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EXISTENCE OF SOLUTIONS FOR SYMMETRIC VECTOR SET-VALUED QUASI-EQUILIBRIUM PROBLEMS WITH APPLICATIONS*

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Abstract: In this paper, an existence theorem of the strong efficient solutions concerning a class of symmetric vector set-valued quasi-equilibrium problems is established by using Kakutani-Fan-Glicksberg fixed point theorem and two existence theorems of the efficient solutions and the weakly efficient solutions for the symmetric vector set-valued quasi-equilibrium problems are derived by using the scalarization method. Some applications to symmetric mixed vector quasi-equilibrium problems, symmetric vector quasi-equilibrium problems and generalized semi-infinite programs with symmetric vector set-valued quasi-equilibrium constraints are also given.

Key words: *symmetric vector set-valued quasi-equilibrium problem, symmetric vector quasi-equilibrium problem, generalized semi-infinite program, existence theorem, scalarization*

Mathematics Subject Classification: *49J53, 54C60, 91B50*

1 Introduction

Since the vector variational inequality was introduced and studied by Giannessi [15] in 1980, various theoretical results, numerical algorithms and applications have been investigated extensively for vector variational inequalities and vector equilibrium problems with their generalizations in the literature (see, for example, [1, 2, 5, 8, 11, 13, 16, 23, 24, 27] and the references therein). Motivations for these come from the fact that these models have many applications in optimization, control theory, mathematical programming, networks, operations research, management science, economics and finance.

On the other hand, as a generalization of the equilibrium problem proposed by Blum and Oettli [4], the symmetric quasi-equilibrium problem was introduced and studied by Noor and Oettli [26]. Inspired by the study in connection with vector variational inequalities and vector equilibrium problems, the symmetric quasi-equilibrium problem was extended to the case of vector-valued bifunctions by Fu [14], known as the symmetric vector quasi-equilibrium problem (in short SVQEP). Farajzadeh [12] used a particular technique to establish an existence theorem of weak solutions for SVQEP and answered an open question raised by Fu [14]. Chen and Gong [9] studied the stability of the set of weak solutions for SVQEP, proved a generic stability theorem and gave an existence theorem for essentially connected components of the

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set of weak solutions for SVQEP. Gong [18] studied existence conditions for strong solutions of SVQEP. Chen et al. [6] established an existence theorem for strong solutions of generalized symmetric vector quasi-equilibrium problems. Chen and Huang [7] gave existence conditions for weak solutions of generalized symmetric vector quasi-equilibrium problems. Han and Gong [20] obtained characterizations for generalized Levitin-Polyak well-posedness of SVQEP by closed graph of the approximating solution mapping. However, to the best of the authors' knowledge, there is no any papers studying the existence of efficient solutions for SVQEP.

We note that, if C is a closed cone in topological vector space Z with $\text{int}C \neq \emptyset$, then it is clear that $Z \setminus (-\text{int}C)$ is closed. However, in many cases, the ordering cone C has an empty interior. For example, in the normed spaces l^p and $L^p(\Omega)$, where $1 < p < +\infty$, the standard ordering cone has an empty interior, but the ordering cone has a base. The fact that $C \setminus \{0\}$ is neither closed nor open makes the study concerning the existence of efficient solutions for SVQEP be more difficult than the study concerning the existences of weak solutions and strong solutions for SVQEP. Thus, it is important and interesting to investigate the existence of efficient solutions for SVQEP with its generalization form.

In this paper, we extended SVQEP to the set-valued version, named a symmetric vector set-valued quasi-equilibrium problem, which provides a unify setting for the study in connection with several generalized (vector) equilibrium problems including symmetric mixed (vector) quasi-equilibrium problems, symmetric (vector) quasi-equilibrium problems, (vector) quasi-equilibrium problems and (vector) equilibrium problems. Under some mild conditions, we show an existence theorem for the strong efficient solutions of the symmetric vector set-valued quasi-equilibrium problem by using Kakutani-Fan-Glicksberg fixed point theorem and two existence theorems of the efficient solutions and the weakly efficient solutions for the symmetric vector set-valued quasi-equilibrium problem by using the scalarization method.

The structure of the paper is as follows. In Section 2, we introduce the symmetric vector set-valued quasi-equilibrium problem and recall some notions and some lemmas. In Section 3, we establish three existence theorems for the strong efficient solutions, the efficient solutions and the weakly efficient solutions of symmetric vector set-valued quasi-equilibrium problems. In Section 4, we give some applications of the main results to symmetric mixed vector quasi-equilibrium problems, symmetric vector quasi-equilibrium problems and generalized semi-infinite programs with symmetric vector set-valued equilibrium constraints.

2 Preliminaries

Throughout this paper, unless otherwise specified, let X, Y and Z be three normed vector spaces and let $E \subseteq X$ and $D \subseteq Y$ be two nonempty, convex, compact subsets. We assume that $C \subseteq Z$ is a convex, closed, pointed cone. Let $S : E \times D \to 2^E$, $H : E \times D \to 2^D$, $F : E \times D \times E \times E \to 2^Z$ and $G : E \times D \times D \times D \to 2^Z$ be four set-valued mappings. We introduce the following symmetric vector set-valued quasi-equilibrium problem (in short, SVSQEP) consisting of finding a point $(x_0, y_0) \in E \times D$ such that

$$\begin{cases} x_0 \in S(x_0, y_0), & F(x_0, y_0, x, x_0) \cap (-\Omega) = \emptyset, \quad \forall x \in S(x_0, y_0), \\ y_0 \in H(x_0, y_0), & G(x_0, y_0, y, y_0) \cap (-\Omega) = \emptyset, \quad \forall y \in H(x_0, y_0), \end{cases}$$

where $\Omega \cup \{0\}$ is a cone in Z.

Let $S(F, G)$ denote the set of all strong efficient solutions of (SVSVEP), i.e. $(x_0, y_0) \in S(F, G)$. Then

$$\begin{cases} x_0 \in S(x_0, y_0), & F(x_0, y_0, x, x_0) \subseteq C, \quad \forall x \in S(x_0, y_0), \\ y_0 \in H(x_0, y_0), & G(x_0, y_0, y, y_0) \subseteq C, \quad \forall y \in H(x_0, y_0). \end{cases}$$

Let $E(F,G)$ denote the set of all efficient solutions of (SVSQEP), i.e. $(x_0,y_0) \in E(F,G)$. Then

$$\begin{cases} x_0 \in S(x_0,y_0), & F(x_0,y_0,x,x_0) \cap (-C\setminus\{0\}) = \emptyset, \quad \forall x \in S(x_0,y_0), \\ y_0 \in H(x_0,y_0), & G(x_0,y_0,y,y_0) \cap (-C\setminus\{0\}) = \emptyset, \quad \forall y \in H(x_0,y_0). \end{cases}$$

Let $W(F,G)$ denote the set of all weak efficient solutions of (SVSQEP), i.e. $(x_0,y_0) \in W(F,G)$. Then

$$\begin{cases} x_0 \in S(x_0,y_0), & F(x_0,y_0,x,x_0) \cap (-\text{int}C) = \emptyset, \quad \forall x \in S(x_0,y_0), \\ y_0 \in H(x_0,y_0), & G(x_0,y_0,y,y_0) \cap (-\text{int}C) = \emptyset, \quad \forall y \in H(x_0,y_0). \end{cases}$$

It is clear that $S(F,G) \subseteq E(F,G) \subseteq W(F,G)$.

Let Z^* be the topological dual space of Z and C^* be defined by

$$C^* = \{f \in Z^* : f(c) \geq 0, \ \forall c \in C\}.$$

Denote the quasi-interior of C^* by $C^\#$, i.e.

$$C^\# = \{f \in Z^* : f(c) > 0, \ \forall c \in C\setminus\{0\}\}.$$

A nonempty convex subset B of C is said to be a base of C, if $C = \text{cone}(B)$ and $0 \notin \text{cl}(B)$. It is easy to see that $C^\# \neq \emptyset$ if and only if C has a base.

Remark 2.1. We can see that if $\text{int}C \neq \emptyset$, then $C^*\setminus\{0\} \neq \emptyset$. Let $f \in C^*\setminus\{0\}$. Then it is easy to see that $f(x) < 0$ provided $x \in -\text{int}C$.

Definition 2.2. Let A be a nonempty convex subset of X. A set-valued mapping $\Phi : A \to 2^Z$ is said to be

(i) [19] C-convex if, for any $x_1, x_2 \in A$ and for any $t \in [0,1]$, one has

$$t\Phi(x_1) + (1-t)\Phi(x_2) \subseteq \Phi(tx_1 + (1-t)x_2) + C.$$

(ii) [22] strictly C-convex if, for any $x_1, x_2 \in A$ with $x_1 \neq x_2$ and for any $t \in (0,1)$, one has

$$t\Phi(x_1) + (1-t)\Phi(x_2) \subseteq \Phi(tx_1 + (1-t)x_2) + \text{int}C.$$

(iii) (Definition 3 of [13]) properly quasi-C-convex if, for any $x_1, x_2 \in A$ and for any $t \in [0,1]$, one has

$$\text{either} \quad \Phi(x_1) \subseteq \Phi(tx_1 + (1-t)x_2) + C \quad \text{or} \quad \Phi(x_2) \subseteq \Phi(tx_1 + (1-t)x_2) + C.$$

(iv) [22] natural quasi C-convex if, for any $x_1, x_2 \in A$ and for any $t \in [0,1]$, there exists $\lambda \in [0,1]$ such that

$$\lambda\Phi(x_1) + (1-\lambda)\Phi(x_2) \subseteq \Phi(tx_1 + (1-t)x_2) + C.$$

(v) [22] natural quasi C-concave if, for any $x_1, x_2 \in A$ and for any $t \in [0,1]$, there exists $\lambda \in [0,1]$ such that

$$\Phi(tx_1 + (1-t)x_2) \subseteq \lambda\Phi(x_1) + (1-\lambda)\Phi(x_2) + C.$$

Remark 2.3. It is clear that if Φ is C-convex, then Φ is natural quasi C-convex. If Φ is properly quasi-C-convex, then Φ is natural quasi C-convex. The class of natural quasi C-convex mappings is strictly larger than both the class of C-convex mappings and the class of properly quasi C-convex mappings. We give an example to illustrate it.

Example 2.4. Let $Z = \mathbb{R}^2$, $C = \mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$, $X = \mathbb{R}$ and $A = [0, \frac{\pi}{2}]$. We denote by B_Z the closed unit ball in Z. Let $\Phi : A \to 2^Y$ be a set-valued mapping defined as follows

$$\Phi(x) = (\sin x, 2 - \sin x) + B_Z.$$

Then it is easy to check that Φ is neither C-convex nor properly quasi-C-convex. But Φ is natural quasi C-convex.

Definition 2.5. Let T and T_1 be two topological vector spaces. A set-valued mapping $\Phi : T \to 2^{T_1}$ is said to be

(i) closed if the set $\{(x, y) \in T \times T_1 : x \in T, y \in \Phi(x)\}$ is closed in $T \times T_1$.

(ii) upper semicontinuous (u.s.c.) at $u_0 \in T$ if, for any neighborhood V of $\Phi(u_0)$, there exists a neighborhood $U(u_0)$ of u_0 such that for every $u \in U(u_0)$, $\Phi(u) \subseteq V$.

(iii) lower semicontinuous (l.s.c.) at $u_0 \in T$ if, for any $x \in \Phi(u_0)$ and any neighborhood V of x, there exists a neighborhood $U(u_0)$ of u_0 such that for every $u \in U(u_0)$, $\Phi(u) \cap V \neq \emptyset$.

We say that Φ is u.s.c. and l.s.c. on T if it is u.s.c. and l.s.c. at each point $u \in T$, respectively. We say that Φ is continuous on T if it is both u.s.c. and l.s.c. on T.

In the following two lemmas, let T and T_1 be two normed vector spaces.

Lemma 2.6 ([3])**.** *A set-valued mapping $\Phi : T \to 2^{T_1}$ is l.s.c. at $u_0 \in T$ if and only if for any sequence $\{u_n\} \subseteq T$ with $u_n \to u_0$ and for any $x_0 \in \Phi(u_0)$, there exists $x_n \in \Phi(u_n)$ such that $x_n \to x_0$.*

Lemma 2.7 ([19])**.** *Let $\Phi : T \to 2^{T_1}$ be a set-valued mapping. For any given $u_0 \in T$, if $\Phi(u_0)$ is compact, then Φ is u.s.c. at $u_0 \in T$ if and only if for any sequence $\{u_n\} \subseteq T$ with $u_n \to u_0$ and for any $x_n \in \Phi(u_n)$, there exist $x_0 \in \Phi(u_0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$.*

Lemma 2.8 (Kakutani-Fan-Glicksberg Fixed Point Theorem [10,17])**.** *Let K be a nonempty compact convex subset of a locally convex Hausdorff topological vector space X and let $F : K \to 2^K$ be an u.s.c. set-valued mapping with nonempty compact convex values. Then there exists $x_0 \in K$ such that $x_0 \in F(x_0)$.*

3 Existence Theorems

In this section, we establish the existence theorem for efficient solutions of (SVQEP) and give an example to illustrate it. Let X^* and Y^* be the topological dual space of X and Y, respectively.

Lemma 3.1 ([21])**.** *Let K be a nonempty compact convex subset of X and $\Phi : K \times K \to 2^Z$ be a set-valued mapping. Assume that*

(i) *for any* $x \in K$, $\Phi(x,x) \subseteq C$;

(ii) *for any* $y \in K$, $\{x \in K : \Phi(x,y) \subseteq C\}$ *is closed;*

(iii) *for any* $x \in K$, $\Phi(x,\cdot)$ *is properly quasi-C-convex.*

Then there exists a point $x_0 \in K$ *such that*

$$\Phi(x_0,y) \subseteq C, \ \ \forall y \in K.$$

Remark 3.2. For any $y \in K$, if $\Phi(\cdot,y)$ is l.s.c., then $\{x \in K : \Phi(x,y) \subseteq C\}$ is closed.

Theorem 3.3. *Assume that the following conditions are satisfied:*

(i) $S : E \times D \to 2^E$ *and* $H : E \times D \to 2^D$ *are continuous,* $S(x,y)$ *and* $H(x,y)$ *are nonempty closed convex subsets for each* $(x,y) \in E \times D$.

(ii) $F : E \times D \times E \times E \to 2^Z$ *and* $G : E \times D \times D \times D \to 2^Z$ *are l.s.c..*

(iii) *For each* $(x,y,\alpha) \in E \times D \times E$, $F(x,y,\alpha,\alpha) \subseteq C$, $F(x,y,\cdot,\alpha)$ *is properly quasi-C-convex and* $F(x,y,\alpha,\cdot)$ *is natural quasi C-concave.*

(iv) *For each* $(x,y,\beta) \in E \times D \times D$, $G(x,y,\beta,\beta) \subseteq C$, $G(x,y,\cdot,\beta)$ *is properly quasi-C-convex and* $G(x,y,\beta,\cdot)$ *is natural quasi C-concave.*

Then $S(F,G) \neq \emptyset$.

Proof. For each $(x,y) \in E \times D$, let

$$A(x,y) = \{u \in S(x,y) : F(x,y,v,u) \subseteq C, \ \forall v \in S(x,y)\},$$

and

$$B(x,y) = \{\eta \in H(x,y) : G(x,y,\alpha,\eta) \subseteq C, \ \forall \alpha \in H(x,y)\}.$$

It follows from Lemma 3.1 that for any fixed $(x,y) \in E \times D$, $A(x,y)$ is nonempty.

We claim that for any fixed $(x,y) \in E \times D$, $A(x,y)$ is a closed subset of E. In fact, for any sequence $\{u_n\} \subseteq A(x,y)$ with $u_n \to u \in E$, since $u_n \in S(x,y)$ and $S(x,y)$ is closed, we have $u \in S(x,y)$. For any $v \in S(x,y)$ and for any $z \in F(x,y,v,u)$, since $F : E \times D \times E \times E \to 2^Z$ is l.s.c., by Lemma 2.6, there exists $z_n \in F(x,y,v,u_n)$ such that $z_n \to z$. Noting that $\{u_n\} \subseteq A(x,y)$, we have $z_n \in C$. It follows from closedness of C that $z \in C$, and so $F(x,y,v,u) \subseteq C$ for any $v \in S(x,y)$. Therefore, $u \in A(x,y)$.

Now, we claim that for any fixed $(x,y) \in E \times D$, $A(x,y)$ is convex. In fact, for any $u_1, u_2 \in A(x,y)$ and for any $t \in [0,1]$, by the definition of $A(x,y)$ and convexity of $S(x,y)$, we have

$$(1-t)u_1 + tu_2 \in S(x,y),$$

and

$$F(x,y,v,u_i) \subseteq C, \ \ i = 1,2, \ \ \forall v \in S(x,y), \tag{3.1}$$

Noting that $F(x,y,v,\cdot)$ is natural quasi C-concave, for any $t \in [0,1]$, there exists $\lambda \in [0,1]$ such that

$$F(x,y,v,tu_1 + (1-t)u_2) \subseteq \lambda F(x,y,v,u_1) + (1-\lambda) F(x,y,v,u_2) + C. \tag{3.2}$$

It follws from (3.1) and (3.2) that

$$F(x,y,v,tu_1+(1-t)u_2)\subseteq \lambda F(x,y,v,u_1)+(1-\lambda)F(x,y,v,u_2)+C\subseteq C.$$

Then $(1-t)u_1+tu_2\in A(x,y)$. Therefore, $A(x,y)$ is convex.

Next, we claim that A is u.s.c. on $E\times D$. Since E is compact, we only have to show that A is a closed mapping. For any sequence $\{(x_n,y_n)\}\subseteq E\times D$ with $(x_n,y_n)\to(x,y)\in E\times D$, let $u_n\in A(x_n,y_n)$ with $u_n\to u$. We will show that $u\in A(x,y)$. It is clear that $u_n\in S(x_n,y_n)$. Since S is u.s.c. at (x,y), by Lemma 2.7, there exist $u_0\in S(x,y)$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k}\to u_0$. Noting that $u_n\to u$, we have $u=u_0\in S(x,y)$. For any $v\in S(x,y)$, since S is l.s.c. at (x,y), by Lemma 2.6, there exists $v_n\in S(x_n,y_n)$ such that $v_n\to v$. Since $u_n\in A(x_n,y_n)$, we have

$$F(x_n,y_n,v_n,u_n)\subseteq C. \tag{3.3}$$

For any $z\in F(x,y,v,u)$, since $F:E\times D\times E\times E\to 2^Z$ is l.s.c., by Lemma 2.6, there exists $z_n\in F(x_n,y_n,v_n,u_n)$ such that $z_n\to z$. It follows from closedness of C and (3.3) that $z\in C$, and so $F(x,y,v,u)\subseteq C$ for any $v\in S(x,y)$. Therefore, $u\in A(x,y)$.

We can see that for any fixed $(x,y)\in E\times D$, $A(x,y)$ is nonempty convex closed subset of E and A is u.s.c. on $E\times D$. Similarly, for any fixed $(x,y)\in E\times D$, $B(x,y)$ is nonempty convex closed subset of D and B is u.s.c. on $E\times D$. We define $\Psi:E\times D\to 2^{E\times D}$ by

$$\Psi(x,y)=(A(x,y),B(x,y)),\quad \text{for all } (x,y)\in E\times D.$$

Then for each $(x,y)\in E\times D$, $\Psi(x,y)$ is nonempty convex closed subset of $E\times D$, and Ψ is u.s.c on $E\times D$. By Lemma 2.8 (Kakutani-Fan-Gilcksberg fixed point theorem), there exists $(x_0,y_0)\in E\times D$ such that $(x_0,y_0)\in\Psi(x_0,y_0)$. By the definition of A and B, we have

$$\begin{cases} x_0\in S(x_0,y_0), & F(x_0,y_0,x,x_0)\subseteq C, \quad \forall x\in S(x_0,y_0),\\ y_0\in H(x_0,y_0), & G(x_0,y_0,y,y_0)\subseteq C, \quad \forall y\in H(x_0,y_0).\end{cases}$$

□

From Corollary 3.2 of [21] and Remark 2.5 of [21], we can get the following lemma.

Lemma 3.4. *Let K be a nonempty compact convex subset of X and $\Psi:K\times K\to 2^{\mathbb{R}}$ be a set-valued mapping. Assume that*

(i) *for any $x\in K$, $\Psi(x,x)\subseteq\mathbb{R}_+$;*

(ii) *for any $y\in K$, $\{x\in K:\Psi(x,y)\subseteq\mathbb{R}_+\}$ is closed;*

(iii) *for any $x\in K$, $\Psi(x,\cdot)$ is $\mathbb{R}_+$-convex.*

Then there exist $x_0\in K$ such that

$$\Psi(x_0,y)\subseteq\mathbb{R}_+,\ \ \forall y\in K.$$

Remark 3.5. For any $y\in K$, if $\Psi(\cdot,y)$ is l.s.c., then $\{x\in K:\Psi(x,y)\subseteq\mathbb{R}_+\}$ is closed.

Similar to the proof of Theorem 3.3, from Lemma 3.4, we can get the following lemma.

Lemma 3.6. *Assume that the following conditions are satisfied:*

(i) *$S : E \times D \to 2^E$ and $H : E \times D \to 2^D$ are continuous, $S(x,y)$ and $H(x,y)$ are nonempty closed convex subsets for each $(x,y) \in E \times D$.*

(ii) *$f : E \times D \times E \times E \to 2^{\mathbb{R}}$ and $g : E \times D \times D \times D \to 2^{\mathbb{R}}$ are l.s.c..*

(iii) *for each $(x,y,\alpha) \in E \times D \times E$, $f(x,y,\alpha,\alpha) \subseteq \mathbb{R}_+$ and $f(x,y,\cdot,\alpha)$ is $\mathbb{R}_+$-convex; for each $(x,y,\beta) \in E \times D \times D$, $g(x,y,\beta,\beta) \subseteq \mathbb{R}_+$ and $g(x,y,\cdot,\beta)$ is $\mathbb{R}_+$-convex.*

(iv) *for each $(x,y) \in E \times D$, the sets*

$$A(x,y) = \{u \in S(x,y) : f(x,y,v,u) \subseteq \mathbb{R}_+, \ \forall v \in S(x,y)\}$$

and

$$B(x,y) = \{\eta \in H(x,y) : g(x,y,\alpha,\eta) \subseteq \mathbb{R}_+, \ \forall \alpha \in H(x,y)\}$$

are convex.

Then there exists $(x_0,y_0) \in E \times D$ such that

$$\begin{cases} x_0 \in S(x_0,y_0), & f(x_0,y_0,x,x_0) \subseteq \mathbb{R}_+, \quad \forall x \in S(x_0,y_0), \\ y_0 \in H(x_0,y_0), & g(x_0,y_0,y,y_0) \subseteq \mathbb{R}_+, \quad \forall y \in H(x_0,y_0). \end{cases}$$

Theorem 3.7. *Let $C^{\#} \neq \emptyset$. Assume that the following conditions are satisfied:*

(i) *$S : E \times D \to 2^E$ and $H : E \times D \to 2^D$ are continuous, $S(x,y)$ and $H(x,y)$ are nonempty closed convex subsets for each $(x,y) \in E \times D$.*

(ii) *$F : E \times D \times E \times E \to 2^Z$ and $G : E \times D \times D \times D \to 2^Z$ are l.s.c..*

(iii) *For each $(x,y,\alpha) \in E \times D \times E$, $F(x,y,\alpha,\alpha) \subseteq C$, $F(x,y,\cdot,\alpha)$ is C-convex and $F(x,y,\alpha,\cdot)$ is natural quasi C-concave.*

(iv) *For each $(x,y,\beta) \in E \times D \times D$, $G(x,y,\beta,\beta) \subseteq C$, $G(x,y,\cdot,\beta)$ is C-convex and $G(x,y,\beta,\cdot)$ is natural quasi C-concave.*

Then $E(F,G) \neq \emptyset$.

Proof. Since $C^{\#} \neq \emptyset$, let $z^* \in C^{\#}$. The composite functions $z^* \circ F : E \times D \times E \times E \to 2^{\mathbb{R}}$ and $z^* \circ G : E \times D \times D \times D \to 2^{\mathbb{R}}$ are defined as follows

$$z^* \circ F(x,y,\alpha,\beta) = \bigcup_{u \in F(x,y,\alpha,\beta)} \{z^*(u)\}, \quad (x,y,\alpha,\beta) \in E \times D \times E \times E,$$

and

$$z^* \circ G(x,y,\alpha,\beta) = \bigcup_{v \in G(x,y,\alpha,\beta)} \{z^*(v)\}, \quad (x,y,\alpha,\beta) \in E \times D \times D \times D.$$

It is clear that $z^* \circ F$ and $z^* \circ G$ are l.s.c., and $z^* \circ F$ and $z^* \circ G$ satisfy condition (iii) of Lemma 3.6. We claim that the composite functions $z^* \circ F$ and $z^* \circ G$ satisfy condition (iv) of Lemma 3.6. We must show that for any fixed $(x,y) \in E \times D$, the sets

$$A(x,y) = \{u \in S(x,y) : z^* \circ F(x,y,v,u) \subseteq \mathbb{R}_+, \ \forall v \in S(x,y)\}$$

and

$$B(x,y) = \{\eta \in H(x,y) : z^* \circ G(x,y,\alpha,\eta) \subseteq \mathbb{R}_+, \ \forall \alpha \in H(x,y)\}$$

are convex.

For any $u_1, u_2 \in A(x,y)$ and for any $t \in [0,1]$, by the definition of $A(x,y)$ and convexity of $S(x,y)$, we know that $(1-t)u_1 + tu_2 \in S(x,y)$ and

$$\begin{cases} z^* \circ F(x,y,v,u_1) \subseteq \mathbb{R}_+, & \forall v \in S(x,y), \\ z^* \circ F(x,y,v,u_2) \subseteq \mathbb{R}_+, & \forall v \in S(x,y). \end{cases} \tag{3.4}$$

Noting that $F(x,y,v,\cdot)$ is natural quasi C-concave, for any $t \in [0,1]$, there exists $\lambda \in [0,1]$ such that

$$F(x,y,v,tu_1 + (1-t)u_2) \subseteq \lambda F(x,y,v,u_1) + (1-\lambda) F(x,y,v,u_2) + C. \tag{3.5}$$

It follows from $z^* \in C^\#$, (3.4) and (3.5) that

$$z^* \circ F(x,y,v,tu_1 + (1-t)u_2) \subseteq \lambda z^* \circ F(x,y,v,u_1) + (1-\lambda) z^* \circ F(x,y,v,u_2) + \mathbb{R}_+ \subseteq \mathbb{R}_+.$$

Then $(1-t)u_1 + tu_2 \in A(x,y)$. Similarly, for any fixed $(x,y) \in E \times D$, $B(x,y)$ is convex. By Lemma 3.6, there exists $(x_0, y_0) \in E \times D$ such that

$$\begin{cases} x_0 \in S(x_0,y_0), & z^* \circ F(x_0,y_0,x,x_0) \subseteq \mathbb{R}_+, & \forall x \in S(x_0,y_0), \\ y_0 \in H(x_0,y_0), & z^* \circ G(x_0,y_0,y,y_0) \subseteq \mathbb{R}_+, & \forall y \in H(x_0,y_0). \end{cases} \tag{3.6}$$

We claim that

$$F(x_0,y_0,x,x_0) \cap (-C\setminus\{0\}) = \emptyset, \;\; \forall x \in S(x_0,y_0).$$

In fact, if not, then there exists $x' \in S(x_0,y_0)$ such that

$$F(x_0,y_0,x',x_0) \cap (-C\setminus\{0\}) \neq \emptyset.$$

Then there exists $\xi \in F(x_0,y_0,x',x_0)$ such that $\xi \in -C\setminus\{0\}$. Noting that $z^* \in C^\#$, we have $z^*(\xi) < 0$, which contradicts (3.6). Similarly, we have

$$G(x_0,y_0,y,y_0) \cap (-C\setminus\{0\}) = \emptyset, \;\; \forall y \in H(x_0,y_0).$$

Therefore, $(x_0,y_0) \in E(F,G)$. □

Similar to the proof of Theorem 3.7, from Remark 2.1, we can get the following theorem. We do not need the assumption of $C^\# \neq \emptyset$.

Theorem 3.8. *Assume that the following conditions are satisfied:*

(i) $S : E \times D \to 2^E$ *and* $H : E \times D \to 2^D$ *are continuous,* $S(x,y)$ *and* $H(x,y)$ *are nonempty closed convex subsets for each* $(x,y) \in E \times D$.

(ii) $F : E \times D \times E \times E \to 2^Z$ *and* $G : E \times D \times D \times D \to 2^Z$ *are l.s.c..*

(iii) *For each* $(x,y,\alpha) \in E \times D \times E$, $F(x,y,\alpha,\alpha) \subseteq C$, $F(x,y,\cdot,\alpha)$ *is* C*-convex and* $F(x,y,\alpha,\cdot)$ *is natural quasi* C*-concave.*

(iv) *For each* $(x,y,\beta) \in E \times D \times D$, $G(x,y,\beta,\beta) \subseteq C$, $G(x,y,\cdot,\beta)$ *is* C*-convex and* $G(x,y,\beta,\cdot)$ *is natural quasi* C*-concave.*

Then $W(F,G) \neq \emptyset$.

Now, we give an example to illustrate Theorems 3.7 and 3.8.

Example 3.9. Let $Z = \mathbb{R}^2$, $C = \mathbb{R}^2_+ = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\right\}$ and $X = Y = \mathbb{R}$. We denote by B_Z the closed unit ball in Z. Let $E = D = [-1, 1]$ and $S : E \times D \to 2^E$ be defined by

$$S(x, y) = \{t \in \mathbb{R} : \sin(xy) - 0.1 \leq t \leq 1\}, \quad (x, y) \in E \times D.$$

Assume that $H : E \times D \to 2^D$ is defined by

$$H(x, y) = \{t \in \mathbb{R} : -1 \leq t \leq \sin(xy) + 0.1\}, \quad (x, y) \in E \times D.$$

Let $F : E \times D \times E \times E \to 2^Z$ be defined by

$$F(x, y, \alpha, \beta) = (f_1(x, y, \alpha, \beta), f_2(x, y, \alpha, \beta)) + B_Z,$$

where

$$f_1(x, y, \alpha, \beta) = x^2 + y + 1 - \cos\left(\frac{\alpha + 1}{2}\right) + \cos\left(\frac{\beta + 1}{2}\right)$$

and

$$f_2(x, y, \alpha, \beta) = \sin x + y^2 + 2 + \alpha^2 - \cos\left(\frac{\beta + 1}{2}\right).$$

Let $G : E \times D \times D \times D \to 2^Z$ be defined by

$$G(x, y, \alpha, \beta) = (g_1(x, y, \alpha, \beta), g_2(x, y, \alpha, \beta)) + B_Z,$$

where

$$g_1(x, y, \alpha, \beta) = x^2 + \cos y - \sin\left(\frac{\alpha + 1}{2}\right) + \sin\left(\frac{\beta + 1}{2}\right)$$

and

$$g_2(x, y, \alpha, \beta) = \cos x + y^2 + 2 - \cos\alpha - \sin\left(\frac{\beta + 1}{2}\right).$$

Then it is easy to check that all conditions of Theorems 3.7 and 3.8 are satisfied. Thus, Theorem 3.7 shows that $E(F, G) \neq \emptyset$ and Theorem 3.8 implies that $W(F, G) \neq \emptyset$.

4 Applications

In this section, we will give some applications of the main results to symmetric mixed vector quasi-equilibrium problems, symmetric vector quasi-equilibrium problems and generalized semi-infinite programs with symmetric vector set-valued equilibrium constraints.

4.1 Symmetric mixed vector quasi-equilibrium problems

Let $L(X, Z)$ and $L(Y, Z)$ denote the space of all continuous linear mappings from X into Z and the space of all continuous linear mappings from Y into Z, respectively. Let $f, g : E \times D \to Z$, $T : E \times D \to L(X, Z)$ and $K : E \times D \to L(Y, Z)$ be four mappings. We introduce the following symmetric mixed vector quasi-equilibrium problem (in short, SMVQEP) consisting of finding a point $(x_0, y_0) \in E \times D$ such that

$$\begin{cases} x_0 \in S(x_0, y_0), & \langle T(x_0, y_0), x - x_0\rangle + f(x, y_0) - f(x_0, y_0) \notin -\Omega, \quad \forall x \in S(x_0, y_0), \\ y_0 \in H(x_0, y_0), & \langle K(x_0, y_0), y - y_0\rangle + g(x_0, y) - g(x_0, y_0) \notin -\Omega, \quad \forall y \in H(x_0, y_0), \end{cases}$$

where $\Omega \cup \{0\}$ is a cone in Z.

Let $E(T,K,f,g)$ denote the set of all efficient solutions of (SMVQEP), i.e. $(x_0,y_0) \in E(T,K,f,g)$. Then we have

$$\begin{cases} x_0 \in S(x_0,y_0), & \langle T(x_0,y_0), x-x_0\rangle + f(x,y_0) - f(x_0,y_0) \notin -C\setminus\{0\}, \quad \forall x \in S(x_0,y_0), \\ y_0 \in H(x_0,y_0), & \langle K(x_0,y_0), y-y_0\rangle + g(x_0,y) - g(x_0,y_0) \notin -C\setminus\{0\}, \quad \forall y \in H(x_0,y_0). \end{cases}$$

Let $W(T,K,f,g)$ denote the set of all weak efficient solutions of (SMVQEP), i.e. $(x_0,y_0) \in W(T,K,f,g)$. Then

$$\begin{cases} x_0 \in S(x_0,y_0), & \langle T(x_0,y_0), x-x_0\rangle + f(x,y_0) - f(x_0,y_0) \notin -\mathrm{int}C, \quad \forall x \in S(x_0,y_0), \\ y_0 \in H(x_0,y_0), & \langle K(x_0,y_0), y-y_0\rangle + g(x_0,y) - g(x_0,y_0) \notin -\mathrm{int}C, \quad \forall y \in H(x_0,y_0). \end{cases}$$

From Theorem 3.7, we can get the following theorem.

Theorem 4.1. *Let $C^{\#} \neq \emptyset$. Assume that the following conditions are satisfied:*

(i) *$S : E \times D \to 2^E$ and $H : E \times D \to 2^D$ are continuous, $S(x,y)$ and $H(x,y)$ are nonempty closed convex subsets for each $(x,y) \in E \times D$.*

(ii) *$f,g : E\times D \to Z$ are continuous, and $T : E\times D \to L(X,Z)$ and $K : E\times D \to L(Y,Z)$ are continuous.*

(iii) *For any $y \in D$, $f(\cdot,y)$ is C-convex; for any $x \in E$, $g(x,\cdot)$ is C-convex.*

Then $E(T,K,f,g) \neq \emptyset$.

From Theorem 3.8, we can get the following theorem.

Theorem 4.2. *Assume that the following conditions are satisfied:*

(i) *$S : E \times D \to 2^E$ and $H : E \times D \to 2^D$ are continuous, $S(x,y)$ and $H(x,y)$ are nonempty closed convex subsets for each $(x,y) \in E \times D$.*

(ii) *$f,g : E\times D \to Z$ are continuous, and $T : E\times D \to L(X,Z)$ and $K : E\times D \to L(Y,Z)$ are continuous.*

(iii) *For any $y \in D$, $f(\cdot,y)$ is C-convex; for any $x \in E$, $g(x,\cdot)$ is C-convex.*

Then $W(T,K,f,g) \neq \emptyset$.

4.2 Symmetric vector quasi-equilibrium problems

Let $f,g : E \times D \to Z$ be two mappings. We consider the following symmetric vector quasi-equilibrium problem (in short, SVQEP) consisting of finding a point $(x_0,y_0) \in E \times D$ such that

$$\begin{cases} x_0 \in S(x_0,y_0), & f(x,y_0) - f(x_0,y_0) \notin -\Omega, \quad \forall x \in S(x_0,y_0), \\ y_0 \in H(x_0,y_0), & g(x_0,y) - g(x_0,y_0) \notin -\Omega, \quad \forall y \in H(x_0,y_0), \end{cases}$$

where $\Omega \cup \{0\}$ is a cone in Z.

Let $S(f,g)$ denote the set of all strong efficient solutions of (SVQEP), i.e. $(x_0,y_0) \in S(f,g)$. Then one has

$$\begin{cases} x_0 \in S(x_0,y_0), & f(x,y_0) - f(x_0,y_0) \in C, \quad \forall x \in S(x_0,y_0), \\ y_0 \in H(x_0,y_0), & g(x_0,y) - g(x_0,y_0) \in C, \quad \forall y \in H(x_0,y_0). \end{cases}$$

Let $E(f,g)$ denote the set of all efficient solutions of (SVQEP), i.e. $(x_0,y_0)\in E(f,g)$. Then

$$\begin{cases} x_0\in S(x_0,y_0), & f(x,y_0)-f(x_0,y_0)\notin -C\setminus\{0\}, \quad \forall x\in S(x_0,y_0),\\ y_0\in H(x_0,y_0), & g(x_0,y)-g(x_0,y_0)\notin -C\setminus\{0\}, \quad \forall y\in H(x_0,y_0).\end{cases}$$

Let $W(f,g)$ denote the set of all weak efficient solutions of (SVQEP), i.e. $(x_0,y_0)\in W(f,g)$. Then we have

$$\begin{cases} x_0\in S(x_0,y_0), & f(x,y_0)-f(x_0,y_0)\notin -\mathrm{int}C, \quad \forall x\in S(x_0,y_0),\\ y_0\in H(x_0,y_0), & g(x_0,y)-g(x_0,y_0)\notin -\mathrm{int}C, \quad \forall y\in H(x_0,y_0).\end{cases}$$

From Theorem 3.3, we can get the following theorem.

Theorem 4.3 ([18])**.** *Assume that the following conditions are satisfied:*

(i) *$S:E\times D\to 2^E$ and $T:E\times D\to 2^D$ are continuous, $S(x,y)$ and $T(x,y)$ are nonempty closed convex subsets for each $(x,y)\in E\times D$.*

(ii) *$f,g:E\times D\to Z$ are continuous.*

(iii) *For any $y\in D$, $f(\cdot,y)$ is properly quasi-C-convex; for any $x\in E$, $g(x,\cdot)$ is properly quasi-C-convex.*

Then $S(f,g)\neq\emptyset$.

Let $T=0$ and $K=0$. By Theorem 4.1, we can get the following theorem.

Theorem 4.4. *Let $C^{\#}\neq\emptyset$. Assume that the following conditions are satisfied:*

(i) *$S:E\times D\to 2^E$ and $T:E\times D\to 2^D$ are continuous, $S(x,y)$ and $T(x,y)$ are nonempty closed convex subsets for each $(x,y)\in E\times D$.*

(ii) *$f,g:E\times D\to Z$ are continuous.*

(iii) *For any $y\in D$, $f(\cdot,y)$ is C-convex; for any $x\in E$, $g(x,\cdot)$ is C-convex.*

Then $E(f,g)\neq\emptyset$.

From Theorem 4.2, we can get the following theorem.

Theorem 4.5. *Assume that the following conditions are satisfied:*

(i) *$S:E\times D\to 2^E$ and $T:E\times D\to 2^D$ are continuous, $S(x,y)$ and $T(x,y)$ are nonempty closed convex subsets for each $(x,y)\in E\times D$.*

(ii) *$f,g:E\times D\to Z$ are continuous.*

(iii) *For any $y\in D$, $f(\cdot,y)$ is C-convex; for any $x\in E$, $g(x,\cdot)$ is C-convex.*

Then $W(f,g)\neq\emptyset$.

In fact, we can obtain $E(f,g)\neq\emptyset$ and $W(f,g)\neq\emptyset$ under weaker conditions. We give the following lemma.

Lemma 4.6. *Assume that the following conditions are satisfied:*

(i) $S : E \times D \to 2^E$ *and* $H : E \times D \to 2^D$ *are continuous,* $S(x,y)$ *and* $H(x,y)$ *are nonempty closed convex subsets for each* $(x,y) \in E \times D$.

(ii) $\varphi, \beta : E \times D \to R$ *are continuous.*

(iii) *for each* $(x,y) \in E \times D$, *the sets*

$$A(x,y) = \{u \in S(x,y) : \varphi(u,y) = \min\{\varphi(v,y) : v \in S(x,y)\}\}$$

and

$$B(x,y) = \{\eta \in H(x,y) : \beta(x,\eta) = \min\{\beta(x,\alpha) : \alpha \in H(x,y)\}\}$$

are convex.

Then there exists $(x_0,y_0) \in E \times D$ *such that*

$$\begin{cases} x_0 \in S(x_0,y_0), & \varphi(x,y_0) \geq \varphi(x_0,y_0), \quad \forall x \in S(x_0,y_0), \\ y_0 \in H(x_0,y_0), & \beta(x_0,y) \geq \beta(x_0,y_0), \quad \forall y \in H(x_0,y_0). \end{cases}$$

Proof. Since $S(x,y)$ is nonempty and compact, it is clear that for any fixed $(x,y) \in E \times D$, $A(x,y)$ is nonempty.

We claim that for any fixed $(x,y) \in E \times D$, $A(x,y)$ is closed subset of E. In fact, for any sequence $\{u_n\} \subseteq A(x,y)$ with $u_n \to u \in E$, since $u_n \in S(x,y)$ and $S(x,y)$ is closed, $u \in S(x,y)$. It follows from $\varphi(u_n,y) \leq \varphi(v,y)$ for all $v \in S(x,y)$ and the continuity of φ that $\varphi(u,y) \leq \varphi(v,y)$ for all $v \in S(x,y)$. Thus $u \in A(x,y)$.

Next, we claim that A is u.s.c on $E \times D$. Since E is compact, we only have to show that A is a closed mapping. For any sequence $\{(x_n,y_n)\} \subseteq E \times D$ with $(x_n,y_n) \to (x,y) \in E \times D$, let $u_n \in A(x_n,y_n)$ and $u_n \to u$. We will show that $u \in A(x,y)$. By Lemma 2.7, there exist $u_0 \in S(x,y)$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to u_0$. Noting that $u_n \to u$, we have $u = u_0 \in S(x,y)$. For any $v \in S(x,y)$, it follows from lower semicontinuity of S and Lemma 2.6 that there exists a sequence $\{v_n\}$ with $v_n \in S(x_n,y_n)$ such that $v_n \to v$. Since $u_n \in A(x_n,y_n)$, we have $\varphi(u_n,y_n) \leq \varphi(v_n,y_n)$. It follows from continuity of φ that $\varphi(u,y) \leq \varphi(v,y)$. Thus $\varphi(u,y) \leq \varphi(v,y)$ for all $v \in S(x,y)$. Therefore, $u \in A(x,y)$.

Similar to the proof of Theorem 3.3, we can see that there exists $(x_0,y_0) \in E \times D$ such that

$$\begin{cases} x_0 \in S(x_0,y_0), & \varphi(x,y_0) \geq \varphi(x_0,y_0), \quad \forall x \in S(x_0,y_0), \\ y_0 \in H(x_0,y_0), & \beta(x_0,y) \geq \beta(x_0,y_0), \quad \forall y \in H(x_0,y_0). \end{cases}$$

□

Theorem 4.7. *Let* $C^\# \neq \emptyset$. *Assume that the following conditions are satisfied:*

(i) $S : E \times D \to 2^E$ *and* $T : E \times D \to 2^D$ *are continuous,* $S(x,y)$ *and* $T(x,y)$ *are nonempty closed convex subsets for each* $(x,y) \in E \times D$.

(ii) $f, g : E \times D \to Z$ *are continuous.*

(iii) *For any* $y \in D$, $f(\cdot,y)$ *is natural quasi* C*-convex; for any* $x \in E$, $g(x,\cdot)$ *is natural quasi* C*-convex.*

Then $E(f,g) \neq \emptyset$.

Proof. Since $C^{\#} \neq \emptyset$, let $z^* \in C^{\#}$. We only prove that the continuous composite functions $z^* \circ f$ and $z^* \circ g$ satisfy condition (iii) of Lemma 4.6. We show that, for any fixed $(x,y) \in E \times D$, the sets

$$A(x,y) = \{u \in S(x,y) : z^* \circ f(u,y) = \min\{z^* \circ f(v,y) : v \in S(x,y)\}\}$$

and

$$B(x,y) = \{\eta \in T(x,y) : z^* \circ g(x,\eta) = \min\{z^* \circ g(x,\alpha) : \alpha \in T(x,y)\}\}$$

are convex.

For any $u_1, u_2 \in A(x,y)$ and for any $t \in [0,1]$, by the definition of $A(x,y)$ and convexity of $S(x,y)$, we have

$$(1-t)u_1 + tu_2 \in S(x,y),$$

and

$$z^* \circ f(u_1,y) = z^* \circ f(u_2,y) = \min\{z^* \circ f(v,y) : v \in S(x,y)\}.$$

Noting that $f(\cdot,y)$ is natural quasi C-convex and $z^* \in C^{\#}$, there exists $\lambda \in [0,1]$ such that

$$\begin{aligned} \min\{z^* \circ f(v,y) : v \in S(x,y)\} &\leq z^* \circ f((1-t)u_1 + tu_2, y) \\ &\leq (1-\lambda) z^* \circ f(u_1,y) + \lambda z^* \circ f(u_2,y) \\ &= \min\{z^* \circ f(v,y) : v \in S(x,y)\}. \end{aligned}$$

Then $(1-t)u_1 + tu_2 \in A(x,y)$. Similarly, for any fixed $(x,y) \in E \times D$, $B(x,y)$ is convex. □

Theorem 4.8 ([12])**.** *Assume that the following conditions are satisfied:*

(i) $S : E \times D \to 2^E$ *and* $T : E \times D \to 2^D$ *are continuous,* $S(x,y)$ *and* $T(x,y)$ *are nonempty closed convex subsets for each* $(x,y) \in E \times D$.

(ii) $f, g : E \times D \to Z$ *are continuous.*

(iii) *For any* $y \in D$, $f(\cdot,y)$ *is natural quasi* C*-convex; for any* $x \in E$, $g(x,\cdot)$ *is natural quasi* C*-convex.*

Then $W(f,g) \neq \emptyset$.

Next, we give an example to illustrate Theorems 4.7 and 4.8.

Example 4.9. Let $Z = \mathbb{R}^2$, $C = \mathbb{R}^2_+ = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$ and $X = Y = \mathbb{R}$. Let $E = D = [-1,1]$ and $S : E \times D \to 2^E$ be defined by

$$S(x,y) = \{t \in \mathbb{R} : \sin(xy) - 0.1 \leq t \leq 1\}, \quad (x,y) \in E \times D.$$

Assume that $H : E \times D \to 2^D$ is defined by

$$H(x,y) = \{t \in \mathbb{R} : -1 \leq t \leq \sin(xy) + 0.1\}, \quad (x,y) \in E \times D$$

and $f : E \times D \to Z$ is defined by

$$f(x,y) = \left((x-y)^2 + 2(x-y) + y^2\cos y + y + 1, x^2 + 3xy - y^2 \sin y + 2y\right).$$

Let $g : E \times D \to Z$ be defined by

$$g(x, y) = \left(2x^2 \cos x - xy + y^2 + \sin x - 3, 3x^3 + x^2 y + 2y^2 + \cos x + 2\right).$$

Then we know that $S : E \times D \to 2^E$ and $H : E \times D \to 2^D$ are continuous, for every $(x, y) \in E \times D$, $S(x, y)$ and $H(x, y)$ are nonempty closed convex subsets, and $f, g : E \times D \to Z$ are continuous. Thus, it is easy to observe that, for any $y \in D$, $f(\cdot, y)$ is natural quasi C-convex; for any $x \in E$, $g(x, \cdot)$ is natural quasi C-convex. Theorem 4.7 shows that $E(f, g) \neq \emptyset$ and Theorem 4.8 implies that $W(f, g) \neq \emptyset$.

4.3 Generalized semi-infinite programs

In this subsection, we give some existence theorems of solutions to generalized semi-infinite programs. Let Λ be a normed vector space ordered by a closed convex pointed cone $P \subseteq \Lambda$ with $\mathrm{int}P \neq \emptyset$ and $\Psi : E \times D \to 2^\Lambda$ be a u.s.c. mapping with nonempty compact values.

Definition 4.10. Let $K \subseteq \Lambda$ be a nonempty set and the set of all weak minimal points of K with respect to the ordering cone P be defined as

$$\mathrm{wMin}_P(K) = \{x \in K : (x - K) \cap (\mathrm{int}P) = \emptyset\}.$$

We consider the following generalized semi-infinite programs.

(GSIP1) Generalized semi-infinite program with constraint $K = S(F, G)$:

$$\mathrm{wMin}_P \Psi(K),$$

where $S(F, G)$ is the set of all strong efficient solutions for (SVSQEP).

(GSIP2) Generalized semi-infinite program with constraint $K = W(F, G)$:

$$\mathrm{wMin}_P \Psi(K),$$

where $W(F, G)$ is the set of all weak efficient solutions for (SVSQEP).

(GSIP3) Generalized semi-infinite program with constraint $K = E(F, G)$:

$$\mathrm{wMin}_P \Psi(K),$$

where $E(F, G)$ is the set of all efficient solutions for (SVSQEP).

Lemma 4.11 ([25]). *Assume that A is a nonempty compact subset of a real topological vector space V and D is a closed convex cone in V with $D \neq V$. Then $\mathrm{wMin}_D(A) \neq \emptyset$.*

Lemma 4.12. *Assume that the following conditions are satisfied:*

(i) *$S : E \times D \to 2^E$ and $H : E \times D \to 2^D$ are continuous, $S(x, y)$ and $H(x, y)$ are nonempty closed convex subsets for each $(x, y) \in E \times D$;*

(ii) *$F : E \times D \times E \times E \to 2^Z$ and $G : E \times D \times D \times D \to 2^Z$ are two l.s.c. set-valued mappings.*

Then $S(F, G)$ is closed. Moreover, $W(F, G)$ is closed.

Proof. For any sequence $\{(x_n, y_n)\} \subseteq S(F,G)$ with $(x_n, y_n) \to (x,y)$, we have

$$\begin{cases} x_n \in S(x_n, y_n), & F(x_n, y_n, x, x_n) \subseteq C, \quad \forall x \in S(x_n, y_n), \\ y_n \in H(x_n, y_n), & G(x_n, y_n, y, y_n) \subseteq C, \quad \forall y \in H(x_n, y_n). \end{cases} \tag{4.1}$$

Noting that $x_n \in S(x_n, y_n)$, since S is u.s.c. at (x,y), by Lemma 2.7, there exist $x_0 \in S(x,y)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$. It follows from $x_n \to x$ that $x = x_0 \in S(x,y)$. For any $v \in S(x,y)$, since S is l.s.c. at (x,y), Lemma 2.6 shows that there exists $v_n \in S(x_n, y_n)$ such that $v_n \to v$. By (4.1), we have

$$F(x_n, y_n, v_n, x_n) \subseteq C. \tag{4.2}$$

For any $z \in F(x,y,v,x)$, since $F : E \times D \times E \times E \to 2^Z$ is l.s.c., by Lemma 2.6, there exists $z_n \in F(x_n, y_n, v_n, x_n)$ such that $z_n \to z$. It follows from closedness of C and (4.2) that $z \in C$, and so $F(x,y,v,x) \subseteq C$ for any $v \in S(x,y)$. Therefore, one has

$$x \in S(x,y), \quad F(x,y,v,x) \subseteq C, \quad \forall v \in S(x,y).$$

By the similar arguments, we have

$$y \in H(x,y), \quad G(x,y,u,y) \subseteq C, \quad \forall u \in H(x,y).$$

Thus, $(x,y) \subseteq S(F,G)$ and so $S(F,G)$ is closed. Similarly, we can show that $W(F,G)$ is closed. □

Lemma 4.13. *Assume that the following conditions are satisfied:*

(i) *For each $(x,y) \in E \times D$, $S(x,y)$ and $H(x,y)$ are nonempty convex subsets;*

(ii) *For each $(x,y,\alpha) \in E \times D \times E$, $F(x,y,\alpha,\alpha) \subseteq C$, $0 \in F(x,y,\alpha,\alpha)$ and $F(x,y,\cdot,\alpha)$ is strictly C-convex;*

(iii) *For each $(x,y,\beta) \in E \times D \times D$, $G(x,y,\beta,\beta) \subseteq C$, $0 \in G(x,y,\beta,\beta)$ and $G(x,y,\cdot,\beta)$ is strictly C-convex.*

Then $E(F,G) = W(F,G)$.

Proof. It suffices to prove that $W(F,G) \subseteq E(F,G)$. Suppose that $W(F,G) \not\subseteq E(F,G)$. Then there exists $(x_0, y_0) \in W(F,G)$ such that $(x_0, y_0) \notin E(F,G)$. Thus,

$$\begin{cases} x_0 \in S(x_0, y_0), & F(x_0, y_0, x, x_0) \cap (-\text{int}C) = \emptyset, \quad \forall x \in S(x_0, y_0), \\ y_0 \in H(x_0, y_0), & G(x_0, y_0, y, y_0) \cap (-\text{int}C) = \emptyset, \quad \forall y \in H(x_0, y_0). \end{cases} \tag{4.3}$$

Noting that $(x_0, y_0) \notin E(F,G)$, without loss of generality, we can assume that there exists $\bar{x} \in S(x_0, y_0)$ such that

$$F(x_0, y_0, \bar{x}, x_0) \cap (-C \setminus \{0\}) \neq \emptyset.$$

Then there exists

$$z_0 \in F(x_0, y_0, \bar{x}, x_0) \tag{4.4}$$

such that

$$z_0 \in -C \setminus \{0\}. \tag{4.5}$$

We claim that $\bar{x} \neq x_0$. In fact, if not, by (4.4) and condition (ii), we know that $z_0 \in F(x_0, y_0, \bar{x}, x_0) \subseteq C$. It follows from (4.5) that $z_0 \in C \cap (-C) = \{0\}$, which contradicts (4.5).

Since $F(x_0, y_0, \cdot, x_0)$ is strictly C-convex, for any $t \in (0,1)$, we have

$$tF(x_0, y_0, \bar{x}, x_0) + (1-t) F(x_0, y_0, x_0, x_0) \subseteq F(x_0, y_0, t\bar{x} + (1-t) x_0, x_0) + \text{int}C. \quad (4.6)$$

It clear that $t\bar{x} + (1-t) x_0 \in S(x_0, y_0)$. It follows from (4.4), (4.6) and $0 \in F(x_0, y_0, x_0, x_0)$ that there exist $z_t \in F(x_0, y_0, t\bar{x} + (1-t) x_0, x_0)$ and $c_0 \in \text{int}C$ such that

$$tz_0 + (1-t) 0 = z_t + c_0.$$

Noting that (4.5) and $c_0 \in \text{int}C$, we have $z_t \in -\text{int}C$. Thus,

$$z_t \in F(x_0, y_0, t\bar{x} + (1-t) x_0, x_0) \cap (-\text{int}C),$$

which contradicts (4.3). □

Theorem 4.14. *Assume that the following conditions are satisfied:*

(i) *$S : E \times D \to 2^E$ and $H : E \times D \to 2^D$ are continuous, $S(x,y)$ and $H(x,y)$ are nonempty closed convex subsets for each $(x,y) \in E \times D$;*

(ii) *$F : E \times D \times E \times E \to 2^Z$ and $G : E \times D \times D \times D \to 2^Z$ are two l.s.c. set-valued mappings;*

(iii) *For each $(x,y,\alpha) \in E \times D \times E$, $F(x,y,\alpha,\alpha) \subseteq C$, $F(x,y,\cdot,\alpha)$ is properly quasi-C-convex and $F(x,y,\alpha,\cdot)$ is natural quasi C-concave;*

(iv) *For each $(x,y,\beta) \in E \times D \times D$, $G(x,y,\beta,\beta) \subseteq C$, $G(x,y,\cdot,\beta)$ is properly quasi-C-convex and $G(x,y,\beta,\cdot)$ is natural quasi C-concave.*

Then (GSIP1) has a solution.

Proof. Theorem 3.3 shows that $S(F,G) \neq \emptyset$. It follows from Lemma 4.12 that $S(F,G)$ is closed. Noting that $S(F,G) \subseteq E \times D$ and $E \times D$ is compact, we can see that $S(F,G)$ is compact. Since $\Psi : E \times D \to 2^\Lambda$ be a u.s.c. mapping with nonempty compact values, we can see that $\Psi(S(F,G))$ is nonempty and compact (see [3] P. 112). It follows from Lemma 4.11 that $\text{wMin}_P \Psi(S(F,G)) \neq \emptyset$. Therefore, (GSIP1) has a solution. □

Similar to the proof of Theorem 4.14, from Theorem 3.8 and Lemma 4.12, we can get the following theorem.

Theorem 4.15. *Assume that the following conditions are satisfied:*

(i) *$S : E \times D \to 2^E$ and $H : E \times D \to 2^D$ are continuous, $S(x,y)$ and $H(x,y)$ are nonempty closed convex subsets for each $(x,y) \in E \times D$;*

(ii) *$F : E \times D \times E \times E \to 2^Z$ and $G : E \times D \times D \times D \to 2^Z$ are two l.s.c. set-valued mappings;*

(iii) *For each $(x,y,\alpha) \in E \times D \times E$, $F(x,y,\alpha,\alpha) \subseteq C$, $F(x,y,\cdot,\alpha)$ is C-convex and $F(x,y,\alpha,\cdot)$ is natural quasi C-concave;*

(iv) *For each* $(x, y, \beta) \in E \times D \times D$, $G(x, y, \beta, \beta) \subseteq C$, $G(x, y, \cdot, \beta)$ *is* C*-convex and* $G(x, y, \beta, \cdot)$ *is natural quasi* C*-concave.*

Then (GSIP2) has a solution.

Similar to the proof of Theorem 4.14, by Theorem 3.7, Lemmas 4.12 and 4.13, we can get the following theorem.

Theorem 4.16. *Let* $C^{\#} \neq \emptyset$. *Assume that the following conditions are satisfied:*

(i) $S : E \times D \to 2^E$ *and* $H : E \times D \to 2^D$ *are continuous,* $S(x, y)$ *and* $H(x, y)$ *are nonempty closed convex subsets for each* $(x, y) \in E \times D$;

(ii) $F : E \times D \times E \times E \to 2^Z$ *and* $G : E \times D \times D \times D \to 2^Z$ *are two l.s.c. set-valued mappings;*

(iii) *For each* $(x, y, \alpha) \in E \times D \times E$, $F(x, y, \alpha, \alpha) \subseteq C$, $0 \in F(x, y, \alpha, \alpha)$, $F(x, y, \cdot, \alpha)$ *is strictly* C*-convex and* $F(x, y, \alpha, \cdot)$ *is natural quasi* C*-concave;*

(iv) *For each* $(x, y, \beta) \in E \times D \times D$, $G(x, y, \beta, \beta) \subseteq C$, $0 \in G(x, y, \beta, \beta)$, $G(x, y, \cdot, \beta)$ *is strictly* C*-convex and* $G(x, y, \beta, \cdot)$ *is natural quasi* C*-concave.*

Then (GSIP3) has a solution.

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