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SHARP BRAUER-TYPE EIGENVALUE INCLUSION THEOREMS FOR TENSORS*

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Abstract: In this paper, some new Brauer-type eigenvalue inclusion theorems are established for general tensors. We show that new eigenvalue inclusion sets are sharper than classical results. As applications, we obtain bounds for the largest eigenvalue of a nonnegative tensor, which achieve tighter bounds than existing bounds. Furthermore, based on these eigenvalue inclusion theorems, we present several sufficient conditions to test positive definiteness and positive semi-definiteness of tensors.

Key words: *eigenvalue inclusion set, largest eigenvalue, positive semi-definite*

Mathematics Subject Classification: *15A18, 15A42,15A69*

1 Introduction

Let $\mathcal{C}$ ($\mathcal{R}$) denote the set of all complex (real) numbers. Consider an m-order n-dimensional tensor $\mathcal{A}$ consisting of n^m entries in $\mathcal{R}$:

$$\mathcal{A} = (a_{i_1 i_2 \dots i_m}), \forall a_{i_1 i_2 \dots i_m} \in \mathcal{R}, 1 \le i_1, i_2, \dots, i_m \le n.$$

$\mathcal{A}$ is called nonnegative (positive) if $a_{i_1 i_2 \dots i_m} \ge 0$ ($a_{i_1 i_2 \dots i_m} > 0$).

The following definitions about eigenvalues of tensors were introduced in [6, 13]. For an n-dimensional column vector $x = [x_1, x_2, \dots, x_n]^T \in \mathcal{R}^n$, real or complex, we define an n-dimensional column vector, whose ith component is

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\dots,i_m=1}^{n} a_{i i_2 \dots i_m} x_{i_2} \dots x_{i_m}.$$

Definition 1.1. Let $\mathcal{A}$ be an m-order n-dimensional tensor. We say that $(\lambda, x) \in \mathcal{C} \times (\mathcal{C}^n \backslash \{0\})$ is an eigenvalue-eigenvector of $\mathcal{A}$ if

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

where $x^{[m-1]} = [x_1^{m-1}, x_2^{m-1}, \dots, x_n^{m-1}]^T$. (λ, x) is called an H-eigenpair if both λ and x are real.

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Tensor eigenvalue problems have attracted a lot of researchers due to their wide applications in medical resonance imaging [1, 13, 14], data analysis [5], higher-order Markov chains [11], positive definiteness of even-order multivariate forms in automatical control [12]. Many researchers focused on investigating Perron-Frobenius theorem for nonnegative tensors [2, 4, 20, 21]. A number of effective algorithms for finding the largest eigenvalue of nonnegative tensors have been presented; for more detailed discussions, see [3, 4, 7, 11, 15, 17, 18, 19, 20, 21, 22, 23, 24]. From these algorithms, we observe that the choice of initial value is very important as it has great influence on the performance of these algorithms. In this context, it is naturel to consider the distribution of eigenvalues. For this purpose, Wang et al. [19] gave upper bounds for the largest eigenvalue of positive tensors. By max-min theorem, Yang et al. [13] proposed bounds of the largest eigenvalue for nonnegative tensors and further results for bounds of the largest eigenvalue have been given in [20]. For general tensors, it is difficult to calculate all eigenvalues of a tensor. Sometimes, we only need to know the distribution range of eigenvalues and we do not need to obtain the eigenvalues. For example, we judge the stability of nonlinear autonomous system by the eigenvalues of the system equation with nonnegative real component in automatic control [12]. To our knowledge, the Gersgorin eigenvalue inclusion theorem proposed in [13] can be considered as a pioneering work for general tensors. As an extension of the theory of [16], Li et al. [8] proposed two new eigenvalue inclusion theorems and showed tighter bounds than results of [13]. Meanwhile, Li et al. [8] raised a question on how to pick S to make $\mathcal{K}^S(\mathcal{A})$ as tight as possible in Theorem 2.2 of [8] when the dimension of $\mathcal{A}$ is large.

Motivated and inspired by the above works, we firstly construct new eigenvalue inclusion set by exploring the largest modulus of the eigenvector and prove that new eigenvalue inclusion set is included by $\mathcal{K}^S(\mathcal{A})$ of Theorem 2.2 of [8], which overcome the drawbacks to pick S [8]. By choosing different components of eigenvector, we give exact characterization of eigenpair, which help us establish sharp eigenvalue inclusion theorems. Furthermore, we discuss relations among different eigenvalue inclusion sets and show the non-substitutability of the eigenvalue inclusion theorems by Example 3.7. As applications, we firstly apply these eigenvalue inclusion theorems to estimate bounds for the largest eigenvalue of nonnegative tensors, which achieve tighter bounds than existing bounds. Secondly, based on eigenvalue inclusion sets, we propose several sufficient conditions to test positive (positive semidefiniteness) definiteness of an even-order real supersymmetric tensor.

This paper is organized as follows. In Section 2, we recall some preliminary results and introduce some existing results. In Section 3, we establish several eigenvalue inclusion theorems, and show that relations among different eigenvalue inclusion sets. We apply these eigenvalue inclusion sets to estimate bounds for the largest eigenvalue of nonnegative tensors and test positive (positive semidefiniteness) definiteness of an even-order real supersymmetric tensor in Section 4.

2 Notation and Preliminaries

In this section, we shall present some definitions and important properties related to eigenvalues of a tensor, which are needed in the subsequent analysis.

Definition 2.1. Let $\mathcal{A}$ and $\mathcal{I}$ be m-order n-dimensional tensors.

(i) We define $\sigma(\mathcal{A})$ as the set of all eigenvalues of $\mathcal{A}$. Assume $\sigma(\mathcal{A}) \neq \emptyset$. Then the spectral radius of $\mathcal{A}$ is denoted by

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

(ii) We say that tensor $\mathcal{A}$ is reducible if there exists a nonempty proper index subset $I \subset \{1, 2, \ldots, n\}$ such that

$$a_{ii_2 \ldots i_m} x_{i_2} \ldots x_{i_m} = 0, \forall i_1 \in I, i_2, \ldots, i_m \notin I.$$

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible.

(iii) We call $\mathcal{I}$ a unit tensor if its entries are

$$\delta_{i_1 i_2 \ldots i_m} = \begin{cases} 1, & if\ i_1 = i_2 = \ldots i_m \\ 0, & otherwise. \end{cases}$$

(iv) [13] Tensor $\mathcal{A}$ is supersymmetric if all the entries $a_{i_1 i_2 \ldots i_m}$ are invariant under any permutation of their indices $\{i_1, i_2, \ldots, i_m\}$.

The Gersgorin eigenvalue inclusion theorems have been established in [13] for real supersymmetric tensors, and further results for general tensors can be found in [20]. Recently, Li et al. [8] established Brauer-type eigenvalue inclusion theorems for general tensors. We summarize the eigenvalue inclusion theorems for general tensors as follows.

Lemma 2.2. *Let $\mathcal{A}$ be a complex tensor of order m and dimension n and S be a nonempty proper subset of $N = \{1, \ldots, n\}$. Then,*

(I) *(Theorem 6 of [13, 20])*

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) = \bigcup_{i \in N} \Gamma_i(\mathcal{A}),$$

where $\Gamma_i(\mathcal{A}) = \{z \in \mathcal{C} : |z - a_{i \ldots i}| \leq r_i(\mathcal{A})\}, r_i(\mathcal{A}) = \sum_{i_2, \ldots i_m \in N, \delta_{ii_2 \ldots i_m} = 0} |a_{ii_2 \ldots i_m}|$.

(II) *(Theorem 2.1 of [8])*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) = \bigcup_{i,j \in N, i \neq j} \mathcal{K}_{i,j}(\mathcal{A}),$$

where $\mathcal{K}_{i,j}(\mathcal{A}) = \{z \in \mathcal{C} : (|z - a_{i \ldots i}| - r_i^j(\mathcal{A}))|z - a_{j \ldots j}| \leq |a_{ij \ldots j}| r_j(\mathcal{A})\}$ *and* $r_i^j(\mathcal{A}) = \sum_{\substack{\delta_{ii_2 \ldots i_m} = 0 \\ \delta_{ji_2 \ldots i_m} = 0}} |a_{ii_2 \ldots i_m}| = r_i(\mathcal{A}) - |a_{ij \ldots j}|$.

(III) *(Theorem 2.2 of [8])*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) = (\bigcup_{i \in S, j \notin S} \mathcal{K}_{i,j}(\mathcal{A})) \bigcup (\bigcup_{i \notin S, j \in S} \mathcal{K}_{i,j}(\mathcal{A})).$$

We end this section with a result for testing positive definiteness (positive semidefinite) of a tensor.

Lemma 2.3 (Theorem 5 of [13])**.** *Let $\mathcal{A}$ be an even-order real supersymmetric tensor. Then,*

(i) *$\mathcal{A}$ is said to be positive definite if all its real eigenvalues are positive.*

(ii) *$\mathcal{A}$ is said to be positive semidefinite if all its real eigenvalues are nonnegative.*

$\boxed{3}$ Eigenvalue Inclusion Theorems

In this section, we will characterize eigenvalues of $\mathcal{A}$ and obtain eigenvalue inclusion sets, which is included in $\mathcal{K}^S(\mathcal{A})$. In some sense, we will answer the question raised by Li et al. [8] on how to pick S to make $\mathcal{K}^S(\mathcal{A})$ as tight as possible when the dimension of $\mathcal{A}$ is large.

Theorem 3.1. *Let $\mathcal{A}$ be a complex tensor of order m and dimension $n \geq 2$. Then, all eigenvalues of $\mathcal{A}$ are located in the union of the following sets:*

$$\sigma(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}) = \bigcup_{i\in N} \bigcap_{j\in N, i\neq j} \mathcal{M}_{i,j}(\mathcal{A}),$$

where $\mathcal{M}_{i,j}(\mathcal{A}) = \{z \in \mathcal{C} : |(z - a_{i\ldots i})(z - a_{j\ldots j}) - a_{ij\ldots j}a_{ji\ldots i}| \leq |(z - a_{j\ldots j})|r_i^j(\mathcal{A}) + |a_{ij\ldots j}|r_j^i(\mathcal{A})$.

Proof. Let λ be an eigenvalue of $\mathcal{A}$ with corresponding eigenvector x, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \tag{3.1}$$

where $x^{[m-1]} = [x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1}]^T$. Since x is an eigenvector, it has at least one nonzero component. Define x_ρ as a component of x with the largest modulus, i.e., $|x_\rho| \geq |x_j|$ for all $j = 1, \ldots n$.

For any $s \neq \rho$, we have

$$\lambda x_\rho^{m-1} = \sum_{\substack{\delta_{\rho i_2\ldots i_m}=0\\ \delta_{si_2\ldots i_m}=0}} a_{\rho i_2\ldots i_m}x_{i_2}\ldots x_{i_m} + a_{\rho\ldots\rho}x_\rho^{m-1} + a_{\rho s\ldots s}x_s^{m-1},$$

$$\lambda x_s^{m-1} = \sum_{\substack{\delta_{\rho i_2\ldots i_m}=0\\ \delta_{si_2\ldots i_m}=0}} a_{si_2\ldots i_m}x_{i_2}\ldots x_{i_m} + a_{s\ldots s}x_\rho^{m-1} + a_{s\rho\ldots\rho}x_s^{m-1},$$

which are equivalent to

$$(\lambda - a_{\rho\ldots\rho})x_\rho^{m-1} - a_{\rho s\ldots s}x_s^{m-1} = \sum_{\substack{\delta_{\rho i_2\ldots i_m}=0\\ \delta_{si_2\ldots i_m}=0}} a_{\rho i_2\ldots i_m}x_{i_2}\ldots x_{i_m},$$

$$(\lambda - a_{s\ldots s})x_s^{m-1} - a_{s\rho\ldots\rho}x_\rho^{m-1} = \sum_{\substack{\delta_{\rho i_2\ldots i_m}=0\\ \delta_{si_2\ldots i_m}=0}} a_{si_2\ldots i_m}x_{i_2}\ldots x_{i_m}.$$

Solving for x_ρ, we obtain

$$\begin{aligned}
&((\lambda - a_{\rho\ldots\rho})(\lambda - a_{s\ldots s}) - a_{\rho s\ldots s}a_{s\rho\ldots\rho})x_\rho^{m-1}\\
&= (\lambda - a_{s\ldots s}) \sum_{\substack{\delta_{\rho i_2\ldots i_m}=0\\ \delta_{si_2\ldots i_m}=0}} a_{\rho i_2\ldots i_m}x_{i_2}\ldots x_{i_m}\\
&+ a_{\rho s\ldots s} \sum_{\substack{\delta_{\rho i_2\ldots i_m}=0\\ \delta_{si_2\ldots i_m}=0}} a_{si_2\ldots i_m}x_{i_2}\ldots x_{i_m}.
\end{aligned} \tag{3.2}$$

Taking the modulus on both sides of equation (3.2) and using the triangle inequality yield

$$\begin{aligned}
&|(\lambda - a_{\rho\ldots\rho})(\lambda - a_{s\ldots s}) - a_{\rho s\ldots s}a_{s\rho\ldots\rho}||x_\rho|^{m-1} \\
&\leq |(\lambda - a_{s\ldots s})| \sum_{\substack{\delta_{\rho i_2\ldots i_m}=0 \\ \delta_{s i_2\ldots i_m}=0}} |a_{\rho i_2\ldots i_m}||x_{i_2}|\ldots|x_{i_m}| \\
&+|a_{\rho s\ldots s}| \sum_{\substack{\delta_{\rho i_2\ldots i_m}=0 \\ \delta_{s i_2\ldots i_m}=0}} |a_{s i_2\ldots i_m}||x_{i_2}|\ldots|x_{i_m}|.
\end{aligned}$$

Since $|x_\rho| > 0$ with $|x_\rho| \geq |x_j|$ for all $j \in N$, we can divide through by $|x_\rho|^{m-1}$ to obtain

$$\begin{aligned}
&|(\lambda - a_{\rho\ldots\rho})(\lambda - a_{s\ldots s}) - a_{\rho s\ldots s}a_{s\rho\ldots\rho}| \\
&\leq |(\lambda - a_{s\ldots s})| \sum_{\substack{\delta_{\rho i_2\ldots i_m}=0 \\ \delta_{s i_2\ldots i_m}=0}} |a_{\rho i_2\ldots i_m}| + |a_{\rho s\ldots s}| \sum_{\substack{\delta_{\rho i_2\ldots i_m}=0 \\ \delta_{s i_2\ldots i_m}=0}} |a_{s i_2\ldots i_m}| \\
&= |(\lambda - a_{s\ldots s})|r_\rho^s(\mathcal{A}) + |a_{\rho s\ldots s}|r_s^\rho(\mathcal{A}),
\end{aligned}$$

which shows $\lambda \in \mathcal{M}_{i,j}(\mathcal{A})$. From the arbitrariness of s, we have $\lambda \in \bigcap\limits_{j\in N, j\neq\rho} \mathcal{M}_{\rho,j}(\mathcal{A})$. Furthermore, $\lambda \in \bigcup\limits_{i\in N} \bigcap\limits_{j\in N, j\neq i} \mathcal{M}_{i,j}(\mathcal{A})$. □

Remark 3.2. (i) When $m = 2$, from Theorem 3.1, we can obtain the eigenvalue inclusion region of matrices of [10].

(ii) By Theorem 6 of [9], we see $\sigma(\mathcal{A}) \subseteq \bigcup\limits_{i,j\in N, i\neq j} \mathcal{M}_{i,j}(\mathcal{A})$. In this paper, we obtain $\sigma(\mathcal{A}) \subseteq \bigcup\limits_{i\in N} \bigcap\limits_{j\in N, i\neq j} \mathcal{M}_{i,j}(\mathcal{A})$. Clearly, $\mathcal{M}(\mathcal{A})$ may localize all eigenvalues of a tensor more precisely than eigenvalue inclusion set in Theorem 6 of [9] and Theorem 2.1 of [8].

In the proof of Theorem 3.1, the choice of x_s is not limited, which is different from x_s as a component of x with the second largest modulus in Theorem 2.1 [8]. The advantage of this characterization is that it provides tight eigenvalue inclusion sets.

Corollary 3.3. *Let $\mathcal{A}$ be a complex tensor of order m and dimension $n \geq 2$. Then,*

$$\sigma(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}),$$

where $\mathcal{K}^S(\mathcal{A}) = (\bigcup\limits_{i\in S, j\notin S} \mathcal{K}_{i,j}(\mathcal{A}))\bigcup(\bigcup\limits_{i\notin S, j\in S} \mathcal{K}_{i,j}(\mathcal{A}))$ is defined in Theorem 2.2 of [8].

Proof. For any $\lambda \in \mathcal{M}(\mathcal{A})$, without loss of generality, there exists $\rho \in N$ such that $\lambda \in \mathcal{M}_{\rho,q}(\mathcal{A})$, i.e.,

$$|(\lambda - a_{\rho\ldots\rho})(\lambda - a_{q\ldots q}) - a_{\rho q\ldots q}a_{q\rho\ldots\rho}| \leq |(\lambda - a_{q\ldots q})|r_\rho^q(\mathcal{A}) + |a_{\rho q\ldots q}|r_q^\rho(\mathcal{A}), \forall q \in N, q \neq \rho.$$

For all $S \subset N$, observe that $\rho \in S$ or $\rho \notin S$. When $\rho \in S$, there exists $q \notin S$ such that

$$\begin{aligned}
&|(\lambda - a_{\rho\ldots\rho})||(\lambda - a_{q\ldots q})| - |a_{\rho q\ldots q}||a_{q\rho\ldots\rho}| = |(\lambda - a_{\rho\ldots\rho})(\lambda - a_{q\ldots q})| - |a_{\rho q\ldots q}a_{q\rho\ldots\rho}| \\
&\leq |(\lambda - a_{\rho\ldots\rho})(\lambda - a_{q\ldots q}) - a_{\rho q\ldots q}a_{q\rho\ldots\rho}| \leq |(\lambda - a_{q\ldots q})|r_\rho^q(\mathcal{A}) + |a_{\rho q\ldots q}|r_q^\rho(\mathcal{A}),
\end{aligned}$$

Furthermore,

$$|(\lambda - a_{\rho\ldots\rho})||(\lambda - a_{q\ldots q})| - |a_{\rho q\ldots q}||a_{q\rho\ldots\rho}| \leq |(\lambda - a_{q\ldots q})|r_\rho^q(\mathcal{A}) + |a_{\rho q\ldots q}|r_q^\rho(\mathcal{A}),$$

equivalently,

$$(|(\lambda - a_{\rho\ldots\rho})| - r_\rho^q(\mathcal{A}))|(\lambda - a_{q\ldots q})| \leq |a_{\rho q\ldots q}|(|r_q^\rho(\mathcal{A}) + |a_{q\rho\ldots\rho}|) = |a_{\rho q\ldots q}|r_q(\mathcal{A}),$$

which implies $\lambda \in \mathcal{K}_{\rho,q}(\mathcal{A})$. It follows from $\mathcal{M}_{\rho}(\mathcal{A}) = \bigcap_{q\in N, q\neq\rho} \mathcal{M}_{\rho,q}$ that

$$\mathcal{M}_{\rho}(\mathcal{A}) \subset \mathcal{K}_{\rho,q}(\mathcal{A}) \subset \mathcal{K}_{\rho\in S, q\notin S}(\mathcal{A}).$$

which implies $\lambda \in \mathcal{K}_{\rho\in S,q\notin S}(\mathcal{A})$ and $\mathcal{M}_{\rho,q}(\mathcal{A}) \subset \mathcal{K}_{\rho\in S,q\notin S}(\mathcal{A})$.

When $\rho \notin S$, there exists $q \in S$. Similarly, we have

$$\mathcal{M}_{\rho}(\mathcal{A}) \subset \mathcal{K}_{\rho,q}(\mathcal{A}) \subset \mathcal{K}_{\rho\notin S, q\in S}(\mathcal{A}).$$

So, $\mathcal{M}(\mathcal{A}) = \bigcup_{i\in N} \mathcal{M}_i(\mathcal{A}) \subset (\bigcup_{i\in S, j\notin S} \mathcal{K}_{i,j}(\mathcal{A})) \bigcup (\bigcup_{i\notin S, j\in S} \mathcal{K}_{i,j}(\mathcal{A})) = \mathcal{K}^S(\mathcal{A})$. □

In the following theorem, based on x_s as a component of x with the second largest modulus, we obtain sharp eigenvalue inclusion theorem.

Theorem 3.4. *Let $\mathcal{A}$ be a complex tensor of order m and dimension $n \geq 2$. Then, all eigenvalues of $\mathcal{A}$ are located in the union of the following sets:*

$$\sigma(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A}) = \bigcup_{i,j\in N, i\neq j} [\mathcal{N}_{i,j}(\mathcal{A}) \bigcap \Gamma_i(\mathcal{A})] \bigcup_{i,j\in N, i\neq j} \mathcal{H}_{i,j}(\mathcal{A}),$$

where $\mathcal{N}_{i,j}(\mathcal{A}) = \{z \in \mathcal{C} : (|z - a_{i\ldots i}| - r_i^j(\mathcal{A}))(|z - a_{j\ldots j}| - P_j^i(\mathcal{A})) \leq |a_{ij\ldots j}|(r_j(\mathcal{A}) - P_j^i(\mathcal{A}))$, $P_j^i(\mathcal{A}) = \sum_{i\notin\{i_2,\ldots,i_m\}} |a_{ji_2\ldots i_m}|$ *and* $\mathcal{H}_{i,j}(\mathcal{A}) = \{z \in \mathcal{C} : |z - a_{i\ldots i}| \leq r_i^j(\mathcal{A}), |z - a_{j\ldots j}| \leq P_j^i(\mathcal{A})\}$.

Proof. Let λ be an eigenvalue of $\mathcal{A}$ with corresponding eigenvector x, i.e., $\mathcal{A}x^{m-1} = \lambda x^{[m-1]}$. Since x is an eigenvector, it has at least one nonzero component. Let $|x_t| \geq |x_s| \geq \{\max |x_k| : k \in N, k \neq s, k \neq t\}$. Obviously, $|x_t| > 0$. Similar to the characterization of inequality (4) of [8], one has

$$(|\lambda - a_{t\ldots t}| - r_t^s(\mathcal{A}))|x_t|^{m-1} \leq |a_{ts\ldots s}||x_s|^{m-1}. \tag{3.3}$$

Obviously, $\lambda \in \Gamma_t(\mathcal{A})$. If $|x_s| = 0$, then $(|\lambda - a_{t\ldots t}| - r_t^s(\mathcal{A})) \leq 0$. For $|z - a_{s\ldots s}| \geq P_s^t(\mathcal{A})$, one has $\lambda \in \mathcal{N}_{t,s}(\mathcal{A})$; For $|z - a_{s\ldots s}| \leq P_s^t(\mathcal{A})$, we have $\lambda \in \mathcal{H}_{t,s}(\mathcal{A})$.

Otherwise, $|x_s| > 0$. Moreover, from (3.1), we get

$$\begin{aligned}
|\lambda - a_{s\ldots s}||x_s|^{m-1} &\leq \sum_{\delta_{si_2\ldots i_m}=0} |a_{si_2\ldots i_m}||x_{i_2}|\ldots|x_{i_m}| \\
&= \sum_{t\in\{i_2,\ldots,i_m\}} |a_{si_2\ldots i_m}||x_{i_2}|\ldots|x_{i_m}| + \sum_{\substack{t\notin\{i_2,\ldots,i_m\}\\ \delta_{si_2\ldots i_m}=0}} |a_{si_2\ldots i_m}||x_{i_2}|\ldots|x_{i_m}| \\
&\leq \sum_{t\in\{i_2,\ldots,i_m\}} |a_{si_2\ldots i_m}|x_t|^{m-1} + \sum_{\substack{t\notin\{i_2,\ldots,i_m\}\\ \delta_{si_2\ldots i_m}=0}} |a_{si_2\ldots i_m}||x_s|^{m-1},
\end{aligned}$$

that is,

$$(|\lambda - a_{s\ldots s}| - \sum_{\substack{t\notin\{i_2,\ldots,i_m\}\\ \delta_{si_2\ldots i_m}=0}} |a_{si_2\ldots i_m}|)|x_s|^{m-1} \leq \sum_{t\in\{i_2,\ldots,i_m\}} |a_{si_2\ldots i_m}|x_t|^{m-1}. \tag{3.4}$$

When $|\lambda - a_{t\ldots t}| \geq r_t^s(\mathcal{A})$ or $|\lambda - a_{s\ldots s}| \geq \sum_{\substack{t\notin\{i_2,\ldots,i_m\}\\ \delta_{si_2\ldots i_m}=0}} |a_{si_2\ldots i_m}|$ holds, multiplying inequalities (3.3) with (3.4), we have

$$(|\lambda - a_{t\dots t}| - r_t^s(\mathcal{A}))(|\lambda - a_{s\dots s}| - \sum_{\substack{t \notin \{i_2,\dots,i_m\} \\ \delta_{si_2\dots i_m}=0}} |a_{si_2\dots i_m}|)|x_t|^{m-1}|x_s|^{m-1}$$
$$\leq |a_{ts\dots s}| \sum_{t\in\{i_2,\dots,i_m\}} |a_{si_2\dots i_m}|x_t|^{m-1}|x_s|^{m-1}.$$

Note that $|x_s| > 0, |x_t| > 0$. Then

$$(|\lambda - a_{t\dots t}| - r_t^s(\mathcal{A}))(|\lambda - a_{s\dots s}| - \sum_{\substack{t \notin \{i_2,\dots,i_m\} \\ \delta_{si_2\dots i_m}=0}} |a_{si_2\dots i_m}|) \leq |a_{ts\dots s}| \sum_{t\in\{i_2,\dots,i_m\}} |a_{si_2\dots i_m}|,$$

equivalently,

$$(|\lambda - a_{t\dots t}| - r_t^s(\mathcal{A}))(|\lambda - a_{s\dots s}| - P_s^t(\mathcal{A})) \leq |a_{ts\dots s}|(r_s(\mathcal{A}) - P_s^t(\mathcal{A})),$$

which implies $\lambda \in \mathcal{N}_{t,s}(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A})$.

When $|\lambda - a_{t\dots t}| \leq r_t^s(\mathcal{A})$ and $|\lambda - a_{s\dots s}| \leq \sum_{\substack{t \notin \{i_2,\dots,i_m\} \\ \delta_{si_2\dots i_m}=0}} |a_{si_2\dots i_m}|$ hold, one has $\lambda \in \mathcal{H}_{t,s}(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A})$. So, the result holds. □

Now, we give a proof to show $\mathcal{N}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$.

Corollary 3.5. *Let $\mathcal{A}$ be a complex tensor of order m and dimension $n \geq 2$. Then,*

$$\sigma(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}),$$

where $\mathcal{K}(\mathcal{A})$ is defined in Theorem 2.2 of [8].

Proof. For any $\lambda \in \mathcal{N}(\mathcal{A})$, without loss of generality, there exists $s \neq t$ such that $\lambda \in \mathcal{N}_{t,s}(\mathcal{A})$ with $\lambda \in \Gamma_t(\mathcal{A})$, that is

$$(|\lambda - a_{t\dots t}| - r_t^s(\mathcal{A}))(|\lambda - a_{s\dots s}| - P_s^t(\mathcal{A})) \leq |a_{ts\dots s}|(r_s(\mathcal{A}) - P_s^t(\mathcal{A})),$$
$$|\lambda - a_{t\dots t}| \leq r_t(\mathcal{A}).$$

Then,

$$\begin{aligned}(|\lambda - a_{t\dots t}| - r_t^s(\mathcal{A}))(|\lambda - a_{s\dots s}|) &\leq (|\lambda - a_{t\dots t}| - r_t^s(\mathcal{A}))P_s^t(\mathcal{A}) + |a_{ts\dots s}|(r_s(\mathcal{A}) - P_s^t(\mathcal{A})) \\ = |a_{ts\dots s}|r_s(\mathcal{A}) + (|\lambda - a_{t\dots t}| - r_t^s(\mathcal{A}) - |a_{ts\dots s}|)P_s^t(\mathcal{A}) &\leq |a_{ts\dots s}|r_s(\mathcal{A}),\end{aligned}$$

since $(|\lambda - a_{t\dots t}| - r_t^s(\mathcal{A}) - |a_{ts\dots s}|) = |\lambda - a_{t\dots t}| - r_t(\mathcal{A}) \leq 0$. This shows $\lambda \in \mathcal{K}_{t,s}(\mathcal{A})$. Otherwise, there exists $s \neq t$ such that $\lambda \in \mathcal{H}_{t,s}(\mathcal{A})$, that is

$$\mathcal{H}_{t,s} = \{\lambda \in \mathcal{C} : |\lambda - a_{t\dots t}| \leq r_t^s(\mathcal{A}), |\lambda - a_{s\dots s}| \leq P_s^t(\mathcal{A})\},$$

which implies $(|\lambda - a_{t\dots t}| - r_t^s(\mathcal{A}))(|\lambda - a_{s\dots s}|) \leq |a_{ts\dots s}|r_s(\mathcal{A})$. Thus, $[\mathcal{N}_{t,s}(\mathcal{A}) \bigcap \Gamma_t(\mathcal{A})] \subseteq \mathcal{K}_{t,s}(\mathcal{A})$ and $\mathcal{H}_{t,s} \subseteq \mathcal{K}_{t,s}(\mathcal{A})$. □

From Theorem 3.1, Theorem 3.4, Theorem 2.3 [8], Corollary 3.3 and Corollary 3.5, there exist inclusion relations among $\mathcal{M}(\mathcal{A}), \mathcal{K}^S(\mathcal{A}), \mathcal{N}(\mathcal{A}), \mathcal{K}(\mathcal{A}), \Gamma(\mathcal{A})$.

Corollary 3.6. *Let $\mathcal{A}$ be a complex tensor of order m and dimension $n \geq 2$. Then,*

$$\sigma(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A}),$$
$$\sigma(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A}).$$

In particular, $\mathcal{M}(\mathcal{A}) = \mathcal{K}^S(\mathcal{A}) = \mathcal{N}(\mathcal{A}) = \mathcal{K}(\mathcal{A})$ when $n = 2$.

The following example shows that Corollary 3.6 holds. It is noteworthy that Theorem 3.1 and Theorem 3.4 are different, since we cannot judge the relations between $\mathcal{M}_{i,j}$ and $\mathcal{N}_{i,j} \bigcup \mathcal{H}_{i,j}$.

Example 3.7. Consider 3 order 3 dimensional tensor $\mathcal{A} = (a_{ijk})$ defined by

$$a_{ijk} = \begin{cases} a_{111} = 1; a_{222} = 2; a_{333} = 3; \\ a_{112} = a_{121} = a_{211} = -1; a_{113} = a_{131} = a_{311} = 1; a_{233} = a_{332} = a_{323} = 2; \\ a_{ijk} = 0, \qquad \textit{otherwise.} \end{cases}$$

By simple computation, we get the eigenpairs of $\mathcal{A}$ as follows

$$\begin{aligned}&\{(\lambda, x) : (\lambda_1 = 1, u_1 = (1, 0, 0)), (\lambda_2 = 2, u_2 = (0, 1, 0)),\\ &(\lambda_3 = 3, u_3 = (0, 0, 1)), (\lambda_4 = -1.5298, u_4 = (1.0000, 0.6325, -0.6325),\\ &(\lambda_5 = 5.8768, u_5 = (0.1050, 0.6448, 0.9007)\}.\end{aligned}$$

For convenience of calculations, we take λ as a real number, where λ is an eigenvalue of $\mathcal{A}$. According to Theorem 6 of [13, 20], we have

$$\lambda \in \Gamma(\mathcal{A}) = \bigcup_{i \in N} \Gamma_i(\mathcal{A}) = \bigcup [-3, 5] \bigcup [-1, 5] \bigcup [-2, 8] = [-3, 8].$$

According to Theorem 2.1 of [8], we have

$$\lambda \in \mathcal{K}(\mathcal{A}) = \bigcup_{i,j \in N, i \neq j} \mathcal{K}_{i,j}(\mathcal{A}) = [-3, 8],$$

where $\mathcal{K}_{1,2} \bigcup \mathcal{K}_{1,3} = [-3, 5] \bigcup [-3, 5] = [-3, 5]$,$\mathcal{K}_{2,1} \bigcup \mathcal{K}_{2,3} = [-2, \frac{5+\sqrt{17}}{2}] \bigcup [2 - \sqrt{11}, 3 + \sqrt{10}] = [-2, 3 + \sqrt{10}]$ and $\mathcal{K}_{3,1} \bigcup \mathcal{K}_{3,2} = [-\sqrt{5}, 4 + \sqrt{13}] \bigcup [-2, 8] = [-\sqrt{5}, 8]$.

According to Theorem 2.2 of [8], choosing $S_1 = \{3, 2\}, \bar{S}_1 = \{1\}$, we have

$$\lambda \in \mathcal{K}^{S_1} = (\mathcal{K}_{2,1} \bigcup \mathcal{K}_{3,1}) \bigcup (\mathcal{K}_{1,2} \bigcup \mathcal{K}_{1,3}) = [-3, 4 + \sqrt{13}]).$$

Similarly, we have

$\mathcal{N}_{1,2}(\mathcal{A}) \bigcap \Gamma_1(\mathcal{A}) = [-3, 0] \bigcup [4, 5]$	$\mathcal{N}_{1,3}(\mathcal{A}) \bigcap \Gamma_1(\mathcal{A}) = [-3, -1] \bigcup \{5\}$
$\mathcal{N}_{2,1}(\mathcal{A}) \bigcap \Gamma_2(\mathcal{A}) = [-1, 1] \bigcup [2, 5]$	$\mathcal{N}_{2,3}(\mathcal{A}) \bigcap \Gamma_2(\mathcal{A}) = [-1, 5]$
$\mathcal{N}_{3,1}(\mathcal{A}) \bigcap \Gamma_3(\mathcal{A}) = [-(1 + \sqrt{2}), \sqrt{2} - 1] \bigcup [1 + \sqrt{2}, 5 + \sqrt{6}]$	$\mathcal{N}_{3,2}(\mathcal{A}) \bigcap \Gamma_3(\mathcal{A}) = [-2, 1] \bigcup [3, 8]$

According to Theorem 3.1, we have

$$\lambda \in \mathcal{M}(\mathcal{A}) = \bigcup_{i \in N} \bigcap_{j \in N, j \neq i} \mathcal{M}_{i,j}(\mathcal{A}) = [-3, 4 + \sqrt{13}],$$

where $\mathcal{M}_{1,2} \bigcap \mathcal{M}_{1,3} = [-3, 5] \bigcap [-3, 5] = [-3, 5]$, $\mathcal{M}_{2,1} \bigcap \mathcal{M}_{2,3} = [-2, \frac{5+\sqrt{17}}{2}] \bigcap [2 - \sqrt{11}, 3 + \sqrt{10}] = [2 - \sqrt{11}, \frac{5+\sqrt{17}}{2}]$ and $\mathcal{M}_{3,1} \bigcap \mathcal{M}_{3,2} = [-\sqrt{5}, 4 + \sqrt{13}] \bigcap [-2, 8] = [-2, 4 + \sqrt{13}]$. It is verified that

$$\mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A}).$$

According to Theorem 3.4, we have

$$\mathcal{N}_{1,3}(\mathcal{A}) \bigcap \Gamma_1(\mathcal{A}) = ([-3, -1] \bigcup [5, 7]) \bigcap [-3, 5] = [-3, -1] \bigcup \{5\} \subset [-3, 5] = \mathcal{K}_{1,3}(\mathcal{A}).$$

Similarly, we have

$\mathcal{N}_{1,2}(\mathcal{A}) \bigcap \Gamma_1(\mathcal{A}) = [-3,0] \bigcup [4,5]$	$\mathcal{N}_{1,3}(\mathcal{A}) \bigcap \Gamma_1(\mathcal{A}) = [-3,-1] \bigcup \{5\}$
$\mathcal{N}_{2,1}(\mathcal{A}) \bigcap \Gamma_2(\mathcal{A}) = [-1,1] \bigcup [2,5]$	$\mathcal{N}_{2,3}(\mathcal{A}) \bigcap \Gamma_2(\mathcal{A}) = [-1,5]$
$\mathcal{N}_{3,1}(\mathcal{A}) \bigcap \Gamma_3(\mathcal{A}) = [-(1+\sqrt{2}), \sqrt{2}-1] \bigcup [1+\sqrt{2}, 5+\sqrt{6}]$	$\mathcal{N}_{3,2}(\mathcal{A}) \bigcap \Gamma_3(\mathcal{A}) = [-2,1] \bigcup [3,8]$

and

$$\bigcup \mathcal{H}_{i,j}(\mathcal{A}) = \mathcal{H}(\mathcal{A}) = [-1,3],$$

where $\mathcal{H}_{1,2} = [-1,3], \mathcal{H}_{1,3} = [1,3], \mathcal{H}_{2,1} = [1,3], \mathcal{H}_{2,3} = [2,3], \mathcal{H}_{3,1} = [1,3], \mathcal{H}_{3,2} = [1,3]$. So,

$$\lambda \in \mathcal{N}(\mathcal{A}) = [-3,8].$$

Clearly,

$$\mathcal{N}_{i,j}(\mathcal{A}) \bigcap \Gamma_i(\mathcal{A}) \subseteq \mathcal{K}_{i,j}(\mathcal{A}),$$
$$\sigma(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A}).$$

It is worth noting that we cannot judge relation between $\mathcal{N}(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$, since there are no inclusion relations between $\mathcal{M}_i$ and $\mathcal{N}_i$, where $\mathcal{M}_i(\mathcal{A}) = \bigcap_{j\in N, j\neq i} \mathcal{M}_{i,j}(\mathcal{A})$ and $\mathcal{N}_i(\mathcal{A}) = \bigcup_{j\in N, j\neq i} [\mathcal{N}_{i,j}(\mathcal{A}) \bigcap \Gamma_i(\mathcal{A})]$. For instance, $\mathcal{N}_3(\mathcal{A}) = [-(1+\sqrt{2}), 1] \bigcup [1+\sqrt{2}, 8] \nsubseteq [-2, 4+\sqrt{13}] = \mathcal{M}_3(\mathcal{A})$.

4 Applications

4.1 Bounds on the Largest Eigenvalue For Nonnegative Tensors

Based on eigenvalue inclusion theorems in Section 3, we give several bounds of the largest eigenvalue of nonnegative tensors, which improve some existing bounds [8, 13, 20]. We start this section with some fundamental results of nonnegative tensors.

Lemma 4.1 (Lemma 5.2 of [20]). *Let $\mathcal{A}$ be a nonnegative tensor with order m and dimension n. Then,*

$$\min_{i\in N} R_i(\mathcal{A}) \le \rho(\mathcal{A}) \le \max_{i\in N} R_i(\mathcal{A}),$$

where $R_i(\mathcal{A}) = \sum_{i_2,\ldots i_m \in N} a_{ii_2\ldots i_m}$.

Lemma 4.2 (Lemma 3.2 of [8]). *Let $\mathcal{A}$ be a nonnegative tensor with order m and dimension $n \ge 2$. Then,*

$$\rho(\mathcal{A}) \ge \max_{i\in N} a_{i\ldots i}.$$

Lemma 4.3 (Theorem 3.1 of [8]). *Let $\mathcal{A}$ be a nonnegative tensor with order m and dimension $n \ge 2$. Then,*

$$\rho(\mathcal{A}) \le w = \max_{i,j\in N, i\neq j} \frac{1}{2}\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\},$$

where $\Delta_{i,j}(\mathcal{A}) = (a_{i\ldots i} - a_{j\ldots j} + r_i^j(\mathcal{A}))^2 + 4a_{ij\ldots j} r_j(\mathcal{A})$.

Lemma 4.4 (Theorem 3.2 of [8]). *Let $\mathcal{A}$ be a nonnegative tensor with order m and dimension $n \ge 2$. Then,*

$$\rho(\mathcal{A}) \le w_S = \max\{w^S, w^{\bar{S}}\},$$

where $w^S = \max\limits_{i\in S, j\in \bar{S}} \frac{1}{2}\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\}$ and $w^{\bar{S}} = \max\limits_{i\in \bar{S}, j\in S} \frac{1}{2}\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\}$

Now, we focus on establishing sharp bounds for the largest eigenvalue of nonnegative tensors.

Theorem 4.5. *Let $\mathcal{A}$ be a nonnegative tensor with order m and dimension $n \geq 2$. Then,*

$$\begin{aligned}&\min_{i\in N}\max_{j\in N, i\neq j}\tfrac{1}{2}\{a_{i\dots i}+a_{j\dots j}+r_i^j(\mathcal{A})+\Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\}=\underline{u}\leq\rho(\mathcal{A})\\ &\leq\bar{u}=\max_{i\in N}\min_{j\in N, i\neq j}\tfrac{1}{2}\{a_{i\dots i}+a_{j\dots j}+r_i^j(\mathcal{A})+\Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\},\end{aligned}$$

where $\Delta_{i,j}(\mathcal{A})=(a_{i\dots i}-a_{j\dots j}+r_i^j(\mathcal{A}))^2+4a_{ij\dots j}r_j(\mathcal{A})$.

Proof. Suppose $\rho(\mathcal{A})$ is the largest eigenvalue of $\mathcal{A}$ associated with eigenvalue x. Without loss of generality, $x_\rho > 0$ with $x_\rho \geq x_j$ for $j \in N$. It follows from Theorem 3.1 that

$$(\rho(\mathcal{A})-a_{\rho\dots\rho})(\rho(\mathcal{A})-a_{s\dots s})-a_{\rho s\dots s}a_{s\rho\dots\rho}\leq(\rho(\mathcal{A})-a_{s\dots s})r_\rho^s(\mathcal{A})+a_{\rho s\dots s}r_s^\rho(\mathcal{A}), \forall s\in N, s\neq\rho,$$

that is,

$$((\rho(\mathcal{A})-a_{\rho\dots\rho})-r_\rho^s(\mathcal{A}))(\rho(\mathcal{A})-a_{s\dots s})\leq a_{\rho s\dots s}r_s(\mathcal{A}).$$

Then, solving for $\rho(\mathcal{A})$, we have

$$\rho(\mathcal{A})\leq\frac{1}{2}(a_{\rho\dots\rho}+a_{s\dots s}+r_\rho^s(\mathcal{A})+\Delta_{\rho,s}^{\frac{1}{2}}(\mathcal{A}))$$

Since $s \in N$ is chosen arbitrarily, it holds

$$\rho(\mathcal{A})\leq\min_{j\in N, j\neq\rho}\frac{1}{2}\{a_{\rho\dots\rho}+a_{j\dots j}+r_\rho^j(\mathcal{A})+\Delta_{\rho,j}^{\frac{1}{2}}(\mathcal{A})\},$$

Furthermore,

$$\rho(\mathcal{A})\leq\max_{i\in N}\min_{j\in N, i\neq j}\frac{1}{2}\{a_{i\dots i}+a_{j\dots j}+r_i^j(\mathcal{A})+\Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\}.$$

On the other hand, set $\mathcal{B} = \mathcal{A} + \mathcal{E}$, where $\mathcal{E}$ is a positive tensor with every entry being ϵ. Obviously, $\mathcal{B} = \mathcal{A}+\mathcal{E}$ is irreducible. Suppose $\rho(\mathcal{A}+\mathcal{E})$ is the largest eigenvalue of $\mathcal{A}+\mathcal{E}$ with corresponding eigenvector x. It follows from Theorem 1.4 of [2] that $x_i > 0, i = 1, 2 \dots, n$. Let $|x_t| \leq |x_j|$, for $j \in N$. Similarly, $\forall s \neq t$, we get

$$[(\rho(\mathcal{A}+\mathcal{E})-(a_{t\dots t}+\epsilon)-r_t^s(\mathcal{A}+\mathcal{E})][\rho(\mathcal{A}+\mathcal{E})-(a_{s\dots s}+\epsilon)]\geq(a_{ts\dots s}+\epsilon)r_s(\mathcal{A}+\mathcal{E}).$$

Then, solving for $\rho(\mathcal{A}+\mathcal{E})$, we have

$$\rho(\mathcal{A}+\mathcal{E})\geq\frac{1}{2}[(a_{t\dots t}+\epsilon)+(a_{s\dots s}+\epsilon)+r_t^s(\mathcal{A}+\mathcal{E})+\Delta_{t,s}^{\frac{1}{2}}(\mathcal{A}+\mathcal{E})].$$

Based on Theorem 2.3 of [20], we notice that $\rho(\mathcal{A})$ is a continuous function of ϵ. So,

$$\begin{aligned}\rho(\mathcal{A})&=\lim_{\epsilon\to0}\rho(\mathcal{A}+\mathcal{E})\geq\lim_{\epsilon\to0}\tfrac{1}{2}[(a_{t\dots t}+\epsilon)+(a_{s\dots s}+\epsilon)+r_t^s(\mathcal{A}+\mathcal{E})+\Delta_{t,s}^{\frac{1}{2}}(\mathcal{A}+\mathcal{E})]\\ &=\tfrac{1}{2}\{a_{t\dots t}+a_{s\dots s}+r_t^s(\mathcal{A})+\Delta_{s,t}^{\frac{1}{2}}(\mathcal{A})\}.\end{aligned}$$

From the arbitrariness of s, we obtain

$$\rho(\mathcal{A})\geq\max_{j\in N, j\neq t}\frac{1}{2}\{a_{t\dots t}+a_{j\dots j}+r_t^j(\mathcal{A})+\Delta_{t,j}^{\frac{1}{2}}(\mathcal{A})\},$$

moreover,

$$\rho(\mathcal{A})\geq\min_{i\in N}\max_{j\in N, j\neq i}\frac{1}{2}\{a_{i\dots i}+a_{j\dots j}+r_i^j(\mathcal{A})+\Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A})\}.$$

$\square$

Corollary 4.6. *Let $\mathcal{A}$ be a nonnegative tensor with order m and dimension $n \geq 2$. Then,*

$$\max\{\max_{i\in N} a_{i\dots i}, \min_{i\in N} R_i(\mathcal{A})\} \leq \underline{u} \leq \rho(\mathcal{A}) \leq \bar{u} \leq w_S \leq \max_{i\in N} R_i(\mathcal{A}),$$

where $\underline{u}, \bar{u}, w_S$ are defined in Theorem 4.5 and Lemma 4.4, respectively.

Proof. We first show $\max_{i\in N} a_{i\dots i} \leq \underline{u}$. Noting that $a_{ij\dots j} r_j(\mathcal{A}) \geq 0$, we have

$$4a_{ij\dots j} r_j(\mathcal{A}) + (a_{i\dots i} - a_{j\dots j} + r_i^j(\mathcal{A}))^2 \geq (a_{j\dots j} - a_{i\dots i} - r_i^j(\mathcal{A}))^2,$$

that is,

$$\sqrt{4a_{ij\dots j} r_j(\mathcal{A}) + (a_{i\dots i} - a_{j\dots j} + r_i^j(\mathcal{A}))^2} \geq \sqrt{(a_{i\dots i} - a_{j\dots j} - r_i^j(\mathcal{A}))^2} \geq a_{j\dots j} - a_{i\dots i} - r_i^j(\mathcal{A}),$$

$$\frac{1}{2}(\sqrt{4a_{ij\dots j} r_j(\mathcal{A}) + (a_{i\dots i} - a_{j\dots j} + r_i^j(\mathcal{A}))^2} + a_{i\dots i} + a_{j\dots j} + r_i^j(\mathcal{A})) \geq a_{j\dots j},$$

furthermore,

$$\min_{i\in N} \frac{1}{2}(\sqrt{4a_{ij\dots j} r_j(\mathcal{A}) + (a_{i\dots i} - a_{j\dots j} + r_i^j(\mathcal{A}))^2} + a_{i\dots i} + a_{j\dots j} + r_i^j(\mathcal{A})) \geq a_{j\dots j},$$

which implies

$$\max_{i\in N, j\neq i} \min_{i\in N} \frac{1}{2}(\sqrt{4a_{ij\dots j} r_j(\mathcal{A}) + (a_{i\dots i} - a_{j\dots j} + r_k^j(\mathcal{A}))^2} + a_{i\dots i} + a_{j\dots j} + r_i^j(\mathcal{A})) \geq \max_{j\in N} a_{j\dots j}.$$

Secondly, we divide the proof into two parts to get $\min_{i\in N} R_i(\mathcal{A}) \leq \underline{u}$.

(i) For $i, j \in N, i \neq j$, if $R_i(\mathcal{A}) \geq R_j(\mathcal{A})$, then

$$a_{ij\dots j} \geq a_{j\dots j} - a_{i\dots i} - r_i^j(\mathcal{A}) + r_j(\mathcal{A}).$$

Similar to the proof of Theorem 3.5 of [8], we obtain

$$\frac{1}{2}\{a_{i\dots i} + a_{j\dots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\} \geq R_j(\mathcal{A}),$$

furthermore,

$$\min_{i\in N} \max_{j\in N, i\neq j} \frac{1}{2}\{a_{i\dots i} + a_{j\dots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\} \geq \min_{j\in N} R_j(\mathcal{A}). \tag{4.1}$$

(ii) For $i, j \in N, i \neq j$, if $R_i(\mathcal{A}) \leq R_j(\mathcal{A})$, then

$$r_j(\mathcal{A}) \geq a_{i\dots i} - a_{j\dots j} + r_i^j(\mathcal{A}) + a_{ij\dots j}.$$

Similar to the proof of Theorem 3.5 of [8], we obtain

$$\frac{1}{2}\{a_{i\dots i} + a_{j\dots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\} \geq R_i(\mathcal{A}),$$

equivalently,

$$\min_{i\in N} \max_{j\in N, i\neq j} \tfrac{1}{2}\{a_{i\dots i} + a_{j\dots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\} \geq \min_{i\in N} R_i(\mathcal{A}).$$

This combined with (4.1) yields $\min_{i\in N} R_i(\mathcal{A}) \le \bar{u}$.

Finally, we only prove $\bar{u} \le w_S$, since $w_S \le \max_{i\in N} R_i(\mathcal{A})$ from Theorem 3.5 of [8]. We rewrite

$$\bar{u} = \max\{\bar{u}^S, \bar{u}^{\bar{S}}\},$$

where $\bar{u}^S = \max_{i\in S}\min_{j\in N, j\neq i} \frac{1}{2}\{a_{i\dots i} + a_{j\dots j} + r_i^j(\mathcal{A}) + \Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A})\}$. Obviously, $\bar{u}^S \le w^S$. Similarly, $\bar{u}^{\bar{S}} \le w^{\bar{S}}$. So,

$$\bar{u} = \max\{\bar{u}^S, \bar{u}^{\bar{S}}\} \le \max\{w^S, w^{\bar{S}}\} = w_S.$$

This completes the proof. □

From Theorem 3.4, we obtain sharp bounds of the largest eigenvalue for nonnegative tensors.

Lemma 4.7. *Let $\mathcal{A}$ be a nonnegative tensor with order m and dimension $n \ge 2$. Then,*

$$\begin{aligned}&\min_{i,j\in N, i\neq j} \tfrac{1}{2}\{[a_{i\dots i} + a_{j\dots j} + r_i^j(\mathcal{A}) + P_j^i(\mathcal{A}) + \Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A})] = \underline{v} \le \rho(\mathcal{A}) \le \\ &\max_{i,j\in N, i\neq j}\{\tfrac{1}{2}[a_{i\dots i} + a_{j\dots j} + r_i^j(\mathcal{A}) + P_j^i(\mathcal{A}) + \Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A})], \min\{r_i^j(\mathcal{A}) + a_{i\dots i}, P_j^i(\mathcal{A}) + a_{j\dots j}\}\},\end{aligned}$$

where $\Lambda_{i,j}(\mathcal{A}) = (a_{i\dots i} - a_{j\dots j} + r_i^j(\mathcal{A}) - P_j^i(\mathcal{A}))^2 + 4[a_{ij\dots j}(r_j(\mathcal{A}) - P_j^i(\mathcal{A}))])$.

Proof. Suppose $\rho(\mathcal{A})$ is the largest eigenvalue of $\mathcal{A}$. From Theorem 3.4, there exist $i_0, j_0 \in N$, $j_0 \neq i_0$ such that $\rho(\mathcal{A}) \in \mathcal{N}_{i_0,j_0}(\mathcal{A})$ or $\rho(\mathcal{A}) \in \mathcal{H}_{i_0,j_0}(\mathcal{A})$. We divide the proof into two parts to show the desired result.

When $\rho(\mathcal{A}) \in \mathcal{N}_{i_0,j_0}(\mathcal{A})$, we have

$$(|\rho(\mathcal{A}) - a_{i_0\dots i_0}| - r_{i_0}^{j_0}(\mathcal{A}))(|\rho(\mathcal{A}) - a_{j_0\dots j_0}| - P_{j_0}^{i_0}(\mathcal{A})) \le |a_{i_0 j_0\dots j_0}|(r_{j_0}(\mathcal{A}) - P_{j_0}^{i_0}(\mathcal{A})).$$

Similar to the proof of Theorem 4.5, one has

$$(\rho(\mathcal{A}) - a_{i_0\dots i_0} - r_{i_0}^{j_0}(\mathcal{A}))(\rho(\mathcal{A}) - a_{j_0\dots j_0} - P_{j_0}^{i_0}(\mathcal{A})) \le a_{i_0 j_0\dots j_0}(r_{j_0}(\mathcal{A}) - P_{j_0}^{i_0}(\mathcal{A})),$$

Then, solving for $\rho(\mathcal{A})$, we get upper bound of $\rho(\mathcal{A})$

$$\begin{aligned}\rho(\mathcal{A}) &\le \frac{1}{2}(a_{i_0\dots i_0} + a_{j_0\dots j_0} + r_{i_0}^{j_0}(\mathcal{A}) + P_{j_0}^{i_0}(\mathcal{A}) + \Lambda_{i_0,j_0}^{\frac{1}{2}}(\mathcal{A})) \\ &\le \max_{i,j\in N, i\neq j} \frac{1}{2}\{a_{i\dots i} + a_{j\dots j} + r_i^j(\mathcal{A}) + P_j^i(\mathcal{A}) + \Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A})\}.\end{aligned} \tag{4.2}$$

Similar to the proof of upper bound of $\rho(\mathcal{A})$, we get lower bound of $\rho(\mathcal{A})$

$$\rho(\mathcal{A}) \ge \min_{i,j\in N, i\neq j} \frac{1}{2}\{a_{i\dots i} + a_{j\dots j} + r_i^j(\mathcal{A}) + P_j^i(\mathcal{A}) + \Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A})\}. \tag{4.3}$$

When $\rho(\mathcal{A}) \in \mathcal{H}_{i_0,j_0}(\mathcal{A})$, one has

$$\rho(\mathcal{A}) - a_{i_0\dots i_0} \le r_{i_0}^{j_0}(\mathcal{A}) \text{ and } \rho(\mathcal{A}) - a_{j_0\dots j_0} \le P_{j_0}^{i_0}(\mathcal{A}),$$

which shows $\rho(\mathcal{A}) \le \min\{r_{i_0}^{j_0}(\mathcal{A}) + a_{i_0\dots i_0}, P_{j_0}^{i_0}(\mathcal{A}) + a_{j_0\dots j_0}\}$. Furthermore,

$$\rho(\mathcal{A}) \le \max_{i,j\in N, i\neq j} \min\{r_i^j(\mathcal{A}) + a_{i\dots i}, P_j^i(\mathcal{A}) + a_{j\dots j}\} \tag{4.4}$$

From (4.2) and (4.3), we get

$$\rho(\mathcal{A}) \leq \max_{i,j\in N, i\neq j}\{\frac{1}{2}[a_{i\dots i}+a_{j\dots j}+r_i^j(\mathcal{A})+P_j^i(\mathcal{A})+\Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A})], \min\{r_i^j(\mathcal{A})+a_{i\dots i}, P_j^i(\mathcal{A})+a_{j\dots j}\}\}.$$

□

Theorem 4.8. *Let $\mathcal{A}$ be a nonnegative tensor with order m and dimension $n \geq 2$. Then,*

$$\begin{aligned}&\min_{i,j\in N, i\neq j} \tfrac{1}{2}\{[a_{i\dots i}+a_{j\dots j}+r_i^j(\mathcal{A})+P_j^i(\mathcal{A})+\Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A})] = \underline{v} \leq \rho(\mathcal{A}) \leq \\ &\bar{v} = \max_{i,j\in N, i\neq j}[\min \tfrac{1}{2}\{[a_{i\dots i}+a_{j\dots j}+r_i^j(\mathcal{A})+P_j^i(\mathcal{A})+\Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A})], 2R_i(\mathcal{A})\}, \\ &\min\{r_i^j(\mathcal{A})+a_{i\dots i}, P_j^i(\mathcal{A})+a_{j\dots j}\}],\end{aligned}$$

where $\Lambda_{i,j}(\mathcal{A}) = (a_{i\dots i}-a_{j\dots j}+r_i^j(\mathcal{A})-P_j^i(\mathcal{A}))^2+4[a_{ij\dots j}(r_j(\mathcal{A})-P_j^i(\mathcal{A}))]$).

Proof. Suppose $\rho(\mathcal{A})$ is the largest eigenvalue of $\mathcal{A}$. It follows from Theorem 3.4 that there exist $i_0, j_0 \in N$, $j_0 \neq i_0$ such that $\rho(\mathcal{A}) \in \mathcal{N}_{i_0,j_0}(\mathcal{A})\bigcap\Gamma_{i_0}(\mathcal{A})$, i.e., $\rho(\mathcal{A}) \leq R_{i_0}(\mathcal{A})$. This combined with (4.8) yields

$$\rho(\mathcal{A}) \leq \bar{v} = \max_{i,j\in N, i\neq j} \min \frac{1}{2}\{[a_{i\dots i}+a_{j\dots j}+r_i^j(\mathcal{A})+P_j^i(\mathcal{A})+\Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A})], 2R_i(\mathcal{A})\}.$$

Using (4.4), we have

$$\begin{aligned}\rho(\mathcal{A}) \leq \bar{v} = \max_{i,j\in N, i\neq j}[&\min \frac{1}{2}\{[a_{i\dots i}+a_{j\dots j}+r_i^j(\mathcal{A})+P_j^i(\mathcal{A})+\Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A})], 2R_i(\mathcal{A})\}, \\ &\min\{r_i^j(\mathcal{A})+a_{i\dots i}, P_j^i(\mathcal{A})+a_{j\dots j}\}].\end{aligned} \tag{4.5}$$

On the other hand, from $\rho(\mathcal{A}) \in \Gamma_i(\mathcal{A})$, we know $a_{i\dots i}-r_i(\mathcal{A}) \leq a_{i\dots i} \leq \rho(\mathcal{A})$, since Lemma 4.2 holds. Similar to the proof of Corollary 4.6, we have

$$\max_{i\in N} a_{i\dots i} \leq \min_{i,j\in N, i\neq j} \frac{1}{2}\{[a_{i\dots i}+a_{j\dots j}+r_i^j(\mathcal{A})+P_j^i(\mathcal{A})+\Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A})] = \underline{v} \leq \rho(\mathcal{A}).$$

So, the conclusion is satisfied. □

Corollary 4.9. *Let $\mathcal{A}$ be a nonnegative tensor with order m and dimension $n \geq 2$. Then,*

$$\max\{\max_{i\in N} a_{i\dots i}, \min_{i\in N} R_i(\mathcal{A})\} \leq \underline{v} \leq \rho(\mathcal{A}) \leq \bar{v} \leq w \leq \max_{i\in N} R_i(\mathcal{A}),$$

where $\underline{v}, \bar{v}, w$ are defined in Theorem 4.8 and Lemma 4.3, respectively.

Proof. Similar to the proof of Corollary 4.6 and Corollary 3.5, we obtain the conclusion holds. □

Now, we give an example to show that the bounds of Theorem 4.5 and Theorem 4.8 are tighter than results in Lemmas 4.1, 4.2 and 4.3.

Example 4.10. Consider 3 order 3 dimensional tensor $\mathcal{A} = (a_{ijk})$ defined by

$$a_{ijk} = \begin{cases} a_{111}=1; a_{222}=2; a_{333}=3; \\ a_{112}=a_{121}=a_{211}=1; a_{113}=a_{131}=a_{311}=1; a_{233}=a_{332}=a_{323}=1; \\ a_{ijk}=0, \qquad otherwise. \end{cases}$$

Since $\mathcal{A}$ is a supersymmetric nonnegative tensor, from Theorem 3.6 of [23], we get the largest eigenvalue $\lambda = 5.1587$ of $\mathcal{A}$ with nonnegative eigenvector $x = (0.6645, 0.5841, 0.7976)$. In the following, we shall estimate the largest eigenvalue of $\mathcal{A}$ according to different Theorems or Lemmas.

According to Theorem 4.5, we have

$$4 = \max\{\max_{i\in N} a_{i\dots i}, \min_i R_i(\mathcal{A})\} \le \underline{u} = 3+\sqrt{3} \le \rho(\mathcal{A}) \le 3+2\sqrt{2} = \bar{u};$$

By Theorem 4.8, we obtain

$$4 = \max\{\max_{i\in N} a_{i\dots i}, \min_i R_i(\mathcal{A})\} \le \underline{v} = 3+\sqrt{2} \le \rho(\mathcal{A}) \le \max\{4+\sqrt{3}, 5\} = 4+\sqrt{3} = \bar{v};$$

According to Lemma 4.1, we have

$$4 \le \rho(\mathcal{A}) \le 6;$$

From Lemma 4.2 and Lemma 4.3, we get

$$3 \le \rho(\mathcal{A}) \le 6;$$

Similar to Example 3.7, by Lemma 4.4,

$$\begin{cases} w_{S_i} = 6, & i = 2,3,4,5 \\ w_{S_i} = 3+2\sqrt{2}, & i = 1,6, \end{cases}$$

where S_i is the same as Example 3.7. This example shows that bounds of Theorem 4.5 and Theorem 4.8 are tighter.

$\boxed{4.2}$ Testing Positive Semidefiniteness and Positive Definiteness of a tensor

By applying the results obtained in Section 3, we give some sufficient conditions for the positive semidefiniteness (positive definiteness) of an even-order real supersymmetric tensor.

Theorem 4.11. *Let $\mathcal{A}$ be an even-order real supersymmetric tensor of order m dimension n with $a_{i\dots i} \ge 0, i \in N$. For $i \in N$, there exists $j \ne i$ such that*

$$a_{i\dots i}a_{j\dots j} - |a_{ij\dots j}a_{ji\dots i}| \ge |a_{ij\dots j}| r_j^i(\mathcal{A}) + a_{j\dots j} r_i^j(\mathcal{A}). \tag{4.6}$$

Then, $\mathcal{A}$ is positive semi-definite.

Proof. Let λ be an H-eigenvalue of $\mathcal{A}$. Suppose that $\lambda < 0$. From Theorem 3.1, we have $\lambda \in \mathcal{M}$, which implies that there exists $i_0 \in N$ such that $\lambda \in \mathcal{M}_{i_0,j}$ for all $j \in N, j \ne i_0$, that is,

$$|(\lambda - a_{i_0\dots i_0})(\lambda - a_{j\dots j}) - a_{i_0 j\dots j}a_{j i_0\dots i_0}| \le |(\lambda - a_{j\dots j})| r_{i_0}^j(\mathcal{A}) + |a_{i_0 j\dots j}| r_j^{i_0}(\mathcal{A}). \tag{4.7}$$

On the other hand, from $a_{i_0\dots i_0} \ge 0$ and (4.6), there exists $j_0 \ne i_0$ such that

$$(a_{i_0\dots i_0} - r_{i_0}^{j_0}(\mathcal{A}))a_{j_0\dots j_0} - |a_{i_0 j_0\dots j_0}a_{j_0 i_0\dots i_0}| \ge |a_{i_0 j_0\dots j_0}| r_{j_0}^{i_0}(\mathcal{A}), \tag{4.8}$$

which implies $a_{i_0\dots i_0} \ge r_{i_0}^{j_0}(\mathcal{A})$. Since $\lambda < 0$ and $a_{i_0\dots i_0} \ge 0$, from (4.8), we obtain

$$(|\lambda - a_{i_0\dots i_0}| - r_{i_0}^{j_0}(\mathcal{A}))(|\lambda - a_{j_0\dots j_0}|) - |a_{i_0 j_0\dots j_0}a_{j_0 i_0\dots i_0}| > |a_{i_0 j_0\dots j_0}| r_{j_0}^{i_0}(\mathcal{A}).$$

Equivalently, we have

$$|\lambda - a_{i_0\dots i_0}||\lambda - a_{j_0\dots j_0}| - |a_{i_0 j_0\dots j_0} a_{j_0 i_0\dots i_0}| - r_{i_0}^{j_0}(\mathcal{A})(|\lambda - a_{j_0\dots j_0}|) > |a_{i_0 j_0\dots j_0}| r_{j_0}^{i_0}(\mathcal{A}).$$

Thus,

$$\begin{aligned}|(\lambda - a_{i_0\dots i_0})(\lambda - a_{j_0\dots j_0}) - a_{i_0 j_0\dots j_0} a_{j_0 i_0\dots i_0}| &\geq |(\lambda - a_{i_0\dots i_0})(\lambda - a_{j_0\dots j_0})| \\ &\quad - |a_{i_0 j_0\dots j_0} a_{j_0 i_0\dots i_0}| \\ &> r_{i_0}^{j_0}(\mathcal{A})(|\lambda - a_{j_0\dots j_0}|) + |a_{i_0 j_0\dots j_0}| r_{j_0}^{i_0}(\mathcal{A}),\end{aligned}$$

which contradicts (4.7). Hence, $\lambda \geq 0$. This shows that $\mathcal{A}$ is positive semi-definite. □

Corollary 4.12. *Let $\mathcal{A}$ be an even-order real supersymmetric tensor of order m dimension n with $a_{k\dots k} > 0, k \in N$. For $i \in N$, there exists $j \neq i$ such that*

$$a_{i\dots i} a_{j\dots j} - |a_{ij\dots j} a_{ji\dots i}| > |a_{ij\dots j}| r_j^i(\mathcal{A}) + a_{j\dots j} r_i^j(\mathcal{A}).$$

Then, $\mathcal{A}$ is positive definite.

Proof. Similar to the proof of Theorem 4.11, we obtain the results. □

Now we use the following example to show how to test positive semi-definiteness and positive definiteness of a tensor by Theorem 4.11 and Corollary 4.12.

Example 4.13. Consider 4 order 3 dimensional tensor $\mathcal{A} = (a_{ijkl}), \mathcal{B} = (b_{ijkl})$ defined by

$$a_{ijkl} = \begin{cases} a_{1111} = 1; a_{2222} = 2; a_{3333} = 1; \\ a_{1122} = a_{1221} = a_{2211} = a_{2112} = -\frac{1}{2}; a_{2233} = a_{2332} = a_{3322} = a_{3223} = -\frac{1}{2}; \\ a_{ijkl} = 0, \qquad \textit{otherwise}, \end{cases}$$

$$b_{ijkl} = \begin{cases} b_{1111} = 1; b_{2222} = 2; b_{3333} = 2; \\ b_{2233} = b_{2332} = b_{3322} = b_{3223} = -\frac{1}{2}; \\ b_{ijkl} = 0, \qquad \textit{otherwise}. \end{cases}$$

It can be verified that $\mathcal{A}, \mathcal{B}$ satisfy all the conditions of Theorem 4.11 and Corollary 4.12, respectively. So, $\mathcal{A}$ is positive semi-definite and $\mathcal{B}$ is positive definite. Indeed, by simple computation, we may compute the smallest eigenvalue $\lambda_{\mathcal{A}} = 0, \lambda_{\mathcal{B}} = 1$.

Based on Theorem 3.4, the conclusions follow immediately.

Theorem 4.14. *Let $\mathcal{A}$ be an even-order real supersymmetric tensor of order m dimension n with $a_{k\dots k} \geq 0, k \in N$. For $i, j \in N, i \neq j$, the following conditions are satisfied*

$$(a_{i\dots i} - r_i^j(\mathcal{A}))(a_{j\dots j} - P_j^i(\mathcal{A})) \geq |a_{ij\dots j}|(r_j(\mathcal{A}) - P_j^i(\mathcal{A})),$$

$$a_{i\dots i} \geq r_i^j(\mathcal{A}) \text{ and } a_{j\dots j} \geq P_j^i(\mathcal{A}).$$

Then, $\mathcal{A}$ is positive semi-definite.

Corollary 4.15. *Let $\mathcal{A}$ be an even-order real supersymmetric tensor of order m dimension n with $a_{k\dots k} > 0, k \in N$. For $i, j \in N, i \neq j$, the following conditions are satisfied*

$$(a_{i\dots i} - r_i^j(\mathcal{A}))(a_{j\dots j} - P_j^i(\mathcal{A})) > |a_{ij\dots j}|(r_j(\mathcal{A}) - P_j^i(\mathcal{A})),$$

$$a_{i\dots i} > r_i^j(\mathcal{A}) \text{ and } a_{j\dots j} > P_j^i(\mathcal{A}).$$

Then, $\mathcal{A}$ is positive definite.

5 Conclusion

In this paper, we have established several Brauer-type eigenvalue inclusion theorems for general tensors, which achieve sharper conclusions than existing results [8, 13]. In some sense, we have answered the question raised in [8]. Furthermore, we obtained some bounds for the largest eigenvalue of a nonnegative tensor which are sharper than that of [8, 13, 20]. In addition, we have given several sufficient conditions to test positive (positive semidefiniteness) definiteness of an even-order real supersymmetric tensor.

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