f, \mathbb{R}^p_{\geqq}) \subseteq \mathcal{E}(Y, f, \mathbb{R}^p_{\geqq})$.*

**Lemma 2.4.** *Let $X$ be a compact subset of $\mathbb{R}^n$, $f$ a continuous function on $X$ and $x$ a non efficient point in $X$ for $f$. Then there exists $\hat{x} \in \mathcal{E}(X, f, \mathbb{R}^p_{\geqq})$ such that $f(\hat{x}) \leq f(x)$.*

**Proposition 2.5.** *Let $f$ be a continuous map, which is an additive function with respect to the decomposition $X = X_1 + X_2$, where $X_1$ and $X_2$ are compact subsets of $\mathbb{R}^n$. Then*

$$\mathcal{E}(X, f, \mathbb{R}^p_{\geqq}) = \mathcal{E}\left(\mathcal{E}(X_1, f, \mathbb{R}^p_{\geqq}) + \mathcal{E}(X_2, f, \mathbb{R}^p_{\geqq}), f, \mathbb{R}^p_{\geqq}\right).$$

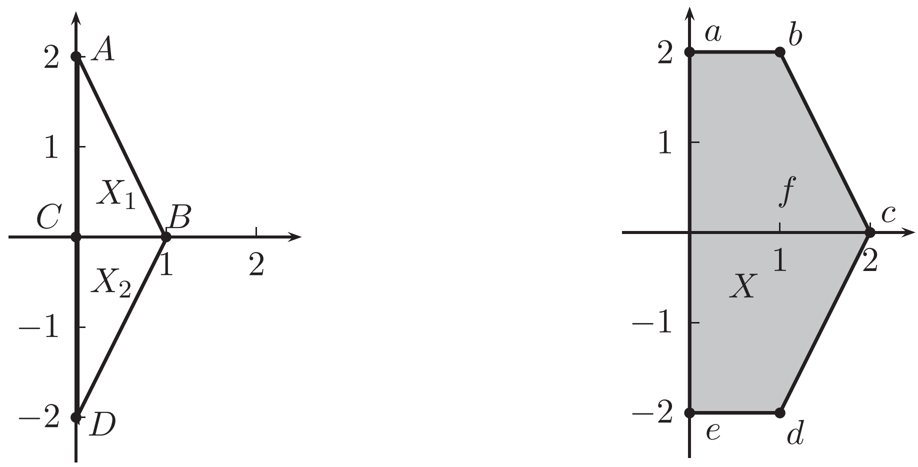


Figure 1: Feasible Sets. Example 2.2.

*Proof.* The inclusion $\mathcal{E}(X, f, \mathbb{R}^p_{\geqq}) \subseteq \mathcal{E}\left(\mathcal{E}(X_1, f, \mathbb{R}^p_{\geqq}) + \mathcal{E}(X_2, f, \mathbb{R}^p_{\geqq}), f, \mathbb{R}^p_{\geqq}\right)$, is an immediate consequence of Lemma 2.3 and Proposition 2.1.

Let $x \in \mathcal{E}\left(\mathcal{E}(X_1, f, \mathbb{R}^p_{\geqq}) + \mathcal{E}(X_2, f, \mathbb{R}^p_{\geqq}), f, \mathbb{R}^p_{\geqq}\right)$, and by contradiction, suppose that $x$ is non efficient in $X$ for $f$. Then there exists $\hat{x} = \hat{x}_1 + \hat{x}_2$, for some $\hat{x}_1 \in X_1$ and $\hat{x}_2 \in X_2$ such that $f(\hat{x}) \leq f(x)$. We observe that the relations $\hat{x}_1 \in \mathcal{E}(X_1, f, \mathbb{R}^p_{\geqq})$ and $\hat{x}_2 \in \mathcal{E}(X_2, f, \mathbb{R}^p_{\geqq})$ cannot occur simultaneously. Indeed, the inequality $f(\hat{x}) \leq f(x)$ contradicts that $x \in \mathcal{E}\left(\mathcal{E}(X_1, f, \mathbb{R}^p_{\geqq}) + \mathcal{E}(X_2, f, \mathbb{R}^p_{\geqq}), f, \mathbb{R}^p_{\geqq}\right)$. So some $\hat{x}_i$, for example, $\hat{x}_2$ is not efficient in $X_2$ for $f$. By Lemma 2.4, there exists $\breve{x}_2$ efficient in $X_2$ for $f$ such that $f(\breve{x}_2) \leq f(\hat{x}_2)$. For $\hat{x}_1$ two possibilities exist: either $\hat{x}_1 \in \mathcal{E}(X_1, f, \mathbb{R}^p_{\geqq})$ or $\hat{x}_1 \notin \mathcal{E}(X_1, f, \mathbb{R}^p_{\geqq})$. If $\hat{x}_1 \in \mathcal{E}(X_1, f, \mathbb{R}^p_{\geqq})$ we make $\breve{x}_1 = \hat{x}_1$. If $\hat{x}_1 \notin \mathcal{E}(X_1, f, \mathbb{R}^p_{\geqq})$ we choose $\breve{x}_1$ efficient in $X_1$ for $f$ such that $f(\breve{x}_1) \leq f(\hat{x}_1)$, which can be guaranteed by Lemma 2.4. Consider $\breve{x} = \breve{x}_1 + \breve{x}_2$, $\breve{x} \in \mathcal{E}(X_1, f, \mathbb{R}^p_{\geqq}) + \mathcal{E}(X_2, f, \mathbb{R}^p_{\geqq})$ and $f(\breve{x}) = f(\breve{x}_1) + f(\breve{x}_2) \leq f(\hat{x}_1) + f(\hat{x}_2) = f(\hat{x})$. Since $f(\hat{x}) \leq f(x)$, we have $f(\breve{x}) \leq f(x)$, which contradicts the assumption that $x$ is efficient in $\mathcal{E}(X_1, f, \mathbb{R}^p_{\geqq}) + \mathcal{E}(X_2, f, \mathbb{R}^p_{\geqq})$ for $f$ and completes the proof. □

According to this proposition, the overall problem can be solved by a two step-procedure. In the initial step, the efficient set of each subproblem (2.2) is obtained. Then, in a second step, the efficient set for $f$ in the efficient sumset, $\mathcal{E}(X_1, f, \mathbb{R}^p_{\geqq}) + \mathcal{E}(X_2, f, \mathbb{R}^p_{\geqq})$, is achieved.

In order to generalize the arguments in the proofs of Proposition 2.1 and Proposition 2.5 to the case in which the feasible set can be decomposed as a finite sum of $N \geqq 3$ compact subsets $X_1, X_2, \ldots, X_N$ of $\mathbb{R}^n$, we require that $X_1, X_2, \ldots, X_N, X \subseteq A$ and the continuous function $f$ to be an additive function with respect to the decomposition $X = \sum_{i=1}^{N} X_i$. Under the above assumptions we obtain

$$\mathcal{E}\left(\sum_{i=1}^{N} X_i, f, \mathbb{R}^p_{\geqq}\right) = \mathcal{E}\left(\sum_{i=1}^{N} \mathcal{E}(X_i, f, \mathbb{R}^p_{\geqq}), f, \mathbb{R}^p_{\geqq}\right).$$

**Example 2.6.** Let us consider the data from Example 2.2. Figure 1(a) shows the feasible sets of the subproblems, and Figure 1(b) shows the feasible set of the global problem.

Solving the subproblems yields $\mathcal{E}(X_1, f, \mathbb{R}^2_{\geqq}) = \{(1-\lambda, 2\lambda) : \lambda \in [0,1]\} = \overline{AB}$, i.e. the segment AB, and $\mathcal{E}(X_2, f, \mathbb{R}^2_{\geqq}) = \{(\mu, 0) : \mu \in [0,1]\} = \overline{BC}$. (See Figure 1(a)). Then $\mathcal{E}(X_1, f, \mathbb{R}^2_{\geqq}) + \mathcal{E}(X_2, f, \mathbb{R}^2_{\geqq}) = \overline{AB} + \overline{BC} = \{(x,y) : 2 \leqq 2x + y \leqq 4, \quad 0 \leqq y \leqq 2\}$, i.e. the parallelogram abcf (see Figure 1(b)).

By Proposition 2.5, $\mathcal{E}(X, f, \mathbb{R}^2_{\geqq}) = \mathcal{E}\left(\mathcal{E}(X_1, f, \mathbb{R}^2_{\geqq}) + \mathcal{E}(X_2, f, \mathbb{R}^2_{\geqq}), f, \mathbb{R}^2_{\geqq}\right) = \{(2-\alpha, 2\alpha) : \alpha \in [0,1]\} \cup \{(\beta, 2) : \beta \in [0,1]\} = \overline{ab} \cup \overline{bc}$, (see Figure 1(b)).

### 2.1.2 Homothety

We now proceed by examining a case where the decision variable is multiplied by a strictly positive real number $\lambda$. The new feasible set $\lambda X$ is obtained from $X$ by applying a homothety of ratio $\lambda$. This system yields the following global problem

$$\min f(y) \text{ subject to } y \in \lambda X. \tag{2.3}$$

The efficient solutions of this problem can be related to the efficient solutions of the general problem MOP (1.1) by assuming the objective function to be positively homogeneous of degree one. In a more explicit way we require that if $\lambda \in \mathbb{R}_>$ and $x \in X$ then $\lambda x \in A$, $\lambda f(x) \in B$ and $f(\lambda x) = \lambda f(x)$. Here the function $f : A \subseteq \mathbb{R}^n \to B \subseteq \mathbb{R}^p$ is said to be *positively homogeneous of degree one on* $X$. One of the most important positively homogeneous functions are the norms. This class of function are used extensively in Mathematical Economics ( [5, 11, 28]).

Under this assumption the following proposition is readily obtained which is illustrated with the example below.

**Proposition 2.7.** *If $f$ is a positively homogeneous function of degree one on $X$, then*

$$\mathcal{E}(\lambda X, f, \mathbb{R}^p_{\geqq}) = \lambda \mathcal{E}(X, f, \mathbb{R}^p_{\geqq}) \text{ for all } \lambda \in \mathbb{R}_>.$$

**Example 2.8.** Let $X = \{(x_1, x_2) : x_1^2 + x_2^2 \leqq 1, x_1 \leqq 0, x_2 \leqq 0\} \subseteq \mathbb{R}^2$, and $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where $f(x) = [x_1, x_2 - x_1]$. By Proposition 2.7, $\mathcal{E}(2X, f, \mathbb{R}^2_{\geqq}) = 2\{(x_1, x_2) : x_1^2 + x_2^2 = 1, x_1 \leqq 0, x_2 \leqq 0\} = \{(x_1, x_2) : x_1^2 + x_2^2 = 4, x_1 \leqq 0, x_2 \leqq 0\}$.

Note that, under the assumption that $f$ is a positively homogeneous function of degree one, all homothety ( dilation, if $\lambda > 1$, or contraction, if $\lambda < 1$) in the feasible set yields the same homothety (same ratio) in the corresponding efficient set.

### 2.1.3 Linear Combination

The system considered in this subsection arises from the last two in a natural way. Here the global variable is a positive linear combination of $N$ given previously. Now the global formulation for this specific case is written as

$$\min f(x) \text{ subject to } x \in X = \sum_{i=1}^{N} \lambda_i X_i, \tag{2.4}$$

where $\lambda_i \in \mathbb{R}_>$ ; $i = 1, 2, ..., N$. Now the following corollary follows from the multiplicative and the additive case results.

**Corollary 2.9.** *If $f : \mathbb{R}^n \to \mathbb{R}^p$ is a linear map, then*

$$\mathcal{E}\left(\sum_{i=1}^{N} \lambda_i X_i, f, \mathbb{R}^p_{\geqq}\right) \subseteq \sum_{i=1}^{N} \lambda_i \mathcal{E}(X_i, f, \mathbb{R}^p_{\geqq}), \tag{2.5}$$

*If, in addition, every $X_i$ is a compact subset of $\mathbb{R}^n$ then*

$$\mathcal{E}\left(\sum_{i=1}^{N} \lambda_i X_i, f, \mathbb{R}^p_{\geqq}\right) = \mathcal{E}\left(\sum_{i=1}^{N} \lambda_i \mathcal{E}(X_i, f, \mathbb{R}^p_{\geqq}), f, \mathbb{R}^p_{\geqq}\right). \tag{2.6}$$

Note that if we consider that $\sum_{i=1}^{N} \lambda_i = 1$, then $\lambda_i$ can be interpreted as the relative importance that is given to the part $X_i$ within the overall set $X$.

## 2.2 Objective Functions Depending on Intermediate Functions

We proceed in this section by including systems that are formed by two sequentially connected problems and a single set of decision variables.

### 2.2.1 Composite Function

We begin by studying the case where the overall objective function depends on the objective function of the first problem.

Let $r : A \subseteq \mathbb{R}^n \longrightarrow B \subseteq \mathbb{R}^q$, $f : B \subseteq \mathbb{R}^q \longrightarrow C \subseteq \mathbb{R}^p$ and $X \subseteq A$. We consider the composite function $F(x) = (f \circ r)(x) = f(r(x))$. One simply takes the output of the first function and uses it as the input of the second function. We refer to function $F$ as the *overall objective* function and to the function $r$ as the *intermediate* function.

This system yields the following global problem

$$\min F(x) = f(r(x)) \text{ subject to } x \in X. \tag{2.7}$$

In the following proposition we developed the first relationship for the system.

**Proposition 2.10.** *If $F = f \circ r$, then*

$$\mathcal{E}(X, F, \mathbb{R}^p_{\geqq}) = \{x : r(x) \in \mathcal{E}(r(X), f, \mathbb{R}^p_{\geqq})\}.$$

*Proof.* Let $x$ be efficient in $X$ for $F$. On the contrary, suppose that $r(x)$ is not efficient in $r(X)$ for $f$. Then there exists $r(\hat{x}) \in r(X)$, such that $F(\hat{x}) = f(r(\hat{x})) \leq f(r(x)) = F(x)$, contrary to the assumption that $x$ is efficient in $X$ for $F$.

Let $x \in X$ such that $r(x)$ is efficient in $r(X)$ for $f$. Suppose that $x$ is not efficient in $X$ for $F$. Then there exists $\hat{x} \in X$ such that $F(\hat{x}) \leq F(x)$. Thus, $f(r(\hat{x})) = F(\hat{x}) \leq F(x) = f(r(x))$ which contradicts the supposition that $r(x)$ is efficient in $r(X)$ for $f$. □

**Example 2.11.** Let $r : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where $r(x_1, x_2) = [x_1, -x_1 + x_2]$, $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where $f(y_1, y_2) = [y_1^2 + y_2^2, y_2]$, and $X = \{(x_1, x_2) : x_1^2 + x_2^2 \leqq 1, x_1 \leqq 0, x_2 \leqq 0\} \subseteq \mathbb{R}^2$. See Figure 2(a).

First we calculate the image of $X$ under $r$: $r(X) = \{(y_1, y_2) : 2y_1^2 + y_2^2 + 2y_1 y_2 \leqq 1, y_1 \leqq 0, y_1 + y_2 \leqq 0\}$. See Figure 2(b). Now, we compute $\mathcal{E}(r(X), f, \mathbb{R}^2_{\geqq}) = \{(y_1, y_2) : y_1 = 0, -1 \leqq y_2 \leqq 0\}$.

By Proposition 2.10, $\mathcal{E}(X, F, \mathbb{R}^2_{\geqq})$, where $F = (f \circ r)$, can be determined using $\mathcal{E}(r(X), f, \mathbb{R}^2_{\geqq})$. Therefore $\mathcal{E}(X, F, \mathbb{R}^2_{\geqq}) = \{(x_1, x_2) : x_1 = 0, -1 \leqq x_2 \leqq 0\}$.

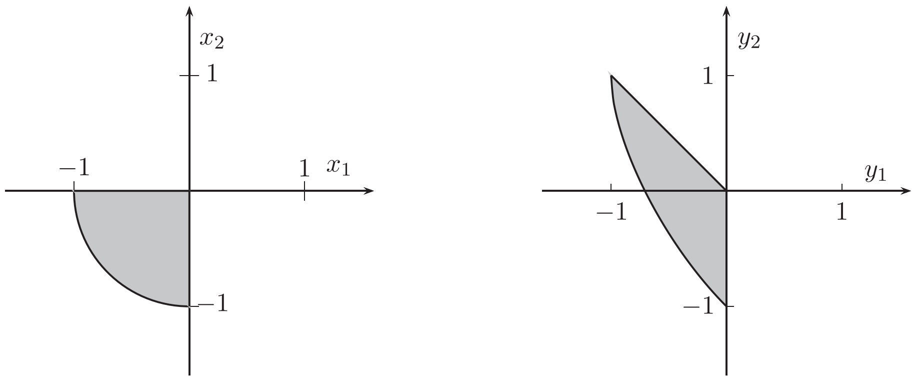


Figure 2: Example 2.11.

In the next proposition, we give conditions to reduce the overall problem to the following subproblem

$$\min r(x) \text{ subject to } x \in X. \tag{2.8}$$

In order to develop the next relationship a strongly increasing objective function $f$ is required. So $f : B \subseteq \mathbb{R}^q \to C \subseteq \mathbb{R}^p$ is said to be a *strongly increasing function* if for all $y_1, y_2 \in B$ such that $y_1 \leq y_2$ then $f(y_1) \leq f(y_2)$. We consider that $f$ is a strongly increasing function on $\tilde{B} \subseteq B$ if its restriction to $\tilde{B}$ is so.

**Proposition 2.12.** *Let $F = f \circ r$. If $f$ is a strongly increasing function on $r(X)$, then the following assertions hold:*

(a) $\mathcal{E}(X, F, \mathbb{R}^p_{\geqq}) \subseteq \mathcal{E}\left(\mathcal{E}(X, r, \mathbb{R}^q_{\geqq}), F, \mathbb{R}^p_{\geqq}\right)$.

(b) *If, furthermore, $X$ is a compact subset of $\mathbb{R}^n$ and $r$ is a continuous function on $X$, then*

$$\mathcal{E}(X, F, \mathbb{R}^p_{\geqq}) = \mathcal{E}\left(\mathcal{E}(X, r, \mathbb{R}^q_{\geqq}), F, \mathbb{R}^p_{\geqq}\right).$$

*Proof.* (a) It is an immediate consecuence of Lemma 2.3 taking $Y = \mathcal{E}(X, r, \mathbb{R}^q_{\geqq})$.
(b) Let $x$ be efficient in $\mathcal{E}(X, r, \mathbb{R}^q_{\geqq})$ for $F$, and suppose that $x$ is not efficient in $X$ for $F$. Then there exists $\hat{x} \in X$ such that $F(\hat{x}) = f(r(\hat{x})) \leq f(r(x)) = F(x)$. We now consider two cases, either $\hat{x} \in X \backslash \mathcal{E}(X, r, \mathbb{R}^q_{\geqq})$ or $\hat{x} \in \mathcal{E}(X, r, \mathbb{R}^q_{\geqq})$.

The second case can be ruled out, because it contradicts the assumption that $x$ is efficient in $\mathcal{E}(X, r, \mathbb{R}^q_{\geqq})$ for $F$.

Suppose $\hat{x} \in X \backslash \mathcal{E}(X, r, \mathbb{R}^q_{\geqq})$, i.e. $\hat{x} \notin \mathcal{E}(X, r, \mathbb{R}^q_{\geqq})$. By Lemma 2.4, there exists $\breve{x} \in \mathcal{E}(X, r, \mathbb{R}^q_{\geqq})$ such that $r(\breve{x}) \leq r(\hat{x})$. Since $f$ is strongly increasing on $r(X)$, $F(\breve{x}) = f(r(\breve{x})) \leq f(r(\hat{x})) \leq f(r(x)) = F(x)$, which contradicts the assumption that $x$ is efficient in $\mathcal{E}(X, r, \mathbb{R}^q_{\geqq})$ for $F$ and completes the proof. □

It should be noted that the previous proposition offers the possibility to find the efficient points of the composite function in the efficient set of the given subproblem. We also observe that the strongly increasing character of the function $f$ is a relatively weak assumption in

Proposition 2.12. Examples of functions of this kind are those in which the component functions $f_i$ are given as in the cases (a) and (b) below.

(a) $f_i(x) = \sum_{j=1}^{q} a_{ij} r_j(x)$ with $a_{ij} > 0$ ; $i = 1, 2, ..., p$.

(b) $f_i(x) = r_1(x) r_2(x) ... r_q(x)$ with $r_j(x) > 0 \quad \forall x \in X$ ; $i = 1, 2, ..., p$.

The following proposition focuses on the efficient points in $X$ for $r$, but this requires stronger assumptions on the objective function. We will say that the function $f$ is a *strongly order reflecting function* if for all $y_1, y_2 \in B$ such that $f(y_1) \leq f(y_2)$ implies $y_1 \leq y_2$. We consider that $f$ is a strongly order reflecting function on $\tilde{B} \subseteq B$ if its restriction to $\tilde{B}$ is so.

**Proposition 2.13.** *Let $F = f \circ r$. If $f$ is a strongly increasing and a strongly order reflecting function on $r(X)$, then*

$$\mathcal{E}(X, F, \mathbb{R}^p_{\geqq}) = \mathcal{E}(X, r, \mathbb{R}^q_{\geqq}).$$

*Proof.* It is an immediate consequence of Proposition 2.12 (a) and the strongly order reflecting character of $f$. □

### 2.2.2 Objective Function Depending on the Input and an Intermediate Output

We continue by studying the case where the overall objective or end function directly depends on the decision variables (input) and also indirectly depends on these variables through an intermediate function.

Let $r : A \subseteq \mathbb{R}^n \longrightarrow B \subseteq \mathbb{R}^q$ and $f : C \subseteq \mathbb{R}^{n+q} \longrightarrow D \subseteq \mathbb{R}^p$ with $A \times r(A) \subseteq C$ and let $X \subseteq A$. We now consider the function $F : A \longrightarrow D$ given by $F(x) = f(x, r(x))$, which is a function of the input and the corresponding output of an intermediate function of the same input. This system yields the following global problem,

$$\min F(x) = f\,(x, r(x)) \text{ subject to } x \in X. \tag{2.9}$$

The dependence of the final function leads us to the following subproblem.

$$\min r\,(x) \text{ subject to } x \in X. \tag{2.10}$$

The relationship or connection between the efficient points for Problem (2.9) and Problem (2.10) requires some assumptions over the objective functions.

**Proposition 2.14.** *Let $F(x) = f\,(x, r(x))$. If $f$ is a strongly increasing function on $\{(x, r(x)) : x \in X\}$ and $r$ is a strongly order reflecting function on $X$, then*

$$\mathcal{E}(X, F, \mathbb{R}^p_{\geqq}) \subseteq \mathcal{E}(X, r, \mathbb{R}^q_{\geqq}).$$

*Proof.* Let $x$ be efficient in $X$ for $F$, and suppose $x$ is not efficient in $X$ for $r$. Then there exists $\hat{x} \in X$ such that $r(\hat{x}) \leq r(x)$. Since $r$ is a strongly order reflecting function on $X$, then $\hat{x} \leq x$. Consequently $(\hat{x}, r(\hat{x})) \leq (x, r(x))$. Since $f$ is strongly increasing on $\{(x, r(x)) : x \in X\}$, we have $F(\hat{x}) = f\,(\hat{x}, r(\hat{x})) \leq f\,(x, r(x)) = F(x)$, which contradicts the assumption that $x$ is efficient in $X$ for $F$. □

For the reverse inclusion, we must modify the assumptions on the functions involved in the problem.

**Proposition 2.15.** *Let $F(x) = f(x, r(x))$. If $r$ is an injective function on $X$ and $f$ is a strongly order reflecting function on $\{(x, r(x)) : x \in X\}$, then*

$$\mathcal{E}(X, r, \mathbb{R}^q_{\geqq}) \subseteq \mathcal{E}(X, F, \mathbb{R}^p_{\geqq}).$$

*Proof.* Let $x$ be efficient in $X$ for $r$, and suppose $x$ is not efficient in $X$ for $F$. Then there exists $\hat{x} \in X$ such that $F(\hat{x}) \leq F(x)$. Equivalently, $f(\hat{x}, r(\hat{x})) \leq f(x, r(x))$. Since $f$ is a strongly order reflecting function on $\{(x, r(x)) : x \in X\}$, we have $(\hat{x}, r(\hat{x})) \leq (x, r(x))$. The injectivity of $r$ on $X$ and the inequality $(\hat{x}, r(\hat{x})) \leq (x, r(x))$ give $r(\hat{x}) \leq r(x)$, which contradicts the efficient character of $x$ in $X$ for $r$. □

From Propositions 2.14 and 2.15, we have the following.

**Corollary 2.16.** *Let $F(x) = f(x, r(x))$. If $f$ is a strongly increasing and a strongly order reflecting function on $\{(x, r(x)) : x \in X\}$ and $r$ is an injective strongly order reflecting function on $X$, then*

$$\mathcal{E}(X, F, \mathbb{R}^p_{\geqq}) = \mathcal{E}(X, r, \mathbb{R}^q_{\geqq}).$$

**Example 2.17.** Let $X = \{(x_1, x_2) : x_1 + x_2 \geqq 1,\ 0 \leqq x_1 \leqq 1,\ x_2 \leqq 1\} \subset \mathbb{R}^2$ (see Figure 3). Let $r : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where $r(x_1, x_2) = [x_1,\ \frac{1}{2}x_1 + x_2]$ and $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^2 \times \mathbb{R}^2_{>}$ where $f(y_1, y_2, y_3, y_4) = [2y_1 + 3,\ \frac{y_2}{4},\ {y_3}^2,\ \exp y_4]$.

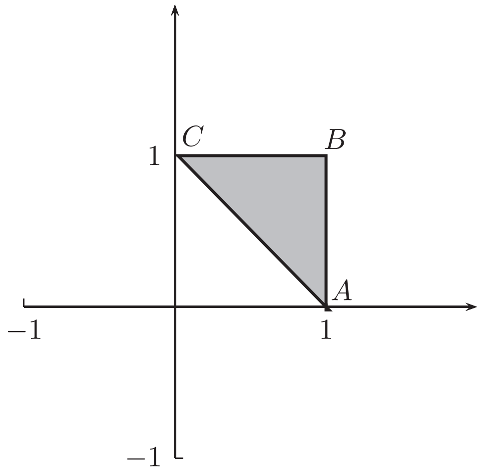


Figure 3: Feasible set. Example 2.17.

Then, we have $F(x_1,\ x_2) = f(x_1,\ x_2,\ r(x_1, x_2)) = (2x_1 + 3,\ \frac{x_2}{4},\ {x_1}^2,\ \exp(\frac{1}{2}x_1 + x_2))$. We can see that $f$ and $r$ satisfy the assumptions of Corollary 2.16, so we obtain $\mathcal{E}(X, F, \mathbb{R}^4_{\geqq}) = \mathcal{E}(X, r, \mathbb{R}^2_{\geqq}) = \overline{AC} = \{(1 - \lambda,\ \lambda) : \lambda \in [0, 1]\}$.

### 2.2.3 Using Intermediate Functions as Objective Functions

In the preceding subsections, we have considered the intermediate function's output as an input in the objective functions. In this subsection, we will consider the intermediate function's output as an additional objective.

Let $r : A \subseteq \mathbb{R}^n \longrightarrow B \subseteq \mathbb{R}^{p_2}$ and $f : B \subseteq \mathbb{R}^{p_2} \longrightarrow C \subseteq \mathbb{R}^{p_1}$ and let $X \subseteq A$. We now consider the function $F : A \to C \times B \subseteq \mathbb{R}^p$ where $p = p_1 + p_2$ and $F(x) = (f(r(x)), r(x))$. The global formulation is as follows,

$$\min F(x) = (f(r(x)), r(x)) \text{ subject to } x \in X. \tag{2.11}$$

Firstly consider the following subproblem for the previous system

$$\min r(x) \text{ subject to } x \in X. \tag{2.12}$$

In order to relate the efficient points in $X$ for $r$ and the efficient points in $X$ for $F$, it is necessary to impose some restrictions on $f$.

**Proposition 2.18.** *If $F = (f \circ r, r)$ and $f$ is a strongly increasing function on $r(X)$, then*

$$\mathcal{E}(X, F, \mathbb{R}^p_{\geqq}) = \mathcal{E}(X, r, \mathbb{R}^{p_2}_{\geqq}).$$

This proposition implies that, under relatively weak conditions on the systems, all efficient solutions for the global problem can be found by computing only efficient solutions for the subproblem. It can be obtained as an immediate consequence of a result from Haimes et al. [9, p. 85] for non-dominated points.

**Example 2.19.** Let $r : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where $r(x_1, x_2) = [\, x_1^2 + x_2^2,\ (x_1 - 1)^2 + (x_2 - 1)^2 \,]$, $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where $f(y_1, y_2) = [\, y_1 \cdot y_2,\ y_1 + y_2 \,]$ and $X$ is given as in Example 2.17. Then $\mathcal{E}(X, r, \mathbb{R}^2_{\geqq}) = \overline{AB} = \{\, (\frac{1}{2} + \frac{\lambda}{2},\ \frac{1}{2} + \frac{\lambda}{2}) : \lambda \in [\,0, 1\,]\}$, (see Figure 4).

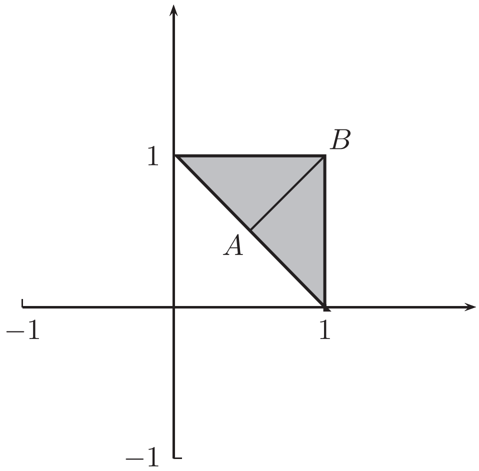


Figure 4: Efficient and Feasible sets. Example 2.19.

The function $f$ is strongly increasing on $r(X)$ and so by Proposition 2.18, $\mathcal{E}(X, F, \mathbb{R}^4_{\geqq}) = \mathcal{E}(X, r, \mathbb{R}^2_{\geqq}) = \overline{AB}$. Note that $F(x_1, x_2) = [\, (x_1^2 + x_2^2) \cdot ((x_1 - 1)^2 + (x_2 - 1)^2),\ (x_1^2 + x_2^2) + ((x_1 - 1)^2 + (x_2 - 1)^2),\ x_1^2 + x_2^2,\ (x_1 - 1)^2 + (x_2 - 1)^2 \,]$. We see that the above example illustrates a typical case where it is easier to obtain the set $\mathcal{E}(X, r, \mathbb{R}^2_{\geqq})$ than the set $\mathcal{E}(X, F, \mathbb{R}^4_{\geqq})$.

We now consider the following new subproblem:

$$\min f\,(r(x)) \text{ subject to } x \in X. \tag{2.13}$$

and we have the following results.

**Proposition 2.20.** *If $F = (f \circ r,\, r)$ and $f$ is an injective function on $r(X)$, then*

$$\mathcal{E}(X, f \circ r, \mathbb{R}^{p_1}_{\geqq}) \subseteq \mathcal{E}(X, F, \mathbb{R}^{p}_{\geqq}).$$

*Proof.* Let $x$ be efficient in $X$ for $(f \circ r)$, and suppose $x$ is not efficient in $X$ for $F = (f \circ r, r)$. Then there exists $\hat{x} \in X$ such that

$$F(\hat{x}) = (f(r(\hat{x})), r(\hat{x})) \leq (f(r(x)), r(x)) = F(x).$$

The possibility that $f\,(r(\hat{x})) = f\,(r(x))$ is ruled out by the injectivity of $f$. So $f\,(r(\hat{x})) \leq f\,(r(x))$, which contradicts the efficient character of $x$ in $X$ for $(f \circ r)$. □

An alternative hypothesis over $f$ yields the following result:

**Proposition 2.21.** *Let $F = (f \circ r,\, r)$. If $f$ is a strongly increasing continuous function on $r(X)$, then*

$$\mathcal{E}(X, f \circ r, \mathbb{R}^{p_1}_{\geq}) \subseteq \mathcal{E}(X, F, \mathbb{R}^{p}_{\geq}).$$

*Proof.* The result follows from Proposition 2.12 (a) and Proposition 2.18. □

Note that the inclusion

$$\{x : r(x) \in \mathcal{E}(r(X), f, \mathbb{R}^{p_1}_{\geqq})\} \subseteq \mathcal{E}(X, F, \mathbb{R}^{p}_{\geqq}), \tag{2.14}$$

by Proposition 2.10, can be viewed as a reformulation of Proposition 2.21. So we see that a subset of the Pareto global set can be obtained by considering the Pareto set for $f$ over the image set $r(X)$.

**Example 2.22.** Let us consider the data from Example 2.19, and we solve it in a new way. Firstly, taking into account that $r(x_1, x_2) = r(x_2, x_1)$ we calculate the image of $X$ under $r$: $r(X) = \{\,(y_1, y_2) : (y_1 - y_2)^2 - 4(y_1 - 1) \geqq 0,\ (y_1 - y_2)^2 - 4(y_1 + y_2) + 4 \leqq 0,\ y_1 - y_2 \geqq 0\} \subseteq \mathbb{R}^2$. See Figure 5.

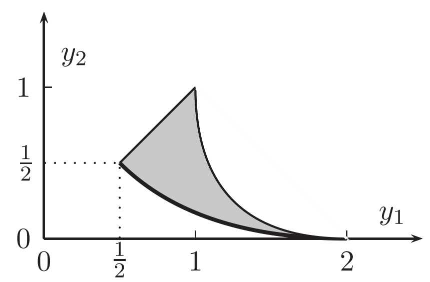


Figure 5: The Image of $X$ under $r$ and Efficient Set. Example 2.22.

Now we compute $\mathcal{E}(r(X), f, \mathbb{R}^2_{\geq}) = \{\,(y_1, y_2) : (y_1 - y_2)^2 - 4(y_1 + y_2) + 4 = 0,\ \frac{1}{2} \leqq y_1 \leqq 2,\ 0 \leqq y_2 \leqq \frac{1}{2}\,\}$. This set is represented by the black line in Figure 5, which is precisely the image of the segment AB (see Figure 4) under the function $r$, i.e. $r(\overline{AB})$. Therefore, $\mathcal{E}(X, F, \mathbb{R}^4_{\geq}) = \overline{AB}$.

In Proposition 2.21 the reverse inclusion does not hold in general as we show in the following example.

**Example 2.23.** In Example 2.22, we replace $f$ by a new function defined by $f(y_1, y_2) = (y_1, y_1 + y_2)$. In this case $\mathcal{E}(r(X), f, \mathbb{R}^2_{\geqq}) = \{(\frac{1}{2}, \frac{1}{2})\}$, while $\mathcal{E}(X, F, \mathbb{R}^4_{\geqq}) = \overline{AB}$, (see Figure 4 and 5).

In light of the above examples we observe that the reverse inclusion, $\mathcal{E}(X, F, \mathbb{R}^p_{\geqq}) \subseteq \mathcal{E}(X, f \circ r, \mathbb{R}^{p_1}_{\geqq})$, from Proposition 2.21 only holds in some special cases. In the next corollary we give some additional conditions under which this occurs.

**Proposition 2.24.** *Let $F = (f \circ r, r)$. If $f$ is a strongly increasing and a strongly order reflecting function on $r(X)$, then*

$$\mathcal{E}(X, f \circ r, \mathbb{R}^{p_1}_{\geqq}) = \mathcal{E}(X, F, \mathbb{R}^p_{\geqq}).$$

*Proof.* It follows from Propositions 2.13 and 2.18. □

## 3 Complex Composite Systems

We use the previously developed relationships from the preceding sections to find the efficient sets of other composite systems, which are referred as complex composite systems. These systems can be thought of as combinations of two or more basic composite systems.

We examine a composite system with two independent blocks of variables. One of them containing all the local decision variables ($x$), each of which directly affects only one of the $N$ subsystems and the other including decision variables ($x_0$) which affect the behavior of the global system and could also influence on the subsystems. Therefore, two cases can be distinguished depending on whether the subsystems have only local information or not. They will be analyzed in the following subsections.

### 3.1 Subsystems with Only Local Information

We start by considering $N$ subsystems each with their own variable, which we denote as $x_i$, and let $r_i : A_i \subseteq \mathbb{R}^{n_i} \to B_i \subseteq \mathbb{R}^{q_i}$ be the objective function for subsystem $i$, or the $i$-th intermediate function. In particular, $X_i \subseteq A_i$.

We consider a map $r : A \subseteq \mathbb{R}^n \to B \subseteq \mathbb{R}^q$, with $n = \sum_{i=1}^N n_i$, $A \subseteq \prod_{i=1}^N A_i$, and the set of the decision vectors $x = (x_1, \ldots, x_N) \in X \subseteq \prod_{i=1}^N X_i \subseteq A$ in order to capture the influence of the different subsystems over the global system. Different possibilities can be employed for the objective function $r$ of a system of $N$-subsystems by mixing the objective functions $r_i$ in many ways. So, for example, each of the following possibilities can be considered:

(1) If $q = q_1 = \cdots = q_N$ and the linear span of the set $\cup_{i=1}^N B_i$ is contained in $B$, then $r$ can be defined as the linear combination of the $r_i$; that is, $r(x) = \sum_{i=1}^N \alpha_i r_i(x_i)$, ($\alpha_i \in \mathbb{R}$, $i = 1, \ldots, N$), for all $x \in X$.

(2) In the case that $q = \sum_{i=1}^N q_i$ and $\prod_{i=1}^N B_i \subseteq B$, then it is possible to define $r$ as the joint vector of the subsystems' objectives, that is, $r(x) = (r_1(x_1), \ldots, r_N(x_N))$, for all $x \in X$.

Since $r$ is the objective function of a system of $N$-subsystems, then the problem for this system can be formulated in the following way:

$$\min r(x) = \text{ subject to } x \in X \subseteq \prod_{i=1}^{N} X_i. \tag{3.1}$$

Let $f : C^0 \times C \subseteq \mathbb{R}^{n_0+q} \to D \subseteq \mathbb{R}^p$ be a function such that $X_0 \subseteq C^0$ and $r(A) \subseteq C$. We can now introduce a new function $F : C^0 \times A \subseteq \mathbb{R}^{n_0+n} \to D \times B \subseteq \mathbb{R}^{p+q}$ defined as $F(x_0, x) = (f(x_0, r(x)), r(x))$. Here the elements of $X_0$ play the role of a global variable for the composite system. We now consider the case that the overall formulation of the problem is as follows,

$$\begin{aligned} &\min F(x_0, x) = (f(x_0, r(x)),\ r(x)) \\ &\text{subject to } (x_0, x) \in Z = X_0 \times X. \end{aligned} \tag{3.2}$$

To develop the next relationship, we require an objective function to be increasing (or strongly increasing) in one of its variables. In a more specific way, given the function $f : C^0 \times C \to D$, we primarily fix the first argument ($s_1 \in C^0$), and we consider the partial function $f_{s_1} : C \longrightarrow D$ given by $f_{s_1}(s_2) = f(s_1, s_2)$. $f$ is said to be an *increasing (resp. strongly increasing) function in the second argument*, when this argument belongs to $\tilde{C} \subseteq C$, if all the restrictions to $\tilde{C}$ of the partial functions $f_{s_1}$ are increasing (resp. strongly increasing).

In a similar way, $f$ is said to be a *continuous function in the second argument* if its restrictions to $\tilde{C}$ of the partial functions $f_{s_1}$ are so.

**Proposition 3.1.** *Let $F(x_0, x) = (f(x_0, r(x)), r(x))$ and $Z = X_0 \times X$, with $X$ a compact subset of $\mathbb{R}^n$. If $r$ is a continuous function on $X$ and $f$ is a increasing function in the second argument, when this argument belongs to $r(X)$, then*

$$\left\{(x_0, x) \in Z :\ (x_0,\ r(x)) \in \mathcal{E}\left(X_0 \times \mathcal{E}(r(X),\ \mathbb{R}^q_{\geqq}),\ f,\ \mathbb{R}^p_{\geqq}\right)\right\} \subseteq \mathcal{E}(Z, F, \mathbb{R}^{p+q}_{\geqq}).$$

*Proof.* Let $(x_0, r(x))$ be efficient in $X_0 \times \mathcal{E}(r(X), \mathbb{R}^q_{\geqq})$ for $f$. To obtain a contradiction,we suppose that $(x_0, x)$ is not efficient in $Z$ for $F$. Then $(\hat{x}_0, \hat{x}) \in Z$ exists such that

$$F(\hat{x}_0, \hat{x}) = (f(\hat{x}_0, r(\hat{x})),\ r(\hat{x})) \leq (f(x_0, r(x)),\ r(x)) = F(x_0, x).$$

We consider two cases, (i) $r(\hat{x}) \leq r(x)$; or (ii) $r(\hat{x}) = r(x)$ and $f(\hat{x}_0, r(\hat{x})) \leq f(x_0, r(x))$.

Case (i) can be ruled out, because it contradicts the assumption that $r(x)$ is non-dominated in $r(X)$. Now we suppose that Case (ii) holds. If $\hat{x}$ is efficient in $X$ for $r$ we make $\breve{x} = \hat{x}$. If $x$ is not efficient, we choose $\breve{x}$ efficient in $X$ for $r$ thereby satisfying $r(\breve{x}) \leq r(\hat{x})$, which can be guaranteed by Lemma 2.4. Since $f$ is an increasing function in the second argument when this argument belongs to $r(X)$, we have that $f(\hat{x}_0, r(\breve{x})) \leqq f(\hat{x}_0, r(\hat{x})) \leq f(x_0, r(x))$, which contradicts the assumption that $(x_0, r(x))$ is efficient in $X_0 \times \mathcal{E}(r(X), \mathbb{R}^q_{\geqq})$ for $f$. □

This proposition implies that a subset of the efficient set for the overall problem can be obtained by a two step-procedure. In the first step, the set of the non-dominated points for $r$ is found. In the second step, the efficient set for $f$ in the Cartesian product of $X_0$ by the set of this non dominated points is calculated.

A second approach to finding efficient solutions to system (3.2) involves fixing the global variable, thereby allowing us the possibility of finding the efficient set for the overall problem by means of finding the efficient sets of the subsystems. This approach is best-suited to problems where the global variable takes a finite number of feasible values.

Let us suppose that the global variable is fixed by setting a value $\bar{x}_0$, $\bar{x}_0 \in X_0$ and we consider the subset defined by

$$Z_{\bar{x}_0} = \{(\bar{x}_0, x) : x \in X\} \tag{3.3}$$

and the partial function $F_{\bar{x}_0} : A \subseteq \mathbb{R}^n \to D \times B \subseteq \mathbb{R}^{p+q}, F_{\bar{x}_0}(x) = F(\bar{x}_0, x)$.

Since $Z = X_0 \times X$, we have $Z_{\bar{x}_0} = \{\bar{x}_0\} \times X$ and by a direct verification we see that

$$\mathcal{E}(Z_{\bar{x}_0}, F, \mathbb{R}^{p+q}_{\geqq}) = \{\bar{x}_0\} \times \mathcal{E}(X, F_{\bar{x}_0}, \mathbb{R}^{p+q}_{\geqq}). \tag{3.4}$$

The following proposition provides a relationship implying that the efficient set of the overall problem can be determined through a multilevel procedure. It is a revised version of a result given by Li and Haimes [16], where the necessary assumption of compactness was not noted.

**Proposition 3.2.** *Let $F(x_0, x) = (f(x_0, r(x)), r(x))$ and $Z = X_0 \times X$, with $X$ a compact subset of $\mathbb{R}^n$. If $r$ is a continuous function on $X$ and $f$ is a continuous function in the second argument, when this argument belongs to $r(X)$, then*

$$\mathcal{E}(Z, F, \mathbb{R}^{p+q}_{\geqq}) = \mathcal{E}\left(\bigcup_{\bar{x}_0 \in X_0} \{\bar{x}_0\} \times \mathcal{E}(X, F_{\bar{x}_0}, \mathbb{R}^{p+q}_{\geqq}), F, \mathbb{R}^{p+q}_{\geqq}\right).$$

*Proof.* We first assert that

$$Z_{\bar{x}_0} \cap \mathcal{E}(Z, F, \mathbb{R}^{p+q}_{\geqq}) \subseteq \{\bar{x}_0\} \times \mathcal{E}(X, F_{\bar{x}_0}, \mathbb{R}^{p+q}_{\geqq}). \tag{3.5}$$

Indeed, let $(\bar{x_0}, x)$ an element of $Z_{\bar{x}_0} \subseteq Z$ which is efficient in $Z$ for $F$. On the contrary, we suppose $(\bar{x_0}, x) \notin \{\bar{x}_0\} \times \mathcal{E}(X, F_{\bar{x}_0}, \mathbb{R}^{p+q}_{\geqq})$, then there exists $\hat{x} \in X$ such that $F_{\bar{x}_0}(\hat{x}) \leq F_{\bar{x}_0}(x)$, which contradicts the assumption that $(\bar{x_0}, x)$ is efficient in $Z$ for $F$.

From (3.5) we obtain $\mathcal{E}(Z, F, \mathbb{R}^{p+q}_{\geqq}) = \bigcup_{\bar{x}_0 \in X_0} \left(Z_{\bar{x_0}} \cap \mathcal{E}(Z, F, \mathbb{R}^{p+q}_{\geqq})\right) \subseteq \bigcup_{\bar{x}_0 \in X_0} \{\bar{x}_0\} \times \mathcal{E}(X, F_{\bar{x}_0}, \mathbb{R}^{p+q}_{\geqq})$. Now by Lemma 2.3 it follows $\mathcal{E}(Z, F, \mathbb{R}^{p+q}_{\geqq}) \subseteq \mathcal{E}(\bigcup_{\bar{x}_0 \in X_0} \{\bar{x}_0\} \times \mathcal{E}(X, F_{\bar{x}_0}, \mathbb{R}^{p+q}_{\geqq}), F, \mathbb{R}^{p+q}_{\geqq})$.

Let us now prove the converse inclusion. Let $(x_0, x)$ be efficient in $\bigcup_{\bar{x}_0 \in X_0} \{\bar{x}_0\} \times \mathcal{E}(X, F_{\bar{x}_0}, \mathbb{R}^{p+q}_{\geqq})$ for $F$ and, by contradiction, suppose that $(x_0, x)$ is not efficient in $Z$ for $F$. Then there exists $(\hat{x_0}, \hat{x}) \in Z = \bigcup_{\bar{x}_0 \in X_0} Z_{\bar{x}_0}$ such that $F(\hat{x_0}, \hat{x}) \leq F(x_0, x)$.

We consider two cases, either $(\hat{x_0}, \hat{x}) \in \{\hat{x}_0\} \times \mathcal{E}(X, F_{\hat{x}_0}, \mathbb{R}^{p+q}_{\geqq})$ or $(\hat{x_0}, \hat{x}) \in \{\hat{x}_0\} \times X$ and $\hat{x} \notin \mathcal{E}(X, F_{\hat{x}_0}, \mathbb{R}^{p+q}_{\geqq})$.

The first case can be ruled out, because this contradicts the assumption that $(x_0, x)$ is efficient in $\bigcup_{\bar{x}_0 \in X_0} \{\bar{x}_0\} \times \mathcal{E}(X, F_{\bar{x}_0}, \mathbb{R}^{p+q}_{\geqq})$ for $F$. Suppose that $\hat{x} \notin \mathcal{E}(X, F_{\hat{x}_0}, \mathbb{R}^{p+q}_{\geqq})$. By Lemma 2.4, there exists $\breve{x} \in \mathcal{E}(X, F_{\hat{x}_0}, \mathbb{R}^{p+q}_{\geqq})$ such that $F_{\hat{x}_0}(\breve{x}) \leq F_{\hat{x}_0}(\hat{x})$. Now, from $F(\hat{x_0}, \hat{x}) \leq F(x_0, x)$ it follows that $(x_0, x)$ is not efficient in $\bigcup_{\bar{x}_0 \in X_0} \{\bar{x}_0\} \times \mathcal{E}(X, F_{\bar{x}_0}, \mathbb{R}^{p+q}_{\geqq})$ for $F$, which is a contradiction. Thus $(x_0, x) \in \mathcal{E}(Z, F, \mathbb{R}^{p+q}_{\geqq})$. $\square$

Propositions 3.1 and 3.2 can be improved when the $N$ subsystems are independent, something which we carry out in the next propositions by a complete reduction of the global problem to each of its components.

Let us explain this new case. If $X, X_1$ and $X_2$ are feasible sets with $X$ a proper subset of $X_1 \times X_2$, then $X$ can be viewed as the subset of the elements $(x_1, x_2)$ of $X_1 \times X_2$ satisfying a certain property $P$. This makes it natural to consider the equality $X = X_1 \times X_2$ as one of requirements for the independence of subsystems. So we will say that the $N$ *subsystems are independent* if $X = \prod_{i=1}^{N} X_i$ and, in addition, the objective function $r(x)$ can be expresed as $r(x) = (r_1(x_1), \ldots, r_N(x_N))$. Then the problem (3.1) can be decomposed into the following subproblems

$$\min r_i(x_i) \text{ subject to } x_i \in X_i,\ i = 1, \ldots, N, \tag{3.6}$$

and the relationship given in Lemma 3.3 can be obtained.

**Lemma 3.3.** *If the subsystems are independent, then*

$$\mathcal{E}(X, r, \mathbb{R}^q_{\geqq}) = \prod_{i=1}^{N} \mathcal{E}(X_i, r_i, \mathbb{R}^{q_i}_{\geqq}).$$

*Proof.* Let $x$ be efficient in $X$ for $r$. On the contrary, suppose that for some $j$, $x_j$ is not efficient in $X_j$ for $r_j$. Then there exists $\hat{x}_j \in X_j$ such that $r_j(\hat{x}_j) \leq r_j(x_j)$, and consequentially for $\check{x} = (x_1, \ldots, x_{j-1}, \hat{x}_j, x_{j+1}, \ldots, x_N) \in X$ it holds $r(\check{x}) \leq r(x)$, contrary to the efficient character of $x$ in $X$ for $r$.

Let $x_i$ be efficient in $X_i$ for $r_i$ for all $i$, and suppose that $x$ is not efficient in $X$ for $r$. Then there exists $\hat{x} \in X$ such that $r(\hat{x}) \leq r(x)$. Then for some $j$, there exists $\hat{x}_j \in X_j$ such that $r_j(\hat{x}_j) \leq r_j(x_j)$, contrary to the assumption that $x_i$ is efficient in $X_i$ for $r_i$ for all $i$. $\square$

This lemma implies that if the subsystems are independent, then problem (3.1) can be solved by considering the resolution of the different subsystems (3.6), in a separate and independent way. So the efficient set for $r$ is obtained as the Cartesian product of the Pareto sets, individually solved, for each subproblem.

In order to improve Proposition 3.1 we use the previous lemma when the subsystems are independent.

**Proposition 3.4.** *Let* $F(x_0, x) = (f(x_0, r(x)), r(x))$ *and* $Z = X_0 \times X$, *with* $X$ *a compact subset of* $\mathbb{R}^n$. *If* $r$ *is a continuous function on* $X$, $f$ *is a increasing function in the second argument when this argument belongs to* $r(X)$ *and the subsystems are independent, then*

$$\left\{ (x_0, x) \in Z : (x_0, r(x)) \in \mathcal{E}\left( X_0 \times \prod_{i=1}^{N} \mathcal{E}(r_i(X_i), \mathbb{R}^{q_i}_{\geqq}), f, \mathbb{R}^p_{\geqq} \right) \right\} \subseteq \mathcal{E}(Z, F, \mathbb{R}^{p+q}_{\geqq}).$$

*Proof.* Using Lemma 3.3 we obtain $\mathcal{E}(r(X), \mathbb{R}^q_{\geqq}) = r\big(\mathcal{E}(X, r, \mathbb{R}^q_{\geqq})\big) = r\big(\prod_{i=1}^{N} \mathcal{E}(X_i, r_i, \mathbb{R}^{q_i}_{\geqq})\big) = \prod_{i=1}^{N} r_i\big(\mathcal{E}(X_i, r_i, \mathbb{R}^{q_i}_{\geqq})\big) = \prod_{i=1}^{N} \mathcal{E}(r_i(X_i), \mathbb{R}^{q_i}_{\geqq})$. Proposition 3.1 now completes the proof. $\square$

**Proposition 3.5.** *Let* $F(x_0, x) = (f(x_0, r(x)), r(x))$ *and* $Z = X_0 \times X$, *with* $X$ *a compact subset of* $\mathbb{R}^n$. *If* $r$ *is a continuous function on* $X$, $f$ *is both an increasing and a continuous function in the second argument when this argument belongs to* $r(X)$ *and the subsystems are independent, then*

$$\mathcal{E}(Z, F, \mathbb{R}^{p+q}_{\geqq}) = \mathcal{E}\left( \bigcup_{\bar{x}_0 \in X_0} \{\bar{x}_0\} \times \prod_{i=1}^{N} \mathcal{E}(X_i, r_i, \mathbb{R}^{q_i}_{\geqq}), F, \mathbb{R}^{p+q}_{\geqq} \right).$$

*Proof.* From Proposition 3.2 and by applying Proposition 2.18 to the partial functions $F_{\bar{x}_0}$, we have $\mathcal{E}(Z, F, \mathbb{R}^{p+q}_{\geqq}) = \mathcal{E}(\bigcup_{\bar{x}_0 \in X_0} \{\bar{x}_0\} \times \mathcal{E}(X, r, \mathbb{R}^q_{\geqq}), F, \mathbb{R}^{p+q}_{\geqq})$. Since the subsystems are independent, we can apply Lemma 3.3 to obtain the stated result. □

It follows from this proposition that the efficient set of the overall problem can be determined through a multilevel procedure. In the lower level, problem (3.1) is solved in each subsystem, to obtain the sets $\mathcal{E}(X_i, r_i, \mathbb{R}^{q_i}_{\geqq}), i = 1, 2, \ldots N$. The Cartesian product of these sets is then constructed and each of its $N$-tuples is augmented by a component taking a fixed value $\bar{x_0}$ of the first variable $x_0$. In the upper level, we obtain the set of efficient solutions for the global problem (3.2) by making use of these $(N+1)$-tuples associated with the different $\bar{x_0}$.

## 3.2 Subsystems with Global and Local Information

In this subsection we study the case that the global variable affects the subsystems.

Let $r_i : A^0 \times A_i \subseteq \mathbb{R}^{n_0+n_i} \to B_i \subseteq \mathbb{R}^{q_i}$ be the objective function for subsystem $i$, which is a function $r_i(x_0, x_i)$ depending on the global variable $x_0 \in A^0$ and the local variable $x_i \in A_i$, for $i = 1, \ldots, N$. In a similar way to Subsection 3.1, let $r : A^0 \times A \subseteq \mathbb{R}^{n_0+n} \to B \subseteq \mathbb{R}^q$ be the function that captures the influence of the subsystems' objective functions on the overall system, similarly $n = \sum_{i=1}^N n_i$ and $A \subseteq \prod_{i=1}^N A_i$. Now let $f : B \subseteq \mathbb{R}^q \to D \subseteq \mathbb{R}^p$ be another function that incorporates the influences of the subsystems' outputs on the overall system, which could be as simple as a linear combination of them, or a more complicated functional dependence. Finally, we introduce the map $F : A^0 \times A \subseteq \mathbb{R}^{n_0+n} \to D \times B \subseteq \mathbb{R}^{p+q}$ defined as $F(x_0, x) = (f(r(x_0, x)), r(x_0, x))$ and we consider particularly the subsets $X_0 \subseteq A^0$ and $\in X_i \subseteq A_i$, for $i = 1, \ldots, N$. We shall consider the following global formulation of the problem:

$$\begin{aligned} &\min F(x_0, x) = (f(r(x_0, x)), r(x_0, x)) \\ &\quad \text{subject to } (x_0, x) \in Z \subseteq \textstyle\prod_{i=0}^N X_i. \end{aligned} \tag{3.7}$$

The results given in Section 2.2.3 can be improved in particular cases, such as when $r$ is the sum of the subsystems' objectives, as well as when $r$ is the joint vector of the subsystems' objectives. More specifically, in the first case we assume $n_0 = n_1 = \cdots = n_N = \frac{n}{N} = q = q_1 = \cdots = q_N$ and $B$ contained in the linear span of the set $\cup_{i=1}^N B_i$, and we consider the function $r : A^0 \times A \subseteq \mathbb{R}^{n_0+n} \to B \subseteq \mathbb{R}^q$ defined by $r(x_0, x) = \sum_{i=1}^N r_i(x_0, x_i)$ and $r_i(x_0, x_i) = \alpha_0^i\, x_0 + \alpha_i\, x_i$ with $\alpha_0^i,\ \alpha_i \in \mathbb{R}_>$, $i = 1, \ldots, N$. So we can write $r(x_0, x) = \sum_{i=0}^N \alpha_i\, x_i$ with $\alpha_0 = \sum_{i=1}^N \alpha_0^i$ and $\alpha_i \in \mathbb{R}_>$, $i = 0, \ldots, N$.

**Corollary 3.6.** *Let $F = (f \circ r, r)$ and $Z = \prod_{i=0}^N X_i$ with $X_i$ a compact subset of $\mathbb{R}^{n_i}$ for $i = 0, \ldots, N$. If $r(x_0, x) = \sum_{i=o}^N \alpha_i\, x_i$, ($\alpha_i \in \mathbb{R}_>$, $i = 0, \ldots, N$) and $f$ is a linear strongly increasing function on $r(Z)$, then*

$$\left\{(x_0, x) : r(x_0, x) \in \mathcal{E}\left(\sum_{i=0}^{N} \alpha_i\, \mathcal{E}(X_i,\, f, \mathbb{R}^p_{\geqq}),\, f,\, \mathbb{R}^p_{\geqq}\right)\right\} \subseteq \mathcal{E}(Z, F, \mathbb{R}^{p+q}_{\geqq}).$$

*Proof.* From Proposition 2.21, reformulated in (2.14), we have $\mathcal{E}(Z, F, \mathbb{R}^{p+q}_{\geqq}) \supseteq \{(x_0, x) : r(x_0, x) \in \mathcal{E}(r(Z), f, \mathbb{R}^p_{\geqq})\}$. Using relationship (2.6), we obtain $\{(x_0, x) : r(x_0, x) \in \mathcal{E}(r(Z), f, \mathbb{R}^p_{\geqq})\} = \{(x_0, x) : r(x_0, x) \in \mathcal{E}(\sum_{i=0}^N \alpha_i X_i, f, \mathbb{R}^p_{\geqq})\} = \{(x_0, x) : r(x_0, x) \in \mathcal{E}\left(\sum_{i=0}^N \alpha_i \mathcal{E}(X_i, f, \mathbb{R}^p_{\geqq}), f, \mathbb{R}^p_{\geqq}\right)\}$. □

We now examine the system (3.7) when $Z = \prod_{i=0}^{N} X_i$ and $r$ is the the joint vector of the subsystems' objectives, in this case $q = \sum_{i=1}^{N} q_i$ and $B \subseteq \prod_{i=1}^{N} B_i$ must hold. Thus, the problem of the $N$ subsystems would have the following form:

$$\begin{aligned} \min r(x_0, x) &= (r_1(x_0, x_1), \ldots, r_N(x_0, x_N)) \\ \text{subject to } (x_0, x) &\in Z = \textstyle\prod_{i=0}^{N} X_i. \end{aligned} \tag{3.8}$$

Due to the global variable $x_0$, the $N$ subsystems are not necessarily independent. Nevertheless, for every fixed $\bar{x}_0 \in X_0$ we can generate a new system constituted by $N$ independent subsystems, and then use the solutions of the resulting subproblems to generate the solutions of the original problem.

Let $\bar{x}_0 \in X_0$ be a fixed element and we consider the subset $Z_{\bar{x}_0}$ as in (3.3). We now consider the partial functions $r_{i\bar{x}_0} : A_i \subseteq \mathbb{R}^{n_i} \to B_i \subseteq \mathbb{R}^{q_i}$ given by $r_{i\bar{x}_0}(x_i) = r_i(\bar{x}_0, x_i)$ for $i = 1, \ldots, N$, and the joint vector of the partial functions $r_{\bar{x}_0}(x) = (r_{1\bar{x}_0}(x_1), \ldots, r_{N\bar{x}_0}(x_N))$. Then, for every fixed $\bar{x}_0 \in X_0$ the problem (3.8) can be decomposed into the following associated subproblems

$$\min r_{i\bar{x}_0}(x_i) \text{ subject to } x_i \in X_i, \ i = 1, \ldots, N, \tag{3.9}$$

and the following relationships can be obtained.

**Lemma 3.7.** *Let $Z = X_0 \times X$, with $X = \prod_{i=1}^{N} X_i$ a compact subset of $\mathbb{R}^n$. If $r = (r_1, \ldots, r_N)$ is a continuous function on $X$, then*

$$\mathcal{E}(Z, r, \mathbb{R}^q_{\geqq}) = \mathcal{E}\left( \bigcup_{\bar{x}_0 \in X_0} \{\bar{x}_0\} \times \prod_{i=1}^{N} \mathcal{E}(X_i, r_{i\bar{x}_0}, \mathbb{R}^{q_i}_{\geqq}), r, \mathbb{R}^q_{\geqq} \right).$$

*Proof.* Firstly, let us prove that $\mathcal{E}(Z, r, \mathbb{R}^q_{\geqq}) = \mathcal{E}(\bigcup_{\bar{x}_0 \in X_0} \{\bar{x}_0\} \times \mathcal{E}(X, r_{\bar{x}_0}, \mathbb{R}^q_{\geqq}), r, \mathbb{R}^q_{\geqq})$.

Let $(x_0^*, x^*) \in \mathcal{E}(Z, r, \mathbb{R}^q_{\geqq})$. Obviously $(x_0^*, x^*) \in Z_{x_0^*}$, and furthermore, $(x_0^*, x^*) \in \{x_0^*\} \times \mathcal{E}(X, r_{x_0^*}, \mathbb{R}^q_{\geqq})$ because otherwise there exists $\hat{x} \in X$ such that $r_{x_0^*}(\hat{x}) \leq r_{x_0^*}(x^*)$, which contradicts that $(x_0^*, x^*)$ is efficient in $Z$ for $r$. Therefore $\mathcal{E}(Z, r, \mathbb{R}^q_{\geqq}) \subseteq \bigcup_{\bar{x}_0 \in X_0} \{\bar{x}_0\} \times \mathcal{E}(X, r_{\bar{x}_0}, \mathbb{R}^q_{\geqq})$. From Lemma 2.3 it follows that $\mathcal{E}(Z, r, \mathbb{R}^q_{\geqq}) \subseteq \mathcal{E}(\bigcup_{\bar{x}_0 \in X_0} \{\bar{x}_0\} \times \mathcal{E}(X, r_{\bar{x}_0}, \mathbb{R}^q_{\geqq}), r, \mathbb{R}^q_{\geqq})$. This proves one inclusion.

Assume that $(x_0, x) \in \mathcal{E}(\bigcup_{\bar{x}_0 \in X_0} \{\bar{x}_0\} \times \mathcal{E}(X, r_{\bar{x}_0}, \mathbb{R}^q_{\geqq}), r, \mathbb{R}^q_{\geqq})$ but $(x_0, x) \notin \mathcal{E}(Z, r, \mathbb{R}^q_{\geqq})$. Then there exists $(\hat{x_0}, \hat{x}) \in Z$ such that $r(\hat{x_0}, \hat{x}) \leq r(x_0, x)$. We assert that $\hat{x} \notin \mathcal{E}(X, r_{\hat{x}_0}, \mathbb{R}^q_{\geqq})$. Indeed, on the contrary, we should have $(\hat{x_0}, \hat{x}) \in \{\hat{x}_0\} \times \mathcal{E}(X, r_{\hat{x}_0}, \mathbb{R}^q_{\geqq})$, with $r(\hat{x_0}, \hat{x}) \leq r(x_0, x)$. This contradicts the efficient character of $(x_0, x)$ in $\bigcup_{\bar{x}_0 \in X_0} \{\bar{x}_0\} \times \mathcal{E}(X, r_{\bar{x}_0}, \mathbb{R}^q_{\geqq})$ for $r$. Therefore, $\hat{x} \notin \mathcal{E}(X, r_{\hat{x}_0}, \mathbb{R}^q_{\geqq})$. Then, by Lemma 2.4, there exists $\breve{x} \in \mathcal{E}(X, r_{\hat{x}_0}, \mathbb{R}^q_{\geqq})$ such that $r_{\hat{x}_0}(\breve{x}) \leq r_{\hat{x}_0}(\hat{x})$, that is, $r(\hat{x}_0, \breve{x}) \leq r(\hat{x}_0, \hat{x})$. Therefore, from $r(\hat{x}_0, \hat{x}) \leq r(x_0, x)$, we again obtain a contradiction and thus $(x_0, x) \in \mathcal{E}(Z, r, \mathbb{R}^q_{\geqq})$. This proves the converse inclusion.

Finally, applying Lemma 3.3 to the set $\mathcal{E}(X, r_{\bar{x}_0}, \mathbb{R}^q_{\geqq})$, the proof is finished. □

This result proves that efficient solutions for the problem (3.8) can be generated by the efficient solutions of $N$ independent subsystems (3.9). Next we show a relationship between these solutions and the original problem (3.7).

**Corollary 3.8.** *Let $F = (f \circ r, r)$ and $Z = X_0 \times X$, with $X = \prod_{i=1}^{N} X_i$ a compact subset of $\mathbb{R}^n$. If $r = (r_1, \ldots, r_N)$ is a continuous function on $X$ and $f$ is a strongly increasing function on $r(Z)$, then*

$$\mathcal{E}(Z, F, \mathbb{R}^{p+q}_{\geqq}) = \mathcal{E}\left(\bigcup_{\bar{x}_0 \in X_0} \{\bar{x}_0\} \times \prod_{i=1}^{N} \mathcal{E}(X_i, r_{i\bar{x}_0}, \mathbb{R}^{q_i}_{\geqq}), r, \mathbb{R}^{q}_{\geqq}\right).$$

*Proof.* By Proposition 2.18, $\mathcal{E}(Z, F, \mathbb{R}^{p+q}_{\geqq}) = \mathcal{E}(Z, r, \mathbb{R}^{q}_{\geqq})$. Applying Lemma 3.7 to the set $\mathcal{E}(Z, r, \mathbb{R}^{q}_{\geqq})$ the proof is completed. □

This result shows, that the vector-valued objective function $F$ can be ignored to calculate the efficient set of the global problem.

**Example 3.9.** Let $r : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ where $r(x_0, , x_1, x_2) = [r_1(x_0, x_1), r_2(x_0, x_2)] = [(x_1 - x_0)^2, \exp(x_2 - x_0)^2]$, and let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ where $f(y_1,\ y_2) = y_2 \sqrt{y_1}$. Let $X = \{(x_1, x_2) : x_1 \geqq 3, 0 \leqq x_2 \leqq 1\} = [3,\ \infty] \times [0, 1] = X_1 \times X_2$, and $X_0 = \{-2, -1, 0, 1, 2\}$.

According to the above functions, we have $F(x_0, x_1, x_2) = [(x_1 - x_0)\exp(x_2 - x_0)^2, (x_1 - x_0)^2, \exp(x_2 - x_0)^2]$. Since the function $f$ is strongly increasing in $r(Z)$, $\mathcal{E}(X_0 \times X_1 \times X_2, F, \mathbb{R}^3_{\geqq})$ can be determined by Corollary 3.8.

Fixing $\bar{x}_0 \in X_0$, and solving $\min r_{1\bar{x}_0}(x_1)$ subject to $x_1 \in X_1$ and $\min r_{2\bar{x}_0}(x_2)$ subject to $x_2 \in X_2$ yields

$$\mathcal{E}\left(X_1, r_{1\bar{x}_0}, \mathbb{R}_{\geqq}\right) = \begin{cases} \bar{x}_0 & \text{if} \quad \bar{x}_0 \geqq 3 \\ 3 & \text{if} \quad \bar{x}_0 < 3 \end{cases}$$

and

$$\mathcal{E}\left(X_2, r_{2\bar{x}_0}, \mathbb{R}_{\geqq}\right) = \begin{cases} \bar{x}_0 & \text{if} \quad 0 \leqq \bar{x}_0 \leqq 1 \\ 0 & \text{if} \quad \bar{x}_0 < 0 \\ 1 & \text{if} \quad \bar{x}_0 > 1 \end{cases}$$

Then $\bigcup_{\bar{x}_0 \in X_0} \{\bar{x}_0\} \times \mathcal{E}(X_1, r_{1\bar{x}_0}, \mathbb{R}_{\geqq}) \times \mathcal{E}(X_2, r_{2\bar{x}_0}, \mathbb{R}_{\geqq}) = \{(-2, 3, 0); (-1, 3, 0); (0, 3, 0); (1, 3, 1); (2, 3, 1)\}$ and $\mathcal{E}\left(\{(-2, 3, 0); (-1, 3, 0); (0, 3, 0); (1, 3, 1); (2, 3, 1)\}, r, \mathbb{R}^2_{\geqq}\right) = \{(1, 3, 1); (2, 3, 1)\}$. Hence $\mathcal{E}(X_0 \times X_1 \times X_2, F, \mathbb{R}^3_{\geqq}) = \{(1, 3, 1); (2, 3, 1)\}$.

## 4 An application for the reliability problem

To show how these methods can be applied to the effective solution of practical problems, let us consider the Redundancy Allocation Problem (RAP) in series-parallel engineering systems and other systems of similar properties. This problem is one of the most important reliability optimization problems in the design phase and it is well known to be a NP-hard problem [2]. The RAP basically involves the determination of the number of components to be allocated in each subsystem, allowing some degree of redundancies, with the purpose of maximizing the system reliability. It has been the subject of many studies, [13, 14, 25, 29].

An engineering system with a series-parallel structure is composed of a fixed number ($N$) of independent engineering subsystems connected in series. For the $i$-th subsystem, it can have up to $n_{max_i}$ functionally equivalent components arranged in parallel. The $n_i$

components are selected from $t_i$ available component types where multiple copies of each type can be selected. $x_{ij}$ denotes the number of $j$-th type components used in subsystem $i$ and $x_i = (x_{i1}, \ldots, x_{it_i})$ is the decision vector of the engineering subsystem $i$. Each component has different levels of cost, weight and reliability.

A subsystem $i$ can work properly if at least one of its components is operational. But it is often advantageous to add redundant components to improve reliability. The use of redundancy increases system reliability but that also increases its cost and weight. The goal is to determine the optimal number to allocate to the redundant components that will maximize system reliability, minimize the total cost and minimize the system weight, for a series-parallel system. This can be reformulated in the following way:

$$\min G(x) = \Big(\frac{1}{\prod_{i=1}^N R_i(x_i)},\ \sum_{i=1}^N C_i(x_i),\ \sum_{i=1}^N W_i(x_i)\Big) \tag{4.1}$$

subject to

$$\begin{aligned} 1 \quad &\le \quad \sum_{j=1}^{t_i} x_{ij} \le n_{\max i}, \quad i = 1, 2, \ldots, N && (4.2)\\ x_{ij} \quad &\in \quad \{0, \ldots, n_{\max i}\}, && (4.3)\end{aligned}$$

being $x_i = (x_{i1}, \ldots, x_{it_i})$ for $i = 1, 2, \ldots, N$, and $x = (x_1, \ldots, x_N)$, where $R_i$, $C_i$ and $W_i$ are the reliability, cost and weight functions of the $i$-th subsystem. Using Proposition 2.13 with $r = G$ and $f : \mathbb{R}_> \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^3$ being the function given by $f(u, v, w) = (\log u, v, w)$, we see that the above optimization problem is equivalent to the following problem:

$$\min \Big( -\sum_i \log R_i(x_i),\ \sum_{i=1}^N C_i(x_i),\ \sum_{i=1}^N W_i(x_i)\Big) \tag{4.4}$$

subject to the same restrictions (4.2) and (4.3).

Now Proposition 2.12 sheds a new light in the resolution of (4.4) subject to the constrains (4.2) and (4.3). Indeed, for each $i$ we consider the subsets $X_i$ of the $x_i = (x_{i1}, \ldots, x_{it_i})$ of $\mathbb{R}^{t_i}$ satisfying (4.2) and (4.3) and we define $r_i : X_i \longrightarrow \mathbb{R}^3$ given by $r_i(x_i) = (-\log R_i(x_i), C_i(x_i), W_i(x_i))$. Let $r : X = \prod X_i \longrightarrow \mathbb{R}^{3^N}$ be the map $r : x = (x_1, \ldots, x_N) \mapsto (r_1(x_1), \ldots, r_1(x_N))$. Using Proposition 2.12 with $f : \mathbb{R}^{3^N} \longrightarrow \mathbb{R}^3$ defined as $f(z_1, \ldots, z_N) = \sum_{i=1}^N z_i$ we can solve (4.4) subject to the constrains (4.2) and (4.3) by reduction to the two different easier problems stated below:

**Problem A.** $\min r(x)$ subject to (4.2) and (4.3).

**Problem B.** (4.4) subject to the restrictions $x \in Y$, where $Y$ is the set of solutions for Problem A.

Furthermore, by Lemma 3.3, Problem A can be descomposed in $N$ optimization problems $A_1, A_2, \ldots A_N$, which for each arbitrary $i$ can be stated in the following way:

**Problem $\mathbf{A_i}$.**

$$\min r_i(x_i) \qquad \text{subject to } x_i \in X_i.$$

In short, we have seen the way that the decomposition methods studied in this paper can be applied in the treatment of the redundancy allocation problem. So we can split these kinds of problems into different sub-problems. At the first level, we consider Problem A, which is treated by descending to the constituent subsystems and solving the corresponding minimization problem in each of them (Problems $A_i$). Now the solution set for Problem A is the starting point for Problem B, since based on this set we must consider the corresponding minimization problem.

This decomposition provides a saving in the computational effort, and could also facilitate the decision process to select a final solution. The efficient set of the subproblem $A_i$ $(i = 1, 2, \ldots, N)$ can be reduced by incorporating preference information stated by the decision maker of this subproblem. Thus, only efficient solutions that are of interest to the decision maker would be generated. Then, by incorporating an interactive procedure (see, for example, [3] and [4]) among the decision makers of each subsystem into the process, a final solution in problem $B$ could be selected. Therefore, decomposition allows each person responsible for a subsystem (with expert knowledge) to be involved in the process of making a final decision.

## 5 Conclusion

This study focuses on multiobjective optimization problems that involve several multiobjective subproblems. In many cases these subproblems can be simpler to treat (they have a lower dimension and/or they are easier to solve, or can be solved in a parallel way,...). In this work various results that relate the efficient set of the overall problem to the efficient sets obtained after the breakdown are offered. These results can be viewed as a theoretical formulation of different decomposition frameworks for the starting problem in subproblems.

The achieved results and methods are considerably general and so they are valuable in the case of non-differentiable problems or with integer variables. The latter occurs in the case of the Redundancy Allocation Problem considered in Section 4, where a decomposition into subproblems is given for its resolution.

We have analyzed different cases depending on the characteristics of certain elements which are related to the structure of the considered problem: decision variables that can be expressed as sum of two or more variables, and can be given as a linear combination of a finite number of variables, or how it affects a change in the unit measure of these variables on the initial problem's solutions; objective functions that depend on intermediate functions, and decision variables that can be local and/or global. In some cases, the decomposition produces independent subunits and solving the corresponding subproblems give the overall solution to the problem. In other cases, the output of the subproblems must be assembled in a second phase in order to obtain the initial problem's solution. We illustrate the theoretical results with some small examples.

We observe that the subproblems studied here each have their own local variables, which are not interconnected. As a future research line, we want to study more general situations where such limitations are dropped. On the other hand, we can see that the existence of some symmetries, such as those underlying Example 2.22, allow us to consider only a certain relevant part of the feasible set, simplifying then the solution of the starting problem. This suggests a more accurate study in a next future of the symmetry concepts to reduce/decompose the overall problem.

**Acknowledgments** The authors wish to express their gratitude to two anonymous referees for their valuable and helpful comments.

## References

[1] M. Balesdent, N. Bérend, P. Dépincé and A. Chriette, A survey of multidisciplinary design optimization methods in launch vehicle design, *Struc. Multi. Optim.* 45 (2012) 619–642.

[2] M.S. Chern, On the computational complexity of reliability redundancy allocation in a series system, *Operations Research Letters* 11 (1992) 309 – 315.

[3] A. Engau, Interactive decomposition-coordination methods for complex decision problems, in *Handbook of Multicriteria Analysis, vol. 103, Applied Optimization*, C. Zopounidis and P. M. Pardalos (eds), Springer, Berlin/Heidelberg, 2010, pp. 329–365.

[4] A. Engau and M.M. Wiecek, Interactive coordination of objective decompositions in multiobjective programming, *Management Science* 54 (2008) 1350–1363.

[5] R. Fare, L. Jansson and C.A.K. Lovell, Modelling scale economies with ray-homothetic production functions. *Review Econ. Statistics* 67 (1985) 624–629.

[6] M. Gardenghi, T. Gómez, F. Miguel and M.M. Wiecek, Algebra of efficient sets for multiobjective complex systems, *J. Optim. Theory Appl.* 149 (2011) 385–410.

[7] M. Gardenghi and M.M. Wiecek, Efficiency for multiobjective multidisciplinary optimization problems with quasiseparable subproblems. *Optim. Engineer.* 13 (2012) 293–318.

[8] T. Gómez, M. González, M. Luque, F. Miguel and F. Ruiz, Multiple objectives decomposition-coordination methods for hierarchical organizations, *Eur. J. Oper. Res.* 133 (2001) 323–341.

[9] Y.Y. Haimes, K. Tarvainen, T. Shima and J. Thadathil, *Hierarchical Multiobjective Analysis of Large-Scale Systems*, Hemisphere Publishing Corporation, New York, 1990.

[10] R. Hartley, On cone-efficiency, cone-convexity and cone-compactness. *SIAM J. Applied Math.* 34 (1978) 211–222.

[11] G.A. Jehle and P.J. Reny, *Advanced Microeconomic Theory*, Prentice Hall, New York, 2011.

[12] J.D. Kelly and D. Zyngier, Hierarchical decomposition heuristic for scheduling: Coordinated reasoning for decentralized and distributed decision-making problems. *Comp. Chemical Engineer.* 32 (2008) 2684–2705.

[13] W. Kuo and V. R. Prasad, An annotated overview of system-reliability optimization. *IEEE Trans. Reliability* 49 (2000) 176–187.

[14] W. Kuo and R. Wan, Recent advances in optimal reliability allocation, *Systems, Man, and Cybernetics-Part A*, vol. 37, 2007, pp. 143–156.

[15] L.S. Lasdon, *Optimization Theory for Large Systems*, Macmillan Company, London, England, 1970.

[16] D. Li and Y. Y. Haimes, The envelope approach for multiobjective optimization problems, *IEEE Transactions on Systems, Man, and Cybernetics*, vol 6, 1987, pp. 1026–1038.

[17] E.R. Lieberman, Hierarchical multiobjective programming: An overview, in *Multiple Criteria Decision Making: Theory and Applications in Business, Industry and Government*, A. Goicoechea, L. Duckstein and S. Zionts (eds), Springer-Verlag /Berlin, 1992, pp. 211–225.

[18] J.R.R.A. Martins and A.B. Lambe, Multidisciplinary design optimization: A survey of architectures, *Amer. Instit. Aeron. Astron.* 51 (2013) 2049–2075.

[19] B. Meijboom and B. Obel, Tactical coordination in a multi-location and multi-stage operations structure: A model and a pharmaceutical company case, *Omega* 35 (2007) 258–273.

[20] F. Miguel, T. Gómez, M. Luque, F. Ruiz and R. Caballero, A decomposition-coordination method for complex multi-objective systems, *Asia-Pacific J. Oper. Res.* 26 (2009) 735–757.

[21] S. Rabeau, P. Dépincé and F. Bennis, Collaborative optimization of complex systems: A multidisciplinary approach, *J. Inter. Design Manu.* 1 (2007) 209–218.

[22] Y. Sawaragi, H. Nakayama and T. Tanino, *Theory of Multiobjective Optimization. Mathematics in Science and Engineering*, vol.176, Academic Press, Inc. New York., 1985.

[23] R. E. Steuer, *Multiple Criteria Optimization: Theory, Computation and Application*, John Wiley and Sons, New York, 1986.

[24] C. Stummer and R. Vetschera, Decentralized planning for multiobjective resource allocation and project selection, Technical Report Working Paper OP 2002—04, Universitt Wien., 2002.

[25] H. A. Taboadaa, F. Baheranwalaa, D.W. Coita and N. Wattanapongsakornb. Practical solutions for multi-objective optimization: An application to system reliability design problems. *Reliability Engineer. System Safety* 92 (2007) 314–322.

[26] R.V. Tappeta, J.E. Renaud and J.F. Rodriguez, An interactive multiobjective optimization design strategy for decision based multidisciplinary design, *Engineer. Optim.* 34 (2002) 523–544.

[27] S. Tosserams, L.F.P. Etman and J.E. Rooda, A classification of methods for distributed system optimization based on formulation structure. *Struc. Mulit. Optim.* 39 (2009) 503–517.

[28] H.R. Varian, The nonparametric approach to production analysis. *Econometrica* 52 (1984) 579–597.

[29] Zai Wang, Tianshi Chen, Ke Tang and Xin Yao, A multi-objective approach to redundancy allocation problem in parallel-series systems, in *2009 IEEE Congress on Evolutionary Computation (CEC 2009*, 2009, pp. 582–589.

[30] R.I. Whitfield, A.H. Duffy, G. Coates, and W. Hills, Distributed design coordination, *Res. Engineer. Design* 13 (2002) 243–252.

*Manuscript received 22 April 2015*
*revised 23 February 2016*
*accepted for publication 22 June 2016*

Francisca Miguel
Universidad de Málaga, Departamento de Economía Aplicada (Matemáticas)
Campus El Ejido s/n, 29071-Málaga, Spain
E-mail address: fcamiguel@uma.es

Trinidad Gómez
Universidad de Málaga, Departamento de Economía Aplicada (Matemáticas)
Campus El Ejido s/n, 29071-Málaga, Spain
E-mail address: trinidad@uma.es