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THE ℓ_q-BALL PROJECTED GRADIENT METHOD FOR QUADRATIC COMPRESSIVE SENSING

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Abstract: In this paper we employ the $\ell_q(0 < q < 1)$-ball constrained least squares method to find the sparse signal in the framework of Quadratic Compressive Sensing which extends the usual Compressive Sensing. We focus on the projected gradient descent algorithm to solve the corresponding optimization problem. Since the projection onto the ℓ_q-ball is nonsmooth, nonconvex and non-Lipschitz, we first transform it into a smooth optimization problem which can be solved by many existing algorithms. Second, we derive a fixed point equation and use it to construct a projected gradient algorithm to compute the ℓ_q-ball constrained least squares problem. Finally, we discuss the convergence of the proposed algorithm.

Key words: *ℓ_q-ball constrained least square, fixed point equation, projected gradient algorithm*

Mathematics Subject Classification: *90C26, 90C30, 90C90*

1 Introduction

Compressive sensing has been intensively studied and widely used in the last decade. The main goal is to reconstruct a sparse signal from the samples. More recently, the theory has been extended to nonlinear compressive sensing. In particular, the so-called Quadratic Compressive Sensing (QCS) aims to find the sparse signal x^* to the problem

$$\min_{x\in\mathbb{R}^n} \|x\|_0 \quad \text{subject to} \quad y_i = x^T A_i x + b_i^T x + c_i, \quad i = 1, \ldots, m, \tag{1.1}$$

where $y_i, c_i \in \mathbb{R}, b_i \in \mathbb{R}^n,\ A_i \in \mathbb{S}^{n\times n}, i = 1, \ldots, m$ are given and $\|x\|_0$ is the number of nonzero entries in x. The literature on nonlinear compressive sensing is very limited and the readers are referred to [2, 3, 11, 14, 15].

It is clear that model (1.1) is the generalization of the compressive sensing problem

$$\min_{x\in\mathbb{R}^n} \|x\|_0 \quad \text{subject to} \quad y_i = b_i^T x, \ \ y_i \in \mathbb{R},\ b_i \in \mathbb{R}^n, \quad i = 1, \ldots, m, \tag{1.2}$$

which is a NP-hard problem. To avoid the combinatorial computation cost of (1.2), a number of approaches to approximate its solution have been proposed. A popular scheme is to replace the ℓ_0-norm of (1.2) with the ℓ_q-norm:

$$\min_{x\in\mathbb{R}^n} \|x\|_q \quad \text{subject to} \quad y_i = b_i^T x, \ \ y_i \in \mathbb{R},\ b_i \in \mathbb{R}^n, \quad i = 1, \ldots, m,$$

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where $q \in (0, 1]$. In particular, as $q = 1$ the ℓ_1-minimization is referred to as basis pursuit [5]. However, ℓ_q-minimization with $q \in (0, 1)$ produces exact reconstruction with fewer measurements [4], and increases robustness to noise and image non-sparsity [13].

An alternative approach to solve (1.2) is to consider the following sparsity-constrained least squares problem,

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{m} (b_i^T x - y_i)^2 \quad \text{subject to} \quad \|x\|_0 \leq s, \tag{1.3}$$

where s is a positive integer. Obviously, the well-know LASSO, i.e.,

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{m} (b_i^T x - y_i)^2 \quad \text{subject to} \quad \|x\|_1 \leq r,$$

can be regard as a convex relaxation of the above minimization. Another is the following non-convex relaxation,

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{m} (b_i^T x - y_i)^2 \quad \text{subject to} \quad \|x\|_q \leq r,$$

where $q \in (0, 1)$ and $r > 0$. To solve the latter, the q-regularized method has attracted many research efforts in the field of optimization, including optimality conditions and algorithms, see, e.g., [6, 8, 16], and references therein. By considering the ℓ_q-constrained least square problems in the framework of Compressive Sensing,

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{m} (b_i^T x - y_i)^2 \quad \text{subject to} \quad \|x\|_q \leq R^*,$$

where $0 \leq q \leq 1$ and $R^* = \|x^*\|_q$ is given, [1] studied the performance of the Projected Gradient Descent(PGD) algorithm and provided its convergence guarantees, that include and generalize the existing results for the Iterative Hard Thresholding algorithm and provide a new accuracy guarantee for the Iterative Soft Thresholding algorithm as special cases. [9] investigated the properties of ℓ_q norm within a projection framework and used its key properties to arrive at an algorithm for ℓ_q norm projection onto the non-negative simplex.

Inspired by their works, we will consider the following ℓ_q-ball constrained least square problem to discuss QCS,

$$\min_{x \in \mathbb{R}^n} f(x), \qquad \text{subject to} \quad \|x\|_q^q \leq r, \tag{1.4}$$

where $f(x) = \sum_{i=1}^{m} \left(x^T A_i x + b_i^T x + a_i - y_i \right)^2$ and $r > 0$. Recall that [11] used a lifting technique and the convex surrogate for the ℓ_0 norm to relax the problem (1.1) into a convex semidefinite program. [2] dealt the problem of minimizing a general continuously differentiable function subject to sparsity-constraints which includes the sparsity-constrained least squares problem for QCS. [3] generalized the iterative hard thresholding algorithm for solving (1.3) to general nonlinear optimization with sparsity-constraint. To the best of our knowledge, there is no literature to discuss problem (1.4). An important issue is how to solve the minimization. Here, we present two different optimality criteria which are based on the notions of normal cone and fixed point theory respectively, and show that the latter is stronger than the former. Subsequently, we employ the fixed point equation to construct a

q-ball projected gradient algorithm. It is worth mentioning that we equivalently transform the q-ball projection into a simply smooth constraint problem and then use the existing algorithm to solve it.

For convenience, we now give some notation as follows. For any n-dimensional vector $x = (x_1, \ldots, x_n)^T$ and $q \in (0, 1)$, denote $|x| = (|x_1|, \ldots, |x_n|)^T, |x|^{\frac{1}{q}} = (|x_1|^{\frac{1}{q}}, \ldots, |x_n|^{\frac{1}{q}})^T$, $\|x\| = (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}}$ and $\|x\|_q = (\sum_{i=1}^n |x_i|^q)^{\frac{1}{q}}$. For any $q \in (0, 1)$ and $r > 0$, denote $\mathcal{B}_{q,r} = \{x \in \mathbb{R}^n : \|x\|_q^q \leq r\}$ and $\partial\mathcal{B}_{q,r} = \{x \in \mathbb{R}^n : \|x\|_q^q = r\}$. Denote $\varphi(x) = \|x\|_q^q$. For a set S, we denote its closure by $\mathrm{cl}(S)$ and its interior by $\mathrm{int}(S)$.

This paper is organized as follows. In section 2, we introduce the projection onto the ℓ_q-ball and transform the corresponding optimization into a smooth problem. Subsequently, we provide two optimality criteria and establish a fixed point equation in Section 3. We construct the projected gradient algorithm and discuss its convergence in Section 4.

2 Projection onto ℓ_q ball

For $q \in (0, 1)$ and $x \in \mathbb{R}^n$, consider

$$\min \frac{1}{2}\|x - u\|^2 \ \text{ subjec to } u \in \mathcal{B}_{q,r}. \tag{2.1}$$

Since the object function is continuous and the constrained set $\mathcal{B}_{q,r}$ is compact, there exists at least one minimizer of the problem (2.1). As mentioned by [1], one can select one of the solutions of the minimization problem (2.1) as a projection operator which is denoted by $P_{\mathcal{B}_{q,r}}(\cdot)$. On the other hand, [1] showed that the operator $P_{\mathcal{B}_{q,r}}(\cdot)$ satisfies the following properties.

Lemma 2.1. *Let $x^\perp = (x_1^\perp, \ldots, x_n^\perp)^T$ be an optimal solution of the problem (2.1). Then,*
(i) $|x_i^\perp| \leq |x_i|$ for all $i = 1, \ldots, n$ while there is at most one $i = 1, \ldots, n$ such that $|x_i^\perp| < \frac{1-q}{2-q}|x_i|$,
(ii) if $|x_i| > |x_j|$ for some $i, j = 1, \ldots, n$ then $|x_i^\perp| \geq |x_j^\perp|$, and
(iii) there exists $\lambda \geq 0$ such that

$$\mathrm{sign}(x_i^\perp)|x_i^\perp|^{1-q}(x_i - x_i^\perp) = q\lambda \tag{2.2}$$

for all $i \in \Gamma^\perp := \mathrm{supp}(x^\perp)$.
(iv) for $|T| \geq T_0 := (2 - q)(\lambda q(1 - q)^{q-1})^{\frac{1}{2-q}}$ the equation

$$\mathrm{sign}(t)|t|^{1-q}(T - t) = q\lambda \tag{2.3}$$

has two roots t_l and t_r satisfying $|t_l| \in (0, \frac{1-q}{2-q}T]$ and $|t_r| \in [\frac{1-q}{2-q}T, \infty)$.

Moreover, [7] stated that there exists an unique implicit function $h_{\lambda,q}(T)$ on (T_0, ∞) such that $h_{\lambda,q}(T)$ satisfying the equation (2.3) when $t \geq 0$ and the properties as follows: the function $h_{\lambda,q}(T)$ is continuous differentiable, $h'_{\lambda,q}(T) = \frac{1}{1+\lambda q(q-1)|h_{\lambda,q}(T)|^{q-2}}$ and $h_{\lambda,q}(T)$ is strictly increasing. Especially, [16] showed that $h_{\lambda,1/2}(T) = \frac{2}{3}T\big(1 + \cos\big(\frac{2\pi}{3} - \frac{2}{3}\phi_\lambda(T)\big)\big)$ and with $\phi_\lambda(T) = \arccos\big(\frac{\lambda}{4}(\frac{|T|}{3})^{-3/2}\big)$.

Obviously, (iv) of Lemma 2.1 states that $x_i^\perp = 0$ if $|x_i| < T_0$. Subsequently, one needs to analyze the selection of $x_i^\perp$ for $|x_i| \geq T_0$. Since the equation (2.2) has two nonzero roots, one can denote the root with the absolute value beyond $\frac{1-q}{2-q}|x_i|$ by x_{i+}; otherwise, by x_{i-}. Moreover, [9] proved that the root x_{i-} corresponding to any x_i is not part of the optimization solution for the minimization (2.1) when it has non-negative constraints $u \in \mathbb{R}^n_+$, except possibly for the smallest x_i among those with non-zero optimal projection. In their paper, they also presented an algorithm outline for $\ell_q(0 < q < 1)$-ball projection onto the non-negative simplex.

In fact, the ℓ_q-ball projection problem can be equivalently transformed into a simply smooth constraint optimization problem, which can be solved by many existing optimization algorithms. In order to give our methods, we first consider the following three minimization problems

$$\min \frac{1}{2}\||x| - u\|^2 \ \text{ subjec to } u \in \mathcal{B}_{q,r}, u \in \mathbb{R}^n_+, \tag{2.4}$$

$$\min \frac{1}{2}\|z - u\|^2 \ \text{ subjec to } u \in \mathcal{B}_{q,r}, u \in \mathbb{R}^n_+ \tag{2.5}$$

and

$$\min \frac{1}{2}\|z - v^{\frac{1}{q}}\|^2 \ \text{ subjec to } \|v\|_1 \leq r,\ v \in \mathbb{R}^n_+ \tag{2.6}$$

for any given $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^n_+$.

Lemma 2.2. *(i) If $x^\perp$ is a minimizer of (2.1), then $|x^\perp|$ is a minimizer of (2.4). Conversely, if $|x|^\perp_+$ is a minimizer of (2.4), then, $|x|^\perp_+ \circ \operatorname{sign}(x)$ is a minimizer of (2.1).*
(ii) If $\hat{u}$ is a minimizer of (2.5), then $\hat{u}^q$ is a minimizer of (2.6). Conversely, if $\hat{v}$ is a minimizer of (2.6), then, $\hat{v}^{\frac{1}{q}}$ is a minimizer of (2.5).

Proof. (i) Since $\||x| - |y|\|^2 \leq \|x - y\|^2$ for any $x, y \in \mathbb{R}^n$, we have for any $u \geq 0$ and $\|u\|_q^q \leq r$,

$$\||x| - |x^\perp|\|^2 \leq \|x - x^\perp\|^2 \leq \|x - u \circ \operatorname{sign}(x)\|^2 = \||x| - u\|^2$$

where the second inequality derives from the assumption that $x^\perp$ is a minimizer of (2.1). Notice that $|x^\perp| \geq 0$ and $\||x^\perp|\|_q^q = \|x^\perp\|_q^q \leq r$. Then, $|x^\perp|$ is a minimizer of (2.4).

Similarly, for any $u \in \mathbb{R}^n$ satisfying $\|u\|_q^q \leq r$, we obtain

$$\|x - |x|^\perp_+ \circ \operatorname{sign}(x)\|^2 = \||x| - |x|^\perp_+\|^2 \leq \||x| - |u|\|^2 \leq \|x - u\|^2,$$

which together with $\||x|^\perp_+ \circ \operatorname{sign}(x)\|_q^q = \||x|^\perp_+\|_q^q \leq r$ implies that $|x|^\perp_+ \circ \operatorname{sign}(x)$ is a minimizer of (2.1). Therefore, the first result (i) holds.

(ii) For any $v \geq 0$ satisfying $\|v\|_1 \leq r$ we have $v^{\frac{1}{q}} \geq 0$ and $\|v^{\frac{1}{q}}\|_q^q = \|v\|_1 \leq r$. Since $\hat{u}$ is a minimizer of (2.5), we have

$$\|z - (\hat{u}^q)^{\frac{1}{q}}\|^2 = \|z - \hat{u}\|^2 \leq \|z - v^{\frac{1}{q}}\|^2,$$

which yields that $\hat{u}^q$ is a minimizer of (2.6). That is, we obtain the first result of (ii).

Similarly, noting that any $u \geq 0$ satisfying $\|u\|_q^q \leq r$ implies that $u^q \geq 0$ and $\|u^q\|_1 = \|u\|_q^q \leq r$, one can conclude from the assumption $\hat{v}$ is a minimizer of (2.6) that

$$\|z - \hat{v}^{\frac{1}{q}}\|^2 \leq \|z - (u^q)^{\frac{1}{q}}\|^2 = \|z - u\|^2,$$

which yields that $\hat{v}^{\frac{1}{q}}$ is a minimizer of (2.5). Then, we get the second of (ii)

□

Remark 2.3. The above lemma implies that for any $x \in \mathbb{R}^n$ one can compute its projection onto ℓ_q-ball by $(\hat{v}(x))^{\frac{1}{q}} \circ \mathrm{sign}(x)$ where $\hat{v}(x)$ is a minimizer of the following problem

$$\min \frac{1}{2}\||x| - v^{\frac{1}{q}}\|^2 \ \text{ subjec to } \|v\|_1 \le r,\, v \in \mathbb{R}^n_+.$$

It is clear that the above problem is a smooth constraint optimization problem which can be computed efficiently by many existing algorithm such as active-set method, trust region method, sequential quadratic programming. Indeed, one can employ the command "fmincon" to compute the optimization problem (2.6) in Matlab.

3 Optimality conditions

In this section, we will provide some optimality conditions to help us construct an efficient algorithm. It is well known that the tangent cone and normal cone of a constrained set are extensively used in optimization and nonlinear analysis. So, we first discuss the Bouligand tangent cone and normal cone of the ℓ_q-ball. Recalling that for any nonempty set $S \subseteq \mathbb{R}^n$, its Bouligand tangent cone $T_S(\bar{x})$ and corresponding normal cone $N_S(\bar{x})$ at the point $\bar{x} \in S$ are defined as [12]:

$$T_S(\bar{x}) := \big\{d \in \mathbb{R}^n : \exists \{x^k\} \subset S, \lim_{k\to\infty} x^k = \bar{x}, \alpha_k \ge 0, k \in \mathbb{N} \text{ such that} \lim_{k\to\infty} \alpha_k(x^k - \bar{x}) = d\big\}$$

and

$$N_S(\bar{x}) := \big\{d \in \mathbb{R}^n : \langle d, z\rangle \le 0, \forall z \in T_S(\bar{x})\big\}.$$

Proposition 3.1. *For any $x \in \mathcal{B}_{q,r}$, it follows that*

$$\mathrm{T}_{\mathcal{B}_{q,r}}(x) = \begin{cases} \{d = (d_\Gamma^T, 0^T)^T \in \mathbb{R}^n : \nabla\varphi(x_\Gamma)^T d_\Gamma \le 0\}, & \text{if } x \in \partial\mathcal{B}_{q,r}; \\ \mathbb{R}^n, & \text{if } x \in \mathrm{int}\mathcal{B}_{q,r}. \end{cases}$$

and

$$\mathrm{N}_{\mathcal{B}_{q,r}}(x) = \begin{cases} \{d = (\lambda\nabla\varphi(x_\Gamma)^T, u^T)^T : \forall\lambda \ge 0, \forall u \in \mathbb{R}^{n-|\Gamma|}\}, & \text{if } x \in \partial\mathcal{B}_{q,r}; \\ \{0\}, & \text{if } x \in \mathrm{int}\mathcal{B}_{q,r}. \end{cases}$$

where $\Gamma = \mathrm{supp}(x)$.

Proof. Obviously, the result holds for the case $x \in \mathrm{int}\mathcal{B}_{q,r}$. We now prove the desired result for the case $x \in \partial\mathcal{B}_{q,r}$. For simplicity, we denote $S_1 = \{d = (d_\Gamma^T, 0^T)^T \in \mathbb{R}^n : \nabla\varphi(x_\Gamma)^T d_\Gamma \le 0\}$. It is not hard to prove the second result when $\mathrm{T}_{\mathcal{B}_{q,r}}(x) = S_1$. So, we only give the proof of the first.

If $|\Gamma| = n$, it is easy to check that

$$\mathrm{T}_{\mathcal{B}_{q,r}}(x) = \{d \in \mathbb{R}^n : \nabla\varphi(x)^T d \le 0\},$$

and then

$$\mathrm{N}_{\mathcal{B}_{q,r}}(x) = \lambda\nabla\varphi(x), \quad \forall\lambda \ge 0.$$

That is, the result holds.

We now prove the desired results when $|\Gamma| < n$. We first prove $\mathrm{T}_{\mathcal{B}_{q,r}}(x) \subseteq S_1$. For any $d \in \mathrm{T}_{\mathcal{B}_{q,r}}(x)$, it follows from the definition of tangent cone that there exist a sequence of real numbers $t_k \to 0^+$ and a sequence of vectors $d_k \to d$ such that $x + t_k d_k \in \mathcal{B}_{q,r}$. Assume that $\nabla\varphi(x_\Gamma)^T d_\Gamma > 0$ or $d_{\Gamma^c} \neq 0$. If $\nabla\varphi(x_\Gamma)^T d_\Gamma > 0$, the limit $d_{k\Gamma} \to d_\Gamma$ implies that there exist a positive number ϵ_1 and a positive integer K_1 such that

$$\nabla\varphi(x_\Gamma)^T d_{k\Gamma} > \epsilon_1, \quad \text{for each}\, k \geq K_1. \tag{3.1}$$

If $d_{\Gamma^c} \neq 0$, the limit $d_{k\Gamma^c} \to d_{\Gamma^c}$ implies that there exist a positive number ϵ_2 and a positive integer K_2 such that

$$\|d_{k\Gamma^c}\|_q^q > \epsilon_2, \quad \text{for each}\, k \geq K_2. \tag{3.2}$$

From (3.1) or (3.2), one can conclude that there exist a positive number ϵ_3 and a positive integer K_3 such that

$$\nabla\varphi(x_\Gamma)^T d_{k\Gamma} + t_k^{-1} o(t_k\|d_{k\Gamma}\|) + t_k^{q-1}\|d_{k\Gamma^c}\|_q^q > \epsilon_3, \quad \text{for each}\, k \geq K_3.$$

Combing this, $x + t_k d_k \in \mathcal{B}_{q,r}$ and $x \in \partial\mathcal{B}_{q,r}$, we have

$$\begin{aligned} r \geq& \|x + t_k d_k\|_q^q \\ =& \|x_\Gamma + t_k d_{k\Gamma}\|_q^q + \|t_k d_{k\Gamma^c}\|_q^q \\ =& \|x_\Gamma\|_q^q + t_k \nabla\varphi(x_\Gamma)^T d_{k\Gamma} + o(t_k\|d_{k\Gamma}\|) + \|t_k d_{k\Gamma^c}\|_q^q \\ =& r + t_k\big(\nabla\varphi(x_\Gamma)^T d_{k\Gamma} + t_k^{-1} o(t_k\|d_{k\Gamma}\|) + t_k^{q-1}\|d_{k\Gamma^c}\|_q^q\big) \\ \geq& r + t_k\epsilon_3 \\ >& r, \end{aligned}$$

which is absurd. So, $\nabla\varphi(x_\Gamma)^T d_\Gamma \leq 0$ and $d_{\Gamma^c} = 0$ which yield that

$$\mathrm{T}_{\mathcal{B}_{q,r}}(x) \subseteq S_1.$$

Next we prove $\mathrm{S}_1 \subseteq \mathrm{T}_{\mathcal{B}_{q,r}}(x)$. For any $d \in \mathrm{int}(S_1)$ which yields that $d = (d_\Gamma^T, 0^T)^T \in \mathbb{R}^n$ and $\nabla\varphi(x_\Gamma)^T d_\Gamma < 0$, we choose

$$d_{k\Gamma} \to d_\Gamma, \quad d_{k\Gamma^c} = 0.$$

Since $d_k := (d_{k\Gamma}^T, 0^T)^T \to (d_\Gamma^T, 0^T)^T$ and $\nabla\varphi(x_\Gamma)^T d_\Gamma < 0$, it follows that there exist a positive number ϵ_4 and a positive integer K_4 such that $\nabla\varphi(x_\Gamma)^T d_{k\Gamma} < -\epsilon_4$ and $|t_k^{-1} o(t_k\|d_{k\Gamma}\|)| \leq \epsilon_4/2$. So,

$$\begin{aligned} \|x + t_k d_k\|_q^q =& \|x_\Gamma + t_k d_{k\Gamma}\|_q^q \\ =& \|x_\Gamma\|_q^q + t_k \nabla\varphi(x_\Gamma)^T d_{k\Gamma} + o(t_k\|d_{k\Gamma}\|) \\ =& r + t_k\big(\nabla\varphi(x_\Gamma)^T d_{k\Gamma} + t_k^{-1} o(t_k\|d_{k\Gamma}\|)\big) \\ \leq& r - t_k\epsilon_4/2 \\ <& r \end{aligned}$$

which leads to $d \in \mathrm{T}_{\mathcal{B}_{q,r}}(x)$. That is, $\mathrm{S}_1 \subseteq \mathrm{T}_{\mathcal{B}_{q,r}}(x)$. Since $\mathrm{cl}\big(\mathrm{int}(S)\big) \subseteq \mathrm{cl}\big(\mathrm{T}_{\mathcal{B}_{q,r}}(x)\big)$ and both S_1 and the Tangent cone are closed set, it follows from $S_1 = \mathrm{cl}\big(\mathrm{int}(S)\big)$ that $S_1 \subseteq \mathrm{T}_{\mathcal{B}_{q,r}}(x)$. Then, we prove $\mathrm{T}_{\mathcal{B}_{q,r}}(x) = S_1$.

□

Remark 3.2. *For the ℓ_q-ball, one can use the similar method to prove that the so-called Clarke tangent cone coincides with the Bouligand tangent cone. So, we only use the latter and the corresponding normal cone.*

From Theorem 6.12 in [12] and Proposition 3.1, one can get immediately the following result.

Theorem 3.3. *Let x^* be a minimizer of problem (1.4). Then,*

$$0 \in \nabla f(x^*) + \mathrm{N}_{\mathcal{B}_{q,r}}(x^*). \tag{3.3}$$

Proposition 3.4. *The generalize equation (3.3) holds if and only if there exists a number $\mu \geq 0$ such that*

$$X^* \nabla f(x^*) + \mu q |x^*|^q = 0, \tag{3.4}$$

where $X^* = \mathrm{diag}(x^*)$.

Proof. Based on Theorem 3.1 and Proposition 3.1, it is not hard to prove the desired result. So, we omit the details. □

Obviously, (3.4) states that there exists a number $\lambda \geq 0$ such that

$$\nabla f(x^*)_\Gamma = -\lambda \nabla \varphi(x^*_\Gamma).$$

For $\nabla f(x^*)_{\Gamma^c}$, however, the equality tells us nothing but $\nabla f(x^*)_{\Gamma^c} \in \mathbb{R}^{n-|\Gamma|}$ which is trivial. A natural question is how to get some further information of $\nabla f(x^*)_{\Gamma^c}$. We next employ a projector onto ℓ_q-ball to discuss another optimality condition which not only helps us to answer the question but also provides an appropriate algorithm to solve the minimization (1.4).

Theorem 3.5. *Let x^* be a minimizer of problem (1.4). For any $\tau \in (0, 1/L_r]$, it follows that*

$$x^* \in P_{\mathcal{B}_{q,r}}(x^* - \tau \nabla f(x^*)), \tag{3.5}$$

where $L_r = \sup_{\|x\|_\infty \leq r^{\frac{1}{q}}} \|\nabla^2 f(x)\|$.

Proof. Since the ℓ_q-ball $\mathcal{B}_{q,r}$ is bounded and closed, the continuous differentiability of f ensures that there exists at least one minimizer of the problem (1.4). For any $\tau > 0$ and $y \in \mathcal{B}_{q,r}$, define the following auxiliary problem

$$\min_{x \in \mathcal{B}_{q,r}} F_\tau(x, y) := f(y) + \nabla f(y)^T (x - y) + \frac{1}{2\tau}\|x - y\|_2^2. \tag{3.6}$$

For any $x, y \in \mathcal{B}_{q,r}$ and $\tau \in (0, 1/L_r]$, it follows that

$$\begin{aligned} f(x) &= f(y) + \nabla f(y)^T (x-y) + \frac{1}{2}(x-y)^T \nabla^2 f(\xi)(x-y) \\ &= F_\tau(x,y) + \frac{1}{2}(x-y)^T \nabla^2 f(\xi)(x-y) - \frac{1}{2\tau}\|x-y\|^2 \\ &\leq F_\tau(x,y) + \frac{1}{2}\|\nabla^2 f(\xi)\|\|x-y\|^2 - \frac{1}{2\tau}\|x-y\|^2 \\ &\leq F_\tau(x,y) + \frac{1}{2}\|x-y\|^2 (L_r - \frac{1}{\tau}) \\ &\leq F_\tau(x,y), \end{aligned} \tag{3.7}$$

where $\xi = y + t(x-y)$ for some $t \in (0,1)$ which implies that $\|\xi\|_\infty < r^{\frac{1}{q}}$ and then $\|\nabla^2 f(\xi)\| \leq L_r$. Denote

$$\bar{x} \in \arg\min_{x \in \mathcal{B}_{q,r}} F_\tau(x, x^*).$$

For any $\tau \in (0, 1/L]$, we conclude from the definition of $\bar{x}$ and (3.7) that

$$f(x^*) = F_\tau(x^*, x^*) \geq F_\tau(\bar{x}, x^*) \tag{3.8}$$

and

$$F_\tau(\bar{x}, x^*) \geq f(\bar{x}) \geq f(x^*), \tag{3.9}$$

which yield that

$$f(x^*) = F_\tau(x^*, x^*) = F_\tau(\bar{x}, x^*) = f(\bar{x}).$$

Then,

$$\bar{x} \in \arg\min_{x \in \mathcal{B}_{q,r}} f(x)$$

and

$$x^* \in \arg\min_{x \in \mathcal{B}_{q,r}} F_\tau(x, x^*).$$

On the other hand, it is easy to check that the problem (3.6) is equivalent to the following minimization problem

$$\min_{x \in \mathcal{B}_{q,r}} \frac{1}{2}\|x - [y - \tau\nabla f(y)]\|^2.$$

By the definition of $P_{\mathcal{B}_{q,r}}$, we get (3.5). $\square$

Proposition 3.6. *Assume that a point $\bar{x}$ satisfies (3.5) with a real number $\tau > 0$. Then, (3.3) holds. Furthermore for each $i \in \bar{\Gamma}^c$, it follows that*

$$|\nabla_i f(\bar{x})| \leq \min_{j \in \bar{\Gamma}} \big(\frac{1}{\tau}|\bar{x}_j| + |\nabla_j f(\bar{x})|\big) \tag{3.10}$$

where $\bar{\Gamma} = \mathrm{supp}(\bar{x})$.

Proof. Suppose $\bar{x}$ satisfies (3.5) which means

$$\bar{x} \in \arg\min_{u \in \mathcal{B}_{q,r}} g(u) := \frac{1}{2}\|u - \big(\bar{x} - \tau\nabla f(\bar{x})\big)\|^2.$$

Similar to Theorem 3.5, we conclude from Theorem 6.12 in [12] and Proposition 3.1 that

$$0 \in \nabla g(\bar{x}) + \mathrm{N}_{\mathcal{B}_{q,r}}(\bar{x}),$$

which together with

$$\nabla g(\bar{x}) = \bar{x} - \big(\bar{x} - \tau\nabla f(\bar{x})\big)$$

yields that

$$0 \in \tau\nabla f(\bar{x}) + \mathrm{N}_{\mathcal{B}_{q,r}}(\bar{x}).$$

Since $\mathrm{N}_{\mathcal{B}_{q,r}}(\bar{x})$ is a cone, then we get (3.3) immediately.

We next prove (3.10). For each $i \in \bar{\Gamma}^c$, we conclude from the result (ii) of Lemma 2.1 that

$$\tau|\nabla_i f(\bar{x})| = |\bar{x}_i - \tau\nabla_i f(\bar{x})| \leq \min_{j \in \bar{\Gamma}} \big(|\bar{x}_j - \tau\nabla_j f(\bar{x})|\big) \leq \min_{j \in \bar{\Gamma}} \big(|\bar{x}_j| + \tau|\nabla_j f(\bar{x})|\big)$$

which together with $\tau > 0$ leads to the desired result. $\square$

Remark 3.7. From Propositions 3.4 and 3.6, one can see that the necessary optimality condition (3.5) is stronger than the condition (3.4). Indeed, the former is similar to the so-called L-stationary property introduced by [2] in the frame work of the sparsity constrained problem. They stated the property is an extension of the concept of stationarity for convex constrained problems. Inspired by this, we here call the point $\bar{x}$ satisfying (3.5) a stationary point. On the other hand, the inequality (3.10) can be used as a stopping criteria for our algorithm.

4 Algorithm and its convergence

By the above analysis, we will use the fixed point equation (3.5) and the inequality (3.10) to provide the following algorithm.

Step 0. Given $q \in (0,1), \lambda > 0, \epsilon \geq 0, \gamma, \alpha \in (0,1), \delta > 0$, choose an arbitrary x^0 and set $k = 0$.
Step 1.
(a) Compute $\nabla f(x^k)$ from $\nabla f(x) = 2\sum_{i=1}^m (x^T A_i x + b_i^T x + c_i - y_i)(2A_i x + b_i)$;
(b) Compute x^{k+1} as

$$x^{k+1} = P_{\mathcal{B}_{q,r}}(x^k - \tau_k \nabla f(x^k)) \tag{4.1}$$

with $\tau_k = \gamma\alpha^{m_k}$ and m_k is the smallest nonnegative integer m such that

$$f(x^k) - f(x^k(\gamma\alpha^m)) \geq \frac{\delta}{2}\|x^k - x^k(\gamma\alpha^m)\|_2^2 \tag{4.2}$$

and $x^k(\tau) = P_{\mathcal{B}_{q,r}}(x^k - \tau\nabla f(x^k))$.
Step 2. Stop if

$$\frac{\|x^{k+1} - x^k\|_2}{\max\{1, \|x^k\|_2\}} \leq \epsilon \;\text{ and }\; |\nabla_i f(\bar{x})| \leq \min_{j\in\bar{\Gamma}} \big(\frac{1}{\tau_k}|\bar{x}_j| + |\nabla_j f(\bar{x})|\big) \text{ for each } i \in (\Gamma^{k+1})^c \tag{4.3}$$

where $\Gamma^{k+1} = \mathrm{supp}(x^{k+1})$. Otherwise, replace k by $k+1$ and go to Step 1.

Notice that Remark 2.3 provides the method for computing the subproblem (4.1). To find the τ_k, we use the so-called Armijo-type line search method in the step (b). It is a natural question whether or not we can find the smallest nonnegative integer m_k. We provide the following lemma to answer the question.

Lemma 4.1. *Let $\bar{L} = \sup_{x\in B}\|\nabla^2 f(x)\|_2$ where $B = \{x \in \mathbb{R}^p : \|x\|_2 \leq r^{1/q}\}$. For any $\delta > 0, \gamma, \alpha \in (0,1)$, define*

$$m_k = \begin{cases} 0, & \text{if } \gamma(\bar{L}+\delta) \leq 1; \\ -\lceil \log_\alpha \gamma(\bar{L}+\delta) \rceil + 1, & \text{otherwise.} \end{cases}$$

Then (4.2) holds.

Proof. From the definition of τ_k and m_k, it is easy to check that

$$\bar{L} - \frac{1}{\tau_k} \leq -\delta. \tag{4.4}$$

Indeed, by taking $\tau_k = \gamma$ we have

$$\bar{L} - \frac{1}{\tau_k} = \frac{\gamma \bar{L} - 1}{\gamma} \leq -\delta,$$

when $\gamma(\bar{L} + \delta) \leq 1$. If $\gamma(\bar{L} + \delta) > 1$,

$$\tau_k = \gamma \alpha^{m_k} \leq \gamma \alpha^{-\log_\alpha \gamma(\bar{L}+\delta)} = \frac{1}{\bar{L} + \delta}$$

which also leads to (4.4).

Note that

$$x^{k+1} \in \arg\min_{x \in \mathbb{R}^n} F_{\tau_k}(x, x^k) \tag{4.5}$$

and

$$x^{k+1} \in \mathcal{B}_{r,q}.$$

Similar to (3.7), we obtain from (4.4) that

$$\begin{aligned} f(x^{k+1}) &\leq F_{\tau_k}(x^{k+1}, x^k) + \frac{1}{2}\|x^{k+1} - x^k\|_2^2\big(\|\nabla^2 f(\xi_k)\|_2 - \frac{1}{\tau_k}\big) \\ &\leq F_{\tau_k}(x^{k+1}, x^k) + \frac{1}{2}\|x^{k+1} - x^k\|_2^2(\bar{L} - \frac{1}{\tau_k}) \\ &\leq F_{\tau_k}(x^{k+1}, x^k) - \frac{\delta}{2}\|x^{k+1} - x^k\|_2^2, \end{aligned}$$

where $\xi_k = x^k + \varrho(x^{k+1} - x^k)$ for some $\varrho \in (0, 1)$ and then $\xi_k \in B$ leads to the second inequality. Combining this and (4.5), we have

$$\begin{aligned} f(x^k) - f(x^{k+1}) &= F_{\tau_k}(x^k, x^k) - f(x^{k+1}) \geq F_{\tau_k}(x^{k+1}, x^k) - f(x^{k+1}) \\ &\geq \frac{\delta}{2}\|x^{k+1} - x^k\|_2^2, \end{aligned}$$

which completes the proof. □

We now consider the convergence of the iterated sequence $\{x^k\}$.

Theorem 4.2. *Let $\{x^k\}$ be the sequence generated by the above algorithm. Then,*

(i) $\{f(x^k)\}$ converges to $f(\tilde{x})$, where $\tilde{x}$ is any accumulation point of $\{x^k\}$;

(ii) $\lim_{k\to\infty} \frac{\|x^{k+1} - x^k\|_2}{\tau_k} = 0$;

(iii) any accumulation point of the sequence $\{x^k\}$ is a stationary point of the minimization problem (1.4).

Proof. (i) Notice that $\{x^k\}$ is bounded which yields that it has at least one accumulation point. Since $\{f(x^k)\}$ is monotonically decreasing and $f(\cdot) \geq 0$, $\{f(x^k)\}$ converges to a constant $\tilde{f}(\geq 0)$. By the continuousness of $f(x)$, we then have $\{f(x^k)\} \to \tilde{f} = f(\tilde{x})$, where $\tilde{x}$ is an accumulation point of $\{x^k\}$ as $k \to \infty$.

(ii) From the definition of x^{k+1} and (4.2), we have

$$\sum_{k=0}^{n} \|x^{k+1} - x^k\|_2^2 \leq \frac{2}{\delta} \sum_{k=0}^{n} [f(x^k) - f(x^{k+1})] = \frac{2}{\delta}[f(\tilde{x}) - f(x^{n+1})] \leq \frac{2}{\delta} f(\tilde{x}).$$

Hence, $\sum_{k=0}^{\infty} \|x^{k+1} - x^k\|_2^2 < \infty$ which implies that $\|x^{k+1} - x^k\|_2 \to 0$ as $k \to \infty$. From Lemma 4.1, we have $\tau_k \in [\gamma\alpha^{\bar{m}}, \gamma]$ where $\bar{m} = \max(0, [-\log_\alpha \gamma(\bar{L} + \delta)] + 1)$. Then, we get the result (ii).

(iii) Notice that both $\{x^k\}$ and $\{\tau_k\}$ are bounded which implies that there exist two subsequences x^{k_j} and τ_{k_j}such that

$$x^{k_j} \to \tilde{x}, \quad \tau_{k_j} \to \tilde{\tau}, \quad \text{as} \quad j \to \infty$$

where $\tilde{x} \in \mathcal{B}_{r,q}$. By the algorithm, we have

$$x^{k_j+1} \in P_{B_{q,r}}[x^{k_j} - \tau_{k_j} \nabla f(x^{k_j})]$$

which means that for any $x \in \mathbb{R}^n$,

$$\|x^{k_j+1} - (x^{k_j} - \tau_{k_j} \nabla f(x^{k_j}))\|^2 \leq \|x - (x^{k_j} - \tau_{k_j} \nabla f(x^{k_j}))\|^2.$$

From $\|x^{k_j+1} - x^{k_j}\| \to 0$ and the continuity of $\nabla f(\cdot)$, we then conclude that for any $x \in \mathbb{R}^n$,

$$\|\tilde{x} - (\tilde{x} - \tilde{\tau}\nabla f(\tilde{x}))\|^2 \leq \|x - (\tilde{x} - \tilde{\tau}\nabla f(\tilde{x}))\|^2$$

which together with the definition of the projection onto the ball $\mathcal{B}_{r,q}$ and $\tilde{x} \in \mathcal{B}_{r,q}$ implies that $\tilde{x} \in P_{\mathcal{B}_{r,q}}\big(\tilde{x} - \tilde{\tau}\nabla f(\tilde{x})\big)$. Then, we complete the proof.

□

5 Numerical Examples

In this section we calculate the following quadratic equations problem to illustrate the proposed algorithm,

$$y_i = \langle a_i, x\rangle^2 + \varepsilon_i, \quad i = 1, \ldots, m, \quad s.t. \ \|x\|_0 \leq s.$$

A noise-free version of this model was considered by [2]. In each simulation 100 Monte Carlo samples are generated and in each case the true value x^* is generated randomly with s nonzero components from the standard Gaussian distribution and the noise $\varepsilon_i \sim N(0, \sigma^2)$ with σ. The vectors $\{a_i\} \in \mathbb{R}^n$ are generated from the standard Gaussian or uniform distribution. Similar to [2] we consider the cases $n = 120$ and $m = 80$ with $s = 3, 4, \ldots, 10$, respectively. The numerical optimization is done using the proposed algorithm with iteration stopping criterion (4.3) or the maximum iterative time of 5000s is reached. We compare the proposed algorithm performed by using $q = 0.8$ and $q = 0.5$ respectively with the greedy sparse-simple method and the partial sparse-simple method in [2] (For convenience, we denote them as GSS and PSS respectively).

To evaluate the selection and estimation accuracy of our method, we calculate the mean squared error (MSE) which is the average of $\|\hat{x} - x^*\|_2^2$. Especially, we report the rate of successful recovery (SR) using the criterion $\|\hat{x} - x^*\|_2^2 \leq 10^{-4}$ when σ is small, for example $\sigma = 0.01$. From Tables 1 and 2, one can see that our SR rates are lower than that of GSS and PSS methods. However, Table 3 shows that our method has certain advantages in the case that σ is relatively large and $\{a_i\}$ are generated from the uniform distribution. The MSE of our method with $q = 0.8$ is significantly smaller that of GSS and PSS, especially for the case $\sigma = 1$. These numerical results show that our method is more robust than GSS and PSS.

Table 1: The rate of SR in the case of Gaussian distribution with $\sigma = 0.01$

	$\|x\|_0$ / method	3	4	5	6	7	8	9	10
SR	GSS	0.96	**0.91**	0.84	**0.71**	**0.55**	0.48	**0.38**	0.23
	PSS	**0.98**	0.90	**0.86**	0.64	0.50	**0.51**	0.23	**0.24**
	$q = 0.5$	0.81	0.72	0.76	0.58	0.44	0.30	0.30	0.24
	$q = 0.8$	0.78	0.70	0.74	0.56	0.36	0.28	0.16	0.18

Table 2: The rate of SR in the case of uniform distribution with $\sigma = 0.01$

	$\|x\|_0$ / method	3	4	5	6	7	8	9	10
SR	GSS	**0.94**	**0.95**	0.98	**1.00**	**1.00**	0.99	0.99	0.99
	PSS	0.89	0.92	**0.99**	0.98	**1.00**	**1.00**	**1.00**	**1.00**
	$q = 0.5$	0.88	0.91	0.97	0.93	0.94	0.88	0.93	0.87
	$q = 0.8$	0.84	0.86	0.84	0.87	0.90	0.88	0.86	0.85

Table 3: The MSE in the case of uniform distribution with $\sigma = 0.5$ and $\sigma = 1$

	$\|x\|_0$ / method	3	4	5	6	7	8	9	10
$\sigma = 0.5$	GSS	0.4492	0.3206	0.2313	0.1280	0.0736	0.0602	0.0555	0.0420
	PSS	0.5260	0.3475	0.2175	0.1759	0.0904	0.0511	0.0625	0.0457
	$q = 0.5$	0.6961	0.3695	0.2173	0.1192	0.0901	0.0755	0.0724	0.0618
	$q = 0.8$	**0.2722**	**0.1553**	**0.0729**	**0.0907**	**0.0430**	**0.0405**	**0.0433**	**0.0406**
$\sigma = 1$	GSS	1.7285	1.3004	1.0335	0.9029	0.5619	0.2972	0.2915	0.2605
	PSS	1.7266	1.3975	1.1799	0.8327	0.5886	0.4039	0.2748	0.2371
	$q = 0.5$	2.6204	1.8076	1.5077	0.9761	0.6281	0.4716	0.3417	0.3031
	$q = 0.8$	**0.5724**	**0.5231**	**0.4304**	**0.3035**	**0.3021**	**0.2619**	**0.1768**	**0.1505**

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