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# OPTIMAL CONTROL PROBLEMS FOR THE VISCOUS MODIFIED NOVIKOV EQUATION*

Lei Zhang and Bin Liu†

**Abstract:** This paper deals with the optimal control problem for the viscous modified Novikov equation. We first investigate the existence and uniqueness of weak solution to the equation, and then prove that the controlled system admits a optimal control. Moreover, we show that the data-to-solution mapping is Gâteaux differentiable. As an application, a necessary optimality condition corresponding to the distributive value observation is established. Finally, an illustrative example is also provided.

**Key words:** *viscous, modified Novikov equation, optimal control, necessary condition*

**Mathematics Subject Classification:** *35D40, 35Q53, 49J20, 49L25, 93C20*

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## 1 Introduction

In this paper, we consider the optimal control problem governed by the following modified Novikov equation [7]:

$$u_t - u_{xxt} + (b+1)u^2u_x - buu_xu_{xx} - u^2u_{xxx} = 0, \quad (x,t) \in (0,1) \times (0,T), \tag{1.1}$$

where $b$ is a real parameter in $\mathbb{R}$, the unknown function $u(x,t)$ represents the fluid velocity at time $t$ and in the spacial direction $x$.

If $b = 3$, the equation (1.1) reduces to the celebrated Novikov equation [8]:

$$u_t - u_{xxt} + 4u^2u_x - 3uu_xu_{xx} - u^2u_{xxx} = 0, \quad (x,t) \in (0,1) \times (0,T), \tag{1.2}$$

which was discovered recently by V. Novikov in his symmetry classification study of nonlocal partial differential equations. After the Novikov equation was derived, many papers were devoted to its study from the mathematical point of view. For example, it is shown that the Novikov equation has a matrix Lax pair, bi-Hamiltonian structure, infinitely many conserved quantities [8], and admits peaked soliton (peakon) solutions [4]. In [15], Tiglay investigated the local well-posedness of (1.2) in the Sobolev spaces. Meanwhile, by using a Cauchy-Kowalevski type theorem, Tiglay established the existence and uniqueness of real analytic solutions. Zhao and Zhou [18] give the symbolic analysis and exact traveling wave solutions to a modified Novikov equation. Taking advantage of the special structure of the equation, Jiang and Ni [5] established sufficient conditions on the initial data to guarantee

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†Corresponding author

the formulation of singularities in finite time. Yan, Li and Zhang [17] studied the local well-posedness of the Novikov equation in the Besov spaces. In [16], the authors established the global existence of strong solutions to the weakly dissipative Novikov equation.

By comparison with the $b$-equation [3]:

$$u_t - u_{xxt} + (b+1)uu_x - bu_x u_{xx} - uu_{xxx} = 0, \quad (x,t) \in (0,1) \times (0,T), \tag{1.3}$$

it is clear that $Eqs.$(1.1) has nonlinear terms that are cubic, rather than quadratic of $b$-equation. Moreover, by taking $b = 2$ and $b = 3$ in $Eqs.$(1.3), respectively, we obtain the famous Camassa-Holm equation and the Degasperis-Procesi equation, which attracted many researches in the area of partial differential equations. For example, in [13], Tian, Shen and Ding studied the optimal control of the viscous Camassa-Holm equation, and in [10], Shen, Gao and Tian considered the control problem of the viscous generalized Camassa-Holm equation. Tian, Shi and Liu [14] studied the boundary control to viscosity Degasperis-Procesi equation. Olivier [9] proved the controllability and asymptotic stabilization of the Camassa-Holm equation, and in [11], Shen and Gao considered the optimal control problem to the viscous weakly dispersive Degasperis-Procesi equation.

From a physical point of view, the nonlinear shallow water wave equations have already been widely applied to some important research fields in physics and engineering. For instance, these equations can be utilized to model the gravity wave in some bounded domain (typically, the surface wave in a man-made pool), and the Rossby and Kelvin waves in the lakes, rivers, oceans and atmosphere. It is worth mentioning that the nonlinear dispersive equations can also be employed in the studying of coastal engineering, wherein the most attractive aspect of this field is to generate the long water waves in laboratories by choosing an effective control strategy. One of the most interesting things that attracts our attention is that the shallow water wave equations mentioned above are closely related to the modelling of the tsunami waves (such as the Indian ocean tsunami happened in 2004). In recent years, people pay much attention to realize the operation mechanism of prototype tsunami in the laboratory (e.g., in the water channel or the swimming pool), and hope to find a really efficient control mechanism to generate expected long water waves in the man-made pool. Naturally, in the course of these researches, an optimization problem need to be considered, that is,

*Question:* How can we generate the exact waves on the water by the hydraulic efficiency servo-system or the other electric engines in an 'optimal way' such that the man-made water waves are close to the expected waves as far as possible?

To the best of our knowledge, it seems that the study of the viscous Novikov equation from the point of view of control theory is a completely open field, especially the optimal control problem. Due to the research interests and being inspired by the previous papers, we are mainly concerned with the 'Optimality Problem' research field mentioned above, the purpose of this paper is to study the optimal distributed control problems involving the following viscous modified Novikov equation

$$u_t - u_{xxt} + (b+1)u^2 u_x - buu_x u_{xx} - u^2 u_{xxx} - \beta(u - u_{xx})_{xx} = 0, \quad (x,t) \in (0,1) \times (0,T), \tag{1.4}$$

where $\beta(u - u_{xx})_{xx}$ is the viscous item and $\beta > 0$ is a real constant. More precisely, the optimal control problem considered in this paper can be stated as follows:

$$\min_{\omega \in \mathscr{U}} \left\{ J(y,\omega) = \frac{1}{2}\|Cy - z_d\|^2_{\mathscr{H}} + \frac{\delta}{2}\|\omega\|^2_{\mathscr{U}} \right\},$$

subject to

$$\begin{cases} y_t - \beta y_{xx} + u^2 y_x + byuu_x = B\omega, \quad (x,t) \in (0,1) \times (0,T), \\ y(x,0) = u(x,0) - u_{xx}(x,0) = \phi, \quad x \in (0,1), \\ u(t)|_{x=0,1} = u_x(t)|_{x=0,1} = u_{xx}(t)|_{x=0,1} = 0, \quad t \in (0,T). \end{cases} \tag{1.5}$$

where the time $T > 0$ is given, $y = u - u_{xx}$, $\phi \in H$, $\mathscr{H}$ and $\mathscr{U}$ are two Hilbert spaces, which are equipped with the norm $\|\cdot\|_{\mathscr{H}}$ and $\|\cdot\|_{\mathscr{U}}$, $\omega$ is the control and $B$ is the control operator. The first term in cost functional measures the physical objective, the second one is the size of the control, where $\delta > 0$ plays the role of a weight. Our aim is to adjust the body force $\omega$ so that the state variable of the system can be driven to a given desired state $z_d$ as much as possible without using too much energy and work, where $\omega$ is a distributed control belonging to the Hilbert space $\mathscr{U}$. Moreover, we prove the existence of optimal control for the controlled system, and establish a necessary optimality condition corresponding to the distributive value observation. Finally, an illustrative example is also provided.

**Notations.** Let $T > 0$, $\Omega = (0,1)$. Denote by $H = L^2(0,1)$ the usual Lebesgue function space equipped with the norm $\|y\|_H = (\int_0^1 |y|^2 dx)^{1/2}$, and the corresponding inner product is defined by $(\cdot,\cdot)$. The space $V = H_0^1(0,1)$ is the closure of $C_0^\infty(0,1)$ in $H^1(0,1)$ with the norm $\|y\|_V = (\|y\|_H^2 + \|y_x\|_H^2)^{1/2}$. $V^* = H^{-1}(0,1)$ and $H^* = L^2(0,1)$ are dual spaces of $V$ and $H$ respectively. It is well known that $V$ is dense in $H$ and $V \hookrightarrow H = H^* \hookrightarrow V^*$.

Define $\|u\|_{H^m(\Omega)} = \|D^m u\|_H$, where $D^m = \partial^m / \partial x^m, m = 0,1,2,\ldots$. A new space $W(0,T;V)$ is introduced as $W(0,T;V) = \{y | y \in L^2(0,T;V), y_t \in L^2(0,T;V^*)\}$, which is a Hilbert space endowed with the norm

$$\|y\|_{W(0,T;V)} = (\|y\|^2_{L^2(0,T;V)} + \|y_t\|^2_{L^2(0,T;V^*)})^{\frac{1}{2}}.$$

Furthermore, we write $L^2(V)$, $C(H)$, $L^2(L^\infty)$ and $W(V)$ in place of $L^2(0,T;V)$, $C(0,T;H)$, $L^2(0,T;L^\infty)$ and $W(0,T;V)$.

The plan of the remaining sections is as follows: Section 2 is devoted to the well-posedness of *Eqs.*(1.1) and the estimate for the norm of weak solution by initial data. In Section 3, we prove that the controlled system admits a optimal control, and show that the data-to-solution mapping is Gâteaux differentiable. In section 4, a necessary condition for optimality is established. In the last section, we give an illustrative example.

## 2 Existence and Uniqueness of the Weak Solution

In this section, we first study the initial/boundary-value problem for the viscous modified Novikov equation without the control item,

$$\begin{cases} u_t - u_{xxt} + (b+1)u^2 u_x - buu_x u_{xx} - u^2 u_{xxx} - \beta(u - u_{xx})_{xx} = f, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x), \quad x \in \Omega, \\ u(t)|_{x=0,1} = u_x(t)|_{x=0,1} = u_{xx}(t)|_{x=0,1} = 0, \quad t \in (0,T). \end{cases} \tag{2.1}$$

By setting $y = u - u_{xx}$, then *Eqs.*(2.1) takes the form of a quasi-linear evolution equation of parabolic type:

$$\begin{cases} y_t - \beta y_{xx} + u^2 y_x + byuu_x = f, \quad (x,t) \in \Omega \times (0,T), \\ y(x,0) = u(x,0) - u_{xx}(x,0) = \phi, \quad x \in \Omega, \\ u(t)|_{x=0,1} = u_x(t)|_{x=0,1} = u_{xx}(t)|_{x=0,1} = 0, \quad t \in (0,T), \end{cases} \tag{2.2}$$

where $b, \beta \in R$, $\beta > 0$, $\phi \in H$, and$f \in L^2(V^*)$.

In order to prove the existence of weak solution to the viscous modified Novikov equation, we give the definition of the weak solution in the space $W(V)$.

**Definition 2.1.** *A function $y(x,t) \in W(V)$ is called a weak solution of Eqs.*(2.2)*, if*

$$\frac{d}{dt}(y, \varphi) + \beta(y_x, \varphi_x) + (u^2 y_x, \varphi) + b(yuu_x, \varphi) = \langle f, \varphi \rangle_{V^*,V},$$

*for all $\varphi \in V$ in the sense of $\mathscr{D}'(0,T)$, and $y(x,0) = \varphi \in H$ is valid.*

**Theorem 2.1.** *Let $\phi \in H$, $f \in L^2(V^*)$, then there exists a unique weak solution to the Eqs.*(2.2) *in the interval [0,T]. Moreover, the solution mapping $\{\phi, f\} \to y$ is continuous.*

*Proof.* We choose $\{\omega_j\}_{j\in N}$ in $V$ as the eigenfunctions of the *Laplacian* operator with one dimension subject to the *Dirichlet* boundary condition

$$\begin{cases} -\ \partial_x^2\ \omega_j = \lambda_j\ \omega_j, \\ \omega_j|_{\partial\Omega} = 0. \end{cases}$$

We also normalize $\omega_j$ such that $\|\omega_j\|_H = 1$. From the elliptic operator theory [2], $\{\omega_j\}_{j\in N}$ forms base functions in $V$. Now, we will use the $Faedo - Garlekin$ method [6] to find the approximate solutions.

Let $m$ be a given positive integer, define the *ansatz* space [11] by

$$V_m = span\{\omega_1, \omega_2, \ldots, \omega_m\} \subseteq V,$$

and set

$$y_m = \sum_{i=1}^{m} g_i^m(t)\omega_i(x)\ .$$

Performing the *Garlekin* procedure to the *Eqs.*(2.2), we have

$$\begin{cases} y_{mt} - \beta y_{mxx} + u_m^2 y_{mx} + b y_m u_m u_{mx} = f, \\ y_m(x,0) = \phi_m(x) \in H, \\ u_m(t)|_{x=0,1} = u_{m,x}(t)|_{x=0,1} = u_{m,xx}(t)|_{x=0,1} = 0, \end{cases} \tag{2.3}$$

where $y_m = u_m - u_{mxx}$, and $\phi_m \to \phi$ strongly in $H$.

Thus (2.3) is reduced to the initial value problem for a system of nonlinear first-order ordinary differential equations (ODE). By applying the theory of ODE [2], we deduce that there is a time $t_m > 0$ such that *Eqs.*(2.3) admits a unique local solution in $[0, t_m]$.

Now, we prove the existence of weak solution by analyzing the limiting behavior of $y_m$ and $u_m$, which implies that the solution is uniformly bounded as $t_m \to T$.

Multiplying the first equation of *Eqs.*(2.3) by $u_m$, and integrating with respect to $x$ on $\Omega$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u_m\|_H^2 + \|u_m\|_V^2) + \beta(\|u_m\|_V^2 + \|u_m\|_{H^2}^2) \\ &\quad = b\int_0^1 u_m^2 u_{mx} u_{mxx} dx + \int_0^1 u_m^3 u_{mxxx} dx + \langle f, u_m \rangle_{V^*,V}. \end{aligned} \tag{2.4}$$

By integrating by parts, we discover

$$\int_0^1 u_m^3 u_{mxxx} dx = -3 \int_0^1 u_m^2 u_{mx} u_{mxx} dx. \tag{2.5}$$

Since $f \in L^2(V^*)$, we can assume that

$$\|f\|_{V^*} \le C_1, \tag{2.6}$$

where $C_1$ is a positive constant.

By Sobolev embedding theorem [2], we have

$$\int_0^1 u_m^2 u_{mx} u_{mxx} dx \le \|u_m\|_{L^\infty}^2 \|u_{mx}\|_H \|u_{mxx}\|_H \le k^2 \|u_m\|_V^3 \|u_m\|_{H^2}, \tag{2.7}$$

where $k > 0$ is a real number, which depends only on $\Omega$. It follows from (2.4) – (2.7) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u_m\|_H^2 + \|u_m\|_V^2\Big) + \beta\Big(\|u_m\|_V^2 + \|u_m\|_{H^2}^2\Big) \\
&\le |b-3| \left| \int_0^1 u_m^2 u_{mx} u_{mxx} dx \right| + \langle f, u_m \rangle_{V^*,V} \\
&\le k^2 |b-3| \|u_m\|_V^3 \|u_m\|_{H^2} + \|f\|_{V^*} \|u_m\|_V \\
&\le \frac{k^4(b-3)^2}{\beta} \|u_m\|_V^6 + \beta \|u_m\|_{H^2}^2 + \frac{1}{\beta}\|f\|_{V^*}^2 + \beta \|u_m\|_V^2 \\
&\le \frac{C_1^2}{\beta} + \beta\Big(\|u_m\|_{H^2}^2 + \|u_m\|_V^2\Big) + \frac{k^4(b-3)^2}{\beta}\|u_m\|_V^6 \\
&\le \frac{C_1^2}{\beta} + \beta\Big(\|u_m\|_{H^2}^2 + \|u_m\|_V^2\Big) + \frac{k^4(b-3)^2}{\beta}\Big(\|u_m\|_V^2 + \|u_m\|_H^2\Big)^3.
\end{aligned}$$

So,

$$\frac{1}{2}\frac{d}{dt}\Big(\|u_m\|_H^2 + \|u_m\|_V^2\Big) \le \frac{C_1^2}{\beta} + \frac{k^4(b-3)^2}{\beta}\Big(\|u_m\|_V^2 + \|u_m\|_H^2\Big)^3. \tag{2.8}$$

By setting

$$h(t) = \|u_m\|_H^2 + \|u_m\|_V^2,$$

since $h(t) \ge 0$, we first multiply both sides of the inequality in (2.8) by $h(t)$, and then use Young inequality [2] to obtain

$$\begin{aligned}
\frac{1}{4}\frac{d}{dt}h^2(t) &\le \frac{C_1^2}{\beta}h(t) + \frac{k^4(b-3)^2}{\beta}h^4(t) \\
&\le \left(\frac{k^4(b-3)^2}{\beta}\right)^{-1/3}\left(\frac{C_1^2}{\beta}\right)^{4/3} + \frac{2k^4(b-3)^2}{\beta}h^4(t) \\
&\le C_2\Big(1 + h^4(t)\Big),
\end{aligned} \tag{2.9}$$

where

$$C_2 = \max\left\{ \left(\frac{k^4(b-3)^2}{\beta}\right)^{-1/3}\left(\frac{C_1^2}{\beta}\right)^{4/3}, \frac{2k^4(b-3)^2}{\beta} \right\}.$$

Then, integrating both sides of (2.9) from 0 to $T$, we get

$$h^2(t) \leq \tan\left\{4C_2T + \arctan\left(\|u_m(0)\|_H^2 + \|u_m(0)\|_V^2\right)\right\} \leq C_3^4, \tag{2.10}$$

where $C_3 \geq 0$ is a constant.

From the above analysis, we obtain $\|u_m\|_H \leq C_3, \quad \|u_m\|_V \leq C_3$, for $\forall t \in [0,T]$.

Multiplying the first equation of $Eqs.$(2.3) by $-u_{mxx}$, and integrating with respect to $x$ on $\Omega$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|u_m\|_V^2 + \|u_m\|_{H^2}^2\right) + \beta\left(\|u_m\|_{H^2}^2 + \|u_m\|_{H^3}^2\right) \\ &= -\langle f, u_{mxx}\rangle_{V^*,V} + (b+1)\int_0^1 u_m^2 u_{mx} u_{mxx} dx \\ &\quad - b\int_0^1 u_{mxx}^2 u_{mx} u_m dx - \int_0^1 u_m^2 u_{mxxx} u_{mxx} dx. \end{aligned} \tag{2.11}$$

By using integration by parts, we discover

$$\int_0^1 u_m^2 u_{mxxx} u_{mxx} dx = -\int_0^1 u_{mxx}^2 u_{mx} u_m dx. \tag{2.12}$$

Due to the Sobolev embedding theorem and Poincaré inequality [2], we deduce that

$$\begin{aligned} \int_0^1 u_m^2 u_{mx} u_{mxx} dx &\leq \|u_m\|_{L^\infty}^2 \int_0^1 |u_{mx} u_{mxx}| dx \\ &\leq k^2\|u_m\|_V^2 \|u_{mx}\|_H \|u_{mxx}\|_H \\ &\leq \frac{k^2}{2}\|u_m\|_V^2\left(\|u_{mx}\|_H^2 + \|u_{mxx}\|_H^2\right) \\ &\leq \frac{k^2}{2}\|u_m\|_V^2\left(\eta^2\|u_{mxx}\|_H^2 + \|u_{mxx}\|_H^2\right) \\ &\leq \frac{k^2(\eta^2+1)}{2}\|u_m\|_V^2\|u_m\|_{H^2}^2 \\ &\leq \frac{k^2(\eta^2+1)C_3^2}{2}\|u_m\|_{H^2}^2, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \int_0^1 u_m^2 u_{mxxx} u_{mxx} dx &\leq \|u_m\|_{L^\infty}^2 \int_0^1 |u_{mxxx} u_{mxx}| dx \\ &\leq k^2\|u_m\|_V^2 \|u_{mxxx}\|_H \|u_{mxx}\|_H \\ &\leq k^2 C_3^2 \|u_m\|_{H^2}\|u_m\|_{H^3}, \end{aligned} \tag{2.14}$$

where $\eta > 0$ is the Poincaré constant, which is a real number which depends only on $\Omega$ [2].

It follows from (2.11) – (2.14) and Young inequality that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|u_m\|_V^2 + \|u_m\|_{H^2}^2\right) + \beta\left(\|u_m\|_{H^2}^2 + \|u_m\|_{H^3}^2\right) \\ &\leq |b-1|\left|\int_0^1 u_m^2 u_{mxxx} u_{mxx} dx\right| + |b+1|\left|\int_0^1 u_m^2 u_{mx} u_{mxx} dx\right| + \|f\|_{V^*}\|u_{mxx}\|_V \\ &\leq |b-1|k^2C_3^2\|u_m\|_{H^2}\|u_m\|_{H^3} + \frac{1}{2}|b+1|k^2(\eta^2+1)C_3^2\|u_m\|_{H^2}^2 + \|f\|_{V^*}\|u_m\|_{H^3} \\ &\leq \frac{2}{\beta}|b-1|^2k^4C_3^4\|u_m\|_{H^2}^2 + \frac{\beta}{2}\|u_m\|_{H^3}^2 + \frac{1}{2}|b+1|k^2(\eta^2+1)C_3^2\|u_m\|_{H^2}^2 \end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{\beta}\|f\|_{V^*}^2+\frac{\beta}{2}\|u_m\|_{H^3}^2\\
&\leq\left(\frac{2}{\beta}|b-1|^2k^4C_3^4+\frac{1}{2}|b+1|k^2(\eta^2+1)C_3^2\right)\|u_m\|_{H^2}^2+\frac{2}{\beta}\|f\|_{V^*}^2\\
&+\beta(\|u_m\|_{H^2}^2+\|u_m\|_{H^3}^2)\\
&\leq C_4(\|u_m\|_V^2+\|u_m\|_{H^2}^2)+\beta(\|u_m\|_{H^2}^2+\|u_m\|_{H^3}^2)+\frac{2C_1^2}{\beta},
\end{aligned}
\tag{2.15}
$$

where $C_4=\frac{2}{\beta}|b-1|^2k^4C_3^4+\frac{1}{2}|b+1|k^2(\eta^2+1)C_3^2$. Then, form (2.15) we obtain

$$
\frac{1}{2}\frac{d}{dt}\Big(\|u_m\|_V^2+\|u_m\|_{H^2}^2\Big)\leq C_4\Big(\|u_m\|_V^2+\|u_m\|_{H^2}^2\Big)+\frac{2}{\beta}C_1^2. \tag{2.16}
$$

By applying Gronwall inequality [2] to (2.16), we obtain

$$
\|u_m\|_V^2+\|u_m\|_{H^2}^2\leq C_5^2, \quad \text{where } C_5\geq 0.
$$

Multiplying the first equation of $Eqs.$(2.3) by $y_m$, and integrating with respect to $x$ on $\Omega$, we get

$$
\frac{1}{2}\frac{d}{dt}\|y_m\|_H^2+\beta\|y_m\|_V^2=\langle f,y_m\rangle_{V^*,V}-b\int_0^1 y_m^2u_mu_{mx}dx-\int_0^1 u_m^2y_{mx}y_mdx. \tag{2.17}
$$

By integrating by parts, we discover

$$
\int_0^1 y_m^2u_mu_{mx}dx=-\int_0^1 u_m^2y_{mx}y_mdx. \tag{2.18}
$$

Due to the Sobolev embedding theorem, we deduce that

$$
\begin{aligned}
\int_0^1 y_m^2u_mu_{mx}dx &\leq \|u_m\|_{L^\infty}\|u_{mx}\|_{L^\infty}\int_0^1 y_m^2dx\\
&\leq k^2\|u_m\|_V\|u_m\|_{H^2}\|y_m\|_H^2\\
&\leq k^2C_3C_5\|y_m\|_H^2.
\end{aligned}
\tag{2.19}
$$

From (2.17) – (2.19), we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|y_m\|_H^2+\beta\|y_m\|_V^2 &\leq \|f\|_{V^*}\|y_m\|_V+|b-1|\int_0^1|y_m^2u_mu_{mx}|dx\\
&\leq \|f\|_{V^*}\|y_m\|_V+|b-1|k^2C_3C_5\|y_m\|_H^2\\
&\leq \frac{2}{\beta}C_1^2+\frac{\beta}{2}\|y_m\|_V^2+|b-1|k^2C_3C_5\|y_m\|_H^2.
\end{aligned}
$$

Thus, we have

$$
\frac{1}{2}\frac{d}{dt}\|y_m\|_H^2\leq\frac{1}{2}\frac{d}{dt}\|y_m\|_H^2+\frac{\beta}{2}\|y_m\|_V^2\leq\frac{2C_1^2}{\beta}+|b-1|k^2C_3C_5\|y_m\|_H^2. \tag{2.20}
$$

By using Gronwall inequality, there exists $C_6\geq 0$, such that

$$
\|y_m\|_H^2\leq C_6^2.
$$

Integrating both sides of (2.20) from 0 to $t$, we obtain

$$\frac{1}{2}\|y_m\|_H^2 + \frac{\beta}{2}\int_0^t \|y_m\|_V^2 ds \le \frac{2TC_1^2}{\beta} + T|b-1|k^2C_3C_5C_6^2 + \frac{1}{2}\|\phi\|_H^2.$$

Hence,

$$\|y_m\|_{L^2(V)}^2 \le \frac{2}{\beta}\left(\frac{2TC_1^2}{\beta} + T|b-1|k^2C_3C_5C_6^2 + \frac{1}{2}\|\phi\|_H^2\right) \triangleq C_7. \tag{2.21}$$

Next, we prove the uniform boundness of the sequence $\{y_{mt}\}$. By *Eqs.*(2.3) and Sobolev embedding theorem, we obtain

$$\begin{aligned}
\|y_{mt}\|_{V^*} &= \sup_{\|\nu\|_V=1} \langle y_{mt}, \nu\rangle_{V^*,V} \\
&\le \sup_{\|\nu\|_V=1} |\langle f + \beta y_{mxx} - u_m^2 y_{mx} - b y_m u_m u_{mx}, \nu\rangle_{V^*,V}| \\
&\le \|f\|_{V^*} + \beta\|y_m\|_V + k^2C_3^2C_6 + |b|k^2C_3^2\|y_m\|_V \\
&\le C_1 + k^2C_3^2C_6 + \Big(\beta + |b|k^2C_3^2\Big)\|y_m\|_V.
\end{aligned}$$

Then, by using Schwarz inequality [2], we obtain

$$\|y_{mt}\|_{V^*}^2 \le 3C_1^2 + 3k^4C_3^4C_6^2 + 3\Big(\beta + |b|k^2C_3^2\Big)^2\|y_m\|_V^2.$$

By performing integration on the interval $[0,T]$, we deduce that

$$\begin{aligned}
\|y_{mt}\|_{L^2(V^*)}^2 &\le 3TC_1^2 + 3Tk^4C_3^4C_6^2 + 3\Big(\beta + |b|k^2C_3^2\Big)^2\|y_m\|_{L^2(V)}^2 \\
&\le 3TC_1^2 + 3Tk^4C_3^4C_6^2 + 3C_7\Big(\beta + |b|k^2C_3^2\Big)^2.
\end{aligned} \tag{2.22}$$

Thus, for a given $T$, the approximate solutions $\{y_m\}_{m\in N}$ is uniformly bounded in $W(V)$. Then we can conclude there exists a subsequence, again denoted by $\{y_m\}$, such that

$$\begin{cases}
y_m \to y & \text{weakly in} \quad L^2(V), \\
y_{mt} \to y_t & \text{weakly in} \quad L^2(V^*), \\
y_m \to y & \text{weakly star} \quad L^\infty(H), \\
y_{mxx} \to y_{xx} & \text{weakly in} \quad L^2(V^*).
\end{cases} \tag{2.23}$$

In order to verify that $y$ is a weak solution to *Eqs.*(2.2), from (2.23), it remains to verify the convergence of the nonlinear terms. By Aubin compactness theorem [1] and (2.23), we get

$$y_m \to y \qquad \text{strongly in} \quad L^2(H). \tag{2.24}$$

Since the space $W(V)$ is compactly imbedded into $C(H)$ [6], we have

$$y_m \to y \qquad \text{strongly in} \quad C(H). \tag{2.25}$$

Similar results can be obtained for the sequences $u_m$, $u_{mx}$ and $u_{mxx}$. Moreover, from [15], it can be shown that

$$y_m \to y \qquad \text{strongly in} \quad L^2(L^\infty). \tag{2.26}$$

From (2.24) $-$ (2.26), for each $\varphi\in L^2(V)$, we have

$$
\begin{aligned}
&\int_0^T\int_0^1\Big(u_m^2y_{mx}-u^2y_x\Big)\varphi dxdt\\
&\le\Big|\int_0^T\int_0^1\Big(u_m^2y_{mx}-u_m^2y_x\Big)\varphi dxdt\Big|+\Big|\int_0^T\int_0^1\Big(u_m^2y_x-u^2y_x\Big)\varphi dxdt\Big|\\
&\le\int_0^T\|u_m\|_{L^\infty}^2\|y_m-y\|_H\|\varphi\|_Vdt+\int_0^T\|u_m^2-u^2\|_{L^\infty}\|y\|_H\|\varphi\|_Vdt\\
&\le k^2C_3^2\|y_m-y\|_{L^2(H)}\|\varphi\|_{L^2(V)}+\|y\|_{C(H)}\|u_m^2-u^2\|_{L^2(L^\infty)}\|\varphi\|_{L^2(V)}\\
&\longrightarrow 0,\quad m\to\infty,
\end{aligned}\tag{2.27}
$$

and

$$
\begin{aligned}
&\int_0^T\int_0^1\Big(y_mu_mu_{mx}-yuu_x\Big)\varphi dxdt\\
&\le\Big|\int_0^T\int_0^1\Big(y_mu_mu_{mx}-y_mu_mu_x\Big)\varphi dxdt\Big|+\Big|\int_0^T\int_0^1\Big(y_mu_mu_x-y_muu_x\Big)\varphi dxdt\Big|\\
&\quad+\Big|\int_0^T\int_0^1\Big(y_muu_x-yuu_x\Big)\varphi dxdt\Big|\\
&\le\int_0^T\|y_m\|_{L^\infty}\|u_m\|_{L^\infty}\|u_m-u\|_H\|\varphi\|_Vdt+\int_0^T\|y_m\|_{L^\infty}\|u_m-u\|_{L^\infty}\|u\|_H\|\varphi\|_Vdt\\
&\quad+\frac12\int_0^T\|y_m-y\|_{L^\infty}\|u\|_H^2\|\varphi\|_Vdt\\
&\le kC_3\|u_m-u\|_{C(H)}\int_0^T\|y_m\|_{L^\infty}\|\varphi\|_Vdt+k\int_0^T\|y_m\|_{L^\infty}\|u_m-u\|_V\|u\|_H\|\varphi\|_Vdt\\
&\quad+\frac12\|u\|_{C(H)}^2\|y_m-y\|_{L^2(L^\infty)}\|\varphi\|_{L^2(V)}\\
&\le kC_3\|u_m-u\|_{C(H)}\|y_m\|_{L^2(L^\infty)}\|\varphi\|_{L^2(V)}+\eta k\|u\|_{C(H)}\|u_{mx}-u_x\|_{C(H)}\\
&\quad\cdot\|y_m\|_{L^2(L^\infty)}\|\varphi\|_{L^2(V)}+\frac12\|u\|_{C(H)}^2\|y_m-y\|_{L^2(L^\infty)}\|\varphi\|_{L^2(V)}\\
&\longrightarrow 0,\quad m\to\infty.
\end{aligned}\tag{2.28}
$$

On the other hand, from (2.23), we have $y_m(0)\ \to y(0)$ strongly in $H$. So, by the uniqueness of the limit, we obtain

$$
y(x,0)=\phi.\tag{2.29}
$$

Therefore, the function $y$ is a weak solution of $Eqs.$(2.2).

Now, it remains to prove the uniqueness. Let $y_1$, $y_2$ be two solutions of $Eqs.$(2.2), and denote $y=y_1-y_2$, $u=u_1-u_2$, then $u$ satisfies

$$
\begin{aligned}
u_t-u_{xxt}-\beta(u-u_{xx})_{xx}\ &=\ u_1^2u_{1xxx}-u_2^2u_{2xxx}-(b+1)(u_1^2u_{1x}-u_2^2u_{2x})\\
&\quad+b(u_1u_{1x}u_{1xx}-u_2u_{2x}u_{2xx}),
\end{aligned}\tag{2.30}
$$

and $u(x,0)=0$. Multiplying (2.30) by $u$, then integrating with respect to $x$ on $\Omega$, we get

$$
\begin{aligned}
&\frac12\frac{d}{dt}\Big(\|u\|_H^2+\|u\|_V^2\Big)+\beta\Big(\|u\|_V^2+\|u\|_{H^2}^2\Big)\\
&=\int_0^1\Big(u_1^2u_{1xxx}-u_2^2u_{2xxx}\Big)udx-(b+1)\int_0^1\Big(u_1^2u_{1x}-u_2^2u_{2x}\Big)udx
\end{aligned}
$$

$$+\, b\int_0^1 \Big(u_1u_{1x}u_{1xx} - u_2u_{2x}u_{2xx}\Big)udx. \tag{2.31}$$

Multiplying *Eqs.*(2.31) by $u$, and use the same technique as that used in the proof of the existence, we deduce that there exist $M_1$, $M_2$, $M_3 \geq 0$, such that

$$\frac{1}{2}\frac{d}{dt}\Big(\|u\|_H^2 + \|u\|_V^2\Big) + \beta\Big(\|u\|_V^2 + \|u\|_{H^2}^2\Big) \leq M_1\|u\|_H\|u\|_V + M_2\|u\|_V\|u\|_{H^2} + M_3\|u\|_H\|u\|_{H^2},$$

then, it follows from Young inequality that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u\|_H^2 + \|u\|_V^2\Big) + \beta\Big(\|u\|_V^2 + \|u\|_{H^2}^2\Big)\\
&\leq \frac{M_1^2 + 2M_3^2}{\beta}\|u\|_H^2 + \frac{2M_2^2}{\beta}\|u\|_V^2 + \beta\Big(\|u\|_V^2 + \|u\|_{H^2}^2\Big)\\
&\leq \max\left\{\frac{M_1^2 + 2M_3^2}{\beta}, \frac{2M_2^2}{\beta}\right\}\Big(\|u\|_H^2 + \|u\|_V^2\Big) + \beta\Big(\|u\|_V^2 + \|u\|_{H^2}^2\Big)\\
&\triangleq M_4\Big(\|u\|_H^2 + \|u\|_V^2\Big) + \beta\Big(\|u\|_V^2 + \|u\|_{H^2}^2\Big).
\end{aligned} \tag{2.32}$$

By using the Gronwall inequality, we get

$$\|u\|_H^2 + \|u\|_V^2 \leq e^{2M_4T}\Big(\|u(0)\|_V^2 + \|u(0)\|_{H^2}^2\Big) \equiv 0,$$

i.e., $u \equiv 0$, and we have proved the uniqueness of the weak solution.

Finally, the continuity of the mapping from $\{\phi, f\}$ to the weak solution $y(t,x)$ is an immediate consequence of the inequality (2.32). Thus the proof is completed. □

In the following, we shall establish the inequality for the norm of weak solution with initial datas, which is necessary in discussing the existence of optimal control.

**Theorem 2.2.** *Let $\phi \in H$, $f \in L^2(V^*)$, then there exist constants $L_1, L_2 \geq 0$, such that*

$$\|y\|_{W(V)}^2 \leq L_1 + L_2\Big(\|f\|_{L^2(V^*)}^2 + \|\phi\|_H^2\Big).$$

*Proof.* Multiplying the first equation of *Eqs.*(2.2) by $u$, and integrating over $\Omega$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big(\|u\|_H^2 + \|u\|_V^2\Big) + \beta\Big(\|u\|_V^2 + \|u\|_{H^2}^2\Big) &= b\int_0^1 u^2u_xu_{xx}dx\\
&\quad + \int_0^1 u^3u_{xxx}dx + \langle f, u\rangle_{V^*,V}.
\end{aligned}$$

As we did in the proof of Theorem 2.1, we can get the estimates: $\|u\|_H \leq C_3$, $\|u\|_V \leq C_3$, where $C_3$ is a constant as same as in Theorem 2.1. Similarly, we obtain $\|u\|_{H^2} \leq C_5$, where $C_5 \geq 0$ is a constant the same as in Theorem 2.1.

Again, multiplying the first equation of *Eqs.*(2.2) by $y$ and integrating over $\Omega$ yields

$$\frac{1}{2}\frac{d}{dt}\|y\|_H^2 + \beta\|y\|_V^2 = \langle f, y\rangle_{V^*,V} - (b-1)\int_0^1 y^2uu_xdx. \tag{2.33}$$

Then, by Young inequality and Sobolev embedding theorem, we have

$$\frac{1}{2}\frac{d}{dt}\|y\|_H^2 + \beta\|y\|_V^2 \leq \frac{2}{\beta}\|f\|_{V^*}^2 + \frac{\beta}{2}\|y\|_V^2 + k^2|b-1|C_3C_5\|y\|_H^2. \tag{2.34}$$

So, from (2.34), we get

$$\frac{1}{2}\frac{d}{dt}\|y\|_H^2 + \frac{\beta}{2}\|y\|_V^2 \le \frac{2}{\beta}\|f\|_{V^*}^2 + k^2|b-1|C_3C_5\|y\|_H^2, \tag{2.35}$$

then, Gronwall inequality and (2.35) imply that

$$\begin{aligned}
\|y\|_H^2 &\le \exp\Big(2k^2|b-1|C_3C_5T\Big)\Big(\|\phi\|_H^2 + \frac{4}{\beta}\|f\|_{L^2(V^*)}^2\Big)\\
&\le \max\Big\{1, \frac{4}{\beta}\Big\}\exp\Big(2k^2|b-1|C_3C_5T\Big)\Big(\|\phi\|_H^2 + \|f\|_{L^2(V^*)}^2\Big)\\
&\triangleq C_8\Big(\|\phi\|_H^2 + \|f\|_{L^2(V^*)}^2\Big).
\end{aligned} \tag{2.36}$$

Integrating both sides of (2.35) on $[0, T]$, we obtain

$$\begin{aligned}
\|y\|_H^2 + \beta\|y\|_{L^2(V)}^2 &\le \frac{4}{\beta}\|f\|_{L^2(V^*)}^2 + 2k^2|b-1|C_3C_5C_8T\Big(\|\phi\|_H^2 + \|f\|_{L^2(V^*)}^2\Big) + \|\phi\|_H^2\\
&\le \Big(1 + \frac{4}{\beta} + 2k^2|b-1|C_3C_5C_8T\Big)\Big(\|f\|_{L^2(V^*)}^2 + \|\phi\|_H^2\Big).
\end{aligned} \tag{2.37}$$

On the other hand, from $Eqs.$(2.2), we have

$$\begin{aligned}
\|y_t\|_{V^*} &= \sup_{\|\nu\|_V=1}\langle y_t, \nu\rangle_{V^*,V}\\
&\le \sup_{\|\nu\|_V=1}|\langle f + \beta y_{xx} - u^2y_x - byuu_x, \nu\rangle_{V^*,V}|\\
&\le \|f\|_{V^*} + \beta\|y\|_V + k^2C_3^2C_6 + |b|k^2C_3^2\|y\|_V\\
&\le C_1 + k^2C_3^2C_6 + \Big(\beta + |b|k^2C_3^2\Big)\|y\|_V,
\end{aligned} \tag{2.38}$$

From (2.27) – (2.38) and integrating (2.39) from 0 to $T$, we have

$$\begin{aligned}
\|y_t\|_{L^2(V^*)}^2 &\le 2T\Big(C_1 + k^2C_3^2C_6\Big)^2 + \Big(\beta + |b|k^2C_3^2\Big)^2\|y\|_{L^2(V)}^2\\
&\le 2T\Big(C_1 + k^2C_3^2C_6\Big)^2 + C_9\Big(\beta + |b|k^2C_3^2\Big)^2\Big(\|f\|_{L^2(V^*)}^2 + \|\phi\|_H^2\Big).
\end{aligned} \tag{2.39}$$

Finally, we deduce from(2.37) and (2.39) that

$$\begin{aligned}
\|y\|_{W(V)}^2 &\le 2T\Big(C_1 + k^2C_3^2C_6\Big)^2 + C_9\Big[1 + \Big(\beta + |b|k^2C_3^2\Big)^2\Big]\Big(\|f\|_{L^2(V^*)}^2 + \|\phi\|_H^2\Big)\\
&\triangleq L_1 + L_2\Big(\|f\|_{L^2(V^*)}^2 + \|\phi\|_H^2\Big).
\end{aligned}$$

This completes the proof of the theorem. □

## $\boxed{3}$ Existence of Optimal Control and its G-Differentiability

In this section, we shall prove the existence of optimal control, and further prove that the solution mapping on control variables is Gâteaux differentiable.

### 3.1. The existence of the optimal control.

Suppose that the Hilbert space $\mathscr{U}$, the space of controls, is equipped with the norm $\|\cdot\|_{\mathscr{U}}$ and suppose that

$$B \in \mathscr{L}(\mathscr{U}, L^2(V^*)). \tag{3.1}$$

We are also given an observation operator $C \in \mathscr{L}(V;\mathscr{H})$, $\mathscr{H}$ is an another Hilbert space with the norm $\|\cdot\|_{\mathscr{H}}$. Moreover, $z_d$ is a fixed desired state in $\mathscr{H}$, and $\delta > 0$ plays the role of a weight.

The optimal control problem that we want to deal with can be represented as follows:

$$(P) \qquad \min_{\omega\in\mathscr{U}} \Big\{ J(y,\omega) = \frac{1}{2}\|Cy - z_d\|^2_{\mathscr{H}} + \frac{\delta}{2}\|\omega\|^2_{\mathscr{U}} \Big\}, \tag{3.2}$$

subject to

$$\begin{cases} y_t - \beta y_{xx} + u^2 y_x + byuu_x = B\omega, & (x,t) \in \Omega\times(0,T), \\ y(x,0) = u(x,0) - u_{xx}(x,0) = \phi, & x\in\Omega, \\ u(t)|_{x=0,1} = u_x(t)|_{x=0,1} = u_{xx}(t)|_{x=0,1} = 0, & t\in(0,T), \end{cases} \tag{3.3}$$

where $y = u - u_{xx}$, $\phi \in H$, $\beta > 0$, $b \in R$ and $B$ is the control operator introduced above.

The following theorem is presented to demonstrate the existence of the optimal control.

**Theorem 3.1.** *For given $\phi \in H$, then there exists an optimal solution $(y^*, \omega^*)$ for the control problem $(P)$.*

*Proof.* From Theorem 2.1, for every given $\omega \in \mathscr{U}$, there exists a unique weak solution $y(\omega)$ to the *Eqs.*(3.3). In view of (3.2), we have

$$J(y,\omega) \geq \frac{\delta}{2}\|\omega\|^2_{\mathscr{U}}. \tag{3.4}$$

We deduce from Theorem 2.2 that

$$\begin{aligned} \|y\|^2_{W(V)} &\leq L_1 + L_2\Big(\|B\omega\|^2_{L^2(V^*)} + \|\phi\|^2_H\Big) \\ &\leq L_1 + L_2\max\Big\{1, \|B\|^2\Big\}\Big(\|\omega\|^2_{\mathscr{U}} + \|\phi\|^2_H\Big), \end{aligned}$$

which implies that $\|y\|_{W(V)} \to +\infty \Longrightarrow \|\omega\|_{\mathscr{U}} \to +\infty$. Then, we have

$$J(y,\omega) \to +\infty \quad \text{as} \quad \|\omega\|_{\mathscr{U}} \to +\infty. \tag{3.5}$$

As the norm is weakly lowered semi-continuous [2], we observe that $J$ is weakly lowered semi-continuous. Since $J(y,\omega) \geq 0$ is bounded below, for all $(y,\omega) \in W(V)\times\mathscr{U}$, there exists a constant $\gamma \geq 0$ such that

$$\gamma = \inf_{\omega\in\mathscr{U}} J(y,\omega). \tag{3.6}$$

Thus, we can deduce from (3.4) – (3.6) that there is a minimizing sequence $\{(y^n,\omega^n)\}_{n\in N}$ such that

$$\gamma = \lim_{n\to+\infty} J(y^n,\omega^n). \tag{3.7}$$

We know from (3.7) that the sequence $\{J(y^n,\omega^n)\}_{n\in N}$ is bounded, and then the sequences $\{\omega^n\}_{n\in N}$ and $\{y^n\}$ are also bounded. Hence, we may extract a subsequence, again denoted by $\{(y^n,\omega^n)\}_{n\in N}$, such that

$$y^n \to y^*, \qquad \text{weakly in } W(V), \tag{3.8}$$

$$\omega^n \to \omega^*, \qquad \text{weakly in } \mathscr{U}. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\lim_{n\to+\infty}\int_0^T \langle y_t^n - y^*, \varphi(t)\rangle_{V^*,V} dt = 0, \qquad \forall\varphi\in L^2(V), \tag{3.10}$$

and

$$\lim_{n\to+\infty}\int_0^T \langle B\omega^n - B\omega^*, \varphi(t)\rangle_{V^*,V} dt = 0, \qquad \forall\varphi\in L^2(V). \tag{3.11}$$

Since $W(V)$ is compactly embedded into $L^2(L^\infty)$ [12], we derive that $y^n \to y^*$ strongly in $L^2(L^\infty)$. Since $W(V)$ is compactly embedded into $C(H)$ [6], we can also derive that $y^n \to y^*$ strongly in $C(H)$. Furthermore, $u^n \to u^*$, $u_x^n \to u_x^*$, $u_{xx}^n \to u_{xx}^*$ strongly in $L^2(L^\infty)$ and $C(H)$ respectively.

In order to verify $y^*$ is a solution to (3.3) corresponding to $\omega^*$, we should analyse the limit in the nonlinear terms. By using $Holder$'s inequality [2] and Poincaré's inequality, for every $\varphi\in L^2(V)$, we have

$$\begin{aligned}
&\int_0^T\int_0^1 \Big((u^n)^2 y_x^n - (u^*)^2 y_x^*\Big)\varphi dxdt\\
&\le \left|\int_0^T\int_0^1 \Big((u^n)^2 - (u^*)^2\Big) y_x^n \varphi dxdt\right| + \left|\int_0^T\int_0^1 (u^*)^2\Big(y_x^n - y_x^*\Big)\varphi dxdt\right|\\
&\le \int_0^T \|(u^n)^2-(u^*)^2\|_{L^\infty}\|y^n\|_H\|\varphi\|_V dt + \int_0^T \|(u^*)^2\|_{L^\infty}\|y^n-y^*\|_H\|\varphi\|_V dt\\
&\le \|y^n\|_{C(H)}\|(u^n)^2-(u^*)^2\|_{L^2(L^\infty)}\|\varphi\|_{L^2(V)} + \|y^n-y^*\|_{C(H)}\|\varphi\|_{L^2(V)}\|(u^*)^2\|_{L^2(L^\infty)}\\
&\longrightarrow 0, \quad n\to\infty,
\end{aligned}\tag{3.12}$$

and

$$\begin{aligned}
&\int_0^T\int_0^1 \Big(y^n u^n u_x^n - y^* u^* u_x^*\Big)\varphi dxdt\\
&\le \left|\int_0^T\int_0^1 \Big(y^n u^n u_x^n - y^n u^n u_x^*\Big)\varphi dxdt\right| + \left|\int_0^T\int_0^1 \Big(y^n u^n u_x^* - y^n u^* u_x^*\Big)\varphi dxdt\right|\\
&\quad + \left|\int_0^T\int_0^1 \Big(y^n u^* u_x^* - y^* u^* u_x^*\Big)\varphi dxdt\right|\\
&\le \int_0^T \|(y^n)\|_{L^\infty}\|(u^n)\|_{L^\infty}\|u^n-u^*\|_H\|\varphi\|_V dt\\
&\quad + \int_0^T \|y^n\|_{L^\infty}\|u^n-u^*\|_{L^\infty}\|u^*\|_H\|\varphi\|_V dt\\
&\quad + \frac{1}{2}\int_0^T \|y^n-y^*\|_{L^\infty}\|(u^*)^2\|_H\|\varphi\|_V dt\\
&\le k\eta\|u^n-u^*\|_{C(H)}\|u_x^n\|_{C(H)}\|y^n\|_{L^2(L^\infty)}\|\varphi\|_{L^2(V)} + k\eta\|u^*\|_{C(H)}\|u_x^n-u_x^*\|_{C(H)}\\
&\quad \|y^n\|_{L^2(L^\infty)}\|\varphi\|_{L^2(V)} + \frac{1}{2}\|(u^*)^2\|_{C(H)}\|y^n-y^*\|_{L^2(L^\infty)}\|\varphi\|_{L^2(V)}\\
&\longrightarrow 0, \quad n\to\infty.
\end{aligned}\tag{3.13}$$

Since $y^n \to y^*$ weakly in $W(V)$, we can infer that $y^n(0)\to y^*(0)$ weakly in $H$. Then we get

$$y^*(0) = \phi. \tag{3.14}$$

Then, it follows from (3.10) – (3.14) that $(y^*, \omega^*)$ satisfies $Eqs.$(3.3), and such that

$$J(y^*, \omega^*) = \min_{\omega \in \mathscr{U}} J(y, \omega).$$

Hence, there exists an optimal control to the control system (3.1) – (3.3). Moreover, due to the relation $u = (1 - \partial_x^2)^{-1} y$, we can find a optimal control for the viscous modified Novikov equation. □

### 3.2. The Gâteaux differentiability of the solution mapping.

Now, we are in a position to prove that the solution mapping is Gâteaux differentiable on control variables. It is well known that if $\omega$ is an optimal control, then it satisfies the necessary optimality condition

$$DJ(\omega)(v - \omega) \geq 0 \quad \text{for all} \;\; v \in \mathscr{U}_{ad}, \tag{3.15}$$

where $DJ(\omega)$ denotes the Gâteaux derivative of $J(v) = J(u(v), v)$ at $v = \omega$. Recall the definition, a mapping $f : D \subset E_1 \to E_2$ is said to be Gâteaux differentiable at $x_0 \in D$, if there exists a linear bounded operator $DJ(x_0) \in \mathcal{L}(E_1, E_2)$ such that

$$\lim_{\lambda \to 0} |\lambda^{-1}(f(x_0 + \lambda\eta) - f(x_0)) - DJ(x_0)\eta| = 0, \quad \text{for all} \;\; \eta \in E_1. \tag{3.16}$$

What we would do is to analyze (3.15) in view of the proper adjoint state system, and prove that the mapping $v \to y(v)$ of $\mathscr{U} \to W(V)$ is Gâteaux differentiable at $v = \omega^*$, and $Dy(\omega^*)\eta$ denotes the derivative in the direction $\eta \in \mathscr{U}$.

**Theorem 3.2.** *The map $y \to y(v)$ of $\mathscr{U} \to W(V)$ is Gâteaux differentiable at $v = \omega^*$, and the Gâteaux derivative of $y(v)$ at $v = \omega^*$ in the direction $v - \omega^* \in \mathscr{U}$, say $z = Dy(\omega^*)(v - \omega^*)$, is a unique weak solution of the following system:*

$$\begin{cases} z_t - \beta z_{xx} + [(2b+2)u^* u_x^* - b u_x^* u_{xx}^* + u_{xxx}^*]\theta \\ \quad + [(u^*)^2 - b u^* u_{xx}^*]\theta_x - b u^* u_x^* \theta_{xx} + (u^*)^2 \theta_{xxx} = Bh, \quad \text{in } \Omega \times (0, T), \\ \theta(t)|_{x=0,1} = \theta_x(t)|_{x=0,1} = \theta_{xx}(t)|_{x=0,1} = 0, \qquad \text{on } (0, T), \\ z(x, 0) = 0, \quad \text{in } \Omega, \end{cases} \tag{3.17}$$

*where $z = \theta - \theta_{xx}$.*

*Proof.* Let us set $h = v - \omega^*$, $u^\lambda = u(\omega^* + \lambda h)$, and

$$z^\lambda = \theta^\lambda - \theta^\lambda_{xx} = \lambda^{-1}[y(\omega^* + \lambda h) - y(\omega^*)], \;\; \theta^\lambda = \lambda^{-1}(u(\omega^* + \lambda h) - u(\omega^*)),$$

where $\lambda \in (-\epsilon, \epsilon)$, $\lambda \neq 0$ and $z^\lambda = \theta^\lambda - \theta^\lambda_{xx}$. It is not difficult to check that $\theta^\lambda$ satisfies:

$$\begin{cases} \theta^\lambda_t - \theta^\lambda_{xxt} - \beta(\theta^\lambda - \theta^\lambda_{xx})_{xx} + \pi^\lambda_1(x,t) + \pi^\lambda_2(x,t) + \pi^\lambda_3(x,t) = Bh, \quad \text{in } \Omega \times (0, T), \\ \theta^\lambda(t)|_{x=0,1} = \theta^\lambda_x(t)|_{x=0,1} = \theta^\lambda_{xx}(t)|_{x=0,1} = 0, \qquad \text{on } (0, T), \\ z^\lambda(x, 0) = 0, \quad \text{in } \Omega, \end{cases} \tag{3.18}$$

where

$$\begin{aligned} \pi^\lambda_1(x,t) &= (b+1)(u^\lambda + u^*)\theta^\lambda u^\lambda_x + (u^*)^2 \theta^\lambda_x, \\ \pi^\lambda_2(x,t) &= -b\theta^\lambda u^\lambda_x u^\lambda_{xx} - b u^* \theta^\lambda_x u^\lambda_{xx} - b u^* u^*_x \theta^\lambda_{xx}, \end{aligned}$$

$$\pi_3^\lambda(x,t) = (u^\lambda + u^*)\theta^\lambda u_{xxx}^\lambda + (u^*)^2\theta_{xxx}^\lambda,$$

Multiplying (3.19) by $\theta^\lambda$, and integrating on $(0,1)$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\theta^\lambda\|_H^2 + \|\theta^\lambda\|_V^2\Big) + \beta\Big(\|\theta^\lambda\|_V^2 + \|\theta^\lambda\|_{H^2}^2\Big)\\&+\int_0^1 (\pi_1^\lambda(x,t) + \pi_2^\lambda(x,t) + \pi_3^\lambda(x,t))\theta^\lambda dx = \int_0^1 \theta^\lambda Bh dx.\end{aligned} \tag{3.19}$$

By using Sobolev inequality and Young inequality, from the estimate corresponding to $y, u$ in section 2 and section 3, we have

$$\begin{aligned}\int_0^1 \pi_1^\lambda(x,t)\theta^\lambda(x,t)dx &= \int_0^1 (b+1)(u^\lambda + u^*)u_x^\lambda|\theta^\lambda|^2 dx - \int_0^1 u^* u_x^* |\theta^\lambda|^2 dx\\&\le [(|b|+1)\|u_x^\lambda\|_\infty(\|u^\lambda\|_\infty\\&\quad +\|u^*\|_\infty) + \|u_x^*\|_\infty\|u^*\|_\infty]\|\theta^\lambda\|_H^2,\end{aligned} \tag{3.20}$$

$$\begin{aligned}\int_0^1 \pi_2^\lambda(x,t)\theta^\lambda(x,t)dx &= \int_0^1 [-b\theta^\lambda u_x^\lambda u_{xx}^\lambda - bu^*\theta_x^\lambda u_{xx}^\lambda - bu^* u_x^*\theta_{xx}^\lambda]\theta^\lambda dx\\&\le \|u_x^\lambda\|_\infty\|u_{xx}^\lambda\|_\infty\|\theta^\lambda\|_H^2 + \|u^*\|_\infty\|u_{xx}^\lambda\|_\infty\|\theta^\lambda\|_H\|\theta^\lambda\|_V\\&\quad + \|u^*\|_\infty\|u_x^*\|_\infty\|\theta^\lambda\|_H\|\theta^\lambda\|_{H^2},\end{aligned} \tag{3.21}$$

$$\begin{aligned}\int_0^1 \pi_3^\lambda(x,t)\theta^\lambda(x,t)dx &= -\int_0^1 u_{xx}^\lambda(u_x^\lambda + u_x^*)|\theta^\lambda|^2 dx - 2\int_0^1 u_{xx}^\lambda(u^\lambda + u^*)\theta^\lambda\theta_x^\lambda dx\\&\le \|u_{xx}^\lambda\|_\infty(\|u_x^\lambda\|_\infty + \|u_x^*\|_\infty)\|\theta^\lambda\|_H\\&\quad + \|u_{xx}^\lambda\|_\infty(\|u^\lambda\|_\infty + \|u^*\|_\infty)\|\theta^\lambda\|_H\|\theta^\lambda\|_V.\end{aligned} \tag{3.22}$$

On the other hand, from Theorem 2.1, we have the following basic properties, that is,

$$\lim_{\lambda\to 0} u(\omega^* + \lambda h) = u(\omega^*), \quad \text{strongly in } \ W(V). \tag{3.23}$$

Then, by (2.25), (2.26) and Young inequality, we obtain from (3.19) – (3.22) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\theta^\lambda\|_H^2 + \|\theta^\lambda\|_V^2\Big) + \beta\Big(\|\theta^\lambda\|_V^2 + \|\theta^\lambda\|_{H^2}^2\Big)\\&\qquad \le M_1\|\theta^\lambda\|_H^2 + \frac{\beta}{2}\|\theta^\lambda\|_V^2 + \frac{\beta}{2}\|\theta^\lambda\|_{H^2}^2 + \|Bh\|_H^2,\end{aligned} \tag{3.24}$$

where $M_1$ is a positive constant. Here and below, we let $M_k (k \in N)$ be some positive constants which depend only dependent on $\Omega, T$. Hence, Gronwall inequality yields that, there is a constant $M_2 > 0$ such that

$$\|\theta^\lambda\|_H^2 + \|\theta^\lambda\|_V^2 + \|\theta^\lambda\|_{L^2(V)}^2 + \|\theta^\lambda\|_{L^2(H^2)}^2 \le M_2. \tag{3.25}$$

Similarly, by using the same argument as in (2.16) and (3.12), there exists a constant $M_3 > 0$ such that

$$\|\theta^\lambda\|_V^2 + \|\theta^\lambda\|_{H^2}^2 \le M_3. \tag{3.26}$$

Now, multiplying the first equation of (3.19) by $z^\lambda$, we have

$$\frac{1}{2}\frac{d}{dt}\|z^\lambda\|_H^2 + \beta\|z^\lambda\|_V^2 + \int_0^1 (\pi_1^\lambda(x,t) + \pi_2^\lambda(x,t) + \pi_3^\lambda(x,t))z^\lambda dx = \int_0^1 z^\lambda Bh dx. \tag{3.27}$$

By (2.25), (2.26), (3.18), (3.19) and Young inequality, we have

$$
\begin{aligned}
&\int_0^1 (\pi_1^\lambda(x,t)+\pi_2^\lambda(x,t)+\pi_3^\lambda(x,t))z^\lambda dx = \int_0^1 [(b+1)(u^\lambda+u^*)\theta^\lambda u_x^\lambda + (u^*)^2\theta_x^\lambda] z^\lambda dx \\
&\quad + \int_0^1 [-b\theta^\lambda u_x^\lambda u_{xx}^\lambda - bu^*\theta_x^\lambda u_{xx}^\lambda - bu^* u_x^* \theta_{xx}^\lambda] z^\lambda dx \\
&\quad + \int_0^1 [(u^\lambda+u^*)\theta^\lambda u_{xxx}^\lambda + (u^*)^2\theta_{xxx}^*] z^\lambda dx \\
&\le (|b|+1)(\|u_x^\lambda\|_\infty + \|u_{xx}^\lambda\|_\infty)(\|u^\lambda\|_\infty + \|u^*\|_\infty)\|\theta^\lambda\|_H \|z^\lambda\|_H \\
&\quad + |b|\|u_x^\lambda\|_\infty \|u_{xx}^\lambda\|_\infty \|\theta^\lambda\|_H \|z^\lambda\|_H \\
&\quad + (|b|\|u^*\|_\infty \|u_{xx}^\lambda\|_\infty + \|u^*\|_\infty^2)\|\theta^\lambda\|_V \|z^\lambda\|_H \\
&\quad + |b|\|u^*\|_\infty \|u_x^*\|_\infty \|\theta^\lambda\|_{H^2} \|z^\lambda\|_H \\
&\quad + \|u_{xx}^\lambda\|_\infty (\|u^\lambda\|_\infty \|u^*\|_\infty)\|\theta^\lambda\|_H \|z^\lambda\|_V + \|u^*\|_\infty \|u_x^*\|_\infty \|\theta^*\|_{H^2} \|z^\lambda\|_H \\
&\quad + \|u^*\|_\infty^2 \|\theta^*\|_{H^2} \|z^\lambda\|_V \\
&\le M_4(\|\theta^\lambda\|_H \|z^\lambda\|_H + \|\theta^\lambda\|_V \|z^\lambda\|_H + \|\theta^\lambda\|_H \|z^\lambda\|_V + \|\theta^\lambda\|_{H^2} \|z^\lambda\|_V) \\
&\le M_5 \|z^\lambda\|_H + \frac{\beta}{4}\|z^\lambda\|_V,
\end{aligned}
$$

Hence, we deduce from (3.27) that

$$
\frac{1}{2}\frac{d}{dt}\|z^\lambda\|_H^2 + \beta\|z^\lambda\|_V^2 \le M_5\|z^\lambda\|_H + \frac{\beta}{2}\|z^\lambda\|_V + \|Bh\|_H^2, \tag{3.28}
$$

by using Gronwall inequality, we obtain that $\|z^\lambda\|_H^2 + \|z^\lambda\|_{L^2(V)}^2 \le M_6$. Then, from (3.19) and (3.28), we get that $\|z^\lambda\|_{L^2(V^*)}^2 \le M_7$. Thus, $z^\lambda$ is bounded in $W(V)$, and there exists a subsequence, still denoted by itself, such that

$$
z^\lambda \to z \quad \text{weakly in} \quad L^2(V), \qquad z_t^\lambda \to z_t \quad \text{weakly in} \quad L^2(V^*). \tag{3.29}
$$

On the other hand, by Aubin theorem, we have that $z^\lambda \to z$ strongly in $L^2(V)\cap C(H)$. Hence, by (3.20), (2.25), (2.26) and the fact that $u^\lambda \to u$ strongly in $W(V)$, we deduce that

$$
\pi_1^\lambda(x,t) \to 2(b+1)u^*u_x^*\theta + (u^*)^2\theta_x, \qquad \text{weakly in} \quad L^2(H), \tag{3.30}
$$

$$
\pi_2^\lambda(x,t) \to -bu_x^*u_{xx}^*\theta - bu^*u_{xx}^*\theta_x - bu^*u_x^*\theta_{xx}, \quad \text{weakly in} \quad L^2(H), \tag{3.31}
$$

$$
\pi_3^\lambda(x,t) \to 2u^*u_{xxx}^*\theta + (u^*)^2\theta_{xxx}, \qquad \text{weakly in} \quad L^2(H), \tag{3.32}
$$

This yields that $z^\lambda \to z = Dy(\omega^*)(v-\omega^*)$ in the weak topology of $W(V)$. However, we shall further prove that $z^\lambda \to z$ in the strong topology of $W(V)$. To this end, we subtract (3.18) from (3.17). By setting $\xi^\lambda = z^\lambda - z$, we obtain

$$
\begin{cases}
(z^\lambda - z)_t - \beta(z^\lambda - z)_{xx} + \Pi_1^\lambda(x,t) + \Pi_2^\lambda(x,t) + \Pi_3^\lambda(x,t) = 0, \quad \text{in } \Omega\times(0,T), \\
(\theta^\lambda - \theta)|_{x=0,1} = (\theta_x^\lambda - \theta_x)|_{x=0,1} = (\theta_{xx}^\lambda - \theta_{xx})|_{x=0,1} = 0, \qquad \text{on } (0,T), \\
\xi^\lambda(x,0) = 0, \quad \text{in } \Omega,
\end{cases} \tag{3.33}
$$

where

$$
\Pi_1^\lambda(x,t) = (b+1)(u^\lambda+u^*)u_x^\lambda\theta^\lambda - 2(b+1)u^*u_x^*\theta + (u^*)^2(\theta_x^\lambda - \theta_x), \tag{3.34}
$$

$$
\Pi_2^\lambda(x,t) = b(u_x^*u_{xx}^*\theta - u_x^\lambda u_{xx}^\lambda\theta^\lambda) + bu^*(u_{xx}^*\theta_x - u_{xx}^\lambda\theta_x^\lambda) + bu^*u_x^*(\theta_{xx} - \theta_{xx}^\lambda), \tag{3.35}
$$

$$
\Pi_3^\lambda(x,t) = (u^\lambda+u^*)u_{xxx}^\lambda\theta^\lambda - 2u^*u_{xxx}^*\theta + (u^*)^2(\theta_{xxx}^\lambda - \theta_{xxx}), \tag{3.36}
$$

since $z^\lambda \to z$ strongly in $L^2(V) \cap C(H)$, and $u^\lambda \to u$ strongly in $W(V)$, we see that

$$\Pi_i^\lambda(x,t) \to 0, \quad \text{strongly in } \ L^2(H), \quad i = 1,2,3. \tag{3.37}$$

Multiplying the first equation of (3.33) by $\xi^\lambda$ and integrating on $(0,1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\xi^\lambda\|_H^2 + \beta\|\xi^\lambda\|_V^2 \le \frac{\beta}{2}\|\xi^\lambda\|_H^2 + M_6(\|\Pi_1^\lambda\|_H^2 + \|\Pi_2^\lambda\|_H^2 + \|\Pi_3^\lambda\|_H^2), \tag{3.38}$$

Hence, by using Gronwall inequality, we deduce from (3.32)and (3.38) that $\xi^\lambda \to 0$ strongly in $L^2(V)$, and $\xi^\lambda \to 0$ strongly in $L^\infty(H)$ as $\lambda \to 0$.

Finally, by means of (3.33) and (3.37) – (3.38), we deduce that $\xi_\lambda \to 0$ strongly in $W(V)$, as $\lambda \to 0$, which implies the Gâteaux differentiability of $y(v)$ at $v = \omega^*$.

Thus, the proof is completed. □

## 4 The Necessary Optimality Condition

Since we have already proved in subsection 3.2 that the mapping $y \to y(v)$ of $\mathscr{U} \to W(V)$ is Gâteaux differentiable at $v = \omega^*$ in the direction $v - \omega^*$, then so is $u$. Hence, we could rewrite the optimality condition (3.15) as follows:

$$(Cy(\omega^*) - z_d, C(Dy(\omega)(v-\omega^*)))_{\mathscr{H}} + (N\omega^*, v-\omega^*)_{\mathscr{U}} \ge 0, \quad \forall v \in \mathscr{U}_{ad}.$$

If we denote

$$\begin{cases} \Lambda = \text{canonical isomorphism of } \mathscr{H} \text{ to } \mathscr{H}', \\ C^* = \text{the adjoint operator of } C. \end{cases}$$

Then, the above formula reduces to

$$\langle C^*\Lambda(y(\omega^*) - z_d), Dy(\omega^*)(v-\omega^*)\rangle_{W(0,T;V)',W(0,T;V)} + (N\omega^*, v-\omega^*)_{\mathscr{U}} \ge 0, \qquad \forall v \in \mathscr{U}_{ad}. \tag{4.1}$$

Now, we are concerned with a type of observation $C$ of distributive. Let $\mathscr{H} = (L^2(H))^2$, and $C \in \mathcal{L}(W(V); \mathscr{H})$, and $q(v) = Cy(v) = (u(v;\cdot), u(v;T)) \in (L^2(H))^2$. In this case, the cost functional that we consider is represented by

$$J(v) = \int_Q |u(v) - z_d|^2 dxdt + \int_\Omega |u(v;T) - m_T|^2 dx + (Nv, v)_{\mathscr{U}}, \ \ \forall v \in \mathscr{U}_{ad}, \tag{4.2}$$

where $z_d \in L^2(H)$, $m_T \in H$ are desired values. Then the optimality condition (4.1) is represented by

$$\int_Q (u(\omega^*) - z_d)\theta dxdt + \int_\Omega (u(\omega^*;T) - m_T)\theta(T)dx + (N\omega^*, v-\omega^*)_{\mathscr{U}} \ge 0, \qquad \forall v \in \mathscr{U}_{ad}, \tag{4.3}$$

where $\theta$ is the solution of (3.17). We introduce the adjoint state $\varphi = \psi - \psi_{xx}$ by

$$\begin{cases} -\varphi_t - \beta\varphi_{xx} + [2bu^*u_x^* + u_{xxx}^* - (3b+6)u_x^*u_{xx}^* - 2u^*u_{xxx}^*]\psi - (u^*)^2\psi_{xxx} \\ \quad -(b+6)u^*u_x^*\psi_{xx} + [(3b+6)u^*u_{xx}^* + 5(u^*)^2 + (u_x^*)^2]\psi_x = u(\omega^*) - z_d, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega \times (0,T), \\ \psi(t)|_{x=0,1} = \psi_x(t)|_{x=0,1} = \psi_{xx}(t)|_{x=0,1} = 0, \qquad \text{on } (0,T), \\ \varphi(x,T) = u(\omega^*;T) - m_T, \quad \text{in } \ \Omega, \end{cases} \tag{4.4}$$

Since $(u(\omega^*)-z_d, u(\omega^*;T)-m_T)\in L^2(H)\times H$. Problem (3.17) admits a unique weak solution in $W(V)$, a fact which follows by applying the linear theory of parabolic equations with the flow of time reversed (change $t$ to $T-t$). Whence, we could provide the character of the optimal control in the following theorem.

**Theorem 4.1.** *The optimal control $\omega^*$ is characterized by the following equations:*

$$\begin{cases}-\varphi_t-\beta\varphi_{xx}+[2bu^*u_x^*+u_{xxx}^*-(3b+6)u_x^*u_{xx}^*-2u^*u_{xxx}^*]\psi-(u^*)^2\psi_{xxx}\\ \;-(b+6)u^*u_x^*\psi_{xx}+[(3b+6)u^*u_{xx}^*+5(u^*)^2+(u_x^*)^2]\psi_x=u(\omega^*)-z_d,\\ \hfill in\ \Omega\times(0,T),\\ \psi(t)|_{x=0,1}=\psi_x(t)|_{x=0,1}=\psi_{xx}(t)|_{x=0,1}=0,\quad on\ (0,T),\\ \varphi(x,T)=u(\omega^*;T)-m_T,\quad in\ \Omega,\end{cases}\tag{4.5}$$

*and the first order necessary optimality condition:*

$$\int_Q\psi(\omega^*)B(v-\omega^*)dxdt+(N\omega^*,v-\omega^*)_{\mathscr{U}}\ge 0,\quad \forall v\in\mathscr{U}_{ad}.\tag{4.6}$$

*where $u(\omega^*)$ is the optimal solution corresponding to the optimal control $\omega^*$ of (3.3).*

*Proof.* We multiply the first equation of (4.5) both sides by $\theta$ and integrate over $[0,T]$. Then, we have

$$\begin{aligned}&-\int_0^T\Big\langle\frac{d}{dt}\varphi,\theta\Big\rangle dt-\beta\int_0^T\Big\langle\varphi_{xx},\theta\Big\rangle dt-\int_0^T\Big\langle(u^*)^2\psi_{xxx}\ +(b+6)u^*u_x^*\psi_{xx},\theta\Big\rangle dt\\ &+\int_0^T\Big\langle[2bu^*u_x^*+u_{xxx}^*-(3b+6)u_x^*u_{xx}^*-2u^*u_{xxx}^*]\psi,\theta\Big\rangle dt\\ &+\int_0^T\Big\langle[(3b+6)u^*u_{xx}^*+5(u^*)^2+(u_x^*)^2]\psi_x,\theta\Big\rangle dt=\int_0^T(u(\omega^*)-z_d,\theta)dt.\end{aligned}\tag{4.7}$$

By integrating by parts, and considering the equations (3.17), we can verify the left-hand side of (4.7) yields

$$\begin{aligned}&-\langle\varphi(\omega^*;T),\theta(T)\rangle+\int_0^T\Big\langle\psi,z_t-\beta z_{xx}\Big\rangle dt+\int_0^T\Big\langle[(2b+2)u^*u_x^*-bu_x^*u_{xx}^*+u_{xxx}^*]\theta\\ &\quad+[(u^*)^2-bu^*u_{xx}^*]\theta_x-bu^*u_x^*\theta_{xx}+(u^*)^2\theta_{xxx},\psi\Big\rangle dt\\ &=-\int_\Omega(u(\omega^*;T)-m_T)\theta(T)dx+\int_0^T(\psi,B(v-\omega^*))dt.\end{aligned}\tag{4.8}$$

Thus, (4.7) and (4.8) yields that

$$\int_0^T(u(\omega^*)-z_d,\theta)dt+\int_\Omega(u(\omega^*;T)-m_T)\theta(T)dx=\int_0^T(\psi,B(v-\omega^*))dt.\tag{4.9}$$

From (4.3) and (4.9), we can deduce that

$$\int_Q\psi(\omega^*)B(v-\omega^*))dxdt+(N\omega^*,v-\omega^*)_{\mathscr{U}}\ge 0,\quad\forall v\in\mathscr{U}_{ad}.$$

This completes the proof of Theorem 4.1. □

## $\boxed{5}$ An Example

In this section, we given an example to show how to solve the optimal control problem for the Novikov equation by means of the above results and the numerical method.

In general, by the necessary optimality condition of optimal control is an effective numerical method for solving the optimal control problems. Basically, there are two ways for numerically solving optimal control problems through necessary conditions. On is the multiple shooting method, which is the most powerful numerical method in seeking the optimal control of the lumped parameter systems through solving a two-point boundary-value problem obtained by the Pontryagin maximum principle. Of course, except for the complexity when the original problem involves inequality constraints of both state variables and controls, the difficulty for shooting method additionally includes the "guess" for the initial data to start the iterative numerical process. It demands that the user understands the essential of the problem well in physics, which is often not a trivial task. To overcome this difficulty, people develop the gradient method; and then the "min-H" approach corrected from the gradient method [19, 20, 21]. In the following, we utilize the min-H iterative method to solve the extremum problem.

Noting that the first-order necessary optimality condition (4.6) can be rewritten as

$$\langle H_\omega(\psi^*, \omega^*), \omega^* \rangle = \min_{\omega \in \mathscr{U}} \langle H_\omega(\psi^*, \omega^*), \omega \rangle, \tag{5.1}$$

where $H(\psi, \omega) = \int_Q \psi B\omega dxdt + \frac{1}{2}(N\omega, \omega)_{\mathscr{U}}$, and $\psi$ is the adjoint function satisfies the adjoint system (4.5). Then the so-called "min-H" iterative algorithm is formulated as follows:

Step1: Give $\omega^0$, determine $u^0(x,t)$ (or $y^0(x,t)$) through the state equation (3.3).

Step2: By $\omega^0$ and $u^0(x,t)$, solve the adjoint equation (4.5) to get $\psi^0(x,t)$.

Step3: By $u^0(x,t)$, $\psi^0(x,t)$ and the Pontryagin maximum principle (5.1), to determine $\omega^1$.

Step4: Calculate $J(\omega^0)$. If it does not reach the minimum, replace $\omega^0$ with $\omega^1$ and redo the steps above until we get the proper $J(\omega^1)$.

However, it is an optimal control problem of the distributed parameter system governed by the nonlinear partial differential equations, to get the numerical solutions for the optimal control-trajectory pair is not an easy job. Here, although we do not give the detailed numerical simulation, the algorithm do give the concrete steps so that people can follow and finish this nontrivial work.

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Lei Zhang
School of Mathematics and Statistics
Hubei Key Laboratory of Engineering Modeling and Scientific Computing
Huazhong University of Science and Technology
Wuhan 430074, Hubei, P. R. China
E-mail address: lzhang890701@163.com

Bin Liu
School of Mathematics and Statistics
Huazhong University of Science and Technology
Wuhan 430074, Hubei, P. R. China
E-mail address: binliu@mail.hust.edu.cn