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A PERTURBATION EXCHANGE ALGORITHM FOR CONVEX SEMI-INFINITE PROGRAMMING WITH APPLICATIONS IN SPARSE BEAMFORMER DESIGN*

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Abstract: In formulating convex semi-infinite programming (CSIP), it is advantageous to have strictly convex objective functions. In this paper, we add a small perturbation to the objective function so that it becomes a strictly convex semi-infinite programming. We prove that the solution of the perturbation CSIP converges to the solution of the original problem when the perturbation diminishes. This provides an efficient way to obtain good approximations to the original problems which do not have strict convexity. Using the developed method, the broadband beamformer design problem is tackled. In particular, we formulate the sparse beamformer design problem and study the successive thinning technique. The perturbation exchange algorithm is employed for the sub-problem ensuring the convexity of the cost function and penalizing unnecessary coefficients at the same time. Numerical results show that the complexity of the designed beamformers can be reduced significantly.

Key words: *convex semi-infinite programming, sparse beamformer design, perturbation exchange algorithm*

Mathematics Subject Classification: *90C34, 65K05, 68U99*

1 Introduction

Semi-infinite programming (SIP) has many important applications in engineering; fast algorithms with low complexity in computation are highly desirable. There are various popular numerical approaches in the literature [8]. One kind is based on discretization of the constraints [21,22]; another kind is the reduction based method [19]. For discretization methods, the complexity is rather high when the number of discretization is very large. On the other hand, the reduction method may need strong assumptions. A third kind of method is based on exact penalty functions [14, 25]. Another important class with low complexity is the exchange method. More recently, [26] proposed a modified variant of the method. In particular, during the exchange process, global optima of the exchange points are not required for addition but only those points with constraint violations. In dropping active points, the Lagrange multipliers are used to remove all inactive constraints. The new adding-dropping rule saves much computational time.

But the convergence analysis of the algorithm associated with the method in [26] needs the assumption that the objective function is strictly convex. However, there are many

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applications which has non-strictly convex objective functions. One application considered here is beamformer design. Beamforming is a spatial filtering technique to enhance the required signal via a sensor array for directional signal transmission or reception [2, 10]. With a fixed configuration, the steering vectors of the desired signals can be estimated together with their direction-of-arrivals [20]. More elaborated physical signal propagation models have also be employed to describe complicated wave phenomena [11]. As a result, the beamformer design problem can be formulated as an optimization problem similar to the design of multidimensional digital filters; various optimization methods, such as linear programming techniques [13], quadratic programming techniques [12] and second-order cone programming [3] have been applied. If the beamformers are applied in the near-field of the speaker, it becomes a broadband design problem and several optimization methods have been developed; these include the use of quadratic programming [17], multicriteria formulation [23], linear programming [24], and semi-definite programming [4, 6]. We have also investigated analytically the performance limit of the optimization when the filter length is long and the number of microphone is large [7]. As the filter length and the number of microphone of the beamformer increase, the optimization problems become large-scale and are difficult to handle even with the state-of-arts optimization software. Also, the complexity of the beamforming system is becoming very high. Thus we need an efficient algorithm to reduce the complexity.

In view of this, for the semi-infinite programming problem which has non-strictly convex objective functions, in order to make use of the low complexity nature of the exchange method proposed in [26], a perturbation exchange algorithm is proposed to tackle problems with non-strictly convex objective function. The idea is to add a small perturbation to the objective function, making it strictly convex. We show that, under certain assumptions, the solution can be obtained in a finite of iterations, giving an approximate solution to the original problem. We prove that the approximate solution of the perturbed problem converges to the true solution as the perturbation diminishes.

We demonstrate the method by tackling the sparse beamformer design problem. In designing beamformers, it is advantageous to have filters with many zeroes. In this way, the implementation complexity can be reduced significantly. Therefore, the design of sparse beamforming filters is of great interest. In solving this l_0-norm problem [18], an often subproblem is to employ l_1-norm as a linear relaxation of the original problem, and iterate on the number of zero via a successive thinning technique [1,9]. When the l_1-norm is employed, the problem becomes a non-strictly convex semi-infinite programming problem. For fast convergence and at the same time reducing the magnitudes of unnecessary filter coefficients, a perturbation exchange algorithm is proposed. The idea is to add a small perturbation to the objective function, making the problem strictly convex. Under the influence of a small perturbation to the objective function, we obtain similar frequency response to the traditional method [24]. Numerical results have shown that it is possible to have a region of approximately equal performance in terms of the maximum stopband error when the sparsity increases. Therefore, the proposed methods can indeed reduce the complexity of the filters significantly.

In the following sections, the perturbation exchange algorithm is described in Section 2. Then, the convergence property of the proposed perturbation exchange algorithm is established in Section 3. In Section 4, the sparse broadband beamformer design problem is studied and the algorithm is described. Finally, numerical results are given in Section 5.

$\boxed{2}$ Perturbation Exchange Algorithm

For the convex semi-infinite programming (CSIP) problem

$$\begin{cases} \min & f(x) \\ \text{s.t.} & g(x,s) \leq 0 \text{ for any } s \in \Omega, \end{cases} \tag{P}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g(\cdot, s) : \mathbb{R}^n \to \mathbb{R}$ are continuous convex functions, and Ω is a given nonempty compact set in $\mathbb{R}^p$ (or in $\mathbb{C}^p$). Associated with each finite set $\mathcal{R} = \{s_j, j = 1, \ldots, m\} \subset \Omega$, the finitely constrained convex programming problem is define by

$$\begin{cases} \min & f(x) \\ \text{s.t.} & g(x,s_j) \leq 0 \text{ for any } j = 1, \ldots, m. \end{cases} \tag{$P(\mathcal{R})$}$$

Similar to [26], we also assume:
(i) f is convex and continuously differentiable on $\mathbb{R}^n$;
(ii) For any $s \in \Omega$, $g(\cdot, s)$ is convex and $\nabla_x g(x,s)$ exists and is continuous on $\mathbb{R}^n$;
(iii) There exists $\hat{x} \in \mathbb{R}^n$ such that $g(\hat{x}, s) < 0$ for all $s \in \Omega$ (Slater constraint qualification);
(iv) There exists a finite subset Ω_0 of Ω such that f is level bounded on the feasible set of $(P(\Omega_0))$, i.e., for every $a \in \mathbb{R}$, the set

$$\mathcal{L}_a^0 := \{x \in \mathbb{R}^n : f(x) \leq a \text{ and } g(x,s) \leq 0 \text{ for all } s \in \Omega_0\}$$

is bounded when it is nonempty.

In order to make the function $f(x)$ strictly convex, we consider a perturbation of the problem (P) as follows:

$$\begin{cases} \min & f_\epsilon(x) \\ \text{s.t.} & g(x,s) \leq 0 \text{ for any } s \in \Omega, \end{cases} \tag{P_ϵ}$$

where

$$f_\epsilon(x) = f(x) + \epsilon \|x\|^2. \tag{2.1}$$

For a given finite set $\mathcal{R} = \{s_j, j = 1, \ldots, m\} \subset \Omega$, we consider the constrained problem:

$$\begin{cases} \min & f_\epsilon(x) \\ \text{s.t.} & g(x,s_j) \leq 0 \text{ for any } j = 1, \ldots, m. \end{cases} \tag{$P_\epsilon(\mathcal{R})$}$$

Remark 2.1. Let $x^* \in \mathbb{R}^n$ be a feasible solution of the problem $(P_\epsilon(\mathcal{R}))$. It is known that x^* is optimal if and only if there exist multipliers $\lambda^* \in \mathbb{R}^m$ such that (x^*, λ^*) satisfies the following Karush-Kuhn-Tucker (KKT) conditions [15]:

$$\begin{aligned} &\nabla f_\epsilon(x) + \sum_{j=1}^m \lambda(s_j) \nabla_x g(x, s_j) = 0, \\ &\lambda(s_j) \geq 0, \ g(x,s_j) \leq 0, \quad \lambda(s_j) g(x, s_j) = 0, \ j = 1, \ldots, m. \end{aligned} \tag{KKT}$$

For given a small $\eta \in (0, 1/2)$, choose $\epsilon = \epsilon(\eta) > 0$ small enough such that

$$\epsilon \sup_{z \in \mathcal{L}_a^0} \|z\|^2 \leq \eta,$$

and $\bar{a} = 1 + f(\bar{z})$ for some given $\bar{z} \in \{x \in \mathbb{R}^n; g(x,s) \leq 0, \forall s \in \Omega\}$. By applying Algorithm 2.1 in [26] to $(P_\epsilon(\mathcal{R}))$, we present a **perturbation exchange algorithm (PEA)** as follows:

Step 0. Choose a finite reference set $\mathcal{R}^0 = \{s_j^0, j = 1, \ldots, m_0\} \subset \Omega$ such that $\Omega_0 \subset \mathcal{R}^0$. Let x^0 be an optimal solution to $(P_\epsilon(\mathcal{R}^0))$. Set $k = 0$.

Step 1. Find a point $s_{new}^k \in \Omega$ such that

$$g(x^k, s_{new}^k) > \eta.$$

If such a point does not exist, then stop. Otherwise, put

$$\bar{\mathcal{R}}^{k+1} = \mathcal{R}^k \cup \{s_{new}^k\}.$$

Step 2. Let x^{k+1} be an optimal solution to $(P_\epsilon(\bar{\mathcal{R}}^{k+1}))$ and let $\{\lambda^{k+1}(s), s \in \bar{\mathcal{R}}^{k+1}\}$ be the set of associated multipliers.

Step 3. Let

$$\mathcal{R}^{k+1} := \left\{ s \in \bar{\mathcal{R}}^{k+1} : s \in \Omega_0 \text{ or } \lambda^{k+1}(s) > 0 \right\}.$$

Set $k = k + 1$, and return to Step 1.

Remark 2.2. It is obvious that the optimal solution x^{k+1} to $(P_\epsilon(\bar{\mathcal{R}}^{k+1}))$ also solves $(P_\epsilon(\mathcal{R}^{k+1}))$.

Let x^k be an optimal solution to $(P_\epsilon(\mathcal{R}^k))$ and let v^k denote the optimal value of $(P_\epsilon(\mathcal{R}^k))$. Let $\Lambda^k = \{\lambda^k(s_j), j = 1, \ldots, m\}$ be the corresponding Lagrange multiplier. Since f_ϵ is strictly convex, by KKT's condition (KKT), if the algorithm (PEA) does not terminate in k iterations, then

$$\begin{aligned} v^{k+1} - v^k &= f_\epsilon(x^{k+1}) - f_\epsilon(x^k) > \nabla f_\epsilon(x^k)^T (x^{k+1} - x^k) \\ &= - \sum_{s_j \in \mathcal{R}^k} \lambda^k(s_j) \nabla_x g(x^k, s_j)^T (x^{k+1} - x^k) \\ &\geq \sum_{s_j \in \mathcal{R}^k} \lambda^k(s_j) \left(g(x^k, s_j) - g(x^{k+1}, s_j) \right) \\ &= - \sum_{s_j \in \mathcal{R}^k} \lambda^k(s_j) g(x^{k+1}, s_j) \geq 0 \end{aligned} \tag{2.2}$$

where the last equality is due to $\sum_{s_j \in \mathcal{R}^k} \lambda^k(s_j) g(x^k, s_j) = 0$. In particular,

$$v^{k+1} - v^k > 0. \tag{2.3}$$

3 Convergence

The following result is a consequence of Theorem 3.1 in [26].

Theorem 3.1. *The algorithm (PEA) terminates in a finite number of iterations.*

Based on the above theorem, our main convergence result can be established by the following theorem.

Theorem 3.2. *For given $\eta > 0$, let x_η^* be the point determinated by the algorithm (PEA). Then*

(1) *Every accumulation point of* $\{x^*_\eta, \eta \to 0\}$ *is an optimal solution of* (P).

(2) $\lim_{\eta\to 0} f_{\epsilon(\eta)}(x^*_\eta) = v^*$, *where* v^* *is the optimal value of* (P).

(3) *For any* $\eta > 0$,

$$0 \le v^*_{\epsilon(\eta)} - f_{\epsilon(\eta)}(x^*_\eta) \le M_1(\eta)\mathrm{dist}\left(\mathcal{F}\cap\{f\le\alpha\}, \mathcal{F}_\eta\cap\{f\le\alpha+\eta\}\right),$$

where $v^*_{\epsilon(\eta)}$ *is the optimal value of* (P_ϵ),

$$\begin{aligned}\{f\le\alpha\} &= \{x : f(x)\le\alpha\} \ \ \textit{and}\ \ \{f\le\alpha+\eta\} = \{x : f(x)\le\alpha+\eta\} \ \ \textit{with}\ \ \alpha\ge v^*,\\ \mathcal{F}_\eta &:= \{x : g(x,s)\le 0 \textit{ for all } s\in\Omega_0 \textit{ and } g(x,s)\le\eta \textit{ for all } s\in\Omega\},\\ \mathcal{F} &= \{x : g(x,s)\le 0 \textit{ for all } s\in\Omega\},\\ M_1(\eta) &:= \sup_{x\in\mathcal{F}_\eta\cap\{f\le\alpha+\eta\}}\left(\|\nabla f(x)\| + 2\epsilon(\eta)\|x\|\right).\end{aligned}$$

(4) *For any* $\eta > 0$,

$$0 \le v^*_{\epsilon(\eta)} - f_{\epsilon(\eta)}(x^*_\eta) \le \frac{\eta M_1(\eta)}{\eta+\rho}\|x^*_\eta - \hat{x}\|,$$

where $\hat{x}$ *is a point such that* $g(\hat{x}, s) < 0$ *for all* $s\in\Omega$, $\rho := -\max_{s\in\Omega} g(\hat{x}, s)$, *and* $\alpha \ge \max\{v^*+\eta, f(\hat{x})\}$.

Proof. Since f is level bounded on the feasible set of $(P(\Omega_0))$, the set $\{x^*_\eta, \eta > 0\}$ is bounded. Therefore, there exists at least an accumulation point x^* of $\{x^*_\eta, \eta\to 0\}$. By $g(x^*_\eta, s)\le\eta$ for all $s\in\Omega$, we have $x^*\in\mathcal{F}$. It is clear that there exists a finite positive integer $N = N_\eta$, such that x^*_η is an optimal solution of the problem

$$\begin{cases}\min\limits_{x\in\mathbb{R}^n} & f_{\epsilon(\eta)}(x)\\ \ \text{s.t.} & g(x,s)\le 0 \text{ for all } s\in\mathcal{R}^N \text{ and } g(x,s)\le\eta \text{ for all } s\in\Omega\backslash\mathcal{R}^N.\end{cases}$$

Therefore, if x is such that $g(x,s)\le 0$ for all $s\in\mathcal{R}^N$ and $g(x,s)\le\eta$ for all $s\in\Omega\backslash\mathcal{R}^N$, then

$$f_{\epsilon(\eta)}(x^*_\eta)\le f_{\epsilon(\eta)}(x).$$

For any subsequence $\eta_n\to 0$ such that $x^*_{\eta_n}\to x^*$, Since $\mathcal{L}^0_{f(x^*)}\supset\{x; f(x)\le f(x^*)\}\cap\mathcal{F}$ and $\{x^*_{\eta_n}\}$ are bounded, we have that as $\eta\to 0$,

$$\sup_{x\in\mathcal{L}^0_{f(x^*)}}|f_{\epsilon(\eta)}(x) - f(x)| \le \epsilon(\eta)\sup_{x\in\mathcal{L}^0_{f(x^*)}}\|x\|^2\to 0,$$

and as $n\to\infty$,

$$|f_{\epsilon(\eta_n)}(x^*_{\eta_n}) - f(x^*)| \le |f(x^*_{\eta_n}) - f(x^*)| + \epsilon(\eta_n)\sup_{k\ge 1}\|x^*_{\eta_k}\|^2\to 0.$$

Hence, $f_{\epsilon(\eta_n)}(x^*_{\eta_n})\to f(x^*)$. Let $\bar{x}$ be the optimal solution of (P), then $f(\bar{x})\le f(x^*)$, for any $x^*\in\mathcal{F}$. And we have $f_{\epsilon(\eta_n)}(x^*_{\eta_n})\le f(\bar{x})$, then $f(x^*)\le f(\bar{x})$. Thus we get $f(x^*) = f(\bar{x})$. Therefore, x^* is an optimal solution of (P), and $\lim_{\eta\to 0} f_{\epsilon(\eta)}(x^*_\eta) = v^*$. (1) and (2) are valid.

Next, let us show (3). Let $\hat{x}^*_\eta$ be the orthogonal projection of x^*_η onto $\mathcal{F}\cap\{f\le\alpha\}$. Then $f_{\epsilon(\eta)}(\hat{x}^*_\eta)\ge v^*_{\epsilon(\eta)}$ and

$$0\le v^*_{\epsilon(\eta)} - f_{\epsilon(\eta)}(x^*_\eta)$$

$$\begin{aligned}
&= v^*_{\epsilon(\eta)} - f_{\epsilon(\eta)}(\hat{x}^*_\eta) + f_{\epsilon(\eta)}(\hat{x}^*_\eta) - f_{\epsilon(\eta)}(x^*_\eta) \\
&\leq f_{\epsilon(\eta)}(\hat{x}^*_\eta) - f_{\epsilon(\eta)}(x^*_\eta) \\
&= \nabla f_{\epsilon(\eta)}(\tilde{x}^*_\eta)(\hat{x}^*_\eta - x^*_\eta) \\
&\leq \left(\|\nabla f(\tilde{x}^*_\eta)\| + 2\epsilon(\eta)\|\tilde{x}^*_\eta\|\right)\|\hat{x}^*_\eta - x^*_\eta\|
\end{aligned}$$

where $\tilde{x}^*_\eta$ is a point of the segment determined by $\hat{x}^*_\eta$ and x^*_η. Noting that $f(x^*_\eta) \leq f_{\epsilon(\eta)}(x^*_\eta) \leq \inf_{x\in\mathcal{F}\cap\mathcal{L}^0_{\bar{a}}}\{f(x) + \epsilon(\eta)\|x\|^2\} \leq v^* + \eta,$, we have that $\tilde{x}^*_\eta \in \mathcal{F}_\eta \cap \{f \leq \alpha + \eta\}$ which is a compact set. Therefore, (3) is valid.

Finally, we prove (4). It is obvious that

$$\begin{aligned}
g\left(\frac{\rho}{\eta+\rho}x^*_\eta + \frac{\eta}{\eta+\rho}\hat{x}, s\right) &\leq \frac{\rho}{\eta+\rho}g(x^*_\eta, s) + \frac{\eta}{\eta+\rho}g(\hat{x}, s) \\
&\leq \frac{\rho}{\eta+\rho} \times \eta + \frac{\eta}{\eta+\rho} \times (-\rho) = 0,
\end{aligned}$$

and so $\hat{z}^*_\eta := \frac{\rho}{\eta+\rho}x^*_\eta + \frac{\eta}{\eta+\rho}\hat{x} \in \mathcal{F}$. Then

$$\begin{aligned}
0 &\leq v^*_{\epsilon(\eta)} - f_{\epsilon(\eta)}(x^*_\eta) \\
&= v^*_{\epsilon(\eta)} - f_{\epsilon(\eta)}(\hat{z}^*_\eta) + f_{\epsilon(\eta)}(\hat{z}^*_\eta) - f_{\epsilon(\eta)}(x^*_\eta) \\
&\leq f_{\epsilon(\eta)}(\hat{z}^*_\eta) - f_{\epsilon(\eta)}(x^*_\eta) \\
&= \nabla f_{\epsilon(\eta)}(\tilde{z}^*_\eta)(\hat{z}^*_\eta - x^*_\eta) \\
&\leq \left(\|\nabla f(\tilde{z}^*_\eta)\| + 2\epsilon(\eta)\|\tilde{z}^*_\eta\|\right)\|\hat{z}^*_\eta - x^*_\eta\|
\end{aligned}$$

where $\tilde{z}^*_\eta$ is a point of the segment determined by $\hat{z}^*_\eta$ and x^*_η, and so,

$$f(\tilde{z}^*_\eta) \leq \max\{f(\hat{z}^*_\eta), f(x^*_\eta)\} \leq \max\{f(x^*_\eta), f(\hat{x})\} \leq \max\{v^* + \eta, f(\hat{x})\}.$$

Thus, $\tilde{z}^*_\eta \in \mathcal{F}_\eta \cap \{f \leq \alpha + \eta\}$ with $\alpha \geq \max\{v^* + \eta, f(\hat{x})\}$. Noting that $\|\hat{z}^*_\eta - x^*_\eta\| \leq \frac{\eta}{\eta+\rho}\|\hat{x} - x^*_\eta\|$, we obtain (4).

□

Remark 3.3. (1) If (P) has a unique optimal solution, denoted by x^*, then by Theorem 3.2 (1), $\lim_{\eta\to 0} x^*_\eta = x^*$, and $\lim_{\eta\to 0} f_\eta(x^*_\eta) = f(x^*)$. Therefore, the perturbation algorithm (PEA) provides an approximate solution.

(2) Theorem 3.2 (3) and (4) provide error bounds for the approximate solution x^*_η.

(3) Since

$$\mathcal{F} \cap \{f \leq \alpha\} \subset \mathcal{F}_\eta \cap \{f \leq \alpha + \eta\} \subset \mathcal{L}^0_{\alpha+\eta},$$

it is obvious that

$$\operatorname{dist}\left(\mathcal{F} \cap \{f \leq \alpha\}, \mathcal{F}_\eta \cap \{f \leq \alpha + \eta\}\right) \leq \sup_{x,y\in\mathcal{L}^0_{\alpha+\eta}} \|x - y\|,$$

and

$$M_1(\eta) = \sup_{x\in\mathcal{F}_\eta\cap\{f\leq\alpha+\eta\}} \left(\|\nabla f(x)\| + 2\epsilon(\eta)\|x\|\right) \leq \sup_{x\in\mathcal{L}^0_{\alpha+\eta}} \left(\|\nabla f(x)\| + 2\epsilon(\eta)\|x\|\right).$$

4 Sparse Beamforming Design Problem

In this section, we apply the proposed method to solve a sequence of non-strictly convex semi-infinite programming problem, namely the sparse broadband beamformer design problem.

4.1 Formulation

In a typical environment, a beamformer contains a series of microphones placed in predefined locations. Behind each microphone, there is an FIR filter attached for processing the received sound signals [23]. Let the beamformer have N microphones and let each FIR filter have L taps. Denote the position vector of the i-th microphone by $\boldsymbol{r}_i$. The transfer function from the source to the i-th microphone is given by

$$A_i(\boldsymbol{r}, f) = \frac{1}{\|\boldsymbol{r} - \boldsymbol{r}_i\|} e^{-j2\pi f \|\boldsymbol{r}-\boldsymbol{r}_i\|/c}. \tag{4.1}$$

The array response is therefore given by

$$\boldsymbol{a}(\boldsymbol{r}, f) = (A_1(\boldsymbol{r}, f), \ldots, A_N(\boldsymbol{r}, f))^\mathsf{T}, \tag{4.2}$$

and the beam response is

$$G(\boldsymbol{r}, f) = \boldsymbol{w}^\mathsf{T} \boldsymbol{d}(\boldsymbol{r}, f)$$

with

$$\boldsymbol{d}(\boldsymbol{r}, f) = \boldsymbol{a}(\boldsymbol{r}, f) \otimes \boldsymbol{d}_0(f),$$

where $\otimes$ is the Kronecker product, and $\boldsymbol{d}_0(f) = \left(1, e^{-j2\pi f/f_s}, \ldots, e^{-j2\pi f(L-1)/f_s}\right)$ is the filter response vector. For a given array configuration, this beamforming design problem can be formulated as a minimax problem:

$$\min_{\boldsymbol{w}\in\mathbb{R}^{NL}} \max_{(\boldsymbol{r},f)\in\Omega} |\boldsymbol{w}^\mathsf{T}\boldsymbol{d}(\boldsymbol{r}, f) - G_d(\boldsymbol{r}, f)|, \tag{4.3}$$

where $G_d(\boldsymbol{r}, f)$ is the specified desired response of the broadband beamformer.

Following [24], we expand the complex functions as

$$\boldsymbol{d}(\boldsymbol{r}, f) = \boldsymbol{d}_1(\boldsymbol{r}, f) + j\boldsymbol{d}_2(\boldsymbol{r}, f),$$

$$G_d(\boldsymbol{r}, f) = G_{d_1}(\boldsymbol{r}, f) + jG_{d_2}(\boldsymbol{r}, f),$$

and denote

$$u(\boldsymbol{r}, f) = \boldsymbol{w}^\mathsf{T}\boldsymbol{d}_1(\boldsymbol{r}, f) - G_{d_1}(\boldsymbol{r}, f),$$

$$v(\boldsymbol{r}, f) = \boldsymbol{w}^\mathsf{T}\boldsymbol{d}_2(\boldsymbol{r}, f) - G_{d_2}(\boldsymbol{r}, f).$$

By introducing a slack variable δ

$$\delta = \max_{(\boldsymbol{r},f)\in\Omega} |u(\boldsymbol{r}, f) + jv(\boldsymbol{r}, f)|,$$

the above minimax problem can be further written as

$$\begin{cases} \displaystyle\min_{\boldsymbol{w}\in\mathbb{R}^{NL},\delta} & \delta \\ \quad \text{s.t.} & |u(\boldsymbol{r}, f) + jv(\boldsymbol{r}, f)| \le \delta \qquad \forall(\boldsymbol{r}, f) \in \Omega. \end{cases} \tag{4.4}$$

Actually, we can control the real part and the imaginary part separately. Using the l_1 norm as a linear relaxation, and introducing two new variables as

$$z_1 = \max_{(\boldsymbol{r},f)\in\Omega} |u(\boldsymbol{r}, f)|, \quad z_2 = \max_{(\boldsymbol{r},f)\in\Omega} |v(\boldsymbol{r}, f)|,$$

we convert the above problem into the following problem:

$$\begin{cases} \min\limits_{\boldsymbol{w}\in\mathbb{R}^{NL},z_1,z_2} & z_1 + z_2 \\ \text{s.t.} & |u(\boldsymbol{r}, f)| \le z_1 \quad \forall(\boldsymbol{r}, f) \in \Omega \\ & |v(\boldsymbol{r}, f)| \le z_2 \quad \forall(\boldsymbol{r}, f) \in \Omega \end{cases} \tag{4.5}$$

which is equivalent to

$$\begin{cases} \min\limits_{\boldsymbol{w}\in\mathbb{R}^{NL},z_1,z_2} & z_1 + z_2 \\ \text{s.t.} & \boldsymbol{w}^\intercal \boldsymbol{d}_1(\boldsymbol{r}, f) - G_{d_1}(\boldsymbol{r}, f) \le z_1 \quad \forall(\boldsymbol{r}, f) \in \Omega \\ & -\boldsymbol{w}^\intercal \boldsymbol{d}_1(\boldsymbol{r}, f) + G_{d_1}(\boldsymbol{r}, f) \le z_1 \quad \forall(\boldsymbol{r}, f) \in \Omega \\ & \boldsymbol{w}^\intercal \boldsymbol{d}_2(\boldsymbol{r}, f) - G_{d_2}(\boldsymbol{r}, f) \le z_2 \quad \forall(\boldsymbol{r}, f) \in \Omega \\ & -\boldsymbol{w}^\intercal \boldsymbol{d}_2(\boldsymbol{r}, f) + G_{d_2}(\boldsymbol{r}, f) \le z_2 \quad \forall(\boldsymbol{r}, f) \in \Omega. \end{cases} \tag{4.6}$$

To summarize, the design problem can be formulated as the following semi-infinite programming problem

$$\begin{cases} \min\limits_{\boldsymbol{z}\in\mathbb{R}^{NL+2}} & \boldsymbol{b}^\intercal \boldsymbol{z} \\ \text{s.t.} & \boldsymbol{H}(\boldsymbol{r}, f)\boldsymbol{z} - \boldsymbol{G}(\boldsymbol{r}, f) \le 0 \quad \forall(\boldsymbol{r}, f) \in \Omega \end{cases} \tag{4.7}$$

where $\boldsymbol{z} = (\boldsymbol{w}, z_1, z_2)^\intercal$, $\boldsymbol{b} = (\boldsymbol{0}, 1, 1)^\intercal$,

$$\boldsymbol{H}(\boldsymbol{r}, f) = \begin{pmatrix} \boldsymbol{d}_1(\boldsymbol{r}, f)^\intercal & -1 & 0 \\ -\boldsymbol{d}_1(\boldsymbol{r}, f)^\intercal & -1 & 0 \\ \boldsymbol{d}_2(\boldsymbol{r}, f)^\intercal & 0 & -1 \\ -\boldsymbol{d}_2(\boldsymbol{r}, f)^\intercal & 0 & -1 \end{pmatrix}, \quad \boldsymbol{G}(\boldsymbol{r}, f) = \begin{pmatrix} G_{d_1}(\boldsymbol{r}, f) \\ -G_{d_1}(\boldsymbol{r}, f) \\ G_{d_2}(\boldsymbol{r}, f) \\ -G_{d_2}(\boldsymbol{r}, f) \end{pmatrix}.$$

Define

$$g(\boldsymbol{z}, (\boldsymbol{r}, f)) = \boldsymbol{H}(\boldsymbol{r}, f)\boldsymbol{z} - \boldsymbol{G}(\boldsymbol{r}, f).$$

Then the above problem (4.7) can be represented by

$$\begin{cases} \min\limits_{\boldsymbol{z}} & \boldsymbol{b}^\intercal \boldsymbol{z} \\ \text{s.t.} & g(\boldsymbol{z}, (\boldsymbol{r}, f)) \le 0 \quad \forall(\boldsymbol{r}, f) \in \Omega. \end{cases} \tag{4.8}$$

In order to reduce implementation complexity, the sparsity of filters should be increased. This can be represented by the constraint $\|w\|_0 \le q$, where q is some integer for controlling the sparsity of the filter. The beamformer design problem can be formulated as

$$\begin{cases} \min & \boldsymbol{b}^\intercal \boldsymbol{z} + \epsilon\|\boldsymbol{z}\|_2^2 \\ \text{s.t.} & g(\boldsymbol{z}, (\boldsymbol{r}, f)) \le 0 \quad \forall(\boldsymbol{r}, f) \in \Omega. \end{cases} \tag{4.9}$$

Note that in the formulation, a small perturbation is added to the objective function, which helps to drive some filter coefficients to much smaller values. These less important coefficients could be identified and could be discarded gradually. Another advantage is that the objective function becomes strictly convex, which enables the use of our perturbation exchange method with a rather low complexity, even for the multi-dimensional design problem. In the previous section we have proved when ϵ is sufficiently small, the approximate solution of the perturbed problem will be converged to the true solution. In the following section, the successive thinning algorithm is described, which composes of an outer cycle to fix the set of zero filter coefficients gradually and (4.9) is a subproblem for this thinning algorithm.

$\boxed{4.2}$ Algorithm

In the sequential thinning algorithm, it maintains a growing list $\mathcal{C}^{(i)} \subset \{1, \ldots, NL - q\}$ of indices, for which w_m is constrained to be zero, where $m \in \mathcal{C}^{(i)}$ and the superscript i denotes the iteration number. In the first iteration, $\mathcal{C}^{(0)}$ is empty and the usual optimal beamformer will be sought by minimizing the maximum error. In subsequent iterations, one new filter coefficient is selected from the previous iteration and constrained to be zero in solving the beamformer design problem (4.8). The major steps of this approach can be summarized as follows:

(1) Initialize $\mathcal{N}^{(0)} = \{1, 2, \ldots, NL\}$ and $\mathcal{C}^{(0)} = \emptyset$. Solve the following augmented beamforming design problem

$$\begin{cases} \min & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{z} + \epsilon\|\boldsymbol{z}\|_2^2 \\ \text{s.t.} & g(\boldsymbol{z}, (\boldsymbol{r}, f)) \leq 0 \qquad \forall (\boldsymbol{r}, f) \in \Omega. \end{cases}$$

(2) For iteration $i = 1, 2, \ldots,$ we use the smallest-coefficient rule to choose the index $p^{(i)}$ of the filter coefficient

$$p^{(i)} = \arg\min_m |w_m^{(i)}|.$$

which should be constrained to zero value in the next iteration. Update

$$\mathcal{N}^{(i)} = \mathcal{N}^{(i-1)} - \{p^{(i)}\}$$

$$\mathcal{C}^{(i)} = \mathcal{C}^{(i-1)} + \{p^{(i)}\},$$

and then solve the following problem

$$\begin{cases} \min & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{z} + \epsilon\|\boldsymbol{z}\|_2^2 \\ \text{s.t.} & g(\boldsymbol{z}, (\boldsymbol{r}, f)) \leq 0 \qquad \forall (\boldsymbol{r}, f) \in \Omega \\ & w_m = 0 \qquad \forall m \in \mathcal{C}^{(i)}. \end{cases} \tag{4.10}$$

In solving the sub-problem (4.10), discretization is employed to convert the solution space into a finite set of points. For a given finite set $\mathcal{R} = \{(\boldsymbol{r}_j, f_j), j = 1, \ldots, m\} \subset \Omega$, we consider the finite problem denoted by $(BP_\epsilon(\mathcal{R}^{(i)}))$:

$$\begin{cases} \min & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{z} + \epsilon\|\boldsymbol{z}\|_2^2 \\ \text{s.t.} & g(\boldsymbol{z}, (\boldsymbol{r}_j, f_j)) \leq 0 \qquad \text{for any } j = 1, \ldots, m \\ & w_m = 0 \qquad \forall m \in \mathcal{C}^{(i)}. \end{cases} \tag{4.11}$$

We use exchange algorithm for this subproblem (4.10). The final algorithm can be summarized as follows:

Step 0:

Choose a finite reference set $\mathcal{R}^{(0,0)} = \{(\boldsymbol{r}_j^{(0,0)}, f_j^{(0,0)}), j = 1, \ldots, m_0\} \subset \Omega$ such that $\Omega_0 \subset \mathcal{R}^0$. Let $\boldsymbol{z}^{(0,0)}$ be an optimal solution to $(BP_\epsilon(\mathcal{R}^{(0,0)}))$ and let $\{\lambda(\boldsymbol{r}_j^{(0,0)}, f_j^{(0,0)}), j = 1, \ldots, m_0\} \in \mathbb{R}^m$ be the set of associated multipliers. Set $\mathcal{C}^{(0)} = \emptyset$, $k = 0$ and $i = 0$.

Step 1:

1.1. Find a set $\{(\boldsymbol{r}_{new}^{(i,k)}, f_{new}^{(i,k)}), new = 1, \ldots, n\} \subset \Omega$ such that

$$g(\boldsymbol{z}^{(i,k)}, (\boldsymbol{r}_{new}^{(i,k)}, f_{new}^{(i,k)})) > \eta.$$

If such a point does not exist, then stop. Otherwise, put $\bar{\mathcal{R}}^{(i,k+1)} = \mathcal{R}^{(i,k)} \cup \{(\boldsymbol{r}_{new}^{(i,k)}, f_{new}^{(i,k)})\}$.

1.2. Let $\boldsymbol{z}^{(i,k+1)}$ be an optimal solution to $(BP_\epsilon(\bar{\mathcal{R}}^{(i,k+1)}))$ and let $\{\lambda^{(i,k+1)}(\boldsymbol{r}, f), (\boldsymbol{r}, f) \in \bar{\mathcal{R}}^{(i,k+1)}\}$ be the set of associated multipliers.

1.3. Let

$$\mathcal{R}^{(i,k+1)} := \left\{(\boldsymbol{r}, f) \in \bar{\mathcal{R}}^{(i,k+1)}; \lambda^{(i,k+1)}(\boldsymbol{r}, f) > 0\right\}.$$

Set $k = k + 1$, and return to step 1.1.

Step 2:

Let $\mathcal{R}^{(i)}$ denote the final reference set and $\boldsymbol{z}^{(i)} = (\boldsymbol{w}^{(i)}, z_1^{(i)}, z_2^{(i)})^\intercal$ be the optimal solution found in Step 1. If $\|\boldsymbol{w}\|_0 = q$, then stop. Otherwise the index $p^{(i)}$ is chosen to correspond to the smallest of the optimal coefficient $\boldsymbol{w}^i$,

$$p^{(i)} = \arg\min_m |w_m^{(i)}|.$$

Put $\mathcal{C}^{(i+1)} = \mathcal{C}^{(i)} \cup \{p^{(i)}\}$.

Step 3:

Set $k = 0$. Let $\mathcal{R}^{(i+1,k)} = \mathcal{R}^{(i)}$, $\boldsymbol{z}^{(i+1,k)}$ be an optimal solution to $(BP_\epsilon(\mathcal{R}^{(i+1,k)})$ and $\{\lambda^{(i+1,k)}(\boldsymbol{r}, f), (\boldsymbol{r}, f) \in \mathcal{R}^{(i+1,k)}\}$ be the set of associated multipliers. Set $i = i + 1$, and return to Step 1.

Note: In step 1.1, we choose multiple points satisfying the adding rule instead of just choosing one. This will reduce the number of iterations and shorten the computational time.

5 Numerical Examples

In this section we provide examples to demonstrate the performance of the algorithm which is implemented in MATLAB. We choose the desired response function as

$$G_d(\boldsymbol{r}, f) = \begin{cases} e^{-j2\pi f\left(\frac{||\boldsymbol{r}-\boldsymbol{r}_c||}{c} + \frac{L-1}{2}T\right)}, & \text{if } (\boldsymbol{r}, f) \text{ is in passband region,} \\ 0, & \text{if } (\boldsymbol{r}, f) \text{ is in stopband region,} \end{cases}$$

where $\boldsymbol{r}_c$ is the reference central microphone location. In this example, we consider an equispaced linear array with five elements that can be seen in Figure 1, where the diamond point denotes the speaker position and the circle points denote the microphone array positions. Here we consider each filter has 7 taps. The passband region is defined as

$$\{(x, f) : -0.4m \le x \le 0.4m,\ 0.5kHz \le f \le 1.5kHz\}$$

while the stopband region is the union of several parts as

$$\{(x, f) : -0.4m \le x \le 0.4m,\ 2.5kHz \le f \le 4kHz\},$$

$$\{(x, f) : 1.5m \le |x| \le 2.5m,\ 0.5kHz \le f \le 1.5kHz\},$$

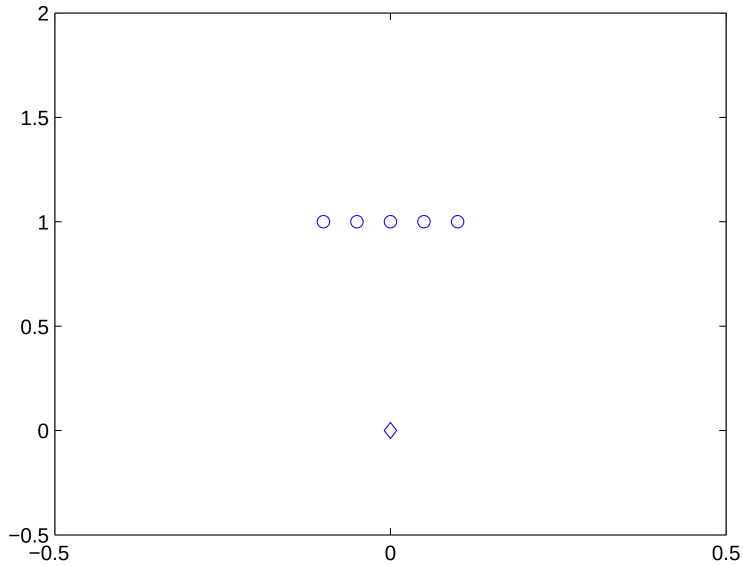

Figure 1: Array configuration (Ex1)

$$\{(x, f) : 1.5m \leq |x| \leq 2.5m,\ 2.5kHz \leq f \leq 4kHz\}.$$

In this example, we fix $\epsilon = 0.05$, Table 1 shows the advantages of using PEA instead of the traditional discretization method and the penalty decomposition method [16]. In terms of computational efficiency, the running time of using PEA is always significantly less than the discretization method, and also with a much lower complexity in the storage requirement. Moreover, if the sparseness of the solution is increased, we can achieve better frequency response functions in even lesser time. This demonstrates the power of the method in tackling the l_0-norm design problem. The amplitude of the actual response $G(\boldsymbol{r}, f)$ using PEA is shown in Figure 2. It is observed in Figure 3 that the performance is not affected greatly even when the sparseness increases to about 45% of zero elements.

percentage of zeros	PEA	SIP	PD
0%	−14.1175(dB) 4.5623(s)	−14.5075(dB) 485.6615(s)	−12.2093(dB) 464.9336(s)
25.71%	−13.8700(dB) 31.6315(s)	−14.4030(dB) 705.8628(s)	−11.6914(dB) 1109.1(s)
45.71%	−12.7979(dB) 52.8846(s)	−11.9340(dB) 853.0882(s)	−11.7307(dB) 1286.1(s)
71.43%	−10.0506(dB) 79.6834(s)	−8.6308(dB) 954.1321(s)	−6.8706(dB) 1141(s)

Table 1: Comparison of the stopband ripple and the running times (Ex1)

In the second example, we extend the length of the filters significantly so that each filter has 26 taps. We continue to fix $\epsilon = 0.05$ and consider the configuration showed in Figure 4

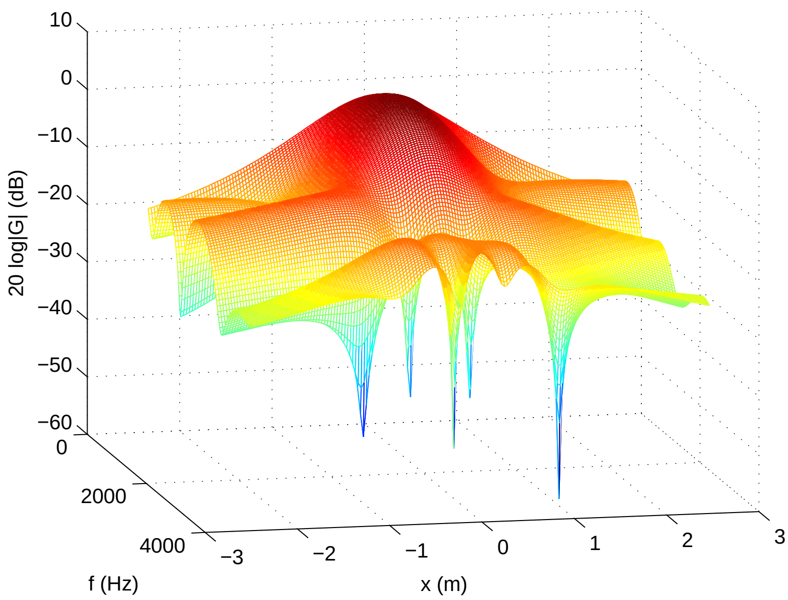

Figure 2: Amplitude of $G(\boldsymbol{r}, f)$ where $N = 5$, $L = 7$, 45.71% of zeroes (Ex1).

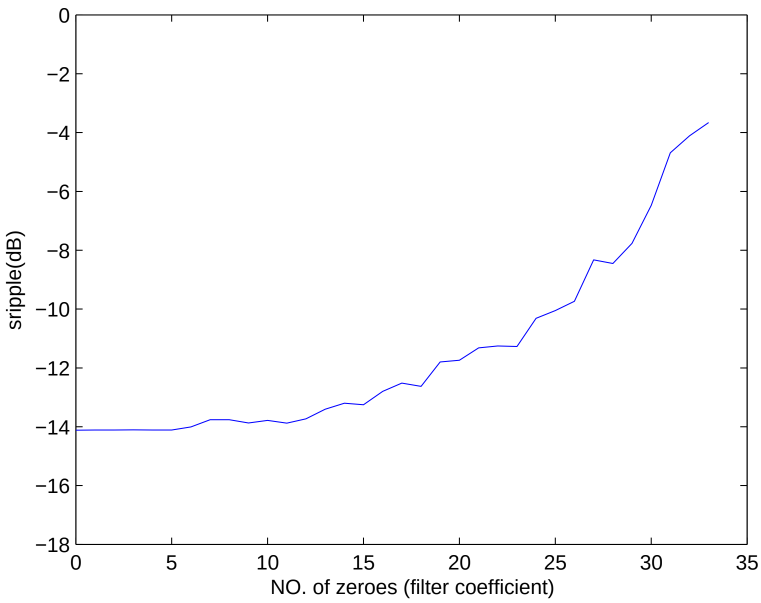

Figure 3: Stopband ripple for filters (Ex1).

for this example. The passband region is defined as

$$\{(x, f) : -0.4m \leq x \leq 0.4m,\ 0.5kHz \leq f \leq 1.5kHz\}$$

while the stopband region is simplified as the union of

$$\{(x, f) : 1.8m \leq |x| \leq 3m,\ 0.5kHz \leq f \leq 1.5kHz\},$$

$$\{(x, f) : -3m \leq |x| \leq 3m,\ 2kHz \leq f \leq 4, kHz\}$$

with fewer transition regions. The amplitude of the actual response $G(\boldsymbol{r}, f)$ using PEA is shown in Figure 5. We can also observe a region of approximately equal performance for this example as shown in Figure 6.

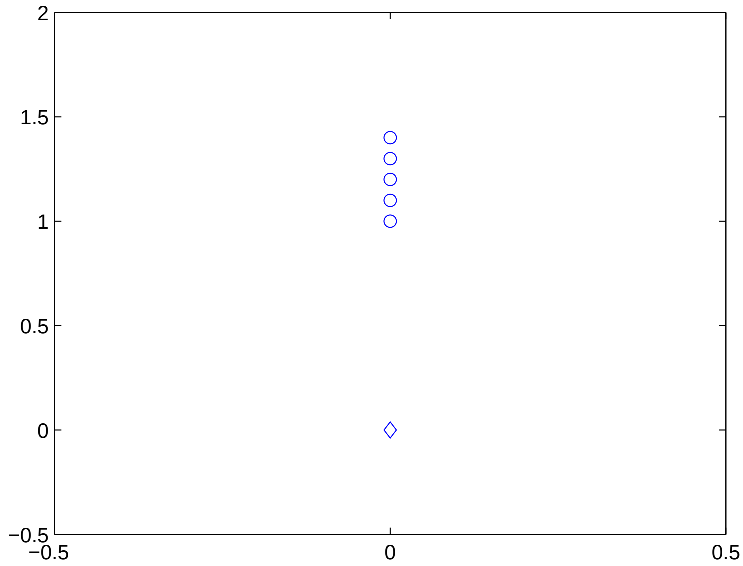

Figure 4: Array configuration (Ex2)

6 Conclusions

In this paper, a new method has been proposed for solving semi-infinite problems with functions not strictly convex. The convergence of the method has been established and a multiple exchange algorithm has been developed for finding the solutions. The method is then employed for designing sparse beamforming system with the successful thinning technique. We have studied the performance of the optimized designs with several examples. Overall, the method has the advantages of having a finite termination and economizing on the storage requirement during iterations. In this way, it has the potential to be extended to other applications as well such as configuration optimization [5]. Furthermore, it would be of interest to study the convergence of the overall successive thinning algorithm as an extension.

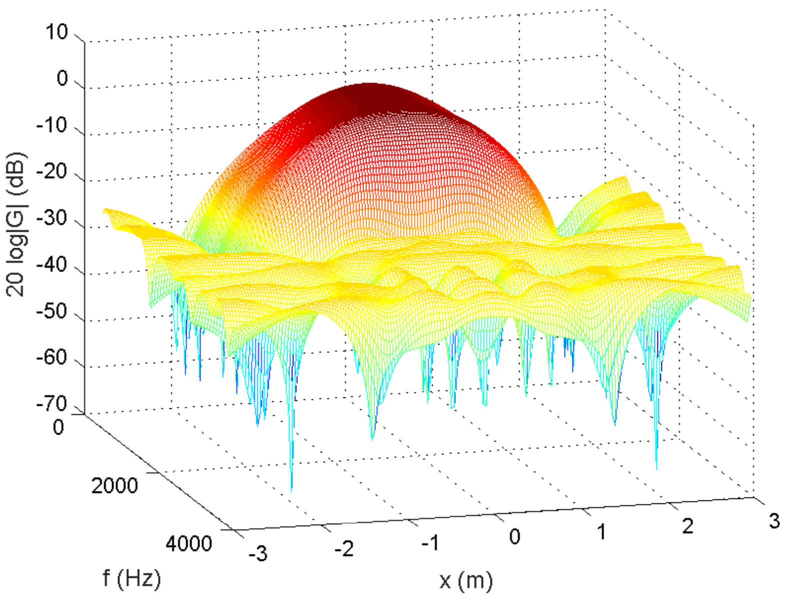

Figure 5: Amplitude of $G(\boldsymbol{r}, f)$ where $N = 5$, $L = 26$, 61.54% of zeroes (Ex2).

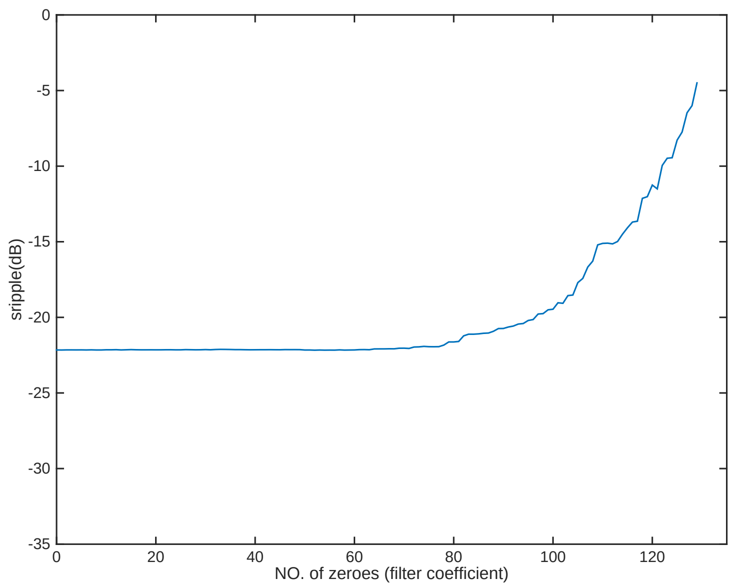

Figure 6: Stopband ripple for filters (Ex2).

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