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AN OUTCOME-SPACE NORMAL CONE METHOD FOR GENERALIZED CONCAVE MULTIPLICATIVE PROBLEMS*

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Abstract: This paper presents an outcome-space outer approximation algorithm for globally solving generalized concave multiplicative problems which involves the maximizing a finite sum of products of concave functions over a nonempty compact convex set. From the relationship between normal cones in decision space and ones in outcome space, the normal vectors of cutting hyperplanes are determined by solving the systems of linear equations which lead the computational efficiency of the proposed algorithm. The convergence of this algorithm is proved and some illustrative examples are reported.

Key words: *generalized multiplicative programming, efficient outcome set, outer approximation, bound and branch algorithm, normal cone method*

Mathematics Subject Classification: *90C29, 90C26*

1 Introduction

Consider the generalized concave multiplicative programming problem

$$\max h(x) = \mathbf{f}_1(x) + \sum_{i=1}^{s} \mathbf{f}_{2i}(x)\mathbf{f}_{2i+1}(x) \ \text{ s.t. } \ x \in X, \tag{MP_X}$$

where $X \subset \mathbb{R}^n$ is a nonempty compact convex set and $\mathbf{f}_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2, \ldots, 2s+1$, are concave functions defined on $\mathbb{R}^n$. It is also assumed that $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_{2s+1}$ are positive functions over X, i.e.

$$\mathbf{f}_j(x) > 0 \ \forall x \in X, \ j = 1, 2, \ldots, 2s+1. \tag{1.1}$$

The applications of Problem (MP_X) include all of the numerous applications of general quadratic programming, bilinear programming and linear zero-one programming [3]. Although each product of two concave positive functions is quasiconcave, the sum of quasiconcave functions is not quasiconcave, in general [3]. Therefore, Problem (MP_X) is a hard global optimization problem, even in special cases such as when the objective function $h(x)$ is the product of two linear functions and X is a nonempty, compact polyhedron [12].

Many algorithms have been proposed for solving Problem (MP_X) when $\mathbf{f}_j, j = 1, 2, \ldots, 2s+1$, are linear and X is polyhedral, such as [5], [7], [9], [14], [17], and the references therein.

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In [3], Benson first studied Problem (MP_X) in general form and proposed the branch and bound algorithm for solving it. As far as we know, recently there are two efficient algorithms to solve globally Problem (MP_X) in general form, which are the branch and bound algorithm [3] and the outcome-space outer approximation algorithm [1]. In [3], the author used the branch and bound technique associated with rectangular partitioning in $\mathbb{R}^s$ to solve a problem that is equivalent to Problem (MP_X) and the required upper bounds are determined by solving ordinary convex programming problems. On the other hand, the authors in [1] solve Problem (MP_X) by solving instead an equivalent semi-infinite optimization problem. The resulting problem is solved by outer approximation algorithm in the outcome space $\mathbb{R}^{2s+1}$. In each typical iteration, the cutting hyperplane is determined by solving a generalized concave maximizing problem and a min-max extremum problem.

In this paper, we present an outer approximation algorithm in the outcome space $\mathbb{R}^{s+1}$ for solving a problem which is equivalent to Problem (MP_X). The outer approximation method used here is similar to the algorithm [1]. However, in each iteration of the algorithm, the cutting hyperplane is easily determined by solving a problem of maximizing a convex function φ over the vertex set of a polytope and finding a nonnegative solution of a linear equation system. This result is obtained based on the relationship between the normal cones to the feasible convex set $X \subset \mathbb{R}^n$ and the normal vectors of the supporting hyperplane to the set $Y \subset \mathbb{R}^{s+1}$, where Y is the image of X under f and $f : \mathbb{R}^n \to \mathbb{R}^{s+1}$ is the vector value function defined in Section 2. Because the outcome space $\mathbb{R}^{s+1}$ typically has much smaller dimension than the desision space $\mathbb{R}^n$ (see [2]), we expect potentially that considerable computational savings could be obtained.

The paper is organized as follows. In Section 2, we first introduce some notations, then we show how to convert the generalized concave multiplicative programming problem (MP_X) into the problem of maximizing a convex function φ over the weakly efficient set $W(Y)$ of Y which is equivalent to Problem (MP_X). Theoretical prerequisites for the algorithm are given in Section 3. The algorithm is described in detail and the convergence of the algorithm is shown in Section 4. Some illustrative examples are reported in Section 5. We draw some conclusions in Section 6.

2 The Problem in Outcome Space

Let

$$f_1(x) = \mathbf{f}_1(x), \qquad f_{i+1}(x) = \sqrt{\mathbf{f}_{2i}(x)\mathbf{f}_{2i+1}(x)}, i = 1, \ldots, s,$$

and

$$p = s + 1.$$

Then the functions $f_j(x), j = 1, \ldots, p$ are also concave on X (see Proposition 2.7 in [16]). It is clear that

$$h(x) = f_1(x) + \sum_{j=2}^{p} f_j^2(x).$$

For each $x \in X$, let $f(x) = (f_1(x), \ldots, f_p(x))^T$. Since f is positive over X, the function h is rewritten as the composition $h(x) = \varphi(f(x))$, where $\varphi : \text{int}\mathbb{R}^{\text{p}}_+ \to \mathbb{R}$ is given by

$$\varphi(y) = y_1 + \sum_{i=2}^{p} y_j^2.$$

It is clear that φ is positive and increasing over $\mathbb{R}^p_+$. It means that for any two vectors $y^1, y^2 \in Y$, if $y^1 \geq y^2$ and $y^1 \neq y^2$ then $\varphi(y^1) > \varphi(y^2)$. Futhermore, by the definition, φ is a quadratic convex function on $\mathbb{R}^p_+$.

Below we show how to convert problem (MP_X) to a problem (WP_Y) in outcome space $\mathbb{R}^p$, where the objective function is the quadratic convex function φ and the feasible solution set is the weakly efficient set of Y, where

$$Y = \{y \in \mathbb{R}^p \mid y = f(x), x \in X\}$$

is the image of X under f and is called the *outcome set* of problem (MP_X). According to (1.1), we have

$$Y \subseteq \text{int}\mathbb{R}^p_+. \tag{2.1}$$

First, let us recall that a point $q^0 \in \mathbb{R}^p$ is called an *efficient point* of a nonempty set $Q \subset \mathbb{R}^p$ when $q^0 \in Q$ and there exists no point $q \in Q$ such that $q \geq q^0$ and $q \neq q^0$. Similarly, a point $q^0 \in \mathbb{R}^p$ is called an *weakly efficient point* of a nonempty set $Q \subset \mathbb{R}^p$ when $q^0 \in Q$ and there exists no point $q \in Q$ such that $q \gg q^0$. Here for any two vectors $a, b \in \mathbb{R}^p$, the notations $a \geq b$ and $a \gg b$ mean $a - b \in \mathbb{R}^p_+$ and $a - b \in \text{int}\mathbb{R}^p_+$ respectively, where $\mathbb{R}^p_+ = \{x \in \mathbb{R}^p \mid x_i \geq 0, i = 1, \ldots, p\}$ is the nonnegative orthant of $\mathbb{R}^p$ and $\text{int}\mathbb{R}^p_+$ is the interior of $\mathbb{R}^p_+$. The set of all efficient points and the set of all weakly efficient points of a nonempty set $Q \subset \mathbb{R}^p$ are denoted $E(Q)$ and $W(Q)$, respectively. By definition, we have

$$E(Q) \subseteq W(Q) \subseteq Q. \tag{2.2}$$

Consider the outcome-space problem

$$\max \varphi(y) \;\; \text{s.t.} \;\; y \in Y. \tag{MP_Y}$$

The following results give some properties of problem (MP_Y) and shows that we can obtain a global optimal solution to problem (MP_Y) from a global optimal solution to problem (WP_Y).

Proposition 2.1. *If y^* is a global optimal solution to problem (MP_Y) then $y^* \in E(Y)$.*

Proof. Let y^* be a global optimal solution to Problem (MP_Y), i.e. $\varphi(y^*) \geq \varphi(y)$ for all $y \in Y$. On the contrary, suppose that $y^* \notin E(Y)$. It means that there is a point $\bar{y} \in Y$ such that $\bar{y} \geq y^*, \bar{y} \neq y^*$. By the increasing monotonicity of the function φ, it follows that $\varphi(\bar{y}) > \varphi(y^*)$, which contradicts the assumption. □

Proposition 2.2. *If y^* is a global optimal solution to problem (MP_Y) then any $x^* \in X$ such that $f(x^*) \geq y^*$ is a global optimal solution to problem (MP_X) and the optimal value of problem (MP_X) is $h(x^*) = \varphi(y^*)$.*

Proof. Suppose that y^* is a global optimal solution to problem (MP_Y) and $x^* \in X$ such that $f(x^*) \geq y^*$. Therefore, $f(x^*) \in y^* + \mathbb{R}^p_+$. From Proposition 2.1, we also have $y^* \in E(Y)$. Combining this fact with the definition of efficient points, we derive $y^* = f(x^*)$ and $h(x^*) = \varphi(f(x^*)) = \varphi(y^*)$. Moreover, since y^* is a global optimal solution to problem (MP_Y), one has $\varphi(y^*) \geq \varphi(y)$ for all $y \in Y$. This shows that $h(x^*) \geq h(x)$ for all $x \in X$, i.e. x^* is an optimal solution of Problem (MP_X). □

Consider the following problem

$$\max \varphi(y) \;\; \text{s.t.} \;\; y \in W(Y). \tag{WP_Y}$$

From (2.2) and Proposition 2.1, it is easily seen that the optimal solution sets of problem (MP_Y) and the problem (WP_Y) are the same. Here, we establish a new outer approximation algorithm for solving problem (WP_Y) to solve globally problem (MP_X).

3 Bases of the Algorithm

For each $j = 1, ..., p$, let $z_j^I \in \mathbb{R}^p$ denote the optimal value of the convex programming

$$\max f_j(x) \;\text{ s.t. }\; x \in X, \qquad (IP_j)$$

and let $z^I = (z_1^I, \ldots, z_p^I)^T$. Let

$$P^0 = \mathbb{R}^p_+ \cap (z^I - \mathbb{R}^p_+) = \{z \in \mathbb{R}^p | 0 \leq z \leq z^I\} \subseteq \mathbb{R}^p_+. \qquad (3.1)$$

It is clear that the box P^0 containing $Y \supset W(Y)$ and the vertex set $V(P^0)$ can be easily determined. Starting with the box P^0, the algorithm will iteratively generate a sequence of nonempty polyhedra $\{P^k\}$, $k = 0, 1, 2, \ldots$ such that

$$P^0 \supset P^1 \supset \cdots \supset P^k \supset \cdots \supset Y \supset W(Y).$$

It is clear that for each $k = 0, 1, 2, \ldots$, the optimal value β_k of the problem

$$\max \;\varphi(y) \text{ s.t. } y \in P^k \qquad (Q(P^k))$$

is an upper bound of problem (WP_Y) . Since φ is a convex function and P^k is a polytope, it is well known that φ achieves its maximum over P^k at a vertex of P^k. It means that

$$\beta_k = \max\{\varphi(y) \text{ s.t. } y \in V(P^k)\}$$

where $V(P^k)$ denotes the vertex set of the polypote P^k.

Let ε be a given sufficiently small positive number. Let $y^* \in W(Y)$. The point y^* is said to be an ε- *optimal solution* to problem (WP_Y) if there is an upper bound β_* for problem (WP_Y) such that $\beta_* - \varphi(y^*) < \varepsilon(|\varphi(y^*)| + 1)$. Then, any $x^* \in X$ such that $f(x^*) \geq y^*$ is called an *approximate optimal solution to problem* (MP_X).

The steps of the outer approximation procedure for solving problem (WP_Y) may be described as follows

- *Step 0.* Choose a sufficiently small number $\varepsilon \geq 0$. Construct a nonempty polytope P^0 containing $W(Y)$ (see (3.1)). Determine the vertex set $V(P^0)$ of the polytope P^0. Set $\alpha_0 = 0$ (*initial lower bound* - see (1.1)). Set $k = 0$.
- *Step 1.* Set $\beta_k = \max\{\varphi(y) \mid y \in V(P^k)\}$ (*currently best upper bound*) and let v^k denote any element of the vertex set $V(P^k)$ for which $\varphi(v^k) = \beta_k$.
- *Step 2.* Find a weakly efficient point $y^k \in W(Y)$ and a point $x^k \in X$ such that $f(x^k) \geq y^k$.
- *Step 3.* **If** $\varphi(y^k) > \alpha_k$ **Then** Update $\alpha_k = \varphi(y^k)$ (*currently best lower bound*); $y^{best} = y^k$ (*currently best feasible point*); $x^{best} = x^k$.
- *Step 4.* **If** $\beta_k - \alpha_k \leq \varepsilon(|\alpha_k| + 1)$ **Then** Terminate the algorithm (y^{best} *is an* ε*-optimal solution to problem* (WP_Y) *and* x^{best} *is an approximate optimal solution to problem* (MP_X)).
- *Step 5.* Set
$$P^{k+1} = P^k \cap \{y \in \mathbb{R}^p | \langle \xi^k, y\rangle \leq \langle \xi^k, y^k\rangle\},$$
where ξ^k is a nonzero vector of $\mathbb{R}^p_+$ and $\langle \xi^k, v^k\rangle > \langle \xi^k, y^k\rangle$. Using the vertex set $V(P^k)$ and the definition of P^{k+1}, determine the vertex set $V(P^{k+1})$. Let $\alpha_{k+1} = \alpha_k$, $k := k + 1$ and go to Step 1.

For each $k = 0, 1, 2, \ldots$, the hyperplane

$$H(y^k) = \{y \in \mathbb{R}^p | \langle \xi^k, y \rangle = \langle \xi^k, y^k \rangle\}$$

is called a *cutting hyperplane*. Each such cutting hyperplane $H(y^k)$ is a supporting hyperplane of Y at $y^k \in W(Y)$, i.e.

$$\langle \xi^k, y \rangle \leq \langle \xi^k, y^k \rangle \quad \forall y \in Y$$

and it is constructed so that P^{k+1} "cut off" a portion of P^k containing v^k in such a way that

$$P^k \supset P^{k+1} \supset Y \supset W(Y).$$

Therefore the sequence of upper bounds $\{\beta_k\}$ is monotonously decreasing.

Finding a weakly efficient point y^k in Step 2 and a point $x^k \in X$ such that $f(x^k) \geq y^k$ will be presented in Subsection 3.1. Determining a normal vector ξ^k of the cutting hyperplane $H(y^k)$ is showed in Subsection 3.2. Using $V(P^k)$ and the definition of P^{k+1} given in Step 5, the vertex set $V(P^{k+1})$ can be determined via one of several special techniques from the global optimization literature; see, for instance, [4, 6, 15].

3.1 Generating a weakly efficient point of Y

Define the set Z by

$$Z = (Y - \mathbb{R}^p_+) = \{z \in \mathbb{R}^p | y \geq z \text{ for some } y \in Y\}.$$

It is easy to show that Z is the nonempty full-dimensional convex set in $\mathbb{R}^p$. Let ∂Z denote the boundary of Z.

Proposition 3.1. i) *Every boundary point of Z belongs to $W(Z)$.*

ii) *Assume that z^* is a point on the boundary of Z and $y^* \in Y$ such that $y^* \geq z^*$. Then y^* is a weakly efficient point of Y.*

Proof. i) Let z^* be an arbitrary boundary point of Z. Assume the contrary, that $z^* \notin W(Z)$. Then, there is a point $z^0 \in Z$ such that $z^0 \gg z^*$. It means that $z^* \in (z^0 - \text{int}\mathbb{R}^p_+)$, i.e., z^* is an interior point of $(z^0 - \mathbb{R}^p_+)$. By definition, we have $(z^0 - \mathbb{R}^p_+) \subset Z$. Then z^* is an interior point of Z. This contradicts the fact that z^* belongs to the boundary of Z.

ii) According (i), we have z^* is a weakly efficient of Z because z^* is the boundary point of Z. It is well known (see for instance [11, p. 91, Theorem 2.10]) that the point z^* of the convex set Z is a weakly efficient if and only if there exists a nonzero vector $\xi \in \mathbb{R}^p_+$ such that

$$\langle \xi, z^* \rangle \geq \langle \xi, z \rangle \text{ for all } z \in Z.$$

Since $Y \subset Z$, it implies that

$$\langle \xi, z^* \rangle \geq \langle \xi, z \rangle \text{ for all } z \in Y.$$

Moreover, we have $\langle \xi, y^* \rangle \geq \langle \xi, z^* \rangle$ because $y^* \geq z^*$ and $\xi \geq 0$. Therefore, one has

$$\langle \xi, y^* \rangle \geq \langle \xi, y \rangle \text{ for all } y \in Y.$$

Now, suppose the contrary that y^* is not a weakly efficient of Y. By the definition, there is a point $\bar{y} \in Y$ such that $\bar{y} \gg y^*$. Combining this fact and $\xi \geq 0$, we have $\langle \xi, \bar{y} \rangle > \langle \xi, y^* \rangle$ and we obtain a contradiction. Thus, $y^* \in W(Y)$. □

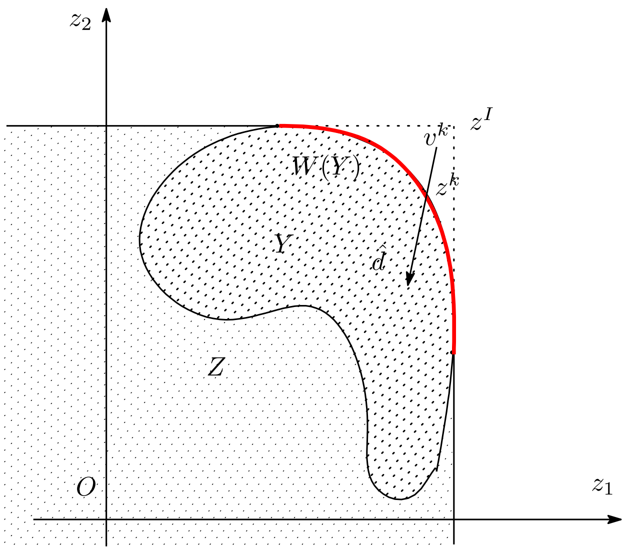

Figure 1: *Generating a weakly efficient point of Y.*

Remark 3.2. Let $\hat{d} \in \mathbb{R}^p$ be a negative vector, i.e., $\hat{d} \in -\text{int}\mathbb{R}^p_+$. In a typical iteration k, we have $v^k \in (\mathbb{R}^p_+ \setminus Z) \cap P^0$, where v^k is the optimal solution of problem $\max\{\varphi(y) | y \in V(P^k)\}$. Let z^k denote the unique point on the boundary of the closed convex set Z that belongs to the ray emanating from v^k along the direction of $\hat{d}$ (see Figure 1). Let (x^k, t_k) and t_k be the optimal solution and the optimal value of the convex programming, respectively,

$$\begin{array}{ll} \min & t \\ \text{s.t.} & f(x) \geq v^k + t\hat{d} \\ & x \in X, t \geq 0. \end{array} \qquad (P(v^k))$$

Then two points $z^* \in \partial Z$ and $y^* \in W(Y)$ described in the Proposition 3.1(ii) can be determined by $z^* = z^k, y^* = y^k$, where

$$z^k = v^k + t_k \hat{d} \text{ and } y^k = f(x^k),$$

respectively. For convenience, the point x^k is called a *feasible solution of problem* (MP_X) *with respect to the weakly point* $y^k \in W(Y)$.

3.2 Determining a cutting hyperplane

In the paper, we will assume henceforth that

$$X = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i = 1, \ldots, m\}, \qquad (3.2)$$

where, for each $i = 1, \ldots, m$, $g_i : \mathbb{R}^n \to \mathbb{R}$ is a finite, convex, differentiable function. Recall that the *(inner) normal cone* to X at $\bar{x}$, denoted by $N_X(\bar{x})$, is defined by

$$N_X(\bar{x}) = \{v \in \mathbb{R}^p | \langle v, \bar{x} \rangle \leq \langle v, x \rangle \text{ for all } x \in X\}.$$

When X is a polyhedron, a formula to calculate the normal cone $N_X(x^0)$ was given in [13, Theorem 6.46] and [8, Lemma 3.1]. A formula to calculate the normal cone to X at $\bar{x} \in X$, where X is convex set determined by (3.2), is described in Proposition 3.3 and is given with full proof for the reader's convenience.

For a system of vectors $\{v^1, \ldots, v^h\} \subset \mathbb{R}^n$, the cone generated by this system, denoted by $\text{cone}\{v^1, \ldots, v^h\}$, consists of all nonnegative combinations $\sum_{i=1}^h t_i v^i$ with $t_i \geq 0$, $i = 1, \ldots, h$. Let $\bar{x} \in X$. Denote by $I(\bar{x})$ the set of *active indices* at $\bar{x} \in X$, i.e.

$$I(\bar{x}) = \{i \in \{1, \ldots, m\} \mid g_i(\bar{x}) = 0\}.$$

Proposition 3.3. *Let $X \subset \mathbb{R}^n$ be a nonempty convex set defined by* (3.2) *and $\bar{x} \in X$. Assume that the Slater condition is satisfied, i.e., there exists a point $\bar{x} \in X$ such that $g_i(\bar{x}) < 0$ for all $i = 1, \ldots, m$. Then, we have*

$$N_X(\bar{x}) = \text{cone}\{-\nabla g_i(\bar{x}), i \in I(\bar{x})\}.$$

Proof. Let $v^* \in N_X(\bar{x})$. Then $\langle v^*, \bar{x}\rangle \leq \langle v^*, x\rangle$ for all $x \in X$ by definition. In other words, $\bar{x} \in X$ is an optimal solution to the following convex programming problem satisfying the Slater condition

$$\begin{aligned} &\min \langle v^*, x\rangle \\ &\quad \text{s.t.} \quad g_i(x) \leq 0, i = 1, \ldots, m. \end{aligned} \tag{SP}$$

By applying Karush-Kuhn-Tucker necessary condition for problem (SP), we derive that the following system is consistent

$$\begin{cases} v^* + \sum\limits_{i=1}^m \eta_i \nabla g_i(\bar{x}) = 0 \\ \eta_i g_i(\bar{x}) = 0, i = 1, \ldots, m \\ \eta_i \geq 0, i = 1, \ldots, m. \end{cases}$$

For each $i \notin I(\bar{x})$, we have $\eta_i = 0$ because $g_i(\bar{x}) \neq 0$. Thus

$$v^* = -\sum_{i \in I(\bar{x})} \eta_i \nabla g_i(\bar{x}), \text{ where } \eta_i \geq 0, i \in I(\bar{x}).$$

Consequently, $v^* \in \text{cone}\{-\nabla g_i(\bar{x}), i \in I(\bar{x})\}$.

Conversely, let $v^* \in \text{cone}\{-\nabla g_i(\bar{x}), i \in I(\bar{x})\}$. By definition, there exists $\eta_i \geq 0, i \in I(\bar{x})$ such that

$$v^* = -\sum_{i \in I(\bar{x})} \eta_i \nabla g_i(\bar{x}).$$

Let $\eta_i = 0$ for all $i \notin I(\bar{x})$. Then it is clear that the following system is consistent

$$\begin{cases} v^* + \sum\limits_{i=1}^m \eta_i \nabla g_i(\bar{x}) = 0 \\ \eta_i g_i(\bar{x}) = 0, i = 1, \ldots, m \\ \eta_i \geq 0, i = 1, \ldots, m. \end{cases}$$

By Karush-Kuhn-Tucker sufficient condition, it implies that $\bar{x}$ is an optimal solution of (SP). Therefore, $v^* \in N_X(\bar{x})$ and the proof is complete. □

Proposition 3.4. *Assume that $\bar{z}$ is a boundary point of the set Z and a point $\bar{x} \in X$ such that $f(\bar{x}) \geq \bar{z}$. Then*

i) *The following system is consistent*

$$\begin{cases} -\sum\limits_{j=1}^{p} \xi_j \nabla f_j(\bar{x}) + \sum\limits_{i\in I(\bar{x})} \eta_i \nabla g_i(\bar{x}) = 0 \\ \eta_i \geq 0,\ i \in I(\bar{x}) \\ \xi_j \geq 0,\ j = 1, \ldots, p \\ \sum\limits_{j=1}^{p} \xi_j = 1 \\ \langle \xi, \bar{z} - f(\bar{x}) \rangle = 0. \end{cases} \tag{3.3}$$

ii) *If* (ξ, η) *is a solution of system* (3.3) *then* ξ *is the normal vector of the cutting hyperplane* $H(\bar{y})$*, where* $\bar{y} = f(\bar{x}) \in W(Y)$*, as well as is the normal vector of the supporting hyperplane to the set* Z *at the point* $\bar{z}$*.*

Proof. i) By Proposition 3.1(i), we have $\bar{z} \in W(Z)$ because $\bar{z}$ is a boundary point of the closed convex set Z. By using analogous arguments in the proof of Proposition 3.1(ii), there exists a vector $\xi = (\xi_1, \ldots, \xi_p) \geq 0, \xi \neq 0$ such that

$$\langle \xi, z \rangle \leq \langle \xi, \bar{z} \rangle \text{ for all } z \in Z. \tag{3.4}$$

From replacing z by $f(\bar{x}) \in Z$ in (3.4), one has

$$\langle \xi, f(\bar{x}) \rangle \leq \langle \xi, \bar{z} \rangle, \text{ i.e. } \langle \xi, f(\bar{x}) - \bar{z} \rangle \leq 0. \tag{3.5}$$

Since $\xi \geq 0$ and $f(\bar{x}) - \bar{z} \geq 0$, we get

$$\langle \xi, f(\bar{x}) - \bar{z} \rangle \geq 0. \tag{3.6}$$

Combining (3.5) and (3.6), we have $\langle \xi, \bar{z} - f(\bar{x}) \rangle = 0$. It means that the last equality of system (3.3) is satisfied.

Combining the fact that $Y \subset Z$ with (3.4) and (3.6), we have $\langle \xi, y \rangle \leq \langle \xi, f(\bar{x}) \rangle$ for all $y \in Y$. It implies that $\langle \xi, f(x) \rangle \leq \langle \xi, f(\bar{x}) \rangle$ for all $x \in X$. Hence

$$\langle \xi, f(x) - f(\bar{x}) \rangle \leq 0$$

Let x be an arbitrary point of X. Since $\bar{x} + t(x - \bar{x}) \in X$ for $0 < t \leq 1$, we have

$$\langle \xi, f(\bar{x} + t(x - \bar{x})) - f(\bar{x}) \rangle \leq 0$$

i.e.

$$\sum_{j=1}^{p} \xi_j \left(f_j(\bar{x} + t(x - \bar{x})) - f_j(\bar{x}) \right) \leq 0.$$

Combining this fact and the differentiability of the function f_j, it is clear that

$$\lim_{t\to 0^+} \frac{f_j(\bar{x} + t(x - \bar{x})) - f_j(\bar{x})}{t} = f_j'(\bar{x}, x - \bar{x}) = \langle \nabla f_j(\bar{x}), x - \bar{x} \rangle \leq 0, \quad \forall \ \ j = 1, \ldots, p,$$

where $f_j'(a, v)$ denotes the directional derivative of f_j at the point $a \in \mathbb{R}^n$ in the direction $v \in \mathbb{R}^n$. Hence,

$$\sum_{j=1}^{p} \xi_j \langle \nabla f_j(\bar{x}), x - \bar{x} \rangle = \left\langle \sum_{j=1}^{p} \xi_j \nabla f_j(\bar{x}), x - \bar{x} \right\rangle \leq 0 \ \forall x \in X. \tag{3.7}$$

Thus, by definition and from Proposition 3.3, one has

$$-\sum_{j=1}^{p} \xi_j \nabla f_j(\bar{x}) \in N_X(\bar{x}) = \text{cone}\{-\nabla g_i(\bar{x}), i \in I(\bar{x})\},$$

which means there exists $\eta_i, i \in I(\bar{x})$ such that

$$-\sum_{j=1}^{p} \xi_j \nabla f_j(\bar{x}) + \sum_{i \in I(\bar{x})} \eta_i \nabla g_i(\bar{x}) = 0.$$

Since $\xi \neq 0$, one has $\sum_{i=1}^{p} \xi_i \neq 0$ and we may assume $\sum_{i=1}^{p} \xi_i = 1$. Therefore, the remaining equalities and inequalities in system (3.3) are satisfied. It means that system (3.3) has the solution (ξ, η).

ii) Since $\bar{z}$ is a boundary point of the set Z and a point $\bar{x} \in X$ such that $\bar{y} = f(\bar{x}) \geq \bar{z}$, we have $\bar{y} \in W(Y)$ (Proposition 3.1(ii)). By Proposition 3.3 and from the two first equations of system (3.3), we have

$$-\sum_{j=1}^{p} \xi_j \nabla f_j(\bar{x}) \in N_X(\bar{x}) = \text{cone}\{-\nabla g_i(\bar{x}), i \in I(\bar{x})\},$$

Therefore,

$$0 \geq \left\langle \sum_{j=1}^{p} \xi_j \nabla f_j(\bar{x}), x - \bar{x} \right\rangle = \sum_{j=1}^{p} \xi_j \left\langle \nabla f_j(\bar{x}), x - \bar{x} \right\rangle \ \forall x \in X. \tag{3.8}$$

For each $j = 1, \ldots, p$, since $f_j(x)$ is concave and differentiable, we have

$$f_j(x) \leq \langle \nabla f_j(\bar{x}), x - \bar{x} \rangle + f_j(\bar{x}) \quad \forall x \in X.$$

Since $\xi \geq 0$ and from (3.8), we have

$$\sum_{j=1}^{p} \xi_j f_j(x) \leq \sum_{j=1}^{p} \xi_j \left\langle \nabla f_j(\bar{x}), x - \bar{x} \right\rangle + \sum_{j=1}^{p} \xi_j f_j(\bar{x}) \leq \sum_{j=1}^{p} \xi_j f_j(\bar{x}) \ \ \forall x \in X.$$

It implies that

$$\langle \xi, y \rangle \leq \langle \xi, \bar{y} \rangle \ \text{ for all } y \in Y. \tag{3.9}$$

Hence, ξ is a normal vector of the supporting hyperplane to Y at $\bar{y}$. By definition, ξ is a normal vector of the cutting hyperplane $H(\bar{y})$.

Let z be an arbitrary point of $Z = Y - \mathbb{R}^p_+$. Then there exists $\hat{y} \in Y$ and $\hat{u} \in \mathbb{R}^p_+$ such that $z = \hat{y} - \hat{u}$. Since $\xi \geq 0$ and $\hat{u} \geq 0$, it follows that

$$\langle \xi, z \rangle = \langle \xi, \hat{y} \rangle - \langle \xi, \hat{u} \rangle \leq \langle \xi, \hat{y} \rangle. \tag{3.10}$$

Furthermore, by (3.9) with $y = \hat{y}$, we have

$$\langle \xi, \hat{y} \rangle \leq \langle \xi, \bar{y} \rangle = \langle \xi, f(\bar{x}) \rangle. \tag{3.11}$$

Combining (3.10) and (3.11), we get $\langle \xi, z \rangle \leq \langle \xi, f(\bar{x}) \rangle$ for all $z \in Z$. Moreover, from the last equation in system (3.3), it implies that

$$\langle \xi, z \rangle \leq \langle \xi, \bar{z} \rangle \text{ for all } z \in Z.$$

This means that ξ is a normal vector of the supporting hyperplane to the set Z at the point $\bar{z} \in Z$. The proof is completed. □

4 The Outer Approximation Algorithm

Now the algorithm can be described in detail as follows.

Outer approximation algorithm for solving Problem (MP_X)
Initialization step.

- (i_1) Choose a sufficiently small number $\varepsilon \geq 0$. Construct the box P^0 described in (3.1). Store the vertex set $V(P^0)$ of P^0.
- (i_2) Choose a fixed vector $\hat{d} \in -\text{int}\mathbb{R}^p_+$.
- (i_3) Set $\alpha_0 = 0$ (*initial lower bound*) and $k = 0$. Go to Iteration k.

Iteration k, $k = 0, 1, 2...$ See Steps k_1 through k_6 below

- (k_1) Set $\beta_k = \max\{\varphi(y) \mid y \in V(P^k)\}$ (*currently best upper bound*) and let v^k denote any element of the vertex set $V(P^k)$ for which $\varphi(v^k) = \beta_k$.
- (k_2) Solve the problem $(P(v^k))$ to obtain an optimal solution (x^k, t_k). Let $z^k = v^k + t_k\hat{d}$ and $y^k = f(x^k)$ (*By Remark 3.2,* $z^k \in \partial Z$ *and* $y^k \in W(Y)$).
- (k_3) **If** $\varphi(y^k) > \alpha_k$ **Then**

  Begin

  $\alpha_k = \varphi(y^k)$ (*currently best lower bound*)

  $y^{best} = y^k$ (*currently best feasible point*)

  $x^{best} = x^k$ (*feasible solution of problem* (MP_X) *with respect to* y^{best})

  End
- (k_4) **If** $\beta_k - \alpha_k \leq \varepsilon(|\alpha_k| + 1)$ **Then** Terminate the algorithm

  (y^{best} *is an* ε*-optimal solution to problem* (WP_Y) *and*

  x^{best} *is an approximate optimal solution to problem* (MP_X))

  Else Solve the system (3.3) with $\bar{x} = x^k$ and $\bar{z} = z^k$ to find a normal vector ξ^k of the cutting hyperplane $H(y^k)$, where $y^k = f(x^k)$ (Proposition 3.4).
- (k_5) Set $P^{k+1} = P^k \cap \{y \in \mathbb{R}^p | \langle \xi^k, y \rangle \leq \langle \xi^k, y^k \rangle\}$ and determine the vertex set $V(P^{k+1})$ of the polytope P^{k+1}.
- (k_6) Let $\alpha_{k+1} = \alpha_k$, $k := k + 1$ and go to Iteration k.

Before proving the convergence of the proposed algorithm, we give some comments on its computational performance. In each typical iteration, we have to implement three main procedures which are solving problem $(P(v^k))$ at Step k_2, solving system (3.3) at Step k_4 and calculating new vertex set $V(P^{k+1})$ at Step k_5. As shown in Section 3.1, problem $(P(v^k))$ is a convex programming problem with $n + 1$ variables and can be solved by some standard numerical methods. By applying Phase I of the two-phase simplex algorithm to find a nonnegative solution of a linear equation system with $n + 2$ equations and $p + I(x^k)$

variables, we can determine the solution (ξ, η) of system (3.3). As mentioned in Section 3, we can use some methods in [4, 6, 15] to determine the vertex set of the polytope P^{k+1} from its inequality representation. For the numerical examples in Section 5, we use the on-line vertex enumerating method in [4]. This method is implemented by adjacency lists and utilizes the vertex set of P^k which is obtained in the previous iteration step. As shown in [2], the performance of this method can be expected to be relatively efficient when the dimension of outcome space is less than or equal to 20, i.e. $p \leq 20$.

Lemma 4.1. *Assume that the algorithm is infinite. Let v^k, z^k be the point generated by Step k_1, k_2 of the algorithm, respectively. Then*

$$\lim_{k\to\infty} \left\| v^k - z^k \right\| = 0.$$

Proof. Denote by $S(v^k)$ the feasible solution set of Problem $(P(v^k))$ and let t_k be its optimal value. Then $S(z^I) \subseteq S(v^k)$ because

$$f(x) \geq z^I + t\hat{d} \geq v^k + t\hat{d} \quad \text{for all} \quad (x,t) \in S(z^I).$$

Hence, $0 \leq t_k \leq t_I$, where t_I is the optimal value of problem $(P(z^I))$. Combining this fact with Remark 3.2, one has $z^k = v^k + t_k\hat{d} \geq t_I\hat{d}$ because $v^k \geq 0$ and $\hat{d} < 0$. Let $z^O = t_I\hat{d}$. It implies that z^k is contained in the box $\hat{P}^0 = [z^O, z^I]$.

At Iteration k, together with the polyhedral set P^{k+1} we also obtain

$$\hat{P}^{k+1} = \left\{ z \in \hat{P}^k \mid \langle \xi^k, z \rangle \leq \langle \xi^k, z^k \rangle \right\}.$$

It is easily seen that $P^k \subseteq \hat{P}^k$ and $\hat{P}^{k+1} \subseteq \hat{P}^k$ for $k \geq 0$. By definition, we have

$$[z^k, v^k] \subseteq \hat{P}^k.$$

Moreover, as ξ^k is nonnegative,

$$(z^k + \text{int}\mathbb{R}^p_+) \cap \hat{P}^{k+1} = \emptyset,$$

which implies $\text{int}[z^k, v^k] \subseteq \hat{P}^k \setminus \hat{P}^{k+1}$. Therefore, the volume of $\hat{P}^k$ satisfies

$$\text{Vol}\hat{P}^k - \text{Vol}\hat{P}^{k+1} \geq \text{Vol}[z^k, v^k]. \tag{4.1}$$

Note that

$$z^k - v^k = t_k\hat{d},$$

which means

$$\text{Vol}[z^k, v^k] = t_k^p \prod_{j=1}^{p} |\hat{d}_j|. \tag{4.2}$$

Combining (4.1) and (4.2), we get

$$\begin{aligned}
\text{Vol}\hat{P}^0 \geq \text{Vol}\hat{P}^0 - \text{Vol}\hat{P}^\ell &= \sum_{k=0}^{\ell} \left(\text{Vol}\hat{P}^k - \text{Vol}\hat{P}^{k+1} \right) \\
&\geq \sum_{i=0}^{\ell} \text{Vol}[z^k, v^k] \geq \left(\sum_{k=0}^{\ell} t_k^p \right) \prod_{j=1}^{p} |\hat{d}_j|
\end{aligned}$$

for every $\ell \geq 1$. Consequently, the positive series $\sum_{k=0}^{\infty} t_k^p$ is convergent. We conclude $\lim_{k\to\infty} t_k = 0$ that means

$$\lim_{i\to\infty} \left\| z^k - v^k \right\| = \lim_{k\to\infty} t_k \left\| \hat{d} \right\| = 0.$$

as requested. □

Theorem 4.2. *For a tolerance $\varepsilon > 0$, the algorithm terminates after finitely many steps and yields an ε-optimal solution to problem (WP_Y).*

Proof. Suppose that the algorithm is infinite. By the construction of α_k and β_k, one has

$$\beta_k - \alpha_k = \varphi(v^k) - \varphi(y^{best}) \leq \varphi(v^k) - \varphi(y^k).$$

Moreover, since φ is increasing and $y^k \geq z^k$, we deduce

$$\beta_k - \alpha_k \leq \varphi(v^k) - \varphi(z^k).$$

By Lemma 4.1, the continuity of φ implies that $\varphi(v^k) - \varphi(z^k) = 0$ when $k \to \infty$. Hence, $(\beta_k - \alpha_k)/(|\alpha_k| + 1) = 0$ when $k \to \infty$, which means there exists an integer $K > 0$ such that $(\beta_k - \alpha_k)/(|\alpha_k| + 1) < \varepsilon$. It contradicts to the assumption. Therefore, the algorithm terminates after a finite number of iterations K and we also obtain that $(\beta_k - \varphi(y^{best}))/(\left|\varphi(y^{best})\right| + 1) < \varepsilon$. Then y^{best} is an ε-optimal solution of (WP$_Y$), which completes the proof. □

5 Illustrative Examples

Now we give some examples to illustrate the algorithm. These examples were performed on a personal computer, using codes written in Matlab R2012a.

Example 5.1. Consider the problem in [1] and [3] as follows

$$\begin{aligned}
\max\ & (x_1 - x_2 + 4) + (5 - 0.25x_1^2)(0.125x_2 + 1) + (0.25x_1 + 1)(4 - 0.125x_2^2) \\
\text{s.t. } & 5x_1 - 8x_2 \geq -24 \\
& 5x_1 + 8x_2 \leq 44 \\
& 6x_1 - 3x_2 \leq 15 \\
& 4x_1 + 5x_2 \geq 10 \\
& x_1 \geq 0.
\end{aligned}$$

It can be verified that the functions $\mathbf{f}_j, j = 1, \ldots, 5$, are positive over the feasible set X. Let the tolerance $\varepsilon = 10^{-4}$. Determine the point $z^I = (2.5000, 2.6425, 2.6045)$.

The algorithm is terminated after 4 iteration steps. The optimal solution of Problem (WP_Y) is $y^{best} = (2.4997, 1.8540, 2.5496)$ and the corresponding solution of Problem (MP_X) is $x^{best} = (2.5003, 0.0006)$. The $\varepsilon-$optimal value of Problem (MP_X) is $h(x^{best}) = 16.4374$. This computational result is the same as one in [3] and [1]. Our algorithm terminates after 4 iterations with $\varepsilon = 10^{-4}$ (see Table 1), while the algorithm in [3] terminates after 77 iterations with $\varepsilon = 0.05$ and the algorithm in [1] terminates after 8 iterations with the same tolerance.

k	y^k	β_k	α_k	Gap	ξ^k
0	$(1.8553, 1.9978, 2.4751)$	20.2661	15.9725	0.3310	$(0.1721, 0.8279, 0)$
1	$(2.4537, 1.8488, 2.5582)$	16.7572	16.4165	0.0254	$(0.1530, 0, 0.8470)$
2	$(2.4866, 1.8572, 2.5481)$	16.4795	16.4285	0.0038	$(0.1923, 0.8077, 0)$
3	$(2.4693, 1.8563, 2.5506)$	16.4608	16.4285	0.0024	$(0.0494, 0.4406, 0.5100)$
4	$(2.4997, 1.8540, 2.5496)$	16.4381	16.4374	5.10^{-5}	$\star$

Table 1: The computational result of Example 5.1

Example 5.2. Consider the problem in [1] and [10] as follows.

$$\begin{aligned} \max\ & (3x_1 - 4x_2 + 15) + (x_1 + 2x_2 - 1.5)(2x_1 - x_2 + 4) \\ & + (x_1 - 2x_2 + 8.5)(2x_1 + x_2 - 1) \\ \text{s.t.}\ & 5x_1 - 8x_2 \geq -24 \\ & 5x_1 + 8x_2 \leq 44 \\ & 6x_1 - 3x_2 \leq 15 \\ & 4x_1 + 5x_2 \geq 10 \\ & x_1 \geq 0. \end{aligned}$$

We can check that the functions $\mathbf{f}_j, j = 1, \ldots, 5$, are positive over the feasible set X. Let the tolerance $\varepsilon = 10^{-5}$. Determine the point $z^I = (22.5000, 8.7464, 8.0829)^T$.

k	y^k	β_k	α_k	Gap	ξ_k
0	$(19.6443, 5.8908, 7.6395)$	164.3333	112.7073	0.4540	$(0.4331, 0.5669, 0)$
1	$(15.5946, 8.4350, 8.0812)$	157.7394	152.0490	0.0372	$(0.3479, 0.6521, 0)$
2	$(15.0366, 8.7276, 8.0640)$	156.8441	156.2356	0.0039	$(0, 0.0842, 0.9158)$
3	$(15.0071, 8.7428, 8.0626)$	156.5115	156.4487	0.0004	$(0.3398, 0.6602, 0)$
4	$(15.0000, 8.7464, 8.0623)$	156.5007	156.4995	8.10^{-6}	

Table 2: The computational result of Example 5.2

The algorithm is terminated after 4 iteration steps (see Table 2). The optimal solution of Problem (WP_Y) is $y^{best} = (15.0000, 8.7464, 8.0623)$ and the corresponding solution of Problem (MP_X) is $x^{best} = (4, 3)$. The ε-optimal value $h(x^{best}) = 156.5$. This computational result is the same as one in [1] and [10].

6 Conclusions

In this paper, we have presented a global optimization algorithm for generalized concave multiplicative programs. By using the suitable reformulation of the original problem in the image space, we have solved an equivalent problem of maximizing an increasing and convex function over the outcome set. In each iteration of algorithm, we established an outer approximation of outcome set by two basic operators such as generating a weakly efficient outcome point and a normal vector of cutting hyperplane by the normal cone method. Because of nice properties of equivalent objective function φ, the problem over outer approximations is solved by an easy way. The convergence of algorithm is proven. In further research, we hope to develop the normal cone method and extend the algorithm for other classes of multiplicative programs, and also apply it to solving practical problems.

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