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LAGRANGIAN-CONIC RELAXATIONS, PART I: A UNIFIED FRAMEWORK AND ITS APPLICATIONS TO QUADRATIC OPTIMIZATION PROBLEMS

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Abstract: In Part I of a series of study on Lagrangian-conic relaxations, we introduce a unified framework for conic and Lagrangian-conic relaxations of quadratic optimization problems (QOPs) and polynomial optimization problems (POPs). The framework is constructed with a linear conic optimization problem (COP) in a finite dimensional Hilbert space, where the cone used is not necessarily convex. By imposing a copositive condition on the COP, we establish fundamental theoretical results for the COP, its (convex-hull) conic relaxations, its Lagrangian-conic relaxations, and their duals. A linearly constrained QOP with complementarity constraints and a general POP can be reduced to the COP satisfying the copositivity condition. Thus the conic and Lagrangian-conic relaxations of such a QOP and POP can be discussed in a unified manner. The Lagrangian-conic relaxation takes a particularly simple form involving only a single equality constraint together with the cone constraint, which is very useful for designing efficient numerical methods. As demonstration of the elegance and power of the unified framework, we present the derivation of the completely positive programming relaxation, and a sparse doubly nonnegative relaxation for a class of a linearly constrained QOPs with complementarity constraints. The unified framework is applied to general POPs in Part II.

Key words: *Lagrangian-conic relaxation, completely positive programming relaxation, doubly nonnegative relaxation, convexification, quadratic optimization problems, exploiting sparsity*

Mathematics Subject Classification: *90C20, 90C25, 90C26*

1 Introduction

We consider a general polynomial optimization problem (POP) of the form:

$$\zeta^* \quad = \quad \inf \left\{ f^0(\boldsymbol{x}) \;\middle|\; \boldsymbol{x} \in \mathbb{J},\ f^k(\boldsymbol{x}) = 0 \ (k = 1, 2, \ldots, m) \right\}, \tag{1.1}$$

where $\mathbb{J}$ denotes a closed (but not necessarily convex) cone in the n-dimensional Euclidean space $\mathbb{R}^n$, and $f^k(\boldsymbol{x})$ a real valued polynomial in $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ $(k = 0, 1, 2, \ldots, m)$. Quadratic optimization problems (QOPs) are a prominent subclass of POPs (1.1). Among

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various QOPs, we are particularly interested in the following linearly constrained QOP with complementarity constraints [22] (see also [2, 3, 9]):

$$\zeta^* = \inf \left\{ \boldsymbol{u}^T \boldsymbol{Q} \boldsymbol{u} + 2\boldsymbol{c}^T \boldsymbol{u} \;\middle|\; \begin{array}{l} \boldsymbol{u} \in \mathbb{R}^n_+, \; \boldsymbol{A}\boldsymbol{u} + \boldsymbol{b} = \boldsymbol{0}, \\ u_i u_j = 0 \; ((i,j) \in \mathcal{E}) \end{array} \right\}, \tag{1.2}$$

where $\boldsymbol{A} \in \mathbb{R}^{q \times n}$, $\boldsymbol{b} \in \mathbb{R}^q$, $\boldsymbol{c} \in \mathbb{R}^n$ and $\mathcal{E} \subset \{(i,j) : 1 \leq i < j \leq n\}$ are given data. Since the binary constraint $u_i(1 - u_i) = 0$ can be converted to a complementarity constraint $u_i v_i = 0$ with a slack variable $v_i = 1 - u_i \geq 0$, QOP (1.2) can represent nonconvex QOPs with linear, binary, and complementarity constraints.

Polynomial optimization problems (POPs) have been recognized as a very important class of optimization problems, attracting a great deal of research in recent years. POPs are nonconvex in general, and many QOPs are known to be NP-hard. While theoretical and numerical studies of POPs have been directed toward developing efficient and effective solution methods, the current solution methods for solving POPs are generally not efficient enough to solve a dense problem beyond a dozen variables. In a series of studies, we aim to address the issue of solving larger POPs efficiently by proposing a unified framework for POPs. Our final goal is to provide efficient numerical methods for solving POPs based on specially designed first-order algorithms under the unified framework.

Convex relaxation techniques over cones such as semidefinite programming (SDP) relaxations [24, 35] have been widely used for solving POPs (1.1) with $\mathbb{J} = \mathbb{R}^n$ or $\mathbb{R}^n_+$. The SDP approach [19,36] implemented via a primal-dual interior-point method [8,15,32,34], however, suffers from numerical inefficiency except for small POPs. It is also numerically challenging to solve QOP (1.2), which includes a special type of QOP which can be cast exactly using the completely positive (CPP) cone as shown by Burer [9]. More precisely, Burer's CPP formulation of a QOP and its extension [2, 4, 11–13, 29, 30] are numerically intractable, despite their theoretical exactness. If the CPP cone is relaxed to the doubly nonnegative (DNN) cone consisting of nonnegative symmetric matrices which are also positive semidefinite, a numerically tractable DNN relaxation is obtained [17,33,37]. However, solving the resulting DNN relaxation by a primal-dual interior-point method is still numerically highly challenging, especially for large scale problems. This is because the DNN relaxation includes a large number of nonnegativity constraints on the entries of the matrix variable $\boldsymbol{X}$, which grow quadratically with the size of X, in addition to the semidefinite constraint on $\boldsymbol{X}$. Thus, it is essential to develop an efficient numerical method beyond the framework of interior-point methods to solve the DNN relaxations of large-scale QOPs and POPs.

1.1 Our framework

We present a unified framework for conic and Lagrangian-conic relaxations of QOPs and POPs. The framework starts with a primal-dual pair of conic optimization problems (COPs) described as follows:

$$\zeta^p(\mathbb{K}) := \inf \left\{ \langle \boldsymbol{Q}^0, \boldsymbol{X} \rangle \;\middle|\; \begin{array}{l} \boldsymbol{X} \in \mathbb{K}, \; \langle \boldsymbol{H}^0, \boldsymbol{X} \rangle = 1, \\ \langle \boldsymbol{Q}^k, \boldsymbol{X} \rangle = 0 \; (k = 1, 2, \ldots, m) \end{array} \right\} \tag{1.3}$$

$$\zeta^d(\mathbb{K}) := \sup \left\{ z_0 \;\middle|\; \boldsymbol{Q}^0 + \sum_{k=1}^m \boldsymbol{Q}^k z_k - \boldsymbol{H}^0 z_0 \in \mathbb{K}^* \right\} \tag{1.4}$$

where $\mathbb{K}$ is a (not necessarily convex nor closed) cone in a finite dimensional vector space $\mathbb{V}$ endowed with an inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. Here $\boldsymbol{X}$ denotes the decision variable, and $\boldsymbol{H}^0$ and $\boldsymbol{Q}^k$ $(k = 1, 2, \ldots, m)$ are given vectors in $\mathbb{V}$. We use the convention of setting $\zeta^p(\mathbb{K}) = \infty$ if the feasible region of (1.3) is empty, and setting $\zeta^d(\mathbb{K}) = -\infty$ if the feasible region of (1.4) is empty. Note that by the standard argument, we have the following weak duality result: $\zeta^d(\mathbb{K}) \leq \zeta^p(\mathbb{K})$.

When applying this framework to QOPs and POPs, we will take $\mathbb{V}$ to be the linear space of symmetric matrices with appropriate dimension. (This is why capital letters are used to denote vectors such as $\boldsymbol{Q}$ and $\boldsymbol{X}$ in the space $\mathbb{V}$.) The primal COP minimizes a linear objective function $\langle \boldsymbol{Q}^0, \boldsymbol{X}\rangle$ subject to three types of constraints: a nonhomogeneous linear equality $\langle \boldsymbol{H}^0, \boldsymbol{X}\rangle = 1$, multiple homogeneous linear equalities $\langle \boldsymbol{Q}^k, \boldsymbol{X}\rangle = 0$ $(k = 1, 2, \ldots, m)$, and a cone constraint $\boldsymbol{X} \in \mathbb{K}$. We should mention that QOP (1.2) is reformulated in the form (1.3) in Section 5 and POP(1.1) will be reformulated in the form (1.3) in Part II.

For subsequent developments, we impose the following condition on the primal-dual pair of COPs (1.3) and (1.4).

Condition (I) The feasible region of (1.3), denoted as

$$F(\mathbb{K}) = \left\{ \boldsymbol{X} \in \mathbb{K} : \langle \boldsymbol{H}^0, \boldsymbol{X}\rangle = 1,\ \langle \boldsymbol{Q}^k, \boldsymbol{X}\rangle = 0\ (k = 1, 2, \ldots, m) \right\}$$

is nonempty, and $\boldsymbol{O} \neq \boldsymbol{H}^0 \in \mathbb{K}^*$ and $\boldsymbol{Q}^k \in \mathbb{K}^*$ $(k = 1, 2, \ldots, m)$.

Here $\mathbb{K}^*$ denotes the dual of $\mathbb{K}$, *i.e.*, $\mathbb{K}^* = \{\boldsymbol{Y} \in \mathbb{V} : \langle \boldsymbol{X}, \boldsymbol{Y}\rangle \geq 0 \text{ for every } \boldsymbol{X} \in \mathbb{K}\}$. We may call an $\boldsymbol{Y} \in \mathbb{V}$ copositive with respect to $\mathbb{K}$ if $\langle \boldsymbol{X}, \boldsymbol{Y}\rangle \geq 0$ for all $\boldsymbol{X} \in \mathbb{K}$ or equivalently $\boldsymbol{Y} \in \mathbb{K}^*$. Thus Condition (I) requires that $\boldsymbol{O} \neq \boldsymbol{H}^0$ and $\boldsymbol{Q}^k$ $(k = 1, 2, \ldots, m)$ are copositive with respect to $\mathbb{K}$. See also the definition of the copositive cone in Section 5.

In theory, we may begin with a more general minimization problem $\inf \left\{ \langle \boldsymbol{Q}^0, \boldsymbol{U}\rangle \mid U \in S \right\}$, where S is a nonempty subset of a linear space $\mathbb{V}'$ and $\boldsymbol{Q}^0 \in \mathbb{V}'$. If we define

$$\mathbb{V} \;=\; \mathbb{R} \times \mathbb{V}',\ \mathbb{K} = \{(\lambda, \lambda \boldsymbol{U}) : \lambda \in \mathbb{R}_+,\ \boldsymbol{U} \in S\},\ \boldsymbol{H}^0 = (1, \boldsymbol{O}) \in \mathbb{V},$$

then we can reformulate the problem as $\inf \left\{ \langle (0, \boldsymbol{Q}^0), \boldsymbol{X}\rangle \mid \boldsymbol{X} \in \mathbb{K},\ \langle \boldsymbol{H}^0, \boldsymbol{X}\rangle = 1 \right\}$. By construction, Condition (I) is satisfied with $F(\mathbb{K}) = \left\{ \boldsymbol{X} \in \mathbb{K} : \langle \boldsymbol{H}^0, \boldsymbol{X}\rangle = 1 \right\}$ and $m = 0$. In practice, many QOPs with linear, binary and complementarity constraints and POPs with equality constraints are formulated as the primal conic optimization problem (1.3) as we discuss in Section 5 and Part II [5]. See also [3, 22].

1.2 Contribution

Many QOPs and POPs can be reformulated in the form of the primal COP (1.3) with a nonconvex cone $\mathbb{K}$ satisfying Condition (I). As a result, the unified framework we propose here allows us to discuss the equivalence between the optimal value of the original nonconvex QOP (or POP) and its convex CPP formulation (or its extended CPP formulation) by proving that $\zeta^p(\mathrm{co}\,\mathbb{K}) = \zeta^p(\mathbb{K})$, where $\mathrm{co}\,\mathbb{K}$ denotes the convex hull of $\mathbb{K}$. We provide a necessary and sufficient condition for this equivalence, which is one of the main theoretical contributions of this paper.

For computational needs, we assume that the cone $\mathbb{K}$ is closed and convex in (1.3) and (1.4) in addition to Condition (I), as in the cases of DNN relaxation of QOP (1.2) or SDP relaxation of POP (1.1). In this case, we reduce the primal-dual pair of COPs (1.3) and (1.4) to an equivalent, but simpler pair of COPs:

$$\eta^p(\mathbb{K}) \;:=\; \inf \left\{ \langle \boldsymbol{Q}^0, \boldsymbol{X}\rangle \mid \langle \boldsymbol{H}^0, \boldsymbol{X}\rangle = 1, \langle \boldsymbol{H}^1, \boldsymbol{X}\rangle = 0, \boldsymbol{X} \in \mathbb{K} \right\} \tag{1.5}$$

$$\eta^d(\mathbb{K}) \quad := \quad \sup \left\{ y_0 \;\middle|\; \boldsymbol{Q}^0 + \boldsymbol{H}^1 y_1 - \boldsymbol{H}^0 y_0 \in \mathbb{K}^* \right\}, \tag{1.6}$$

where the homogeneous linear equality constraints $\langle \boldsymbol{Q}^k, \boldsymbol{X} \rangle = 0$ $(k = 1, \ldots, m)$ in (1.3) are combined into a single a homogeneous linear equality $\langle \boldsymbol{H}^1, \boldsymbol{X} \rangle = 0$ with $\boldsymbol{H}^1 = \sum_{k=1}^m \boldsymbol{Q}^k$. Applying the Lagrangian relaxation to the simplified COP (1.5), we then obtain the Lagrangian-conic relaxation of the COP (1.3) and its dual

$$\eta^p(\lambda, \mathbb{K}) \quad := \quad \inf \left\{ \langle \boldsymbol{Q}^0 + \lambda \boldsymbol{H}^1, \boldsymbol{X} \rangle \;\middle|\; \boldsymbol{X} \in \mathbb{K},\ \langle \boldsymbol{H}^0, \boldsymbol{X} \rangle = 1 \right\}, \tag{1.7}$$

$$\eta^d(\lambda, \mathbb{K}) \quad := \quad \sup \left\{ y_0 \;\middle|\; \boldsymbol{Q}^0 + \lambda \boldsymbol{H}^1 - \boldsymbol{H}^0 y_0 \in \mathbb{K}^* \right\}, \tag{1.8}$$

where $\lambda \in \mathbb{R}$ denotes the Lagrangian multiplier for the homogeneous equality $\langle \boldsymbol{H}^1, \boldsymbol{X} \rangle = 0$ in (1.5). As the relations between the pairs (1.3)-(1.4), (1.5)-(1.6) and (1.7)-(1.8) can be delicate, we summarize the most elegant case (when Condition (I) and Conditions (II)–(III) to be described later hold) among the relations between these three pairs below for the convenience of the reader:

$$\begin{array}{ccc} \eta^p(\lambda, \mathbb{K}) & \uparrow & \eta^p(\mathbb{K}) = \zeta^p(\mathbb{K}) \\ || & & || \\ \eta^d(\lambda, \mathbb{K}) & \uparrow & \eta^d(\mathbb{K}) = \zeta^d(\mathbb{K}), \end{array}$$

where $\eta^p(\lambda, \mathbb{K}) \uparrow \eta^p(\mathbb{K})$ means that $\eta^p(\lambda, \mathbb{K})$ converges monotonically to $\eta^p(\mathbb{K})$ as λ increases. We would like to emphasize that the primal-dual pair (1.7)-(1.8) above satisfies some nice properties which make it conducive for one to design effective algorithms to solve them. In particular:

(a) The common optimal value $\eta^p(\lambda, \mathbb{K}) = \eta^d(\lambda, \mathbb{K})$ serves as a lower bound for the optimal value $\zeta^d(\mathbb{K})$ of the original dual COP (or in the QOP case, the optimal value of the dual of the DNN relaxation of a nonconvex QOP), and it monotonically converges to $\zeta^d(\mathbb{K})$ as λ tends to ∞.

To compute $\eta^d(\lambda, \mathbb{K})$, we further reformulate (1.8) as a one-dimensional maximization problem with $\eta^d(\lambda, \mathbb{K}) := \sup \ \{y_0 \mid g_\lambda(y_0) = 0\}$. Here $g_\lambda : \mathbb{R} \to [0, \infty)$ is defined by $\|\Pi_{\mathbb{K}}(\boldsymbol{H}^0 y_0 - \lambda \boldsymbol{H}^1 - \boldsymbol{Q}^0)\|$, where $\Pi_{\mathbb{K}}(\cdot)$ denotes the metric projection onto $\mathbb{K}$, and it satisfies

(b) $g_\lambda : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function such that $g_\lambda(y_0) = 0$ if and only if $y_0 \le \eta^d(\lambda, \mathbb{K})$, and it is continuously differentiable, strictly increasing and convex in the interval $(\eta^d(\lambda, \mathbb{K}), \infty)$.

Property (a) ensures that solving (1.7) and/or (1.8) with a sufficiently large λ can generate a tight lower bound $\eta^d(\lambda, \mathbb{K})$ for $\zeta^d(\mathbb{K})$. Property (b) provides the theoretical support to design efficient numerical methods for computing $\eta^d(\lambda, \mathbb{K})$. In fact, it was property (b) that made it possible to apply a bisection method combined with first-order methods efficiently, stably, and effectively to solve the Lagrangian-DNN relaxation of QOPs in [22]. In addition to the bisection method, property (b) allows us to apply a globally convergent 1-dimensional Newton iteration from any initial point y_0 with $g_\lambda(y_0) > 0$ for computing $\eta^d(\lambda, \mathbb{K}) = \max\{y_0 : g_\lambda(y_0) = 0\}$. Furthermore the Newton iteration generates as a byproduct, a sequence of feasible solutions of (1.7) whose objective values monotonically tend to $\eta^p(\lambda, \mathbb{K}) = \eta^d(\lambda, \mathbb{K})$. As we shall see in Section 6, the numerical efficiency can further be improved if the method proposed in [16, 26] for exploiting structured sparsity in SDPs is incorporated into the Lagrangian-DNN relaxations for QOPs.

1.3 Related work

The Lagrangian-DNN relaxation of linearly constrained QOPs with complementarity constraints was proposed in [22] and an efficient method based on first-order algorithms was also designed to solve these Lagrangian-DNN relaxation problems. The Lagrangian-DNN relaxation was originally suggested in [3] as a numerically tractable method for approximately solving the NP-hard Lagrangian-CPP relaxation. The numerical results in [22] demonstrated that with an appropriately designed algorithm which is based on a bisection framework combined with the proximal alternating direction multiplier method [14] and the accelerated proximal gradient method [7], one can efficiently solve the Lagrangian-DNN relaxation of various classes of test problems, including maximum stable set and quadratic assignment problems.

1.4 Paper outline

In Section 2, we discuss three primal-dual pairs of COPs over a cone $\mathbb{K}$ (not necessarily convex nor closed). The first pair is the primal-dual COPs (1.3) and (1.4) with the objective values $\zeta^p(\mathbb{K})$ and $\zeta^d(\mathbb{K})$, and it will be used as a unified model to represent nonconvex QOPs and general POPs as well as their convex relaxations in the subsequent discussions. The second pair is the simplified COPs (1.5) and (1.6) with the objective values $\eta^p(\mathbb{K})$ and $\eta^d(\mathbb{K})$. It is equivalent to the first pair under the copositivity condition (Condition (I)). The third pair is the Lagrangian-conic relaxation (1.7) and (1.8) with the objective values $\eta^p(\lambda, \mathbb{K})$ and $\eta^p(\lambda, \mathbb{K})$. We investigate the relationships among their optimal values $\zeta^p(\mathbb{K})$, $\zeta^d(\mathbb{K})$, $\eta^p(\mathbb{K})$, $\eta^d(\mathbb{K})$, $\eta^p(\lambda, \mathbb{K})$ and $\eta^d(\lambda, \mathbb{K})$ in details.

In Section 3, the COP satisfying the copositive condition is considered for a nonconvex cone $\mathbb{K}$. We establish a necessary and sufficient condition for $\zeta^p(\mathbb{K}) = \zeta^p(\text{co}\,\mathbb{K})$. This identity indicates that the optimal value of the COP over the nonconvex cone $\mathbb{K}$ is attained by its convexification, *i.e.*, by replacing the nonconvex cone $\mathbb{K}$ by its convex hull co $\mathbb{K}$. The result in Section 3 is applied to QOP (1.2) in Section 4.2 and to POP (1.1) in Part II [5, Section 3].

In Section 4, we convert the dual Lagrangian-conic relaxation problem into a maximization problem involving the function $g_\lambda(y_0)$ in a single real variable y_0, and present some fundamental properties on the function g_λ for the bisection and 1-dimensional Newton methods to find the optimal solution efficiently.

In Section 5, we deal with a class of linearly constrained QOPs (1.2) with complementarity constraints, and derive some fundamental properties of their CPP and DNN relaxations. The results in this section are closely related to, but more general than, the ones obtained in [22] where the same class of QOPs was studied. In order to further improve the numerical efficiency of our method, in Section 6, we describe how to exploit structured sparsity in the DNN and the Lagrangian-DNN relaxations for QOP (1.2). Section 7 is devoted to concluding remarks.

2 Basic Analysis of the Unified Framework

We discuss fundamental properties of the three primal-dual paris of COPs introduced in Section 1, the primal-dual COPs (1.3)–(1.4), the simplified primal-dual COPs (1.5)–(1.6) and the Lagrangian conic relaxation of the primal COP and its dual (1.7)–(1.8), and the relationship between them. We note that the cone $\mathbb{K}$ involved there is not necessary convex.

2.1 Notation and symbols

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, and $\mathbb{Z}_+$ the set of nonnegative integers. We use the following notation and symbols throughout the paper.

$$\begin{aligned}
\mathbb{V} &= \text{a finite dimensional vector space endowed with an inner product} \\
&\quad \langle \boldsymbol{Q}, \boldsymbol{X} \rangle \text{ and a norm } \|\boldsymbol{X}\| = \sqrt{\langle \boldsymbol{X}, \boldsymbol{X} \rangle} \text{ for every } \boldsymbol{Q},\ \boldsymbol{X} \in \mathbb{V}, \\
\mathbb{K} &= \text{a nonempty (but not necessarily convex nor closed) cone in } \mathbb{V}, \\
&\quad \text{where } \mathbb{K} \text{ is called a cone if } \alpha \boldsymbol{X} \in \mathbb{K} \text{ for each } \boldsymbol{X} \in \mathbb{K} \text{ and } \alpha \geq 0, \\
\mathbb{L} &= \text{the subspace of } \mathbb{V} \text{ generated by } \mathbb{K}, \\
&\quad \text{(the minimal subspace of } \mathbb{V} \text{ that contains } \mathbb{K}), \\
\mathbb{K}^* &= \{\boldsymbol{Y} \in \mathbb{V} : \langle \boldsymbol{X}, \boldsymbol{Y} \rangle \geq 0 \text{ for every } \boldsymbol{X} \in \mathbb{K}\} \ \text{(the dual of } \mathbb{K}), \\
\text{co } \mathbb{K} &= \text{the convex hull of } \mathbb{K}, \\
\boldsymbol{H}^0,\ \boldsymbol{Q}^k &\in \mathbb{V}\ (k = 0, 1, 2, \ldots, m), \\
F(\mathbb{K}) &= \left\{ \boldsymbol{X} \in \mathbb{V} \ \middle|\ \boldsymbol{X} \in \mathbb{K},\ \langle \boldsymbol{H}^0, \boldsymbol{X} \rangle = 1,\ \langle \boldsymbol{Q}^k, \boldsymbol{X} \rangle = 0\ (k = 1, 2, \ldots, m) \right\}.
\end{aligned}$$

For illustration, we consider the following QOP.

Example 2.1.

$$\zeta^* = \inf\{\boldsymbol{x}^T \boldsymbol{Q}^0 \boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{R}^n_+,\ \boldsymbol{x}^T \boldsymbol{x} = 1,\ x_k x_{k+1} = 0\ (k = 1, 2, \ldots, n-1)\}. \tag{2.1}$$

Here $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ denotes a column vector variable, $\boldsymbol{x}^T$ the transposition of $\boldsymbol{x}$, $\boldsymbol{Q}^0 \in \mathbb{V} = \mathbb{S}^n$ (the linear space of $n \times n$ symmetric matrices). Let

$$\begin{aligned}
\boldsymbol{H}^0 &= \text{the } n \times n \text{ identity matrix}, \\
\boldsymbol{e}^k &= \text{the } k\text{th unit column coordinate vector of } \mathbb{R}^n\ (k = 1, 2, \ldots, n), \\
\boldsymbol{Q}^k &= \boldsymbol{e}^k (\boldsymbol{e}^{k+1})^T + \boldsymbol{e}^{k+1} (\boldsymbol{e}^k)^T \in \mathbb{S}^n\ (k = 1, 2, \ldots, n-1), \\
\boldsymbol{\Gamma} &= \{\boldsymbol{x}\boldsymbol{x}^T \mid \boldsymbol{x} \in \mathbb{R}^n_+\} \subset \mathbb{S}^n.
\end{aligned}$$

Then, we can rewrite QOP (2.1) as in the primal COP (1.3) with $\mathbb{K} = \boldsymbol{\Gamma}$, *i.e.*, $\zeta(\boldsymbol{\Gamma}) = \inf\{\langle \boldsymbol{Q}^0, \boldsymbol{X} \rangle \mid \boldsymbol{X} \in \boldsymbol{F}(\boldsymbol{\Gamma})\}$. We note that $\boldsymbol{\Gamma}$ forms a closed cone and is nonconvex when $n \geq 2$.

2.2 Combining the homogeneous equalities of (1.3) in a single equality

The following lemma shows the equivalence between the two primal-dual pairs, the primal-dual COPs (1.3)–(1.4) and the simplified primal-dual COPs (1.5)–(1.6) under Condition (I).

Lemma 2.1. *Suppose that Condition (I) is satisfied. Then, the following assertions hold.*

(i) $\boldsymbol{X} \in F(\mathbb{K})$ *if and only if*

$$\boldsymbol{X} \in \mathbb{K},\ \langle \boldsymbol{H}^0, \boldsymbol{X} \rangle = 1,\ \langle \boldsymbol{H}^1, \boldsymbol{X} \rangle = 0, \tag{2.2}$$

Hence, $F(\mathbb{K}) = \left\{\boldsymbol{X} \in \mathbb{K} : \langle \boldsymbol{H}^0, \boldsymbol{X} \rangle = 1,\ \langle \boldsymbol{H}^1, \boldsymbol{X} \rangle = 0\right\}$, *where* $\boldsymbol{H}^1 = \sum_{k=1}^m \boldsymbol{Q}^k$.

(ii) $\zeta^p(\mathbb{K}) = \eta^p(\mathbb{K})$.

(iii) $\zeta^d(\mathbb{K}) = \eta^d(\mathbb{K})$.

Proof. We prove (i) and (iii) since (ii) follows directly from (i). (i) The "only if" part follows from the definitions of $F(\mathbb{K})$ and $\boldsymbol{H}^1$. Assume that (2.2) holds. Thus,

$$0 \;=\; \langle \boldsymbol{H}^1,\, \boldsymbol{X}\rangle = \textstyle\sum_{k=1}^m \langle \boldsymbol{Q}^k,\, \boldsymbol{X}\rangle.$$

By Condition (I) and $\boldsymbol{X} \in \mathbb{K}$, we know that $\langle \boldsymbol{Q}^k,\, \boldsymbol{X}\rangle \geq 0$ $(k = 1, 2, \ldots, m)$. Therefore, $\langle \boldsymbol{Q}^k,\, \boldsymbol{X}\rangle = 0$ $(k = 1, 2, \ldots, m)$, and $\boldsymbol{X} \in F(\mathbb{K})$.

(iii) If (y_0, y_1) is a feasible solution of the simplified dual COP (1.6) with the objective value y_0, then $(z_0, z_1, \ldots, z_m) = (y_0, y_1, \ldots, y_1)$ is a feasible solution of the dual COP (1.4) with the same objective value. Conversely if $(z_0, z_1, \ldots, z_m)$ is a feasible solution of (1.4) with the objective value z_0, then

$$\begin{aligned}\mathbb{K}^* \quad &\ni \quad \boldsymbol{Q}^0 + \sum_{k=1}^m \boldsymbol{Q}^k z_k - \boldsymbol{H}^0 z_0 + \sum_{k=1}^m \boldsymbol{Q}^k \left(\max_j z_j - z_k\right) \text{ (by Condition (I))} \\ &= \quad \boldsymbol{Q}^0 + \left(\sum_{k=1}^m \boldsymbol{Q}^k\right) \max_j z_j - \boldsymbol{H}^0 z_0 = \boldsymbol{Q}^0 + \boldsymbol{H}^1 \max_j z_j - \boldsymbol{H}^0 z_0.\end{aligned}$$

Thus, $(y_0, y_1) = (z_0, \max_j z_j)$ is a feasible solution of (1.6) with the same objective value. Consequently, $\zeta^d(\mathbb{K}) = \eta^d(\mathbb{K})$ holds. □

If $\boldsymbol{H}^0$, $\boldsymbol{Q}^k$ $(k = 1, 2, \ldots, n-1)$ and $\boldsymbol{\Gamma}$ are given as in Example 2.1, Condition (I) is obviously satisfied with $\mathbb{K} = \boldsymbol{\Gamma}$. Hence all assertions (i), (ii) and (iii) in Lemma 2.1 hold with $\mathbb{K} = \boldsymbol{\Gamma}$.

2.3 Applying the Lagrangian relaxation to the simplified primal COP (1.5)

We now focus on the primal-dual pair (1.7)–(1.8), which have been derived as a Lagrangian relaxation of the simplified primal COP (1.5) and its dual.

Lemma 2.2. *Suppose that Condition (I) is satisfied. Then, the following assertions hold.*

(i) $\eta^d(\lambda, \mathbb{K}) \leq \eta^p(\lambda, \mathbb{K})$ *for every* $\lambda \in \mathbb{R}$.

(ii) $\eta^p(\lambda_1, \mathbb{K}) \leq \eta^p(\lambda_2, \mathbb{K}) \leq \eta^p(\mathbb{K})$ *if* $\lambda_1 < \lambda_2$.

(iii) $\eta^d(\lambda_1, \mathbb{K}) \leq \eta^d(\lambda_2, \mathbb{K}) \leq \eta^d(\mathbb{K})$ *if* $\lambda_1 < \lambda_2$, *and* $\lim_{\lambda\to\infty} \eta^d(\lambda, \mathbb{K}) = \eta^d(\mathbb{K})$.

Proof. Since the weak duality relation (i) is straightforward, we only prove assertions (ii) and (iii).

(ii) The first inequality follows from the inequality $\langle \boldsymbol{H}_1,\, \boldsymbol{X}\rangle \geq 0$ for every $\boldsymbol{X} \in \mathbb{K}$. To show the second inequality, suppose that $\boldsymbol{X} \in \mathbb{K}$ is a feasible solution of the simplified primal COP (1.5) with objective value $\langle \boldsymbol{Q}^0,\, \boldsymbol{X}\rangle$. Then, it is a feasible solution of the Lagrangian-conic relaxation (1.7) with the same objective value for any $\lambda \in \mathbb{R}$. Hence $\eta^p(\lambda_2, \mathbb{K}) \leq \eta^p(\mathbb{K})$.

(iii) Suppose that $\lambda_1 < \lambda_2$. If y_0 is a feasible solution of the dual of the Lagrangian-conic relaxation(1.8) with $\lambda = \lambda_1$, then it is a feasible solution of (1.8) with $\lambda = \lambda_2$ because $\boldsymbol{H}^1 \in \mathbb{K}^*$. This implies the first inequality. To show the second inequality, suppose that y_0 is a feasible solution of (1.8) with $\lambda = \lambda_2$. Then (y_0, y_1) with $y_1 = \lambda_2$ is a feasible solution of the simplified dual COP (1.6), and the second inequality follows. If (y_0, y_1) is a feasible solution of (1.6), then y_0 is a feasible solution of (1.8) with $\lambda = y_1$. Therefore, we obtain $\lim_{\lambda\to\infty} \eta^d(\lambda, \mathbb{K}) \geq \eta^d(\mathbb{K})$. □

2.4 Strong duality relations

We assume the following condition to discuss the strong duality between the Lagrangian-conic relaxation and its dual (1.7) and (1.8), and between the primal-dual COPs (1.3) and (1.4) in this subsection.

Condition (II) $\mathbb{K}$ is closed and convex.

Lemma 2.3. *Suppose that Conditions (I) and (II) are satisfied. Then, the following assertions hold.*

(i) $\eta^d(\lambda, \mathbb{K}) = \eta^p(\lambda, \mathbb{K})$ *for every* $\lambda \in \mathbb{R}$. *Moreover, if* $\eta^p(\lambda, \mathbb{K})$ *is finite, then* (1.8) *has an optimal solution with the objective value* $\eta^p(\lambda, \mathbb{K})$.

(ii) $\left(\eta^d(\lambda, \mathbb{K}) = \eta^p(\lambda, \mathbb{K})\right)\uparrow = \eta^d(\mathbb{K}) = \zeta^d(\mathbb{K})$. *Here* $\uparrow$ *means "increases monotonically as* $\lambda \to \infty$*".*

Proof. Assertion (ii) follows from assertion (i) and Lemma 2.2. Thus, we only have to show (i). Let $\lambda \in \mathbb{R}$ be fixed. We know by the weak duality that $\eta^d(\lambda, \mathbb{K}) \leq \eta^p(\lambda, \mathbb{K})$. By $F(\mathbb{K}) \neq \emptyset$ from Condition (I), we have $\eta^p(\lambda, \mathbb{K}) < \infty$. If $\eta^p(\lambda, \mathbb{K}) = -\infty$, then it is obvious that the equality holds. Thus, we assume that $\eta^p(\lambda, \mathbb{K})$ takes a finite value, and prove the assertion by the duality theorem. We notice, however, that $\mathbb{K}$ may not have an interior point with respect to $\mathbb{V}$. In this case, the standard duality theorem cannot be applied directly (see, for example, Theorem 4.2.1 in [27]). Let $\mathbb{L}$ denote the subspace of $\mathbb{V}$ generated by $\mathbb{K}$, *i.e.*, the minimal subspace of $\mathbb{V}$ that contains $\mathbb{K}$. Then $\mathbb{K}$ has an interior-point with respect to $\mathbb{L}$. Now, the Lagrangian-conic relaxation (1.7) and its dual (1.8) can be converted into conic optimization problems within the space $\mathbb{L}$:

$$\hat{\eta}^p(\lambda, \mathbb{K}) := \inf\left\{\langle \widehat{\boldsymbol{Q}}^0 + \lambda \widehat{\boldsymbol{H}}^1, \boldsymbol{X}\rangle \;\middle|\; \boldsymbol{X} \in \mathbb{K},\ \langle \widehat{\boldsymbol{H}}^0, \boldsymbol{X}\rangle = 1 \right\}, \tag{2.3}$$

$$\hat{\eta}^d(\lambda, \mathbb{K}) := \sup\left\{y_0 \mid \widehat{\boldsymbol{Q}}^0 + \lambda \widehat{\boldsymbol{H}}^1 - \widehat{\boldsymbol{H}}^0 y_0 \in \mathbb{K}^* \cap \mathbb{L}\right\}, \tag{2.4}$$

where $\widehat{\boldsymbol{Q}}^0$, $\widehat{\boldsymbol{H}}^0$ and $\widehat{\boldsymbol{H}}^1$ are the metric projections of $\boldsymbol{Q}^0$, $\boldsymbol{H}^0$ and $\boldsymbol{H}^1$ onto $\mathbb{L}$, respectively. Then all the coefficient matrices $\widehat{\boldsymbol{Q}}^0$, $\widehat{\boldsymbol{H}}^1$, $\widehat{\boldsymbol{H}}^0$ and the cone $\mathbb{K}$ of the problem (2.3) are included in the linear space $\mathbb{L}$, (2.3) and (2.4) form a primal-dual pair in the space $\mathbb{L}$. Since the identity $\langle \widehat{\boldsymbol{A}}, \boldsymbol{X}\rangle = \langle \boldsymbol{A}, \boldsymbol{X}\rangle$ holds if $\widehat{\boldsymbol{A}}$ is the metric projection of $\boldsymbol{A} \in \mathbb{V}$ onto $\mathbb{L}$ and $\boldsymbol{X} \in \mathbb{L}$, (1.7) is equivalent to (2.3). We also see that

$$\begin{aligned}&\boldsymbol{Q}^0 + \lambda \boldsymbol{H}^1 - \boldsymbol{H}^0 y_0 \in \mathbb{K}^* \\ &\qquad i.e., \langle \boldsymbol{Q}^0 + \lambda \boldsymbol{H}^1 - \boldsymbol{H}^0 y_0, \boldsymbol{X}\rangle \geq 0 \text{ for every } \boldsymbol{X} \in \mathbb{K}\end{aligned}$$

if and only if

$$\begin{aligned}&\widehat{\boldsymbol{Q}}^0 + \lambda \widehat{\boldsymbol{H}}^1 - \widehat{\boldsymbol{H}}^0 y_0 \in \mathbb{K}^* \cap \mathbb{L} \\ &\qquad i.e., \langle \widehat{\boldsymbol{Q}}^0 + \lambda \widehat{\boldsymbol{H}}^1 - \widehat{\boldsymbol{H}}^0 y_0, \boldsymbol{X}\rangle \geq 0 \text{ for every } \boldsymbol{X} \in \mathbb{K} \text{ and} \\ &\qquad\quad \widehat{\boldsymbol{Q}}^0 + \lambda \widehat{\boldsymbol{H}}^1 - \widehat{\boldsymbol{H}}^0 y_0 \in \mathbb{L}.\end{aligned}$$

Thus, (1.8) is equivalent to (2.4). It suffices to show by the duality theorem that $\hat{\eta}^p(\lambda, \mathbb{K}) = \hat{\eta}^d(\lambda, \mathbb{K})$. By $F(\mathbb{K}) \neq \emptyset$ from Condition (I), there exists an $\widehat{\boldsymbol{X}} \in \mathbb{K}$ such that $\langle \widehat{\boldsymbol{H}}^0, \widehat{\boldsymbol{X}}\rangle > 0$.

We can take such an $\widehat{\boldsymbol{X}}$ from the interior of $\mathbb{K}$ with respect to $\mathbb{L}$ because $\mathbb{L}$ is the subspace of $\mathbb{V}$ generated by $\mathbb{K}$. Then, $\widehat{\boldsymbol{X}}/\langle\widehat{\boldsymbol{H}}^0,\, \widehat{\boldsymbol{X}}\rangle$ is an interior feasible solution of (2.3). Recall that $\hat{\eta}^p(\lambda,\mathbb{K}) = \eta^p(\lambda,\mathbb{K})$ is assumed to be finite. By the duality theorem, the dual problem (2.4) (hence (1.8)) has an optimal solution with the objective value $\hat{\eta}^p(\lambda,\mathbb{K})$. □

The following lemma shows the difficulty of proving the strong duality for the primal-dual COPs (1.3)–(1.4) and the simplified primal-dual COPs (1.5)–(1.6) in the same way as in the proof above for the Lagrangian-conic relaxation and its dual (1.7)–(1.8) by the duality theorem.

Lemma 2.4. *Suppose that Conditions (I) and (II) are satisfied and that $F(\mathbb{K})$ is a proper subset of $\{\boldsymbol{X} \in \mathbb{K} : \langle \boldsymbol{H}^0,\, \boldsymbol{X}\rangle = 1\}$. Then, the feasible region $F(\mathbb{K})$ of the primal COP (1.3) (and (1.5)) contains no interior point of $\mathbb{K}$ with respect to $\mathbb{L}$ (= the subspace of $\mathbb{V}$ generated by $\mathbb{K}$).*

Proof. We assume that $F(\mathbb{K}) \neq \emptyset$ since otherwise the assertion is trivial. By Condition (I) and the assumption that $F(\mathbb{K})$ is a proper subset of $\{\boldsymbol{X} \in \mathbb{K} : \langle \boldsymbol{H}^0,\, \boldsymbol{X}\rangle = 1\}$, there exists $k \in \{1, 2, \ldots, m\}$ and $\boldsymbol{X} \in \mathbb{K}$ such that $\langle \boldsymbol{Q}^k,\, \boldsymbol{X}\rangle > 0$. Let $\widehat{\boldsymbol{Q}}^k$ be the metric projection of $\boldsymbol{Q}^k$ onto $\mathbb{L}$. Then, $\widehat{\boldsymbol{Q}}^k \in \mathbb{K}^* \cap \mathbb{L}$ and $\langle \widehat{\boldsymbol{Q}}^k,\, \boldsymbol{X}\rangle = \langle \boldsymbol{Q}^k,\, \boldsymbol{X}\rangle > 0$. Let $\widetilde{\boldsymbol{X}}$ be an arbitrary interior point of $\mathbb{K}$ with respect to $\mathbb{L}$. Then, there exists a positive number ϵ such that $\widetilde{\boldsymbol{X}} - \epsilon\widehat{\boldsymbol{Q}}^k$ remains in $\mathbb{K}$. Thus, $\langle \widehat{\boldsymbol{Q}}^k,\, \widetilde{\boldsymbol{X}} - \epsilon\widehat{\boldsymbol{Q}}^k\rangle \geq 0$. It follows that

$$\langle \boldsymbol{Q}^k,\, \widetilde{\boldsymbol{X}}\rangle \;=\; \langle \widehat{\boldsymbol{Q}}^k,\, \widetilde{\boldsymbol{X}}\rangle > \langle \widehat{\boldsymbol{Q}}^k,\, \widetilde{\boldsymbol{X}}\rangle - \epsilon\langle \widehat{\boldsymbol{Q}}^k,\, \widehat{\boldsymbol{Q}}^k\rangle \;= \langle \widehat{\boldsymbol{Q}}^k,\, \widetilde{\boldsymbol{X}} - \epsilon\widehat{\boldsymbol{Q}}^k\rangle \geq 0.$$

Therefore, any interior point of $\mathbb{K}$ with respect to $\mathbb{L}$ cannot be contained in $F(\mathbb{K})$. □

We need an additional condition to ensure the strong duality between the primal-dual COPs (1.3) and (1.4).

Condition (III) $\left\{\boldsymbol{X} \in F(\mathbb{K}) : \langle \boldsymbol{Q}^0,\, \boldsymbol{X}\rangle \leq \tilde{\zeta}\right\}$ is nonempty and bounded for some $\tilde{\zeta} \in \mathbb{R}$.

Lemma 2.5. *Suppose that Conditions (I), (II) and (III) are satisfied. Then, the following assertions hold.*

(i) $\lim_{\lambda\to\infty} \eta^p(\lambda,\mathbb{K}) = \eta^p(\mathbb{K})$.

(ii) $\left(\eta^d(\lambda,\mathbb{K}) = \eta^p(\lambda,\mathbb{K})\right)\!\uparrow\, = \eta^d(\mathbb{K}) = \zeta^d(\mathbb{K}) = \eta^p(\mathbb{K}) = \zeta^p(\mathbb{K})$.

Proof. Assertion (ii) follows from assertion (i) and Lemma 2.3, thus we only prove (i). We first show that the set $L(\lambda) = \{\boldsymbol{X} \in \mathbb{K} : \langle \boldsymbol{H}^0,\, \boldsymbol{X}\rangle = 1,\ \langle \boldsymbol{Q}^0 + \lambda\boldsymbol{H}^1,\, \boldsymbol{X}\rangle \leq \tilde{\zeta}\}$ is nonempty, closed, and bounded (hence $-\infty < \eta^p(\lambda,\mathbb{K})$) for every sufficiently large λ. The closedness of $L(\lambda)$ follows from Condition (II). By Conditions (I) and (III), we see that

$$\emptyset \neq \left\{\boldsymbol{X} \in F(\mathbb{K}) : \langle \boldsymbol{Q}^0,\, \boldsymbol{X}\rangle \leq \tilde{\zeta}\right\} \subset L(\lambda_2) \subset L(\lambda_1) \text{ if } 0 < \lambda_1 < \lambda_2.$$

Next, we show that $L(\lambda)$ is bounded for every sufficiently large $\lambda > 0$. Assume on the contrary that there exists a sequence $\left\{(\lambda^k, \boldsymbol{X}^k) \in \mathbb{R}_+ \times \mathbb{K}\right\}$ such that $\boldsymbol{X}^k \in L(\lambda^k)$, $0 < \lambda^k \to \infty$ and $0 < \|\boldsymbol{X}^k\| \to \infty$ as $k \to \infty$. Then, we have

$$\frac{\boldsymbol{X}^k}{\|\boldsymbol{X}^k\|} \in \mathbb{K},\ \langle \boldsymbol{H}^1,\, \frac{\boldsymbol{X}^k}{\|\boldsymbol{X}^k\|}\rangle \geq 0,\ \langle \boldsymbol{Q}^0,\, \frac{\boldsymbol{X}^k}{\|\boldsymbol{X}^k\|}\rangle \leq \frac{\tilde{\zeta}}{\|\boldsymbol{X}^k\|},$$

$$\langle \boldsymbol{H}^0, \frac{\boldsymbol{X}^k}{\|\boldsymbol{X}^k\|}\rangle = \frac{1}{\|\boldsymbol{X}^k\|} \text{ and } \langle \boldsymbol{Q}^0, \frac{\boldsymbol{X}^k}{\lambda^k\|\boldsymbol{X}^k\|}\rangle + \langle \boldsymbol{H}^1, \frac{\boldsymbol{X}^k}{\|\boldsymbol{X}^k\|}\rangle \leq \frac{\tilde{\zeta}}{\lambda^k\|\boldsymbol{X}^k\|}.$$

We may assume without loss of generality that $\boldsymbol{X}/\|\boldsymbol{X}^k\|$ converges to a nonzero $\boldsymbol{D} \in \mathbb{K}$. By taking the limit as $k \to \infty$, we obtain that

$$\boldsymbol{O} \neq \boldsymbol{D} \in \mathbb{K},\ \langle \boldsymbol{H}^0, \boldsymbol{D}\rangle = 0,\ \langle \boldsymbol{H}^1, \boldsymbol{D}\rangle = 0,\ \langle \boldsymbol{Q}^0, \boldsymbol{D}\rangle \leq 0.$$

Thus, if we choose an $\boldsymbol{X}$ from the set $\left\{\boldsymbol{X} \in F(\mathbb{K}) : \langle \boldsymbol{Q}^0, \boldsymbol{X}\rangle \leq \tilde{\zeta}\right\}$, then $\{\boldsymbol{X} + \mu\boldsymbol{D} : \mu \geq 0\}$ forms an unbounded ray contained in the set by Condition (II). This contradicts Condition (III). Therefore, we have shown that $L(\tilde{\lambda})$ is bounded for some sufficiently large $\tilde{\lambda} > 0$ and $\emptyset \neq L(\lambda) \subset L(\tilde{\lambda})$ for every $\lambda \geq \tilde{\lambda}$.

Let $\{\lambda^k \geq \tilde{\lambda}\}$ be a divergent sequence to ∞. Since the nonempty and closed level set $L(\lambda^k)$ is contained in a bounded set $L(\tilde{\lambda})$, the Lagrangian-conic relaxation (1.7) with each $\lambda = \lambda^k$ has an optimal solution $\boldsymbol{X}^k$ with the objective value $\eta^p(\lambda^k) = \langle \boldsymbol{Q}^0 + \lambda^k\boldsymbol{H}^1, \boldsymbol{X}^k\rangle$ in the level set $L(\tilde{\lambda})$. We may assume without loss of generality that $\boldsymbol{X}^k$ converges to some $\widetilde{\boldsymbol{X}} \in L(\tilde{\lambda})$. Since $\eta^p(\lambda^k, \mathbb{K}) \leq \eta^p(\mathbb{K})$ by Lemma 2.2, it follows that

$$\langle \boldsymbol{H}^0, \boldsymbol{X}^k\rangle = 1,\ \langle \frac{\boldsymbol{Q}^0}{\lambda^k} + \boldsymbol{H}^1, \boldsymbol{X}^k\rangle \leq \frac{\eta^p(\mathbb{K})}{\lambda^k},\ \langle \boldsymbol{H}^1, \boldsymbol{X}^k\rangle \geq 0,\ \langle \boldsymbol{Q}^0, \boldsymbol{X}^k\rangle \leq \eta^p(\mathbb{K}).$$

By taking the limit as $k \to \infty$, we obtain that

$$\widetilde{\boldsymbol{X}} \in \mathbb{K},\ \langle \boldsymbol{H}^0, \widetilde{\boldsymbol{X}}\rangle = 1,\ \langle \boldsymbol{H}^1, \widetilde{\boldsymbol{X}}\rangle = 0,\ \langle \boldsymbol{Q}^0, \widetilde{\boldsymbol{X}}\rangle \leq \eta^p(\mathbb{K}).$$

This implies that $\widetilde{\boldsymbol{X}}$ is an optimal solution of the problem (1.5), hence, $\langle \boldsymbol{Q}^0, \boldsymbol{X}^k\rangle$ converges to $\eta^p(\mathbb{K})$ as $k \to \infty$. We also see from

$$\langle \boldsymbol{Q}^0, \boldsymbol{X}^k\rangle \leq \eta^p(\lambda^k, \mathbb{K}) = \langle \boldsymbol{Q}^0 + \lambda^k\boldsymbol{H}^1, \boldsymbol{X}^k\rangle \leq \eta^p(\mathbb{K})$$

that $\eta^p(\lambda^k, \mathbb{K})$ converges to $\eta^p(\mathbb{K})$ as $k \to \infty$. Thus, we have shown assertion (i). □

Remark 2.2. The strong duality result in Lemma 2.5 can alternatively be established by incorporating the linear constraint $\langle \boldsymbol{H}^1, \boldsymbol{X}\rangle = 0$ into the cone $\mathbb{K}$ for the simplified primal COP (1.5). Define $\mathbb{M} = \{\boldsymbol{X} \in \mathbb{V} : \langle \boldsymbol{H}^1, \boldsymbol{X}\rangle = 0\}$ and $\widetilde{\mathbb{K}} = \mathbb{K} \cap \mathbb{M}$, and consider the following primal-dual pair:

$$\begin{aligned} \tilde{\eta}^p &= \inf \left\{ \langle \boldsymbol{Q}^0, \boldsymbol{X}\rangle \;\middle|\; \boldsymbol{X} \in \widetilde{\mathbb{K}},\ \langle \boldsymbol{H}^0, \boldsymbol{X}\rangle = 1 \right\}. \\ \tilde{\eta}^d &= \sup \left\{ y_0 \;\middle|\; \boldsymbol{Q}^0 - \boldsymbol{H}^0 y_0 \in \widetilde{\mathbb{K}}^* \right\}. \end{aligned}$$

By the same argument as in the proof of Lemma 2.3, we can prove that there is no duality gap between these problems, *i.e.*, $\tilde{\eta}^p = \tilde{\eta}^d$, and that if their common optimal value is finite, then the dual problem has an optimal solution. Some readers may find this proof to be more elegant than the one presented in Lemma 2.5. However, it should be mentioned that the dual problem is not equivalent to the simplified dual COP (1.6) although the primal problem is equivalent to the simplified primal COP (1.5). In fact, we know that $\widetilde{\mathbb{K}}^* = \mathrm{cl}(\mathbb{K}^* + \mathbb{M}^\perp)$ while (1.6) is equivalent to the dual problem above by replacing $\widetilde{\mathbb{K}}^*$ by $\mathbb{K}^* + \mathbb{M}^\perp$. But note that in general, $\mathbb{K}^* + \mathbb{M}^\perp$ may not be closed and may be a proper subset of $\widetilde{\mathbb{K}}^*$. Such an example was given in Section 3.3 of [3].

The following theorem summarizes the results in this section.

Theorem 2.6.

(i) $\eta^d(\lambda,\mathbb{K})\uparrow = \eta^d(\mathbb{K}) = \zeta^d(\mathbb{K}) \le \eta^p(\mathbb{K}) = \zeta^p(\mathbb{K})$ *and* $\left(\eta^d(\lambda,\mathbb{K}) \le \eta^p(\lambda,\mathbb{K})\right)\uparrow \le \eta^p(\mathbb{K})$ *under Condition (I).*

(ii) $\left(\eta^d(\lambda,\mathbb{K}) = \eta^p(\lambda,\mathbb{K})\right)\uparrow = \eta^d(\mathbb{K}) = \zeta^d(\mathbb{K}) \le \eta^p(\mathbb{K}) = \zeta^p(\mathbb{K})$ *under Conditions (I) and (II).*

(iii) $\left(\eta^d(\lambda,\mathbb{K}) = \eta^p(\lambda,\mathbb{K})\right)\uparrow = \eta^d(\mathbb{K}) = \zeta^d(\mathbb{K}) = \eta^p(\mathbb{K}) = \zeta^p(\mathbb{K})$ *under Conditions (I), (II) and (III).*

We cannot apply any of Lemmas 2.3, 2.4 and 2.5 to Example 2.1 since $\boldsymbol{\Gamma}$ is nonconvex there if $n \ge 2$. To present an example that satisfies the assumptions of the lemmas, we replace $\boldsymbol{\Gamma} = \{\boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x} \in \mathbb{R}^n_+\}$ by its convex hull co $\boldsymbol{\Gamma}$, which forms a completely positive cone. Then the resulting problem $\zeta(\text{co}\,\boldsymbol{\Gamma}) = \inf\{\langle \boldsymbol{Q}^0, \boldsymbol{X}\rangle \mid \boldsymbol{X} \in \boldsymbol{F}(\text{co}\,\boldsymbol{\Gamma})\}$, which we call a convexification of the problem with $\boldsymbol{\Gamma}$, satisfies not only Condition (II) and but also Condition (III) with $\mathbb{K} = \text{co}\,\boldsymbol{\Gamma}$. Hence all assertions in the lemmas are valid for the problem with $\mathbb{K} = \text{co}\,\boldsymbol{\Gamma}$. Since co $\boldsymbol{\Gamma} \supset \boldsymbol{\Gamma}$, $\zeta(\text{co}\,\boldsymbol{\Gamma}) \le \zeta(\boldsymbol{\Gamma})$ holds. In Section 3, whether $\zeta(\text{co}\,\boldsymbol{\Gamma}) = \zeta(\boldsymbol{\Gamma})$ holds is discussed in a more general setting. As co $\boldsymbol{\Gamma}$ is numerically intractable, a doubly nonnegative (DNN) cone $\mathbb{S}^n_+ \cap \mathbb{N}^n \supset \text{co}\boldsymbol{\Gamma}$ needs to be introduced to illustrate the convergence of $\eta^d(\lambda,\mathbb{K})$ to $\zeta^p(\mathbb{K})$ in Lemma 2.5. This leads to a DNN relaxation of the QOP in Example 2.1

$$\zeta^p(\mathbb{S}^n_+ \cap \mathbb{N}^n) \quad = \quad \left\{ \langle \boldsymbol{Q}^0,\, \boldsymbol{X}\rangle \;\middle|\; \begin{array}{l} \boldsymbol{X} \in \mathbb{S}^n_+ \cap \mathbb{N}^n,\ \langle \boldsymbol{H}^0,\, \boldsymbol{X}\rangle = 1, \\ \langle \boldsymbol{Q}^k,\, \boldsymbol{X}\rangle = 0 \ (k = 1, 2, \ldots, n) \end{array} \right\}.$$

Figure 1 displays the converging trend of computed $\eta^d(\lambda, \mathbb{S}^n_+ \cap \mathbb{N}^n)$ to $\zeta^p(\mathbb{S}^n_+ \cap \mathbb{N}^n)$ as λ increases. All the DNN problems were converted into the standard form of SDPs and were solved by SeDuMi [32]. We present numerical methods, including the bisection and projection method [22], for solving the the Lagrangian-conic relaxation and its dual (1.7) and (1.8) in Section 4, and discuss DNN and Lagrangian-DNN relaxations of a class of QOPs, which includes Example 2.1 as a special case, in Section 5. We refer to the paper [22] for extensive numerical results on the bisection and projection method applied to the class of QOPs.

3 Convexification

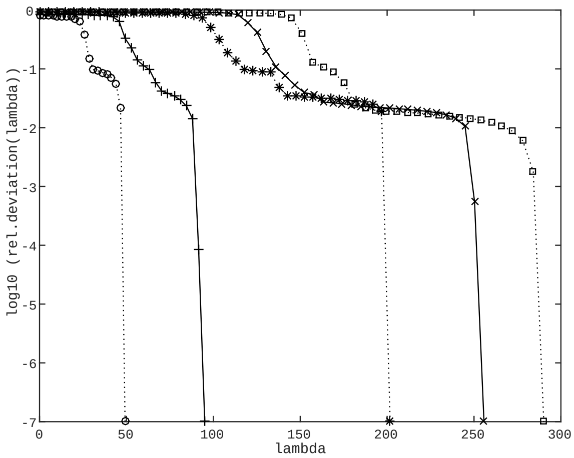
We focus on the primal COP (1.3) with a nonconvex cone $\boldsymbol{\Gamma}$ in $\mathbb{V}$ in this section. Assuming that Condition (I) holds for $\mathbb{K} = \boldsymbol{\Gamma}$, we derive a necessary and sufficient condition for the equivalence between the primal COP (1.3) with $\mathbb{K} = \boldsymbol{\Gamma}$ and the primal COP (1.3) with $\mathbb{K} = \text{co}\,\boldsymbol{\Gamma}$. We call the second COP a convexification of the first.

Since $\boldsymbol{\Gamma} \subset \text{co}\,\boldsymbol{\Gamma}$, we immediately see that $\zeta^p(\boldsymbol{\Gamma}) \ge \zeta^p(\text{co}\,\boldsymbol{\Gamma})$. Suppose that Condition (I) is satisfied for $\mathbb{K} = \boldsymbol{\Gamma}$. Then it also holds for $\mathbb{K} = \text{co}\,\boldsymbol{\Gamma}$. As a result, we can consistently define the simplified primal COP (1.5), the Lagrangian-conic relaxation (1.7) and their duals for $\mathbb{K} = \text{co}\,\boldsymbol{\Gamma}$, and all results established in Lemmas 2.1 and 2.2 remain valid for $\mathbb{K} = \text{co}\,\boldsymbol{\Gamma}$.

To characterize the convexification of the primal COP (1.3) with $\mathbb{K} = \boldsymbol{\Gamma}$, we introduce the following COP:

$$\zeta^p_0(\mathbb{K}) \quad = \quad \inf\ \left\{ \langle \boldsymbol{Q}^0,\, \boldsymbol{X}\rangle \;\middle|\; \boldsymbol{X} \in F_0(\mathbb{K}) \right\}, \tag{3.1}$$

Figure 1: The decrease of the relative deviation of $\eta^d(\lambda,\mathbb{K})$ with respect to $\zeta^p(\mathbb{K})$ to 0 as λ increases. Here $\mathbb{K}=\mathbb{S}^n_+ \cap \mathbb{N}^n$. The vertical axis indicates $\log_{10}\left((\zeta^p(\mathbb{K})-\eta^d(\lambda,\mathbb{K}))/\,|\zeta^p(\mathbb{K})|\right)$, and the horizontal axis λ. The coefficient matrix $\boldsymbol{Q}^0 \in \mathbb{S}^n$ of the QOP in Example 2.1 was chosen randomly and the dimension n was varied from 20 to 60. The lines $\cdots\circ\cdots$, —+—, $\cdots * \cdots$, —x— and $\cdots\Box\cdots$ correspond to $n = 20, 30, 40, 50$, and 60, respectively.



where $F_0(\mathbb{K}) = \left\{\boldsymbol{X}\in\mathbb{K} : \langle \boldsymbol{H}^0,\, \boldsymbol{X}\rangle = 0,\ \langle \boldsymbol{Q}^k,\, \boldsymbol{X}\rangle = 0\ (k=1,2,\ldots,m)\right\}$ and $\mathbb{K}$ denotes a cone in $\mathbb{V}$. We will assume Condition (I) and the following condition for $\mathbb{K}=\boldsymbol{\Gamma}$ to ensure that $\zeta^p(\text{co}\ \boldsymbol{\Gamma}) = \zeta^p(\boldsymbol{\Gamma})$ in Theorem 3.2.

Condition (IV) $\langle \boldsymbol{Q}^0,\, \boldsymbol{X}\rangle \geq 0$ for every $\boldsymbol{X}\in F_0(\mathbb{K})$.

Lemma 3.1. *Assume that Condition (I) holds. Then,*

$$\zeta^p_0(\mathbb{K}) = \zeta^p_0(co\ \mathbb{K}) = \begin{cases} 0 & \textit{if Condition (IV) holds,} \\ -\infty & \textit{otherwise.}\end{cases}$$

Proof. We first prove that Condition (IV) is equivalent to the condition

$$\langle \boldsymbol{Q}^0,\, \boldsymbol{X}\rangle \geq 0 \text{ for every } \boldsymbol{X}\in F_0(\text{co}\ \mathbb{K}). \tag{3.2}$$

Since the condition above implies Condition (IV), we only need to show that Condition (IV) implies the condition above. Assume that $\boldsymbol{X}\in F_0(\text{co}\ \mathbb{K})$. Then there are $\boldsymbol{X}^i\in\mathbb{K}$ $(i=1,2,\ldots,r)$ such that

$$\boldsymbol{X} = \textstyle\sum_{i=1}^r \boldsymbol{X}^i, \quad 0 \ = \ \langle \boldsymbol{H}^0,\, \boldsymbol{X}\rangle = \sum_{i=1}^r \langle \boldsymbol{H}^0,\, \boldsymbol{X}^i\rangle,$$

$$0 = \langle \boldsymbol{Q}^k,\, \boldsymbol{X}\rangle = \textstyle\sum_{i=1}^r \langle \boldsymbol{Q}^k,\, \boldsymbol{X}^i\rangle\ (k=1,2,\ldots,m).$$

By Condition (I), we know that $\langle \boldsymbol{H}^0,\, \boldsymbol{X}^i\rangle \geq 0$ and $\langle \boldsymbol{Q}^k,\, \boldsymbol{X}^i\rangle \geq 0$ $(i=1,2,\ldots,r,\ k=1,2,\ldots,m)$. Thus, each $\boldsymbol{X}^i$ $(i=1,2,\ldots,r)$ satisfies

$$\boldsymbol{X}^i\in\mathbb{K},\ \langle \boldsymbol{H}^0,\, \boldsymbol{X}^i\rangle = 0,\ \langle \boldsymbol{Q}^k,\, \boldsymbol{X}^i\rangle = 0\ (k=1,2,\ldots,m),$$

or $\boldsymbol{X}^i \in F_0(\mathbb{K})$ $(i = 1, 2, \ldots, r)$. By Condition (IV), $\langle \boldsymbol{Q}^0, \boldsymbol{X}\rangle = \sum_{i=1}^r \langle \boldsymbol{Q}^0, \boldsymbol{X}^i\rangle \geq 0$ holds.

Since the objective function of the problem (3.1) is linear and its feasible region forms a cone, we know that $\zeta_0^p(\mathbb{K}) = 0$ or $-\infty$ and that $\zeta_0^p(\mathbb{K}) = 0$ if and only if the objective value is nonnegative for all feasible solutions, *i.e.*, Condition (IV) holds. Similarly, $\zeta_0^p(\text{co } \mathbb{K}) = 0$ or $-\infty$, and $\zeta_0^p(\text{co } \mathbb{K}) = 0$ if and only if the condition (3.2), which has been shown to be equivalent to Condition (IV), holds. □

Before presenting the main result of this section, we show a simple illustrative example.

Example 3.1. Let $\mathbb{V} = \mathbb{R}^2$, $\boldsymbol{Q}^0 = (0, \alpha)$, $\boldsymbol{H}^0 = (1, 0)$, $m = 0$, and

$$\boldsymbol{\Gamma} = \left\{(x_1, x_2) \in \mathbb{R}^2_+ : x_2 - x_1 = 0 \text{ or } x_1 = 0\right\},$$

where α denotes a parameter to be specified. Then, the primal COP (1.3) with $\mathbb{K} = \boldsymbol{\Gamma}$ is of the form

$$\begin{aligned} \zeta^p(\boldsymbol{\Gamma}) &= \inf \left\{\alpha x_2 \;\middle|\; (x_1, x_2) \in \boldsymbol{\Gamma},\ \langle \boldsymbol{H}^0, (x_1, x_2)\rangle = x_1 = 1\right\} \\ &= \inf \left\{\alpha x_2 \;\middle|\; (x_1, x_2) \in \mathbb{R}^2_+,\ x_2 - x_1 = 0,\ x_1 = 1\right\}. \end{aligned}$$

Thus, the feasible region of the primal COP (1.3) with $\mathbb{K} = \boldsymbol{\Gamma}$ consists of a single point $\boldsymbol{x} = (1, 1)$. We further see that

$$\begin{aligned} \text{co } \boldsymbol{\Gamma} &= \left\{(x_1, x_2) \in \mathbb{R}^2_+ : x_2 - x_1 \geq 0\right\}, \\ \zeta_0^p(\boldsymbol{\Gamma}) &= \inf \left\{\alpha x_2 \;\middle|\; (x_1, x_2) \in \mathbb{R}^2_+,\ x_1 = 0\right\}, \\ \zeta^p(\text{co } \boldsymbol{\Gamma}) &= \inf \left\{\alpha x_2 \;\middle|\; (x_1, x_2) \in \mathbb{R}^2_+,\ x_2 - x_1 \geq 0,\ x_1 = 1\right\}. \end{aligned}$$

If $\alpha < 0$, then Condition (IV) is not satisfied for $\mathbb{K} = \boldsymbol{\Gamma}$, and $-\infty = \zeta^p(\text{co } \boldsymbol{\Gamma}) < \zeta^p(\boldsymbol{\Gamma}) = \alpha$. Otherwise, Condition (IV) is satisfied for $\mathbb{K} = \boldsymbol{\Gamma}$, and $\zeta^p(\text{co } \boldsymbol{\Gamma}) = \zeta^p(\boldsymbol{\Gamma}) = \alpha$.

Theorem 3.2. *Suppose that Condition (I) holds for* $\mathbb{K} = \boldsymbol{\Gamma}$. *Then,*

(i) $F(co\ \boldsymbol{\Gamma}) = co\ F(\boldsymbol{\Gamma}) + co\ F_0(\boldsymbol{\Gamma})$.

(ii) $\zeta^p(co\ \boldsymbol{\Gamma}) = \zeta^p(\boldsymbol{\Gamma}) + \zeta_0^p(\boldsymbol{\Gamma})$.

(iii) $\zeta^p(co\ \boldsymbol{\Gamma}) = \zeta^p(\boldsymbol{\Gamma})$ *if and only if Condition (IV) holds for* $\boldsymbol{\Gamma}$ *or* $\zeta^p(\boldsymbol{\Gamma}) = -\infty$.

Proof. (i) To show the inclusion $F(\text{co } \boldsymbol{\Gamma}) \subset \text{co } F(\boldsymbol{\Gamma}) + \text{co } F_0(\boldsymbol{\Gamma})$, assume that $\boldsymbol{X} \in F(\text{co } \boldsymbol{\Gamma})$. Then there exist $\boldsymbol{X}^i \in \boldsymbol{\Gamma}$ $(i = 1, 2, \ldots, r)$ such that

$$\boldsymbol{X} = \textstyle\sum_{i=1}^r \boldsymbol{X}^i,\ 1 = \langle \boldsymbol{H}^0, \boldsymbol{X}\rangle = \sum_{i=1}^r \langle \boldsymbol{H}^0, \boldsymbol{X}^i\rangle$$
$$0 = \langle \boldsymbol{Q}^k, \boldsymbol{X}\rangle = \textstyle\sum_{i=1}^r \langle \boldsymbol{Q}^k, \boldsymbol{X}^i\rangle \quad (k = 1, 2, \ldots, m).$$

Without loss of generality, we may assume $\boldsymbol{X}^r = \boldsymbol{O}$ for the consistency of the proof. Hence, I_0 defined in the following is nonempty. Let

$$\begin{aligned} & I_+ = \left\{i : \langle \boldsymbol{H}^0, \boldsymbol{X}^i\rangle > 0\right\},\ I_0 = \left\{j : \langle \boldsymbol{H}^0, \boldsymbol{X}^j\rangle = 0\right\}, \\ & \boldsymbol{Y} = \sum_{i \in I_+} \boldsymbol{X}^i,\ \lambda_i = \langle \boldsymbol{H}^0, \boldsymbol{X}^i\rangle,\ \boldsymbol{Y}^i = (1/\lambda_i)\boldsymbol{X}^i \in \boldsymbol{\Gamma}\ (i \in I_+), \\ & \boldsymbol{Z} = \sum_{j \in I_0} \boldsymbol{X}^j,\ \lambda_j = 1/|I_0|,\boldsymbol{Z}^j = (1/\lambda_j)\boldsymbol{X}^j \in \boldsymbol{\Gamma}\ (j \in I_0), \end{aligned}$$

where $|I_0|$ denotes the number of elements in I_0. Then $\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z}$, and

$$\begin{aligned} \lambda_i &> 0 \ (i \in I_+), \ 1 = \textstyle\sum_{i \in I_+} \lambda_i, \ \boldsymbol{Y} = \textstyle\sum_{i \in I_+} \lambda_i \boldsymbol{Y}^i, \\ \lambda_j &> 0 \ (j \in I_0), \ 1 = \textstyle\sum_{j \in I_0} \lambda_j, \ \boldsymbol{Z} = \textstyle\sum_{j \in I_0} \lambda_j \boldsymbol{Z}^j, \end{aligned}$$

which imply that $\boldsymbol{Y}$ and $\boldsymbol{Z}$ lie in the convex hull of $\boldsymbol{Y}^i$ $(i \in I_+)$ and $\boldsymbol{Z}^j$ $(j \in I_0)$, respectively. To see $\boldsymbol{Y}^i \in F(\boldsymbol{\Gamma})$ $(i \in I_+)$ and $\boldsymbol{Z}^j \in F_0(\boldsymbol{\Gamma})$ $(j \in I_0)$, we observe that

$$\begin{aligned} 1 &= \langle \boldsymbol{H}^0, \, (1/\lambda_i)\boldsymbol{X}^i \rangle = \langle \boldsymbol{H}^0, \, \boldsymbol{Y}^i \rangle \ \ (i \in I_+), \\ 0 &= \langle \boldsymbol{H}^0, \, (1/\lambda_j)\boldsymbol{X}^j \rangle = \langle \boldsymbol{H}^0, \, \boldsymbol{Z}^j \rangle \ \ (j \in I_0), \\ 0 &= \langle \boldsymbol{Q}^k, \, \boldsymbol{X} \rangle = \textstyle\sum_{i \in I_+} \lambda_i \langle \boldsymbol{Q}^k, \, \boldsymbol{Y}^i \rangle + \textstyle\sum_{j \in I_0} \lambda_j \langle \boldsymbol{Q}^k, \, \boldsymbol{Z}^j \rangle \ (k = 1, 2, \ldots, m). \end{aligned} \tag{3.3}$$

Since $\boldsymbol{Y}^i \in \mathbb{K}$ $(i \in I_+)$, $\boldsymbol{Z}^j \in \mathbb{K}$ $(j \in I_0)$ and $\boldsymbol{Q}^k \in \mathbb{K}^*$ $(k = 1, 2, \ldots, m)$ by Condition (I), we also see that $\langle \boldsymbol{Q}^k, \, \boldsymbol{Y}^i \rangle \geq 0$ $(i \in I_+)$ and $\langle \boldsymbol{Q}^k, \, \boldsymbol{Z}^j \rangle \geq 0$ $(j \in I_0)$. Hence, it follows from $\lambda_i > 0$ $(i \in I_+)$, $\lambda_j > 0$ $(j \in I_0)$ and the identity (3.3) that $\langle \boldsymbol{Q}^k, \, \boldsymbol{Y}^i \rangle = 0$ and $\langle \boldsymbol{Q}^k, \, \boldsymbol{Z}^j \rangle = 0$ $(i \in I_+, \ j \in I_0, \ k = 1, 2, \ldots, m)$. Thus we have shown that $\boldsymbol{Y}^i \in F(\boldsymbol{\Gamma})$ $(i \in I_+)$ and $\boldsymbol{Z}^j \in F_0(\boldsymbol{\Gamma})$ $j \in I_0$. Therefore $\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z} = \sum_{i \in I_+} \lambda_i \boldsymbol{Y}^i + \sum_{j \in I_0} \lambda_j \boldsymbol{Z}^j \in \text{co } F(\boldsymbol{\Gamma}) + \text{co } F_0(\boldsymbol{\Gamma})$.

To show the converse inclusion, suppose that $\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z}$ for some $\boldsymbol{Y} \in \text{co } F(\boldsymbol{\Gamma})$ and $\boldsymbol{Z} \in \text{co } F_0(\boldsymbol{\Gamma})$. Then we can represent $\boldsymbol{Y} \in \text{co } F(\boldsymbol{\Gamma})$ as

$$\begin{aligned} \boldsymbol{Y} \ &= \ \sum_{i=1}^{p} \lambda_i \boldsymbol{Y}^i, \ \sum_{i=1}^{p} \lambda_i = 1, \ \lambda_i > 0, \ \boldsymbol{Y}^i \in \boldsymbol{\Gamma}, \ \langle \boldsymbol{H}^0, \, \boldsymbol{Y}^i \rangle = 1, \ \langle \boldsymbol{Q}^k, \, \boldsymbol{Y}^i \rangle = 0 \\ &\quad\ (i = 1, 2, \ldots, p, \ k = 1, 2, \ldots, m), \end{aligned}$$

and $\boldsymbol{Z} \in \text{co } F_0(\boldsymbol{\Gamma})$ and

$$\begin{aligned} \boldsymbol{Z} \ &= \ \sum_{j=1}^{q} \lambda_i \boldsymbol{Z}^j, \ \sum_{j=1}^{q} \lambda_j = 1, \ \lambda_j > 0, \ \boldsymbol{Z}^j \in \boldsymbol{\Gamma}, \ \langle \boldsymbol{H}^0, \, \boldsymbol{Z}^j \rangle = 0, \ \langle \boldsymbol{Q}^k, \, \boldsymbol{Z}^j \rangle = 0 \\ &\quad\ (j = 1, 2, \ldots, q, \ k = 1, 2, \ldots, m). \end{aligned}$$

Since co $\boldsymbol{\Gamma}$ is a cone, it follows from $\sum_{i=1}^{p} \lambda_i \boldsymbol{Y}^i \in \text{co } \boldsymbol{\Gamma}$ and $\sum_{j=1}^{q} \lambda_j \boldsymbol{Z}^j \in \text{co } \boldsymbol{\Gamma}$ that $\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z} = \sum_{i=1}^{p} \lambda_i \boldsymbol{Y}^i + \sum_{j=1}^{q} \lambda_j \boldsymbol{Z}^j \in \text{co } \boldsymbol{\Gamma}$. We also see that

$$\begin{aligned} \langle \boldsymbol{H}^0, \, \boldsymbol{X} \rangle \ &= \ \textstyle\sum_{i=1}^{p} \lambda_i \langle \boldsymbol{H}^0, \, \boldsymbol{Y}^i \rangle + \textstyle\sum_{j=1}^{q} \lambda_j \langle \boldsymbol{H}^0, \, \boldsymbol{Z}^j \rangle = \textstyle\sum_{i=1}^{p} \lambda_i + 0 = 1 \\ \langle \boldsymbol{Q}^k, \, \boldsymbol{X} \rangle \ &= \ \textstyle\sum_{i=1}^{p} \lambda_i \langle \boldsymbol{Q}^k, \, \boldsymbol{Y}^i \rangle + \textstyle\sum_{j=1}^{q} \lambda_j \langle \boldsymbol{Q}^k, \, \boldsymbol{Z}^j \rangle = 0 \ (k = 1, 2, \ldots, m). \end{aligned}$$

Thus we have shown that $\boldsymbol{X} \in F(\text{co } \boldsymbol{\Gamma})$.

(ii) Since the objective function is linear, we see from (i)

$$\begin{aligned} \zeta^p(\text{co } \boldsymbol{\Gamma}) \ &= \ \inf \ \left\{ \langle \boldsymbol{Q}^0, \, \boldsymbol{X} \rangle \ | \ \boldsymbol{X} \in F(\text{co } \boldsymbol{\Gamma}) \right\} \\ &= \ \inf \ \left\{ \langle \boldsymbol{Q}^0, \, \boldsymbol{Y} + \boldsymbol{Z} \rangle \ | \ \boldsymbol{Y} \in \text{co } F(\boldsymbol{\Gamma}), \ \boldsymbol{Z} \in \text{co } F_0(\boldsymbol{\Gamma}) \right\} \\ &= \ \inf \ \left\{ \langle \boldsymbol{Q}^0, \, \boldsymbol{Y} \rangle + \langle \boldsymbol{Q}^0, \, \boldsymbol{Z} \rangle \ | \ \boldsymbol{Y} \in \text{co } F(\boldsymbol{\Gamma}), \ \boldsymbol{Z} \in \text{co } F_0(\boldsymbol{\Gamma}) \right\} \\ &= \ \inf \ \left\{ \langle \boldsymbol{Q}^0, \, \boldsymbol{Y} \rangle \ | \ \boldsymbol{Y} \in \text{co } F(\boldsymbol{\Gamma}) \right\} + \inf \ \left\{ \langle \boldsymbol{Q}^0, \, \boldsymbol{Z} \rangle \ | \ \boldsymbol{Z} \in \text{co } F_0(\boldsymbol{\Gamma}) \right\} \\ &= \ \zeta^p(\boldsymbol{\Gamma}) + \zeta^p_0(\boldsymbol{\Gamma}). \end{aligned}$$

(iii) By Condition (I) with $\mathbb{K} = \mathbf{\Gamma}$, $F(\mathbf{\Gamma})$ is nonempty. Hence we have $\zeta^p(\mathbf{\Gamma}) < \infty$. If $\zeta^p(\mathbf{\Gamma}) = -\infty$, then $\zeta^p(\text{co}\,\mathbf{\Gamma}) = \zeta^p(\mathbf{\Gamma}) = -\infty$. Now suppose that $\zeta^p(\mathbf{\Gamma})$ is finite. If Condition (IV) holds, then $\zeta^p_0(\mathbf{\Gamma}) = 0$ follows from Lemma 3.2. Hence $\zeta^p(\text{co}\,\mathbf{\Gamma}) = \zeta^p(\mathbf{\Gamma})$ by (ii). Conversely, if $\zeta^p(\text{co}\,\mathbf{\Gamma}) = \zeta^p(\mathbf{\Gamma})$, then $\zeta^p_0(\mathbf{\Gamma}) = 0$ follows from (ii), which implies that Condition (IV) holds by Lemma 3.2. Thus we have shown (iii). □

If $\boldsymbol{H}^0 \in \mathbb{S}^n$, $\boldsymbol{Q}^k \in \mathbb{S}^n$ $(k = 1, 2, \ldots, n-1)$ and $\mathbf{\Gamma} \subset \mathbb{S}^n$ are given as in Example 2.1, $\{\boldsymbol{O}\} \subset F_0(\mathbf{\Gamma}) \subset \{\boldsymbol{X} \in \mathbf{\Gamma} \mid \langle \boldsymbol{H}^0, \boldsymbol{X}\rangle = 0\} = \{\boldsymbol{O}\}$. Hence Conditions (I) and (IV) are satisfied. Hence $\zeta^p(\text{co}\,\mathbf{\Gamma}) = \zeta(\mathbf{\Gamma})$ by Lemma 3.1 and Theorem 3.2.

In [2, 4], the following condition was assumed for a COP of the form (1.3) with a nonconvex cone $\mathbb{K}$ induced from various classes of QOPs and POPs, in addition to Condition (I).

Condition (IV)′

$$F_0(\mathbb{K}) \quad \subset \quad F(\mathbb{K})^\infty \;=\left\{ \boldsymbol{D} \in \mathbb{V} : \begin{array}{l} \exists\, (\mu_r, \boldsymbol{X}_r) \in \mathbb{R}_+ \times F(\mathbb{K}) \ (r = 1, 2, \ldots,\ ) \\ \text{satisfying } (\mu_r, \mu_r \boldsymbol{X}_r) \to (0, \boldsymbol{D}) \text{ as } r \to \infty \end{array} \right\}$$

(the horizontal cone generated by $F(\mathbb{K})$)

Assume that Condition (I) holds for $\mathbb{K} = \mathbf{\Gamma}$. When $\zeta^p(\mathbb{K})$ is finite, Condition (IV) is a necessary and sufficient for the identity $\zeta^p(\mathbb{K}) = \zeta^p(\text{co}\,\mathbb{K})$, while Condition (IV)′ is merely a sufficient condition for the identity. Thus, Condition (IV)′ implies Condition (IV). In particular, if Condition (IV)′ holds, then Condition (IV) is satisfied for any $\boldsymbol{Q}^0 \in \mathbb{V}$. In fact, we can prove the following lemma which shows that Condition (IV) is much weaker than Condition (IV)′.

Lemma 3.3. *Suppose that $\zeta^p(\mathbb{K})$ is finite. Then, Condition (IV)′ implies Condition (IV) for any $\boldsymbol{Q}^0 \in \mathbb{V}$.*

Proof. Assume on the contrary that $\langle \boldsymbol{Q}^0, \boldsymbol{D}\rangle < 0$ for some $\boldsymbol{O} \neq \boldsymbol{D} \in F_0(\mathbb{K})$. By Condition (IV)′, there exists a sequence $\{(\mu_r, \boldsymbol{X}_r) \in \mathbb{R}_+ \times F(\mathbb{K}) : r = 1, 2, \ldots\}$ such that $(\mu_r, \mu_r \boldsymbol{X}_r)$ converges to $(0, \boldsymbol{D})$ as $r \to \infty$. Thus, there is a positive number δ such that $\langle \boldsymbol{Q}^0, \mu_r \boldsymbol{X}_r\rangle < -\delta$ for every sufficiently large r. Therefore, $\langle \boldsymbol{Q}^0, \boldsymbol{X}_r\rangle < -\delta/\mu_r \to -\infty$ along a sequence $\{\boldsymbol{X}_r : r = 1, 2, \ldots\}$ of feasible solutions of COP (1.3). But this contradicts the assumption that $\zeta^p(\mathbb{K})$ is finite. □

The results in this section are summarized as the following theorem.

Theorem 3.4. *$\zeta^p(\mathbb{K}) = \zeta^p(co\ \mathbb{K})$ under Conditions (I) and (IV).*

4 Numerical Methods for Solving the Primal-Dual Pair of the Lagrangian-Conic Relaxation (1.7) and (1.8)

In this section, we take $\mathbb{K}$ to be a closed convex cone. For every $\boldsymbol{G} \in \mathbb{V}$, let $\Pi(\boldsymbol{G})$ and $\Pi^*(\boldsymbol{G})$ denote the metric projection of $\boldsymbol{G}$ onto the cone $\mathbb{K}$ and $\mathbb{K}^*$, respectively:

$$\begin{aligned} \Pi(\boldsymbol{G}) &= \operatorname{argmin}\{\|\boldsymbol{G} - \boldsymbol{X}\| \mid \boldsymbol{X} \in \mathbb{K}\}, \\ \Pi^*(\boldsymbol{G}) &= \operatorname{argmin}\{\|\boldsymbol{G} - \boldsymbol{Z}\| \mid \boldsymbol{Z} \in \mathbb{K}^*\}. \end{aligned}$$

In addition to Conditions (I), (II) and (III), we assume the following condition throughout this section.

Condition (V) For every $\boldsymbol{G} \in \mathbb{V}$, $\Pi(\boldsymbol{G})$ can be computed.

Under these four conditions, we briefly present two types of numerical methods for solving the primal-dual pair of the Lagrangian-conic relaxation (1.7) and (1.8) with a fixed λ. The first is based on a bisection method, which was proposed in [22], and the second is an 1-dimensional Newton method, which is newly proposed in this paper.

Remark 4.1. When the unified framework is applied to QOPs in Section 5, and to POPs in Part II [5, Section 4], the cone $\mathbb{K}$ is given as the intersection of two closed convex cones $\mathbb{K}_1$ and $\mathbb{K}_2$ in the space of symmetric matrices. In such cases, we can adapt the accelerated proximal gradient method [7] to compute the metric projection onto $\mathbb{K} = \mathbb{K}_1 \cap \mathbb{K}_2$ based on those metric projections onto $\mathbb{K}_1$ and $\mathbb{K}_2$ individually as in Algorithm C of [22].

Let $\lambda \in \mathbb{R}$ be a fixed scalar. For every $y_0 \in \mathbb{R}$, define

$$\begin{aligned} \boldsymbol{G}_\lambda(y_0) &= \boldsymbol{Q}^0 + \lambda \boldsymbol{H}^1 - \boldsymbol{H}^0 y_0, \\ g_\lambda(y_0) &= \min\{\|\boldsymbol{G}_\lambda(y_0) - \boldsymbol{Z}\| \mid \boldsymbol{Z} \in \mathbb{K}^*\}, \\ \widehat{\boldsymbol{Z}}_\lambda(y_0) &= \Pi^*(\boldsymbol{G}_\lambda(y_0)), \quad \widehat{\boldsymbol{X}}_\lambda(y_0) = \Pi(-\boldsymbol{G}_\lambda(y_0)) \end{aligned}$$

By the decomposition theorem of Moreau [25], we know that

$$\widehat{\boldsymbol{Z}}_\lambda(y_0) - \widehat{\boldsymbol{X}}_\lambda(y_0) = \boldsymbol{G}_\lambda(y_0), \quad \langle \widehat{\boldsymbol{X}}_\lambda(y_0), \widehat{\boldsymbol{Z}}_\lambda(y_0)\rangle = 0. \tag{4.1}$$

Hence for every y_0,

$$g_\lambda(y_0) = \|\boldsymbol{G}_\lambda(y_0) - \widehat{\boldsymbol{Z}}_\lambda(y_0)\| = \|\widehat{\boldsymbol{X}}_\lambda(y_0)\|, \tag{4.2}$$

$$(g_\lambda(y_0))^2 = \|\widehat{\boldsymbol{X}}_\lambda(y_0)\|^2 = \langle -\boldsymbol{G}_\lambda(y_0), \widehat{\boldsymbol{X}}_\lambda(y_0)\rangle = \langle \boldsymbol{H}^0, \widehat{\boldsymbol{X}}_\lambda(y_0)\rangle y_0 - \langle \boldsymbol{Q}^0 + \lambda\boldsymbol{H}^1, \widehat{\boldsymbol{X}}_\lambda(y_0)\rangle. \tag{4.3}$$

By definition, $g_\lambda(y_0) \geq 0$ for all $y_0 \in \mathbb{R}$, and y_0 is a feasible solution of the dual of the Lagrangian-conic relaxation (1.8) if and only if $g_\lambda(y_0) = 0$. Therefore we can rewrite (1.8) as

$$\eta^d(\lambda, \mathbb{K}) := \sup \{y_0 \mid g_\lambda(y_0) = 0\}.$$

Thus we can easily design a standard bracketing and bisection method for computing $\eta^d(\lambda, \mathbb{K})$. We omit the details here but refer the reader to [22].

To describe the 1-dimensional Newton method for computing $\eta^d(\lambda, \mathbb{K})$, we need the following lemma, which exhibits some fundamental properties of the function g_λ.

Lemma 4.1. *Let $\lambda \in \mathbb{R}$ be fixed. Assume that $\eta^d(\lambda, \mathbb{K}) > -\infty$.*

(i) *$g_\lambda : \mathbb{R} \to \mathbb{R}_+$ is continuous and convex.*

(ii) *If $y_0 > \eta^d(\lambda, \mathbb{K})$, then $\langle \boldsymbol{H}^0, \widehat{\boldsymbol{X}}_\lambda(y_0)\rangle > 0$.*

(iii) *If $y_0 > \eta^d(\lambda, \mathbb{K})$, then $dg_\lambda(y_0)/dy_0 = \langle \boldsymbol{H}^0, \widehat{\boldsymbol{X}}_\lambda(y_0)\rangle / g_\lambda(y_0) > 0$; hence $g_\lambda : (\eta^d(\lambda, \mathbb{K}), \infty) \to \mathbb{R}$ is continuously differentiable and strictly increasing.*

(v) *Assume that $\boldsymbol{G}_\lambda(\bar{z}_0)$ lies in the interior of $\mathbb{K}^*$ for some $\bar{z}_0$. Then $\dfrac{g_\lambda(y_0) - g_\lambda(\eta^d(\lambda, \mathbb{K}))}{y_0 - \eta^d(\lambda, \mathbb{K})}$ converges to a positive value as $y_0 \downarrow \eta^d(\lambda, \mathbb{K})$; hence the right derivative of $g_\lambda(y_0)$ at $y_0 = \eta^d(\lambda, \mathbb{K})$ is positive.*

Proof. Consider the distance function $\theta(\boldsymbol{x}) = \min\{\|\boldsymbol{x}-\boldsymbol{y}\| : \boldsymbol{y} \in C\}$ from $\boldsymbol{x} \in \mathbb{V}$ to a closed convex subset C of $\mathbb{V}$ and the metric projection $\boldsymbol{P}(\boldsymbol{x}) = \operatorname{argmin}\{\|\boldsymbol{x}-\boldsymbol{y}\| : \boldsymbol{y} \in C\}$ of $\boldsymbol{x} \in \mathbb{V}$ onto C in general. It is well-known and also easily proved that θ is convex and continuous (see for example [20, 38]). It is also known that $\theta^2(\cdot)$ is continuously differentiable with $\nabla\theta^2(\boldsymbol{x}) = 2(\boldsymbol{x} - \boldsymbol{P}(\boldsymbol{x}))$ (see for example [31, Proposition 2.2]).

Since $g_\lambda(y_0) = \theta(\boldsymbol{G}_\lambda(y_0))$ and $\boldsymbol{G}_\lambda(y_0)$ is linear with respect to $y_0 \in \mathbb{R}$, the assertion (i) follows. In addition, we have that

$$\frac{dg_\lambda^2(y_0)}{dy_0} = 2\langle \boldsymbol{G}_\lambda(y_0) - \Pi^*(\boldsymbol{G}_\lambda(y_0)),\, -\boldsymbol{H}^0\rangle = 2\langle \widehat{\boldsymbol{X}}_\lambda(y_0),\, \boldsymbol{H}^0\rangle. \tag{4.4}$$

Next we prove assertion (ii) for $y_0 > \eta^d(\lambda,\mathbb{K})$. Note that by the definition of $\eta^d(\lambda,\mathbb{K})$, $g_\lambda(y_0) > 0$. Assume on the contrary that $\langle \boldsymbol{H}^0,\, \widehat{\boldsymbol{X}}_\lambda(y_0)\rangle = 0$. Then we see by (4.3) that

$$\langle \boldsymbol{Q}^0 + \lambda\boldsymbol{H}^1,\, \widehat{\boldsymbol{X}}_\lambda(y_0)\rangle = -\|\widehat{\boldsymbol{X}}_\lambda(y_0)\|^2 = -g_\lambda(y_0)^2 < 0.$$

Hence $\widehat{\boldsymbol{X}}_\lambda(y_0) \neq \boldsymbol{O}$ is a direction along which the objective function of (1.7) tends to $-\infty$. This contradicts to the assumption $-\infty < \eta^d(\lambda,\mathbb{K}) = \eta^p(\lambda,\mathbb{K})$. Therefore $\langle \boldsymbol{H}^0,\, \widehat{\boldsymbol{X}}_\lambda(y_0)\rangle > 0$.

We now prove assertion (iii). Again, note that $g_\lambda(y_0) > 0$ for $y_0 > \eta^d(\lambda,\mathbb{K})$. By (4.4), we get

$$\frac{dg_\lambda(y_0)}{dy_0} = \frac{\langle \widehat{\boldsymbol{X}}_\lambda(y_0),\, \boldsymbol{H}^0\rangle}{g_\lambda(y_0)} > 0.$$

From here, the remaining assertions follow.

Finally, we prove assertion (iv). Let L denote the line $\{\boldsymbol{G}_\lambda(y_0) : y_0 \in \mathbb{R}\}$ in $\mathbb{V}$. Then $L \cap \mathbb{K}^*$ forms a half-line with the end point $\boldsymbol{G}_\lambda(\eta^d(\lambda,\mathbb{K}))$. Since $\boldsymbol{G}_\lambda(\eta^d(\lambda,\mathbb{K}))$ is on the boundary of the closed convex cone $\mathbb{K}^*$, there exists a supporting hyperplane

$$T = \left\{\boldsymbol{Y} \in \mathbb{V} \;|\; \langle \boldsymbol{N},\, \boldsymbol{Y}\rangle = \left\langle \boldsymbol{N},\, \boldsymbol{G}_\lambda(\eta^d(\lambda,\mathbb{K}))\right\rangle\right\}$$

such that

$$\mathbb{K}^* \quad \subset \quad \left\{\boldsymbol{Y} \in \mathbb{V} \;|\; \langle \boldsymbol{N},\, \boldsymbol{Y}\rangle \leq \left\langle \boldsymbol{N},\, \boldsymbol{G}_\lambda(\eta^d(\lambda,\mathbb{K}))\right\rangle\right\}.$$

By the assumption, $\boldsymbol{G}_\lambda(\bar{z}_0)$ lies in the interior of $\mathbb{K}^*$. Hence

$$\langle \boldsymbol{N},\, \boldsymbol{G}_\lambda(\bar{z}_0)\rangle < \left\langle \boldsymbol{N},\, \boldsymbol{G}_\lambda(\eta^d(\lambda,\mathbb{K}))\right\rangle.$$

Geometrically, the line L transversally intersects with the supporting hyperplane T of the cone $\mathbb{K}^*$ at $\boldsymbol{G}_\lambda(\eta^d(\lambda,\mathbb{K}))$. It follows that

$$\langle \boldsymbol{N},\, \boldsymbol{G}_\lambda(\bar{z}_0)\rangle < \left\langle \boldsymbol{N},\, \boldsymbol{G}_\lambda(\eta^d(\lambda,\mathbb{K}))\right\rangle < \langle \boldsymbol{N},\, \boldsymbol{G}_\lambda(y_0)\rangle \text{ if } \bar{z}_0 < \eta^d(\lambda,\mathbb{K}) < y_0. \tag{4.5}$$

For every $y_0 \in \mathbb{R}$, define

$$h_\lambda(y_0) \quad = \quad \min\{\|\boldsymbol{G}_\lambda(y_0) - \boldsymbol{Z}\| \;|\; \boldsymbol{Z} \in T\}.$$

Then there exists a linear projection operator $\boldsymbol{P}$ from $\mathbb{V}$ onto the hyperplane T, and we see that

$$h_\lambda(y_0) \quad = \quad \|\boldsymbol{G}_\lambda(y_0) - \boldsymbol{P}\boldsymbol{G}_\lambda(y_0)\|$$

$$\begin{aligned}
&= \left\| \boldsymbol{G}_\lambda(y_0) - \boldsymbol{P}\left(\boldsymbol{G}_\lambda(\eta^d(\lambda,\mathbb{K})) + \boldsymbol{G}_\lambda(y_0) - \boldsymbol{G}_\lambda(\eta^d(\lambda,\mathbb{K}))\right)\right\| \\
&= \left\| \boldsymbol{G}_\lambda(y_0) - \boldsymbol{P}\boldsymbol{G}_\lambda(\eta^d(\lambda,\mathbb{K})) - \boldsymbol{P}\left(\boldsymbol{G}_\lambda(y_0) - \boldsymbol{G}_\lambda(\eta^d(\lambda,\mathbb{K}))\right)\right\| \\
&= \left\| \boldsymbol{H}^0(\eta^d(\lambda,\mathbb{K}) - y_0) - \boldsymbol{P}\boldsymbol{H}^0(\eta^d(\lambda,\mathbb{K}) - y_0)\right\| \\
&\quad \text{(because } \boldsymbol{G}_\lambda(\eta^d(\lambda,\mathbb{K})) \in \mathbb{K}^* \cap T \text{ and } \boldsymbol{P}\boldsymbol{G}_\lambda(\eta^d(\lambda,\mathbb{K})) = \boldsymbol{G}_\lambda(\eta^d(\lambda,\mathbb{K}))) \\
&= \left| y_0 - \eta^d(\lambda,\mathbb{K})\right| \left\| (\boldsymbol{I} - \boldsymbol{P})\boldsymbol{H}^0\right\|.
\end{aligned}$$

Here $\boldsymbol{I} - \boldsymbol{P}$ represents the linear projection operator from $\mathbb{V}$ onto $T^\perp = \{w\boldsymbol{N} \mid w \in \mathbb{R}\}$. By the above identities and (4.5), we obtain that

$$h_\lambda(y_0) = (y_0 - \eta^d(\lambda,\mathbb{K}))\left\|(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{H}^0\right\| > 0 \text{ for every } y_0 > \eta^d(\lambda,\mathbb{K}).$$

On the other hand, we know that

$$\begin{aligned}
g_\lambda(y_0) &= \min\{\|\boldsymbol{G}_\lambda(y_0) - \boldsymbol{Z}\| \mid \boldsymbol{Z} \in \mathbb{K}^*\} \\
&\geq h_\lambda(y_0) = \min\{\|\boldsymbol{G}_\lambda(y_0) - \boldsymbol{Z}\| \mid \boldsymbol{Z} \in T\} \text{ for every } y_0 > \eta^d(\lambda,\mathbb{K}).
\end{aligned}$$

To see this inequality, assume that $y_0 > \eta^d(\lambda,\mathbb{K})$ and $g_\lambda(y_0) = \left\|\boldsymbol{G}_\lambda(y_0) - \widetilde{\boldsymbol{Z}}\right\|$ for some $\widetilde{\boldsymbol{Z}} \in \mathbb{K}^*$. Then

$$\langle \boldsymbol{N}, \widetilde{\boldsymbol{Z}}\rangle \leq \left\langle \boldsymbol{N}, \boldsymbol{G}_\lambda(\eta^d(\lambda,\mathbb{K}))\right\rangle < \langle \boldsymbol{N}, \boldsymbol{G}_\lambda(y_0)\rangle.$$

Thus there is a $\boldsymbol{Z} \in \mathbb{V}$ on the line segment joining $\widetilde{\boldsymbol{Z}}$ and $\boldsymbol{G}_\lambda(y_0)$ such that $\langle \boldsymbol{N}, \boldsymbol{Z}\rangle = \left\langle \boldsymbol{N}, \boldsymbol{G}_\lambda(\eta^d(\lambda,\mathbb{K}))\right\rangle$, *i.e.*, $\boldsymbol{Z} \in T$. Therefore, we have shown that

$$g_\lambda(y_0) = \left\|\boldsymbol{G}_\lambda(y_0) - \widetilde{\boldsymbol{Z}}\right\| \geq \|\boldsymbol{G}_\lambda(y_0) - \boldsymbol{Z}\| \geq h_\lambda(y_0).$$

for every $y_0 > \eta^d(\lambda,\mathbb{K})$. Hence, for every $y_0 > \eta^d(\lambda,\mathbb{K})$,

$$\frac{g_\lambda(y_0) - g_\lambda(\eta^d(\lambda,\mathbb{K}))}{y_0 - \eta^d(\lambda,\mathbb{K})} \geq \frac{h_\lambda(y_0) - h_\lambda(\eta^d(\lambda,\mathbb{K}))}{y_0 - \eta^d(\lambda,\mathbb{K})} = \left\|(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{H}^0\right\| > 0.$$

(Note that $g_\lambda(\eta^d(\lambda,\mathbb{K})) = h_\lambda(\eta^d(\lambda,\mathbb{K})) = 0$ since $\boldsymbol{G}_\lambda(\eta^d(\lambda,\mathbb{K})) \in \mathbb{K}^* \cap T$.) Therefore, taking the limit as $\eta^d(\lambda,\mathbb{K}) < y_0 \to \eta^d(\lambda,\mathbb{K})$, we obtain the desired result. (We note that $\dfrac{g_\lambda(y_0) - g_\lambda(\eta^d(\lambda,\mathbb{K}))}{y_0 - \eta^d(\lambda,\mathbb{K})}$ decreases monotonically as $\eta^d(\lambda,\mathbb{K}) < y_0 \to \eta^d(\lambda,\mathbb{K})$ since $g_\lambda : \mathbb{R} \to \mathbb{R}$ is a continuous convex function). □

Remark 4.2. For the proof of (iv) of Lemma 4.1, the existence of a supporting hyperplane $T \subset \mathbb{V}$ of the cone $\mathbb{K}^*$ at $\boldsymbol{G}_\lambda(\eta^d(\lambda,\mathbb{K}))$ that transversally intersects the line $L = \{\boldsymbol{G}_\lambda(y_0) \mid y_0 \in \mathbb{R}\}$ is essential. Thus, the assumption of (iv) can be weaken to the existence of such a supporting hyperplane. Suppose that $\eta^d(\lambda,\mathbb{K})$ is finite. When the right derivative of g_λ at $\eta^d(\lambda,\mathbb{K})$ is positive and large, we can expect the numerical approximation of $\eta^d(\lambda,\mathbb{K})$ to be easier. Conversely, if it was zero, then accurate numerical approximation of $\eta^d(\lambda,\mathbb{K})$ would be more difficult. Even in this case, if we perturb $\boldsymbol{Q}^0$ to $\boldsymbol{Q}^0 + \boldsymbol{F}$ with any $\boldsymbol{F} \in \text{int}\,\mathbb{K}^*$, then

$$\begin{aligned}
\boldsymbol{Q}^0 + \boldsymbol{F} + \lambda\boldsymbol{H}^1 - \boldsymbol{H}^0\eta^d(\lambda,\mathbb{K}) &\in \text{int}\,\mathbb{K}^*, \\
\sup\left\{y_0 \mid \boldsymbol{Q}^0 + \boldsymbol{F} + \lambda\boldsymbol{H}^1 - \boldsymbol{H}^0 y_0 \in \mathbb{K}^*\right\} &\to \eta^d(\lambda,\mathbb{K}) \text{ as } \|\boldsymbol{F}\| \to 0.
\end{aligned}$$

Suppose that $g_\lambda(\bar{y}_0) > 0$ for some $\bar{y}_0 \in \mathbb{R}$. Then the Newton iteration for computing $\eta^d(\lambda, \mathbb{K})$ is given by

$$\begin{aligned} \bar{y}_0^+ &= \bar{y}_0 - \frac{g_\lambda(\bar{y}_0)}{dg_\lambda(\bar{y}_0)/dy_0} = \bar{y}_0 - \frac{\langle \widehat{\boldsymbol{X}}_\lambda(\bar{y}_0), \widehat{\boldsymbol{X}}_\lambda(\bar{y}_0)\rangle}{\langle \boldsymbol{H}^0, \widehat{\boldsymbol{X}}_\lambda(\bar{y}_0)\rangle} \\ &= \bar{y}_0 - \frac{\langle \widehat{\boldsymbol{Z}}(\bar{y}_0) - \boldsymbol{G}_\lambda(\bar{y}_0), \widehat{\boldsymbol{X}}_\lambda(\bar{y}_0)\rangle}{\langle \boldsymbol{H}^0, \widehat{\boldsymbol{X}}_\lambda(\bar{y}_0)\rangle} \quad \text{(by (4.1))} \\ &= \left\langle \boldsymbol{Q}^0 + \lambda \boldsymbol{H}^1, \widetilde{\boldsymbol{X}}_\lambda(\bar{y}_0) \right\rangle \geq \eta^p(\lambda, \mathbb{K}). \end{aligned}$$

where $\widetilde{\boldsymbol{X}}_\lambda(\bar{y}_0) = \widehat{\boldsymbol{X}}_\lambda(\bar{y}_0)/\langle \boldsymbol{H}^0, \widehat{\boldsymbol{X}}_\lambda(\bar{y}_0)\rangle$ denotes a feasible solution of the Lagrangian-conic relaxation (1.7). It should be noted that the sequence of y_0's generated by the Newton iteration monotonically decreases and converges to $\eta^d(\lambda, \mathbb{K})$ by (i), (ii) and (iii) of Lemma 4.1. Furthermore, if the assumption in (iv) of Lemma 4.1 is satisfied, then the generated sequence converges quadratically to $\eta^d(\lambda, \mathbb{K})$ (see page 451 of [28]). In general, the interior of $\mathbb{K}^*$ can be empty even under Conditions (I) through (V). When $\mathbb{K}$ is a copositive cone or a doubly nonnegative cone as assumed in Section 5, the interior of $\mathbb{K}^*$ is nonempty.

We may also use the relation

$$\frac{dg_\lambda(y_0)}{dy_0} = \frac{\left\langle \boldsymbol{H}^0, \widehat{\boldsymbol{X}}_\lambda(y_0) \right\rangle}{g_\lambda(y_0)} = \frac{\left\langle \boldsymbol{H}^0, \widehat{\boldsymbol{X}}_\lambda(y_0) \right\rangle}{\|\widehat{\boldsymbol{X}}_\lambda(y_0)\|} = \frac{1}{\|\widetilde{\boldsymbol{X}}_\lambda(\bar{y}_0)\|} > 0 \text{ for every } y_0 > \eta^d(\lambda, \mathbb{K})$$

for more stable and accurate computation of $\eta^d(\lambda, \mathbb{K})$. The details are omitted here.

5 Applications to a Class of Quadratic Optimization Problems

In this section, we demonstrate how to apply the unified framework described in the primal-dual COPs (1.3)–(1.4) to the class of linearly constrained QOP with complementarity constraints.

We take $\mathbb{V}$ to be the linear space of $(1+n) \times (1+n)$ symmetric matrices $\mathbb{S}^{1+n}$ with the inner product $\langle \boldsymbol{Q}, \boldsymbol{X}\rangle = \text{Trace}\, \boldsymbol{Q}^T\boldsymbol{X} = \sum_{i=0}^n \sum_{j=0}^n Q_{ij}X_{ij}$. We assume that the row and column indices of each matrix in $\mathbb{S}^{1+n}$ range from 0 to n. We are particularly interested in the following cones in the space of $\mathbb{S}^{1+n}$:

$$\begin{aligned} \mathbb{S}^{1+n}_+ &= \text{the cone of positive semidefinite matrices in } \mathbb{S}^{1+n}, \\ \mathbb{N}^{1+n} &= \left\{\boldsymbol{X} \in \mathbb{S}^{1+n} : X_{ij} \geq 0 \ (1 \leq i \leq j \leq 1+n)\right\}, \\ \mathbb{C}^{1+n} &= \left\{\boldsymbol{A} \in \mathbb{S}^{1+n} : \langle \boldsymbol{A}, \boldsymbol{x}\boldsymbol{x}^T\rangle \geq 0 \text{ for every } \boldsymbol{x} \in \mathbb{R}^{1+n}_+\right\} \\ &= \text{(the copositive cone)}, \\ (\mathbb{C}^{1+n})^* &= \left\{\sum_{i=1}^r \boldsymbol{x}_i\boldsymbol{x}_i^T : \boldsymbol{x}_i \in \mathbb{R}^{1+n}_+ \ (i = 1, 2, \ldots, r), \ r \in \mathbb{Z}_+\right\} \\ &= \text{(the compeletely positive (CPP) cone)}, \\ \mathbb{D}^{1+n} &= \mathbb{S}^{1+n}_+ \cap \mathbb{N}^{1+n} \text{ (the doubly nonnegative cone)}. \end{aligned}$$

5.1 Representation of QOP (1.2) as a COP over a nonconvex cone and its convexification

Recall the linearly constrained QOP (1.2) with complementarity constraints introduced in Section 1. We assume that the feasible region of QOP (1.2) is nonempty. To convert QOP (1.2) to COP (1.3), we define

$$
\begin{aligned}
\boldsymbol{Q}^0 &= \begin{pmatrix} 0 & \boldsymbol{c} \\ \boldsymbol{c} & \boldsymbol{Q} \end{pmatrix} \in \mathbb{S}^{1+n},\ \boldsymbol{Q}^1 = \begin{pmatrix} \boldsymbol{b}^T\boldsymbol{b} & \boldsymbol{b}^T\boldsymbol{A} \\ \boldsymbol{A}^T\boldsymbol{b} & \boldsymbol{A}^T\boldsymbol{A} \end{pmatrix} \in \mathbb{S}^{1+n}, \qquad (5.1)\\
\boldsymbol{C}^{ij} &= \text{the } n\times n \text{ matrix with 1 at the } (i,j)\text{th element, and 0 elsewhere } ((i,j)\in\mathcal{E}),\\
\boldsymbol{Q}^{ij} &= \begin{pmatrix} 0 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{C}^{ij}+(\boldsymbol{C}^{ij})^T \end{pmatrix} \in \mathbb{S}^{1+n}\ ((i,j)\in\mathcal{E}),\\
\boldsymbol{H}^0 &= \text{the } (1+n)\times(1+n) \text{ matrix whose } (0,0)\text{th element is 1, and 0 elsewhere.}\\
\boldsymbol{\Delta}_1 &= \left\{ \boldsymbol{U} = \begin{pmatrix} 1 \\ \boldsymbol{u} \end{pmatrix}\begin{pmatrix} 1 \\ \boldsymbol{u} \end{pmatrix}^T = \begin{pmatrix} 1 & \boldsymbol{u}^T \\ \boldsymbol{u} & \boldsymbol{u}\boldsymbol{u}^T \end{pmatrix} \in \mathbb{S}^{1+n} : \boldsymbol{u}\in\mathbb{R}^n_+ \right\}.
\end{aligned}
$$

We renumber the superscript ij of $\boldsymbol{Q}^{ij}$ $((i,j)\in\mathcal{E})$ to $2,\ldots,m$ for some m. Then, we can rewrite QOP (1.2) as follows:

$$
\zeta^* = \inf\left\{ \langle \boldsymbol{Q}^0, \boldsymbol{U}\rangle \;\middle|\; \boldsymbol{U}\in\boldsymbol{\Delta}_1,\ \langle \boldsymbol{Q}^k, \boldsymbol{U}\rangle = 0\ (k=1,2,\ldots,m) \right\}. \qquad (5.2)
$$

By definition, we know that

$$
\boldsymbol{O} \neq \boldsymbol{H}^0 \in \mathbb{S}^{1+n}_+ + \mathbb{N}^{1+n} = \left(\mathbb{D}^{1+n}\right)^*,\ \boldsymbol{Q}^k \in \left(\mathbb{D}^{1+n}\right)^*\ (k=1,2,\ldots,m). \qquad (5.3)
$$

We embed $\boldsymbol{\Delta}_1$ in a nonconvex cone by homogenizing $\boldsymbol{\Delta}_1$ as

$$
\boldsymbol{\Gamma} = \left\{ \boldsymbol{X} = \begin{pmatrix} x_0 \\ \boldsymbol{x} \end{pmatrix}\begin{pmatrix} x_0 \\ \boldsymbol{x} \end{pmatrix}^T = \begin{pmatrix} x_0^2 & x_0\boldsymbol{x}^T \\ x_0\boldsymbol{x} & \boldsymbol{x}\boldsymbol{x}^T \end{pmatrix} \in \mathbb{S}^{1+n} : \begin{pmatrix} x_0 \\ \boldsymbol{x} \end{pmatrix} \in \mathbb{R}^{1+n}_+ \right\}.
$$

Obviously, $\boldsymbol{\Gamma}$ forms a cone in $\mathbb{S}^{1+n}$.

Now, we consider the primal COP (1.3) for $\mathbb{K} = \boldsymbol{\Gamma}$ and co $\boldsymbol{\Gamma}$. The feasible region $F(\boldsymbol{\Gamma})$ of the first COP coincides with the feasible region of QOP (5.2), resulting in $\zeta^p(\boldsymbol{\Gamma}) = \zeta^*$. The second COP with a convex feasible region $F(\text{co}\,\boldsymbol{\Gamma})$ corresponds to a convexification of the first COP. In particular, co $\boldsymbol{\Gamma}$ coincides with *the completely positive programming* (CPP) cone $\left(\mathbb{C}^{1+n}\right)^*$ and the COP with $\mathbb{K} = \text{co}\,\boldsymbol{\Gamma} = \left(\mathbb{C}^{1+n}\right)^*$ is called *a completely positive programing* (CPP) relaxation of QOP (1.2) (or QOP (5.2)) [2,3,9,22]. Since co $\boldsymbol{\Gamma} \supset \boldsymbol{\Gamma}$, we have that

$$
\zeta^p(\text{co}\,\boldsymbol{\Gamma}) \le \zeta^*
$$

where ζ^* is the optimal value of the QOP (1.2).

Next we describe a condition that characterizes the equivalence between $\zeta^p(\text{co}\,\boldsymbol{\Gamma})$ and ζ^*. By construction, $\boldsymbol{\Gamma}$, co $\boldsymbol{\Gamma} \subset \mathbb{D}^{1+n}$. From the nonemptiness of the feasible region of QOP (1.2) and (5.3), we see that Condition (I) is satisfied for $\mathbb{K} = \boldsymbol{\Gamma}$ and $\mathbb{K} = \text{co}\,\boldsymbol{\Gamma}$. Thus, we can consistently define the simplified primal COP (1.5), the Lagrangian-conic relaxation (1.7) and their duals for $\mathbb{K} = \boldsymbol{\Gamma}$ and $\mathbb{K} = \text{co}\,\boldsymbol{\Gamma}$, and apply Lemmas 2.1 and 2.2.

To present main results of this section, we consider the following problem:

$$
\zeta_0^* = \inf\left\{ \boldsymbol{u}^T\boldsymbol{Q}\boldsymbol{u} \;\middle|\; \begin{array}{l} \boldsymbol{u}\in\mathbb{R}^n_+,\ \boldsymbol{A}\boldsymbol{u} = \boldsymbol{0}, \\ u_iu_j = 0\ ((i,j)\in\mathcal{E}) \end{array} \right\}. \qquad (5.4)
$$

The set of feasible solutions of this problem forms a cone in $\mathbb{R}^n_+$. Hence, we see that $\zeta_0^* = 0$ or $\zeta_0^* = -\infty$, and that $\zeta_0^* = 0$ if and only if

$$\boldsymbol{u}^T\boldsymbol{Q}\boldsymbol{u} \geq 0 \text{ for every feasible solution } \boldsymbol{u} \text{ of (5.4).} \tag{5.5}$$

Assume that the set $\{\boldsymbol{u} \in \mathbb{R}^n_+ : \boldsymbol{A}\boldsymbol{u}+\boldsymbol{b} = \boldsymbol{0}\}$ of vectors satisfying the linear constraints in QOP (1.2) is bounded. Then $\{\boldsymbol{u} \in \mathbb{R}^n_+ : \boldsymbol{A}\boldsymbol{u} = \boldsymbol{0}\} = \{\boldsymbol{0}\}$, which implies that the condition (5.5) holds trivially. Furthermore, for every cone $\mathbb{K}$ satisfying $\mathbb{K} \subset \mathbb{D}^{1+n}$, we see that

$$\begin{aligned} \boldsymbol{O} \;&\in\; \left\{\boldsymbol{X} \in \mathbb{K} : \langle \boldsymbol{H}^0,\, \boldsymbol{X}\rangle = 0,\ \langle \boldsymbol{Q}^k,\, \boldsymbol{X}\rangle = 0\ (k = 1, 2, \ldots, m)\right\} \\ &\subset\; \left\{\boldsymbol{X} \in \mathbb{D}^{1+n} : \langle \boldsymbol{H}^0,\, \boldsymbol{X}\rangle = 0,\ \langle \boldsymbol{Q}^k,\, \boldsymbol{X}\rangle = 0\ (k = 1, 2, \ldots, m)\right\} = \{\boldsymbol{O}\}, \end{aligned} \tag{5.6}$$

where the last identity of (5.6)) is proved in Lemma 2.1 of [22]. This implies that the feasible region $F(\mathbb{K})$ of COP (1.3) is bounded. As a result, Condition (III) holds.

Lemma 5.1.

(i) *Assume that* ζ^* *is finite. Then,*

$$\zeta^p(co\ \boldsymbol{\Gamma}) = \zeta^* + \zeta_0^* = \begin{cases} \zeta^* & \textit{if the condition } (5.5) \textit{ holds,} \\ -\infty & \textit{otherwise.} \end{cases}$$

(ii) *Assume that the set* $\{\boldsymbol{u} \in \mathbb{R}^n_+ : \boldsymbol{A}\boldsymbol{u} + \boldsymbol{b} = \boldsymbol{0}\}$ *is bounded and the feasible region of QOP* (1.2) *is nonempty. Then,*

$$\big(\eta^d(\lambda, co\ \boldsymbol{\Gamma}) = \eta^p(\lambda, co\ \boldsymbol{\Gamma})\big)\!\uparrow = \zeta^d(co\ \boldsymbol{\Gamma}) = \zeta^p(co\ \boldsymbol{\Gamma}) = \zeta^*.$$

Proof. (i) We apply (iii) of Lemma 3.2. Observe that $\boldsymbol{X} \in \boldsymbol{\Gamma}$ and $\langle \boldsymbol{H}^0,\, \boldsymbol{X}\rangle = 1$ if and only if $\boldsymbol{X} = \begin{pmatrix} 1 \\ \boldsymbol{u} \end{pmatrix}\begin{pmatrix} 1 \\ \boldsymbol{u} \end{pmatrix}^T$ for some $\boldsymbol{u} \in \mathbb{R}^n_+$. With this correspondence, we see that $\boldsymbol{u}^T\boldsymbol{Q}\boldsymbol{u} = \langle \boldsymbol{Q}^0,\, \boldsymbol{X}\rangle$, and that $\boldsymbol{u}$ is a feasible solution of (5.4) if and only if $\boldsymbol{X}$ is a feasible solution of (3.1) with $\mathbb{K} = \boldsymbol{\Gamma}$. Thus, $\zeta_0^* = \zeta_0^p(\boldsymbol{\Gamma})$ and the condition (5.5) corresponds to Condition (IV) with $\mathbb{K} = \boldsymbol{\Gamma}$. The desired result follows from (iii) of Theorem 3.2.

(ii) The CPP cone co $\boldsymbol{\Gamma} = (\mathbb{C}^{1+n})^*$ is known to be a closed convex cone. Thus Condition (II) holds for $\mathbb{K} =$ co $\boldsymbol{\Gamma}$. We have observed that the assumption implies that Condition (III) holds for $\mathbb{K} = \boldsymbol{\Gamma}$ and that F(co $\mathbb{K}$) is bounded. The desired result follows from (iii) of Theorem 2.6 and assertion (i). ☐

Remark 5.1. We can extend Lemma 5.1 to a more general QOP [2]

$$\zeta^* \;=\; \inf\left\{\boldsymbol{u}^T\boldsymbol{Q}\boldsymbol{u} + 2\boldsymbol{c}^T\boldsymbol{u} \;\middle|\; \begin{array}{l} \boldsymbol{u} \in D,\ \boldsymbol{A}\boldsymbol{u} + \boldsymbol{b} = \boldsymbol{0}, \\ u_iu_j = 0\ ((i,j) \in \mathcal{E}) \end{array}\right\}, \tag{5.7}$$

where D is a closed subset of $\mathbb{R}^n$. If the definition of the cone $\boldsymbol{\Gamma}$ is replaced by

$$\boldsymbol{\Gamma} \;=\; \left\{\boldsymbol{X} = x_0\begin{pmatrix} 1 \\ \boldsymbol{x} \end{pmatrix}\begin{pmatrix} 1 \\ \boldsymbol{x} \end{pmatrix}^T = \begin{pmatrix} x_0 & x_0\boldsymbol{x}^T \\ x_0\boldsymbol{x} & x_0\boldsymbol{x}\boldsymbol{x}^T \end{pmatrix} \in \mathbb{S}^{1+n} : \begin{pmatrix} x_0 \\ \boldsymbol{x} \end{pmatrix} \in \mathbb{R}_+ \times D\right\},$$

then QOP (5.7) can be reformulated as the primal COP (1.3) with $\mathbb{K} = \boldsymbol{\Gamma}$. If we assume that $u_iu_j \geq 0$ $((i,j) \in \mathcal{E})$ for every $\boldsymbol{u} \in D$, the copositivity condition in Condition (I) is satisfied and Lemma 5.1 remains valid. In [2], the equivalence of QOP (5.7) to its convexification, *i.e.*, $\zeta^p(\mathbf{\Gamma}) = \zeta^p(\text{co}\,\mathbf{\Gamma})$, was discussed as an extension of Burer's CPP reformulation [9] of a class of QOPs with linear and binary constraints under a hierarchy of copositivity condition. Dickinson, Eichfelder and J. Povh [12] (see also [13]) extended Burer's CPP reformulation [9] to QOPs with an additional nonconvex constraint $\boldsymbol{u} \in D$. Since we can reduce their QOPs to QOP (5.7) by introducing slack variables and replacing the binary constraint with the complementarity constraint, we can also derive their extension from Lemma 3.2. See Remarks 2.1 and Section 5 of [2] for more details.

5.2 DNN and Lagrangian-DNN relaxations

In this subsection, we present the DNN and Lagrangian-DNN relaxations of QOP (5.2). First, we let $\mathbb{K}$ be *the doubly nonnegative* cone $\mathbb{D}^{1+n} = \mathbb{S}^{1+n}_+ \cap \mathbb{N}^{1+n}$. Note the relation

$$\text{co}\,\mathbf{\Gamma} \;=\; \left(\mathbb{C}^{1+n}\right)^* \subset \mathbb{D}^{1+n} \subset \mathbb{S}^{1+n}_+ + \mathbb{N}^{1+n} = \left(\mathbb{D}^{1+n}\right)^* \subset \mathbb{C}^{1+n}.$$

Hence, $\zeta^p(\mathbb{D}^{1+n}) \leq \zeta^p(\left(\mathbb{C}^{1+n}\right)^*) \leq \zeta^*$. By (5.3), Condition (I) is satisfied for $\mathbb{K} = \mathbb{D}^{1+n}$. As a result, we can introduce the simplified primal COP (1.5), the Lagrangian-conic relaxation (1.7) and their duals for $\mathbb{K} = \mathbb{D}^{1+n}$.

Lemma 5.2.

(i) *Assume that the set* $\{\boldsymbol{u} \in \mathbb{R}^n_+ : \boldsymbol{A}\boldsymbol{u} + \boldsymbol{b} = \boldsymbol{0}\}$ *is bounded. Then,*

$$\zeta^* \;\geq\; \zeta^d(\mathbb{D}^{1+n}) = \zeta^p(\mathbb{D}^{1+n}) \;=\; \left(\eta^d(\lambda, \mathbb{D}^{1+n}) = \eta^p(\lambda, \mathbb{D}^{1+n})\right)\uparrow$$

(ii) *Assume that the condition* (5.5) *does not hold. Then,* $\zeta^p(\mathbb{D}^{1+n}) = -\infty$.

Proof. (i) Since Conditions (I), (II) and (III) hold for $\mathbb{K} = \mathbb{D}^{1+n}$, the desired result follows from Theorem 2.6. Note that Condition (III) was verified in the paragraph above Lemma 5.1.

(ii) Since co $\mathbf{\Gamma} \subset \mathbb{D}^{1+n}$, we know that $\zeta^* \geq \zeta^p(\text{co}\,\mathbf{\Gamma}) \geq \zeta^p(\mathbb{D}^{1+n})$. Hence $\zeta^p(\mathbb{D}^{1+n}) = -\infty$ follows from (i) of Lemma 5.1. □

We can apply all the discussions in Section 4 to the Lagrangian-conic relaxation and its dual (1.7)–(1.8) with $\mathbb{K} = \mathbb{D}^{1+n}$. In particular, the computation of $\eta^d(\lambda, \mathbb{D}^{1+n})$ is reduced to the simple problem: $\eta^d(\lambda, \mathbb{D}^{1+n}) = \sup\{y_0 : g_\lambda(y_0) = 0\}$, where $g_\lambda : \mathbb{R} \to \mathbb{R}_+$ is defined as the norm of the metric projection of $-(\boldsymbol{Q}^0 + \lambda \boldsymbol{H}^1 - \boldsymbol{H}^0 y_0)$ onto $\mathbb{D}^{1+n}$. To compute $\eta^d(\lambda, \mathbb{D}^{1+n})$, we can employ the numerical methods such as the bisection method and the 1-dimensional Newton method described in Section 4. In fact, Kim, Kojima and Toh [22] implemented the bisection method combined with the proximal alternating direction multiplier method [14] and the accelerated proximal gradient method [7], for solving the Lagrangian-conic relaxation and its dual (1.7)–(1.8) arising from QOPs of the form (1.2). It was assumed in [22] that the linear constraint set $\{\boldsymbol{u} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{u} + \boldsymbol{b} = \boldsymbol{0}\}$ is bounded. Thus, by Lemma 5.2, a common optimal value $\eta^p(\lambda, \mathbb{D}^{1+n}) = \eta^d(\lambda, \mathbb{D}^{1+n})$ of the Lagrangian-conic relaxation and its dual (1.7)–(1.8) converges to the optimal value $\zeta^p(\mathbb{D}^{1+n})$ of the DNN relaxation of QOP (1.2). In addition, the primal Lagrangian-conic relaxation problem, the Lagrangian-conic relaxation (1.7) with $\mathbb{K} = \mathbb{D}^{1+n}$ is strictly feasible (*i.e.*, its feasible region intersect with the interior of the DNN cone). These properties contributed to the effectiveness, efficiency, and stability of their numerical method. Notice that Assertion (ii) of Lemma 5.2 shows that their method will not work if condition (5.5) does not hold.

6 Exploiting Sparsity in the DNN and Lagrangian-DNN Relaxations for QOP (1.2)

As another demonstration of the usefulness and power of the unified framework introduced through the primal-dual COPs (1.5)–(1.6) and their Lagrangian-conic relaxations (1.7)–(1.8), here we show how to apply the framework to derive sparse DNN and sparse Lagrangian-DNN relaxations for the QOP (1.2).

6.1 Notation, symbols and basics

Let $N_0 = \{0, 1, \ldots, n\}$. We say that a subset $\mathcal{G} \subset N_0 \times N_0$ is symmetric if $(i, j) \in \mathcal{G}$ implies $(j, i) \in \mathcal{G}$. For every symmetric subset $\mathcal{G}$ of $N_0 \times N_0$ and every cone $\mathbb{J} \subset \mathbb{S}^{1+n}$, let

$$\begin{aligned}
\mathcal{G}^c &= \{(i, j) \in N_0 \times N_0 : (i, j) \notin \mathcal{G}\}, \\
\mathbb{S}^{1+n}(\mathcal{G}, 0) &= \left\{\boldsymbol{X} \in \mathbb{S}^{1+n} : X_{ij} = 0 \text{ if } (i, j) \notin \mathcal{G}\right\}, \\
\mathbb{J}(\mathcal{G}, 0) &= \mathbb{J} \cap \mathbb{S}^{1+n}(\mathcal{G}, 0), \\
\mathbb{J}(\mathcal{G}, ?) &= \mathbb{J} + \mathbb{S}^{1+n}(\mathcal{G}^c, 0) \;=\; \left\{\boldsymbol{X} \in \mathbb{S}^{1+n} : X_{ij} = \bar{X}_{ij} \ ((i, j) \in \mathcal{G}) \text{ for some } \bar{\boldsymbol{X}} \in \mathbb{J}\right\}.
\end{aligned}$$

Obviously, $\mathbb{S}^{1+n}(\mathcal{G}, 0)$ forms a linear subspace of $\mathbb{S}^{1+n}$. $\mathbb{J}(\mathcal{G}, 0)$ and $\mathbb{J}(\mathcal{G}, ?)$ are cones in $\mathbb{S}^{1+n}$, and

$$\mathbb{S}^{1+n}(\mathcal{G}, 0)^{\perp} = \mathbb{S}^{1+n}(\mathcal{G}, 0)^* = \mathbb{S}^{1+n}(\mathcal{G}^c, 0) \ \text{ and } \mathbb{J}(\mathcal{G}, 0) \subset \mathbb{J} \subset \mathbb{J}(\mathcal{G}, ?). \tag{6.1}$$

We denote the dual of $\mathbb{J}(\mathcal{G}, 0)$ by $\mathbb{J}(\mathcal{G}, 0)^*$.

Lemma 6.1. *Let $\mathcal{G}$ be a symmetric subset $N_0 \times N_0$ and $\mathbb{J}$ a cone in $\mathbb{S}^{1+n}$. Then, the following assertions hold.*

(i) $\mathbb{J}(\mathcal{G}, ?)^* = \mathbb{J}^* \cap \mathbb{S}^{1+n}(\mathcal{G}, 0)$.

(ii) *Moreover, if $\mathbb{J}$ is a closed convex cone, then* $\left(\mathbb{J}^* \cap \mathbb{S}^{1+n}(\mathcal{G}, 0)\right)^* = cl(\mathbb{J}(\mathcal{G}, ?))$.

Proof. (i) Suppose that $\boldsymbol{X} \in \mathbb{J}^* \cap \mathbb{S}^{1+n}(\mathcal{G}, 0)$. Then, for every $\boldsymbol{Y} + \boldsymbol{Z} \in \mathbb{J}(\mathcal{G}, ?)$ with $\boldsymbol{Y} \in \mathbb{J}$ and $\boldsymbol{Z} \in \mathbb{S}^{1+n}(\mathcal{G}^c, 0)$, we see that $\langle \boldsymbol{X}, \boldsymbol{Y}+\boldsymbol{Z}\rangle = \langle \boldsymbol{X}, \boldsymbol{Y}\rangle + \langle \boldsymbol{X}, \boldsymbol{Z}\rangle \geq 0$. Hence, $\boldsymbol{X} \in \mathbb{J}(\mathcal{G}, ?)^*$. Now suppose that $\boldsymbol{X} \in \mathbb{S}^{1+n}$ and $\boldsymbol{X} \notin \mathbb{J}^* \cap \mathbb{S}^{1+n}(\mathcal{G}, 0)$. Then we have either

$$\boldsymbol{X} \notin \mathbb{J}^* \ \text{ and } \boldsymbol{X} \in \mathbb{S}^{1+n}(\mathcal{G}, 0) \tag{6.2}$$

$$\text{or} \quad \boldsymbol{X} \notin \mathbb{S}^{1+n}(\mathcal{G}, 0), \ \textit{i.e.}, \ X_{ij} \neq 0 \text{ for some } (i, j) \in \mathcal{G}^c. \tag{6.3}$$

In the case of (6.2), there exists a $\boldsymbol{Y} \in \mathbb{J} \subset \mathbb{J}(\mathcal{G}, ?)$ such that $\langle \boldsymbol{X}, \boldsymbol{Y}\rangle < 0$. Thus, $\boldsymbol{X} \notin \mathbb{J}(\mathcal{G}, ?)^*$. For the case of (6.3), let $\boldsymbol{Y} \in \mathbb{S}^{1+n}$ be such that

$$Y_{ij} = \begin{cases} 0 & \text{if } (i, j) \in \mathcal{G}, \\ -X_{ij} & \text{if } (i, j) \in \mathcal{G}^c. \end{cases}$$

Then, $\boldsymbol{Y} \in \mathbb{J} + \mathbb{S}^{1+n}(\mathcal{G}^c, 0) = \mathbb{J}(\mathcal{G}, ?)$ and

$$\langle \boldsymbol{X}, \boldsymbol{Y}\rangle \;=\; \textstyle\sum_{(i,j)\in\mathcal{G}^c} X_{ij} Y_{ij} = \sum_{(i,j)\in\mathcal{G}^c} X_{ij}(-X_{ij}) = -\sum_{(i,j)\in\mathcal{G}^c} X_{ij}^2 < 0.$$

Consequently, $\boldsymbol{X} \notin \mathbb{J}(\mathcal{G}, ?)^*$.

(ii) It is known in general that $(A^*)^* = A$ and $(A \cap B)^* = \mathrm{cl}\,(A^* + B^*)$ if A and B are closed convex cone in $\mathbb{S}^{1+n}$. Thus, we obtain that

$$\left(\mathbb{J}^* \cap \mathbb{S}^{1+n}(\mathcal{G},0)\right)^* = \mathrm{cl}\left((\mathbb{J}^*)^* + \mathbb{S}^{1+n}(\mathcal{G},0)^\perp\right) \quad = \quad \mathrm{cl}\left(\mathbb{J} + \mathbb{S}^{1+n}(\mathcal{G}^c,0)\right) = \mathrm{cl}\left(\mathbb{J}(\mathcal{G},?)\right).$$

□

6.2 Sparse DNN and Lagrangian relaxation

We can utilize the last inclusion relation $\mathbb{J} \subset \mathbb{J}(\mathcal{G},?)$ of (6.1) with $\mathbb{J} = \mathbb{D}^{1+n}$ to construct sparse DNN and Lagrangian-DNN relaxations of QOP (1.2). As we have seen in the previous section, the primal COP (1.3) with $\mathbb{K} = \mathbb{D}^{1+n}$ serves as the DNN relaxation of QOP (5.2), which has been shown to be equivalent to QOP (1.2). If $\mathcal{G}$ is a symmetric subset of $N_0 \times N_0$ and $\mathbb{K}_1$ is a convex cone in $\mathbb{S}^{1+n}$ satisfying $\mathbb{D}^{1+n}(\mathcal{G},?) \subset \mathbb{K}_1$, then the primal COP (1.3) with $\mathbb{K} = \mathbb{K}_1$ serves as a sparse DNN relaxation. To derive effective and efficient sparse DNN and Lagrangian-DNN relaxations of QOP (5.2), some additional restrictions on $\mathcal{G}$ and $\mathbb{K}_1$ are necessary. In particular, Condition (I) with $\mathbb{K} = \mathbb{K}_1$ is necessary for the sparse Lagrangian-DNN relaxation (1.7) with $\mathbb{K} = \mathbb{K}_1$. We also want to choose a symmetric subset $\mathcal{G}$ of $N_0 \times N_0$ so that it properly reflects the sparsity of the matrices $\boldsymbol{Q}^k$ $(k = 0, 1, \ldots, m)$ for the resulting DNN and Lagrangian-DNN relaxations to be solved efficiently.

For such a symmetric subset $\mathcal{G}$ of $N_0 \times N_0$, we introduce the sparsity pattern (undirected) graph $G(N_0, \mathcal{E}_0)$ such that

$$\mathcal{E}_0 \quad = \quad \left\{(i,j) \in N_0 \times N_0 : i \neq j,\ Q^k_{ij} \neq 0 \ \text{ for some } k \in \{0,1,\ldots,m\}\right\}.$$

We identify $(i,j) \in \mathcal{E}_0$ and $(j,i) \in \mathcal{E}_0$ so that $G(N_0, \mathcal{E}_0)$ forms an undirected graph. Let $\bar{G}(N_0, \bar{\mathcal{E}}_0)$ be a chordal extension of $G(N_0, \mathcal{E}_0)$. Consider the set of maximal cliques $C_1, \ldots, C_r$ of $\bar{G}(N_0, \bar{\mathcal{E}}_0)$, where each maximal clique is denoted by a subset of N_0. It is known that the number r of the maximal cliques is not greater than the size $1+n$ of the node set N_0, and that the maximal cliques can be renumbered so that they satisfy the running intersection property

$$\forall p \in \{1, \ldots, r-1\},\ \exists q > p;\ C_p \cap (C_{p+1} \cup \cdots \cup C_r) \subset C_q. \tag{6.4}$$

Let $\square C_p = C_p \times C_p$ $(p = 1, \ldots, p)$ and $\mathcal{E} = \bigcup_{p=1}^r \square C_p$. In this case, we can apply the following lemma to determine whether a matrix $\boldsymbol{X} \in \mathbb{S}^{1+n}$ belongs to $\mathbb{S}^{1+n}_+(\mathcal{E},?)$ and $\mathbb{S}^{1+n}_+(\mathcal{E},0)$. If $\boldsymbol{X} \in \mathbb{S}^{1+n}_+(\mathcal{E},?)$, then X_{ij} $((i,j) \in \mathcal{E}^c)$ may be regarded as elements with undetermined values, but their values can be assigned so that the completed matrix belongs to $\mathbb{S}^{1+n}_+$. The technique for assigning appropriate values is known as positive semidefinite matrix completion in the literature [18]. Techniques for exploiting sparsity in SDPs based on the positive semidefinite matrix completion were proposed in [16, 26]; see also [21, 23]. We can utilize those techniques for an efficient implementation of the method proposed in this section.

For later discussions, we use the notation $(X_{ij} : \square C_p)$ to denote the submatrix extracted from a given $\boldsymbol{X} \in \mathbb{S}^{1+n}$ by extracting the elements at $(i,j) \in \square C_p$. We also define

$$\mathbb{D}(\mathcal{E}) \ = \ \mathbb{S}^{1+n}_+(\mathcal{E},?) \cap \mathbb{N}^{1+n}(\mathcal{E},?).$$

Lemma 6.2. *Let* $\boldsymbol{X} \in \mathbb{S}^{1+n}$.

(i) $\boldsymbol{X} \in \mathbb{S}^{1+n}_+(\mathcal{E}, ?)$ *if and only if*

$$\boldsymbol{X} \in \bigcap_{p=1}^{r} \mathbb{S}^{1+n}_+(\Box C_p, ?) = \bigcap_{p=1}^{r} \left\{\boldsymbol{X} \in \mathbb{S}^{1+n} : (X_{ij} : \Box C_p) \text{ is positive semidefinite}\right\}.$$

(ii) $\boldsymbol{Q} \in \mathbb{S}^{1+n}_+(\mathcal{E}, 0)$ *if and only if* $\boldsymbol{Q} \in \sum_{p=1}^{r} \mathbb{S}^{1+n}_+(\Box C^p, 0)$.

(iii) $\boldsymbol{X} \in \mathbb{N}^{1+n}_+(\mathcal{E}, ?)$ *if and only if*

$$\boldsymbol{X} \in \bigcap_{p=1}^{r} \mathbb{N}^{1+n}_+(\Box C_p, ?) = \bigcap_{p=1}^{r} \left\{\boldsymbol{X} \in \mathbb{N}^{1+n} : (\boldsymbol{X}_{ij} : \Box C_p) \text{ is a nonnegative matrix}\right\}.$$

(ii) $\boldsymbol{Q} \in \mathbb{N}^{1+n}(\mathcal{E}, 0)$ *if and only if* $\boldsymbol{Q} \in \sum_{p=1}^{r} \mathbb{N}^{1+n}(\Box C^p, 0)$.

Proof. See [18] and [1] for assertions (i) and (ii), respectively. Assertions (iii) and (iv) are straightforward to verify. ☐

Note that assertion (ii) may be regarded as a dual of (i) since

$$\begin{aligned}
\mathbb{S}^{1+n}_+(\mathcal{E}, 0) &= \left(\mathbb{S}^{1+n}_+(\mathcal{E}, ?)\right)^* \text{ (by (i) of Lemma 6.1)} \\
&= \left(\bigcap_{p=1}^{r} \mathbb{S}^{1+n}_+(\Box C_p, ?)\right)^* \text{ (by (i))} \\
&= \text{cl}\left(\sum_{p=1}^{r} \mathbb{S}^{1+n}_+(\Box C_p, 0)\right) = \sum_{p=1}^{r} \mathbb{S}^{1+n}_+(\Box C_p, 0).
\end{aligned}$$

Note that the closedness of the cone $\sum_{p=1}^{r} \mathbb{S}^{1+n}_+(\Box C_p, 0)$ can be proved easily.

As a sparse DNN relaxation of QOP (5.2), we employ the primal-dual pair of COPs (1.3)–(1.4) with the convex cone $\mathbb{K} = \mathbb{D}(\mathcal{E}) = \mathbb{S}^{1+n}_+(\mathcal{E}, ?) \cap \mathbb{N}^{1+n}(\mathcal{E}, ?)$, *i.e.*,

$$\zeta^p(\mathbb{D}(\mathcal{E})) := \inf\left\{\langle \boldsymbol{Q}^0, \boldsymbol{X}\rangle \;\middle|\; \begin{array}{l} \boldsymbol{X} \in \mathbb{D}(\mathcal{E}),\ \langle \boldsymbol{H}^0, \boldsymbol{X}\rangle = 1, \\ \langle \boldsymbol{Q}^k, \boldsymbol{X}\rangle = 0 \ (k = 1, 2, \ldots, m) \end{array}\right\} \quad (6.5)$$

$$\zeta^d(\mathbb{D}(\mathcal{E})) := \sup\left\{z_0 \;\middle|\; \boldsymbol{Q}^0 + \sum_{k=1}^{m} \boldsymbol{Q}^k z_k - \boldsymbol{H}^0 z_0 \in \mathbb{D}(\mathcal{E})^* \right\} \quad (6.6)$$

Lemma 6.3.

(i) $\mathbb{D}(\mathcal{E}) = \bigcap_{p=1}^{r} \left(\left\{\boldsymbol{X} \in \mathbb{S}^{1+n} : (X_{ij} : \Box C_p) \text{ is doubly nonnegative}\right\}\right)$, *and* $\mathbb{D}(\mathcal{E})$ *is closed.*

(ii) $\mathbb{D}(\mathcal{E})^* = \mathbb{S}^{1+n}_+(\mathcal{E}, 0) + \mathbb{N}^{1+n}(\mathcal{E}, 0) = \sum_{p=1}^{r} \left(\mathbb{S}^{1+n}_+(\Box C_p, 0) + \mathbb{N}^{1+n}(\Box C_p, 0)\right)$, *and* $\mathbb{D}(\mathcal{E})^*$ *is closed.*

Proof. (i) By Lemma 6.2,

$$\begin{aligned}
\mathbb{D}(\mathcal{E}) &= \mathbb{S}^{1+n}_+(\mathcal{E}, ?) \cap \mathbb{N}(\mathcal{E}, ?) \\
&= \bigcap_{p=1}^{r} \left(\mathbb{S}^{1+n}_+(\Box C_p, ?) \cap \mathbb{N}^{1+n}(\Box C_p, ?)\right)
\end{aligned}$$

$$= \bigcap_{p=1}^{r} \left(\left\{ \boldsymbol{X} \in \mathbb{S}^{1+n} : (X_{ij} : \square C_p) \text{ is doubly nonegative} \right\} \right).$$

The closedness of $\mathbb{D}(\mathcal{E})$ follows from the above identity.

(ii) By definition, Lemmas 6.1, 6.2 and assertion (i),

$$\begin{aligned} \mathbb{D}(\mathcal{E})^* &= \left(\mathbb{S}_+^{1+n}(\mathcal{E}, ?) \cap \mathbb{N}^{1+n}(\mathcal{E}, ?) \right)^* \\ &= \text{cl } \left(\mathbb{S}_+^{1+n}(\mathcal{E}, 0) + \mathbb{N}^{1+n}(\mathcal{E}, 0) \right) \\ &= \mathbb{S}_+^{1+n}(\mathcal{E}, 0) + \mathbb{N}^{1+n}(\mathcal{E}, 0) \\ &= \sum_{p=1}^{r} \left(\mathbb{S}_+^{1+n}(\square C_p, 0) + \mathbb{N}^{1+n}(\square C_p, 0) \right). \end{aligned}$$

We need to prove the closedness of $\mathbb{S}_+^{1+n}(\mathcal{E}, 0) + \mathbb{N}^{1+n}(\mathcal{E}, 0)$ for the third identity above. Suppose that $\boldsymbol{X}^s = \boldsymbol{Y}^s + \boldsymbol{Z}^s$, $\boldsymbol{Y}^s \in \mathbb{S}_+^{1+n}(\mathcal{E}, 0)$, $\boldsymbol{Z}^s \in \mathbb{N}^{1+n}(\mathcal{E}, 0)$ $(s = 1, 2, \dots)$ and $\boldsymbol{X}^s \to \overline{\boldsymbol{X}}$ for some $\overline{\boldsymbol{X}} \in \mathbb{S}^{1+n}$ as $s \to \infty$. We will prove that $\overline{\boldsymbol{X}} = \overline{\boldsymbol{Y}} + \overline{\boldsymbol{Z}}$ for some $\overline{\boldsymbol{Y}} \in \mathbb{S}_+^{1+n}(\mathcal{E}, 0)$ and $\overline{\boldsymbol{Z}} \in \mathbb{N}^{1+n}(\mathcal{E}, 0)$, so that $\overline{\boldsymbol{X}} \in \mathbb{S}_+^{1+n}(\mathcal{E}, 0) + \mathbb{N}^{1+n}(\mathcal{E}, 0)$. First we show that the sequence $\{\boldsymbol{Y}^s\} \subset \mathbb{S}_+^{1+n}(\mathcal{E}, 0)$ is bounded. Assume on the contrary that there is a subsequence of $\{\boldsymbol{Y}^s\}$ along which $\|\boldsymbol{Y}^s\|$ diverges. We may assume without loss of generality that $\boldsymbol{Y}^s/\|\boldsymbol{Y}^s\| \to \overline{\boldsymbol{Y}}$ for some nonzero $\overline{\boldsymbol{Y}} \in \mathbb{S}_+^{1+n}(\mathcal{E}, 0)$ as $s \to \infty$. Then $\mathbb{N}^{1+n}(\mathcal{E}, 0) \ni \boldsymbol{Z}^s/\|\boldsymbol{Y}^s\| = \boldsymbol{X}^s/\|\boldsymbol{Y}^s\| - \boldsymbol{Y}^s/\|\boldsymbol{Y}^s\| \to -\overline{\boldsymbol{Y}}$ as $s \to \infty$. Since $\mathbb{N}^{1+n}(\mathcal{E}, 0)$ is closed, we obtain $-\overline{\boldsymbol{Y}} \in \mathbb{N}^{1+n}(\mathcal{E}, 0)$, which implies that all the diagonal elements of $\overline{\boldsymbol{Y}} \in \mathbb{S}_+^{1+n}(\mathcal{E}, 0)$ vanish. Therefore $\overline{\boldsymbol{Y}} = \boldsymbol{O}$, which is a contradiction. Thus $\{\boldsymbol{Y}^s\}$ is bounded. As a result $\{\boldsymbol{Z}^s\}$ is also bounded. From here, the required result follows. □

By the construction of $\mathcal{E}$, it follows that

$$\begin{aligned} &\boldsymbol{O} \neq \boldsymbol{H}^0 \in \mathbb{S}_+^{1+n}(\mathcal{E}, 0) \cap \mathbb{N}^{1+n}(\mathcal{E}, 0) \subset \mathbb{D}(\mathcal{E})^*, \\ &\boldsymbol{Q}^k \in \mathbb{S}_+^{1+n}(\mathcal{E}, 0) + \mathbb{N}^{1+n}(\mathcal{E}, 0) = \mathbb{D}(\mathcal{E})^* \ (k = 1, 2, \dots, m). \end{aligned}$$

Hence COP (6.5) (*i.e.*, COP (1.3) with $\mathbb{K} = \mathbb{D}(\mathcal{E})$) serves as a sparse DNN relaxation of QOP (5.2) satisfying Conditions (I) and (II) with $\mathbb{K} = \mathbb{D}(\mathcal{E})$. Lemmas 2.1, 2.2 and 2.3 can be applied for $\mathbb{K} = \mathbb{D}(\mathcal{E})$. In particular, we obtain the relation

$$\left(\eta^d(\lambda, \mathbb{D}(\mathcal{E})) = \eta^p(\lambda, \mathbb{D}(\mathcal{E})) \right) \uparrow = \zeta^d(\mathbb{D}(\mathcal{E})) \leq \zeta^p(\mathbb{D}(\mathcal{E})). \tag{6.7}$$

In addition, if Condition (III) is satisfied with $\mathbb{K} = \mathbb{D}(\mathcal{E})$, then the equality $\zeta^d(\mathbb{D}(\mathcal{E})) = \zeta^p(\mathbb{D}(\mathcal{E}))$ holds. See Lemma 2.5. For every $p = 1, 2, \dots, r$, let $(x_k : C_p)$ denote a column vector of the elements x_k $(k \in C_p \backslash \{0\})$ of a given $\boldsymbol{x} \in \mathbb{R}^n$.

Lemma 6.4. *Assume that the sets* $\{(x_k : C_p) : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ (x_k : C_p) \geq \boldsymbol{0}\}$ $(p = 1, 2, \dots, r)$ *are all nonempty and bounded. Then* $F(\mathbb{D}(\mathcal{E}))$ *is bounded and hence Condition (III) with* $\mathbb{K} = \mathbb{D}(\mathcal{E})$ *is satisfied.*

Proof. Assume on the contrary that there exists an unbounded sequence $\{\boldsymbol{X}^s\} \subset F(\mathbb{D}(\mathcal{E}))$. We may assume without loss of generality that $\boldsymbol{X}^s / \|\boldsymbol{X}^s\|$ converges to a nonzero $\boldsymbol{D} \in \mathbb{D}(\mathcal{E})$ as $s \to \infty$, which satisfies

$$(D_{ij} : \square C_p) \text{ is a positive semidefinite and nonnegative matrix } (p = 1, 2, \dots, r),$$

$D_{ij} = \overline{D}_{ij}$ $((i,j) \in \mathcal{E})$ for some nonzero $\overline{\boldsymbol{D}} \in \mathbb{S}^{1+n}_+$,
$\overline{\boldsymbol{D}} \neq \boldsymbol{O}$, $\langle \boldsymbol{H}^0, \overline{\boldsymbol{D}} \rangle = 0$ and $\langle \boldsymbol{Q}^1, \overline{\boldsymbol{D}} \rangle = 0$.

Represent $\overline{\boldsymbol{D}} \in \mathbb{S}^{1+n}_+$ as $\overline{\boldsymbol{D}} = \begin{pmatrix} \overline{D}_{00} & \boldsymbol{d}^T \\ \boldsymbol{d} & \widetilde{\boldsymbol{D}} \end{pmatrix}$ for some $\overline{D}_{00} \in \mathbb{R}$, $\boldsymbol{d} \in \mathbb{R}^n$ and $\widetilde{\boldsymbol{D}} \in \mathbb{S}^n_+$. By the construction of $\boldsymbol{H}^0$, we know that $0 = \langle \boldsymbol{H}^0, \overline{\boldsymbol{D}} \rangle = \overline{D}_{00}$. Thus $\boldsymbol{d} = \boldsymbol{0}$ and $\widetilde{\boldsymbol{D}} \neq \boldsymbol{O}$. It follows from $\langle \boldsymbol{Q}^1, \overline{\boldsymbol{D}} \rangle = 0$ and (5.1) that $0 = \langle \boldsymbol{Q}^1, \overline{\boldsymbol{D}} \rangle = \langle \boldsymbol{A}^T\boldsymbol{A}, \widetilde{\boldsymbol{D}} \rangle$. Since both $\boldsymbol{A}^T\boldsymbol{A}$ and $\widetilde{\boldsymbol{D}}$ are positive semidefinite, we see that $\boldsymbol{A}^T\boldsymbol{A}\widetilde{\boldsymbol{D}} = \boldsymbol{O}$, which implies $\boldsymbol{A}\widetilde{\boldsymbol{D}} = \boldsymbol{O}$. Let $j \in \{1, 2, \ldots, n\}$ be fixed arbitrary, and let $\boldsymbol{u} \in \mathbb{R}^n$ denote the jth column of $\widetilde{\boldsymbol{D}}$. Then we can take a $p \in \{1, 2, \ldots, r\}$ such that $j \in C_p$, and we have $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{0}$ and $(u_k : C_p) \geq \boldsymbol{0}$. If $(u_k : C_p) \neq \boldsymbol{0}$, then $(u_k : C_p)$ forms an unbounded direction of the set $\{(x_k : C_p) : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ (x_k : C_p) \geq \boldsymbol{0}\}$. This contradicts the assumption. Thus, $(u_k : C_p) = \boldsymbol{0}$, in particular, $u_j = \widetilde{D}_{jj} = 0$. Since we have chosen $j \in \{1, 2, \ldots, n\}$ arbitrarily, we can conclude that all the diagonal elements of the nonzero positive semidefinite matrix $\widetilde{\boldsymbol{D}}$ are zero. This is a contradiction. □

Remark 6.1. In general, the boundedness of $\{\boldsymbol{x} \in \mathbb{R}^n_+ : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$, which has been assumed in (i) of Lemma 5.2, is weaker than the assumption of Lemma 6.4. It is not clear whether $F(\mathbb{D}(\mathcal{E}))$ is still bounded under the weaker assumption. We are not able to prove it nor show a counter example of unbounded $F(\mathbb{D}(\mathcal{E}))$ under the weaker assumption.

6.3 The applications of the bisection and 1-dimensional Newton methods to the sparse Lagrangian-DNN relaxation problems

We recall that the computation of the metric projection of each $\boldsymbol{G} \in \mathbb{V}$ onto $\mathbb{K}$, which is assumed in Condition (V), is the key to apply the bisection and the 1-dimensional Newton methods presented in Section 4 to the primal-dual pair of Lagrangian-conic relaxation problems (1.7)–(1.8); see also Remark 4.1. For the sparse Lagrangian-DNN relaxations of the sparse DNN conic problems (6.5)–(6.6) with $\mathbb{K} = \mathbb{D}(\mathcal{E})$, we would need to transform them into a form for which the metric projection can be computed easily. The purpose of this subsection is to describe the transformation to facilitate the efficient computation of the metric projection.

We note that since the variables X_{ij} $((i,j) \notin \mathcal{E})$ are redundant in both the equality constraints and the cone constraint $\boldsymbol{X} \in \mathbb{D}(\mathcal{E})$ of the primal COPs (1.3), (1.5) and (1.7) with $\mathbb{K} = \mathbb{D}(\mathcal{E})$, those variables can be eliminated from the primal COPs. On the other hand, all matrices $\boldsymbol{Q}^k$ $(k = 0, 1, \ldots, m)$, $\boldsymbol{H}^0$, $\boldsymbol{H}^1$ and the cone $\mathbb{D}(\mathcal{E})^*$ are contained in $\mathbb{S}^{1+n}(\mathcal{E}, 0)$ in the dual COPs (1.4), (1.6) and (1.8) with $\mathbb{K} = \mathbb{D}(\mathcal{E})$. This implies that the elements H^0_{ij}, H^1_{ij}, Q^k_{ij} $(k = 0, 1, 2, \ldots, m)$ $((i,j) \notin \mathcal{E})$ are redundant in the inclusion constraints of the dual COPs. Furthermore, checking whether $\boldsymbol{X} \in \mathbb{S}^{1+n}$ belongs to $\mathbb{S}^{1+n}_+(\mathcal{E}, ?)$ can be determined by checking whether its sub matrices $(X_{ij} : (i,j) \in \square C_p)$ $(p = 1, 2, \ldots, r)$ are all positive semidefinite (Lemma 6.3). We note that some elements may appear in a pair of these submatrices, *i.e.*, $\square C_p \cap \square C_q \neq \emptyset$ for some p, q. To continue the discussion, let

$$
\begin{aligned}
\mathbb{S}^{C_p} &= \left\{ \boldsymbol{Y}^p = (Y^p_{ij} : (i,j) \in \square C_p) : Y^p_{ij} = Y^p_{ji} \in \mathbb{R} \right\} \ (p = 1, 2, \ldots, r), \\
\mathbb{S}^{C_p}_+ &= \left\{ \boldsymbol{Y}^p \in \mathbb{S}^{C_p} : \text{positive semidefinite} \right\} \ (p = 1, 2, \ldots, r), \\
\mathbb{S}^{\mathcal{E}} &= \prod_{p=1}^{r} \mathbb{S}^{C_p} = \left\{ \boldsymbol{Y} = (\boldsymbol{Y}^1, \boldsymbol{Y}^2, \ldots, \boldsymbol{Y}^r) : \boldsymbol{Y}^p \in \mathbb{S}^{C_p} \ \ (p = 1, 2, \ldots, r) \right\},
\end{aligned}
$$

$$\begin{aligned}
\mathbb{S}^{\mathcal{E}}_+ &= \prod_{p=1}^{r} \mathbb{S}^{C_p}_+ = \left\{ \boldsymbol{Y} = (\boldsymbol{Y}^1, \boldsymbol{Y}^2, \ldots, \boldsymbol{Y}^r) : \boldsymbol{Y}^p \in \mathbb{S}^{C_p}_+ \ \ (p = 1, 2, \ldots, r) \right\}, \\
\mathbb{L}^{\mathcal{E}} &= \left\{ \boldsymbol{Y} = (\boldsymbol{Y}^1, \boldsymbol{Y}^2, \ldots, \boldsymbol{Y}^r) \in \mathbb{S}^{\mathcal{E}} : \ Y^p_{ij} = Y^q_{ij} \text{ if } (i,j) \in \square C_p \cap \square C_q \ \right\}, \\
\mathbb{K}_1 &= \mathbb{S}^{\mathcal{E}}_+, \\
\mathbb{K}_2 &= \left\{ \boldsymbol{Y} = (\boldsymbol{Y}^1, \boldsymbol{Y}^2, \ldots, \boldsymbol{Y}^r) \in \mathbb{S}^{\mathcal{E}} : \begin{array}{l} Y^p_{ij} \geq 0 \ (i,j) \in \square C_p \\ (p = 1, 2, \ldots, r) \end{array} \right\} \cap \mathbb{L}^{\mathcal{E}}.
\end{aligned}$$

Observe that each $\boldsymbol{Y} \in \mathbb{S}^{\mathcal{E}}$ may be regarded as a block diagonal matrix with diagonal blocks $\boldsymbol{Y}^1, \boldsymbol{Y}^2, \ldots, \boldsymbol{Y}^r$. We use $\langle \widetilde{\boldsymbol{U}}, \boldsymbol{Y} \rangle = \sum_{p=1}^{r} \langle \widetilde{\boldsymbol{U}}^p, \boldsymbol{Y}^p \rangle$ for the inner product of $\widetilde{\boldsymbol{U}} = (\widetilde{\boldsymbol{U}}^1, \widetilde{\boldsymbol{U}}^2, \ldots, \widetilde{\boldsymbol{U}}^r)$, $\boldsymbol{Y} = (\boldsymbol{Y}^1, \boldsymbol{Y}^2, \ldots, \boldsymbol{Y}^r) \in \mathbb{S}^{\mathcal{E}}$.

We now associate each $\boldsymbol{X} \in \mathbb{S}^{1+n}(\mathcal{E}, ?)$ with $\widetilde{\boldsymbol{X}} = (\widetilde{\boldsymbol{X}}^1, \widetilde{\boldsymbol{X}}^2, \ldots, \widetilde{\boldsymbol{X}}^r) \in \mathbb{S}^{\mathcal{E}}$ where

$$\widetilde{\boldsymbol{X}}^p = (X_{ij} : (i,j) \in \square C_p) \ (p = 1, 2, \ldots, r).$$

This correspondence yields that $\boldsymbol{X} \in \mathbb{D}(\mathcal{E})$ if and only if $\widetilde{\boldsymbol{X}} \in \mathbb{K}_1 \cap \mathbb{K}_2$. It is also possible to choose $\widetilde{\boldsymbol{Q}}^0$, $\widetilde{\boldsymbol{H}}^0$, $\widetilde{\boldsymbol{H}}^1 \in \mathbb{S}^{\mathcal{E}}$ such that

$$\langle \widetilde{\boldsymbol{Q}}^0, \widetilde{\boldsymbol{X}} \rangle = \langle \boldsymbol{Q}^0, \boldsymbol{X} \rangle, \ \langle \widetilde{\boldsymbol{H}}^0, \widetilde{\boldsymbol{X}} \rangle = \langle \boldsymbol{H}^0, \boldsymbol{X} \rangle \text{ and } \langle \widetilde{\boldsymbol{H}}^0, \widetilde{\boldsymbol{X}} \rangle = \langle \boldsymbol{H}^0, \boldsymbol{X} \rangle.$$

Consequently, we obtain the following primal-dual pair of sparse Lagrangian-DNN relaxation COPs, which are equivalent to the original primal-dual pair of sparse Lagrangian-DNN relaxation COPs with $\mathbb{K} = \mathbb{D}(\mathcal{E})$:

$$\tilde{\zeta}^p(\lambda) = \inf \left\{ \langle \widetilde{\boldsymbol{Q}}^0 + \lambda \widetilde{\boldsymbol{H}}^1, \widetilde{\boldsymbol{X}} \rangle \;\middle|\; \begin{array}{l} \widetilde{\boldsymbol{X}} = (\widetilde{\boldsymbol{X}}^1, \widetilde{\boldsymbol{X}}^2, \ldots, \widetilde{\boldsymbol{X}}^r) \in \mathbb{K}_1 \cap \mathbb{K}_2, \\ \langle \widetilde{\boldsymbol{H}}^0, \widetilde{\boldsymbol{X}} \rangle = 1 \end{array} \right\} \tag{6.8}$$

$$\tilde{\zeta}^d(\lambda) = \sup \left\{ y_0 \;\middle|\; \widetilde{\boldsymbol{Q}}^0 + \lambda \widetilde{\boldsymbol{H}}^1 - \widetilde{\boldsymbol{H}}^0 y_0 \in \mathbb{K}_1^* + \mathbb{K}_2^* \right\}. \tag{6.9}$$

For the above pair of transformed sparse Lagrangian-DNN relaxation COPs, the metric projections Π_i from $\mathbb{S}^{\mathcal{E}}$ onto $\mathbb{K}_i$ $(i = 1, 2)$ are expressed as

$$\begin{aligned}
\Pi_i(\widetilde{\boldsymbol{X}}) &= (\Pi_{i1}(\widetilde{\boldsymbol{X}}), \Pi_{i2}(\widetilde{\boldsymbol{X}}), \ldots, \Pi_{ir}(\widetilde{\boldsymbol{X}})) \ (i = 1, 2), \\
\Pi_{1p}(\widetilde{\boldsymbol{X}}) &= \text{the metric projection of } \widetilde{\boldsymbol{X}}^p \in \mathbb{S}^{C_p} \text{ onto } \mathbb{S}^{C_p}_+ \ (p = 1, 2, \ldots, r), \\
\left(\Pi_{2p}(\widetilde{\boldsymbol{X}})\right)_{ij} &= \max \left\{ \frac{\sum_{p \in P(i,j)} \widetilde{\boldsymbol{X}}^p_{ij}}{\# P(i,j)}, 0 \right\} \quad ((i,j) \in \square C_p, \ p = 1, 2, \ldots, r),
\end{aligned}$$

where $P(i,j) = \{p : (i,j) \in \square C_p\}$ $((i,j) \in \mathcal{E})$. We refer to [16, 26] for details on the conversion from the primal-dual pair (1.7)–(1.8) with $\mathbb{K} = \mathbb{D}(\mathcal{E})$ to the primal-dual pair (6.8) and (6.9), and [7, 22] for the numerical methods for computing the metric projection onto $\mathbb{K} = \mathbb{K}_1 \cap \mathbb{K}_2$.

7 Concluding Remarks

We have provided a unified framework expressed in a primal-dual pair of COPs, which provides a convenient and effective tool to develop the theory and methods originated from the completely positive programming relaxation of QOPs. By imposing Condition (I) on the primal-dual pair of COPs, equivalent but simpler primal-dual pair of COPs and their Lagrangian-conic relaxations have been derived. We have investigated some essential theoretical properties of the three primal-dual pairs of COPs and the conditions which yield the equivalence for the optimal values of the COPs. When the cone $\mathbb{K}$ involved in the first primal-dual pair of COPs is nonconvex, we have provided a necessary and sufficient condition for the equivalence between the primal COP and its convexification, *i.e.*, the COP obtained by replacing the nonconvex cone $\mathbb{K}$ by its convex hull. This result has been applied to a class of linearly constrained QOPs with complementarity constraints.

In our recent papers [6, 22], some promising numerical results were reported on the Lagrangian-DNN relaxation approach to approximately solve QOPs using a bisection method in conjunction with efficient algorithms for computing the metric projection onto the DNN cone. But any sparsity was not utilized there. In the current paper, we have proposed the sparse Lagrangian-DNN relaxation for the same class of QOPs, and the 1-dimensional Newton method for solving the primal-dual pair of Lagrangian-conic relaxation problems (1.7)–(1.8). One can expect that if sparsity in the data is exploited, and the 1-dimensional Newton method is incorporated with the bisection method, then we will be able to solve large scale QOPs much more efficiently. Our next goal is to present numerical results on this topic in the future.

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