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HIGHER-ORDER KUHN-TUCKER OPTIMALITY CONDITIONS FOR SET-VALUED OPTIMIZATION*

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Abstract: A new kind of higher-order tangent set is introduced, with which a new kind of higher-order tangent derivative, higher-order M-tangent derivative, is introduced for set-valued maps. Some properties of them are discussed. When both the objective function and constraint function are higher-order $M-$derivable, under the assumption of near cone-subconvexlikeness, by applying a separation theorem for convex sets, higher-order Fritz John and Kuhn-Tucker necessary optimality conditions are obtained for a point pair to be a weak minimizer of set-valued optimization problem. Under the assumption of lower semicontinuity, a higher-order Kuhn-Tucker sufficient optimality condition is obtained for a point pair to be a weak minimizer of set-valued optimization problem.

Key words: *set-valued optimization, Higher-order tangent derivative, Kuhn-Tucker condition, weak minimizer, near cone-subconvexlikeness*

Mathematics Subject Classification: *90C29, 90C46, 49J53, 46G05*

1 Introduction

In the last several decades, nonsmooth set-valued vector optimization has attracted increasing attentions. To consider optimality conditions, many generalized derivatives have been introduced with fruitful applications [4, 5, 6, 7, 9]. Corley [7] established optimality conditions for maximization of set-valued maps by derivatives for the set-valued maps in real normed linear spaces. Bigi and Castellani [6] addressed a generalized concept of $K-$epiderivative and employed it to develop a quite general scheme for necessary optimality conditions in set-valued problems. For many of the mentioned notions, second-order optimality conditions in set-valued optimization have also been proposed [4, 8, 11, 12, 16, 18, 26]. For example, Jahn, Khan and Zeilinger [11] proposed second-order epiderivatives for set-valued maps, by using these concepts, they gave second-order necessary optimality conditions and a sufficient optimality condition for set optimization. Zhu, Li and Teo [26] proposed the concept of second-order composed contingent derivative for set-valued maps, by virtue of the second-order composed contingent derivative, they established some second-order Karush-Kuhn-Tucker necessary and sufficient optimality conditions for a set-valued optimization problem. Rather few results on higher-order optimality conditions have been developed [1, 2, 3, 13, 14, 15, 20, 21]. Li, Teo and Yang [15] established higher-order Fritz John type

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necessary and sufficient optimality conditions for a set-valued optimization problem. Khanh and Tuan [13] proposed the concept of higher-order variational sets for set-valued maps, by virtue of these sets, they established higher-order necessary and sufficient conditions. In classic Kuhn-Tucker optimality conditions (see Theorems 4.2.10, 4.2.11, 4.3.6, 4.3.7 in [5]), the gradient of the objective function is a linear combination of those of constraint functions. However, in Refs. [1, 2, 3, 6, 7, 9, 13, 14, 15, 20, 21, 26] in the expressions of optimality conditions, the derivatives of objective and constraint functions are not separated, thus the properties for the derivative of objective function are not easily obtained from those of constraint function.

On the other hand, various generalizations of convex functions have appeared in the literature [17, 19, 22, 24, 25]. Sometimes, the driving force has been the fact that convexity plays a key role in optimization theory. Yang, Li and Wang [24] introduced a new class of generalized convexity, termed near cone-subconvexlikeness, for set-valued functions, which is a generalization of cone-subconvexlikeness introduced by Li [17]. Xu [22] has demonstrated that near cone-subconvexlikeness is also a generalization of generalized cone-subconvexlikeness introduced by Yang, Yang and Chen [25]. Sach [19] introduced a new convexity notion for set-valued maps, called ic-cone-convexlikeness. Xu and Song [23] obtained the following results: (i) when the ordering cone has nonempty interior, ic-cone-convexness is equivalent to near cone-subconvexlikeness; (ii) when the ordering cone has empty interior, ic-cone-convexness implies near cone-subconvexlikeness, a counter example was given to show that the converse implication is not true. Hence, near cone-subconvexlikeness is a kind of very generalized convexity.

The above discussions motivate the aim of this paper: to introduce a new kind of higher-order tangent set and with which to introduce a new kind of higher-order tangent derivative for a set-valued function. Also we will use it to investigate weak minimizers for set-valued optimization problem under the assumption of near cone-subconvexlikeness.

2 Basic Concepts

Throughout the paper, let X, Y and Z be three real normed linear spaces, $0_X, 0_Y$ and 0_Z denote the original points of X, Y and Z, respectively. Let C and D be closed convex pointed cones in Y and Z, respectively, Y^* and Z^* be the topological dual spaces of Y and Z, respectively. The dual cones of C and D be defined respectively by

$$C^* = \{f \in Y^* : f(c) \geq 0, \forall c \in C\},$$

$$D^* = \{f \in Z^* : f(d) \geq 0, \forall d \in D\}.$$

Let M be a nonempty subset of Y, the interior, closure and cone hull of M are denoted by $\mathrm{int}M$, $\mathrm{cl}M$ and $\mathrm{cone}M$, respectively. The cone hull of M is defined by

$$\mathrm{cone}M = \{tm : t \geq 0, m \in M\}.$$

Let $F : X \to 2^Y$ be a set-valued map. The domain image, graph and epigraph of F are defined respectively by

$$\mathrm{dom}F := \{x \in X : F(x) \neq \emptyset\},$$

$$\mathrm{Im}F := \{y \in Y : y \in F(X)\},$$

$$\mathrm{graph}F := \{(x, y) \in X \times Y : y \in F(x)\},$$

$$\mathrm{epi}F := \{(x, y) \in X \times Y : y \in F(x) + C\}.$$

Definition 2.1 ([10])**.** Let $F : X \to 2^Y$ be a set-valued map and S be a nonempty convex subset of X. Then F is said to be $C-$convex on S if only if for all $x_1, x_2 \in S$ and $\lambda \in [0,1]$, we have

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + C.$$

Remark 2.2. If F is $C-$convex on a convex set S, $y \in Y$, then $F - y$ is also $C-$convex on S.

Definition 2.3 ([24])**.** Let $E \subset X$. A set-valued map $F : X \to 2^Y$ is said to be nearly $C-$subconvexlike on E if $\mathrm{clcone}(F(E) + C)$ is convex.

Remark 2.4 ([24])**.** If F is $C-$convex on a convex set S, then F is nearly $C-$subconvexlike on S.

Definition 2.5 ([4])**.** Let $S \subset X$. A set-valued map $F : S \to 2^Y$ is called lower semicontinuous at $\hat{x} \in S$ if for any $x_n \in S$ with $x_n \to \hat{x}$ and any $\hat{y} \in F(\hat{x})$, there exists a sequence $y_n \in F(x_n)$ such that $y_n \to \hat{y}$.

F is called lower semicontinuous on S if F is lower semicontinuous at any $x \in S$.

Definition 2.6 ([4])**.** Let A be a nonempty subset of Y, $\hat{y} \in \mathrm{cl}A$, $v_1, v_2, \ldots, v_{m-1} \in Y$. The $m^{th}-$order contingent set of A at $(\hat{y}, v_1, \ldots, v_{m-1})$, denoted by $T^{(m)}(A, \hat{y}, v_1, \ldots, v_{m-1})$, is given by

$$T^{(m)}(A, \hat{y}, v_1, \ldots, v_{m-1}) := \limsup_{h \to 0^+} \frac{A - \hat{y} - hv_1 - \cdots - h^{m-1}v_{m-1}}{h^m}. \tag{2.1}$$

Remark 2.7 ([4])**.** (2.1) is equivalent to

$$\begin{aligned} T^{(m)}(A, \hat{y}, v_1, \ldots, v_{m-1}) := \quad & \{y \in Y : \exists t_n \to 0^+, y_n \to y, \text{such that} \\ & \hat{y} + t_n v_1 + t_n^2 v_2 + \cdots + t_n^{m-1} v_{m-1} + t_n^m y_n \in A, \forall n \in N\}. \end{aligned}$$

Definition 2.8 ([4])**.** Let $F : X \to 2^Y$ be a set-valued map, $(\hat{x}, \hat{y}) \in \mathrm{graph}F$. The $m^{th}-$order contingent derivative of F at $(\hat{x}, \hat{y})$ for vectors $(u_1, v_1), \ldots, (u_{m-1}, v_{m-1})$ is defined by

$$\mathrm{graph}D^{(m)}F(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1}) := T^{(m)}\left(\mathrm{graph}F, \hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1}\right).$$

In the following, we introduce a new kind of higher-order tangent set.

Definition 2.9. Let Q be a nonempty subset of $X \times Y$, $(\hat{x}, \hat{y}) \in \mathrm{cl}Q$. The higher-order $M-$tangent set of Q at $(\hat{x}, \hat{y})$ for vectors $(u_1, v_1), \ldots, (u_{m-1}, v_{m-1})$, denoted by $M^{(m)}(Q, \hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})$, is given by

$$\begin{aligned} M^{(m)}(Q, \hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1}) := \{&(u, v) \in X \times Y : \forall t_n \to 0^+, \forall u_n' \to u, \exists v_n' \to v, \\ & \text{such that } (\hat{x} + t_n u_1 + \cdots + t_n^{m-1} u_{m-1} + t_n^m u_n', \\ & \hat{y} + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m v_n') \in Q\}. \end{aligned}$$

$\boxed{3}$ The Second-Order M-Composed Tangent Derivative

In this setion, by virtue of the higher-order $M-$tangent set, we will introduce the concept of higher-order $M-$tangent derivative. Furthermore, some important properties of higher-order $M-$tangent derivative will be proposed.

Definition 3.1. Let $F : X \to 2^Y$ be a set-valued map, $(\hat{x}, \hat{y}) \in \mathrm{graph}F$. The $m^{th}-$order $M-$tangent derivative of F at $(\hat{x}, \hat{y})$ for vectors $(u_1, v_1), \ldots, (u_{m-1}, v_{m-1})$ is defined by

$$\mathrm{graph}M^{(m)}F(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1}) := M^{(m)}\left(\mathrm{epi}F, \hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1}\right).$$

If $M^{(m)}\left(\mathrm{epi}F, \hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1}\right) \neq \emptyset$, then F is called higher-order $M-$ derivable at $(\hat{x}, \hat{y})$ for vectors $(u_1, v_1), \ldots, (u_{m-1}, v_{m-1})$, or the higher-order $M-$tangent derivative of F at $(\hat{x}, \hat{y})$ for vectors $(u_1, v_1), \ldots, (u_{m-1}, v_{m-1})$ exists.

Remark 3.2. From Remark 2.7 and Definition 2.9, it follows that

$$\mathrm{graph}M^{(m)}F(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1}) \subset \mathrm{graph}D^{(m)}F_+(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1}),$$

where $F_+ = F(x) + C,\ \forall x \in \mathrm{dom}F$. However, the inverse inclusion is not necessarily true, as is shown in the following example.

Example 3.3. Let R be the set of real numbers, $X = Y = R$, $C = \{t \in R : t \geq 0\}$. A set-valued map $F : X \to 2^Y$ is defined by

$$F(x) = \begin{cases} \{y \in R : y \geq x\}, & \text{if } x \geq 0, \\ \{y \in R : y \geq -\sqrt{-x}\}, & \text{otherwise.} \end{cases}$$

Let $(\hat{x}, \hat{y}) = (0, 0)$, a direct calculation gives

$$T^{(1)}(\mathrm{epi}F, (0,0)) = \{(x, y) : x > 0, y \geq x\} \cup \{(x, y) : x \leq 0, y \in R\},$$

$$M^{(1)}(\mathrm{epi}F, (0,0)) = \{(x, y) : x \geq 0, y \geq x\} \cup \{(x, y) : x < 0, y \in R\},$$

$$D^{(1)}F_+(0,0)(x) = \begin{cases} \{y : y \geq x\}, & x > 0, \\ R, & x \leq 0, \end{cases}$$

$$M^{(1)}F(0,0)(x) = \begin{cases} \{y : y \geq x\}, & x \geq 0, \\ R, & x < 0. \end{cases}$$

Consequently,

$$\mathrm{graph}D^{(1)}F_+(0,0) \not\subset \mathrm{graph}M^{(1)}F(0,0).$$

The following property of higher-order $M-$tangent derivative is of importance to establish higher-order necessary optimality condition in the next Theorem 4.1.

Proposition 3.4. *Suppose that $E \subset X$, the higher-order M-tangent derivative of $F : X \to 2^Y$ at $(\hat{x}, \hat{y}) \in \mathrm{graph}F$ for vectors $(u_1, v_1), \ldots, (u_{m-1}, v_{m-1})$ exists, then*

$$\begin{aligned} &M^{(m)}F(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})\left(T^{(m)}(E, \hat{x}, u_1, \ldots, u_{m-1})\right) \\ &\subset \mathrm{clcone}\left(\mathrm{cone}(\cdots \mathrm{cone}(\mathrm{cone}(F(E) + C - \hat{y}) - v_1) - \cdots - v_{m-2}) - v_{m-1}\right), \end{aligned}$$

where the number of "cone"in above expression is m.

Proof. Let $v \in M^{(m)}F(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})\left(T^{(m)}(E, \hat{x}, u_1, \ldots, u_{m-1})\right)$, then there exists a $u \in T^{(m)}(E, \hat{x}, u_1, \ldots, u_{m-1})$ such that

$$v \in M^{(m)}F(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(u).$$

Thus

$$(u, v) \quad \in \mathrm{graph}M^{(m)}F(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})$$

$$= M^{(m)}\left(\mathrm{epi}F, \hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1}\right). \tag{3.1}$$

From $u \in T^{(m)}(E, \hat{x}, u_1, \ldots, u_{m-1})$, it follows that there exist sequences $t_n \to 0^+$, $u_n' \to u$ such that

$$\hat{x} + t_n u_1 + \cdots + t_n^{m-1} u_{m-1} + t_n^m u_n' \in E.$$

For above t_n and u_n', it follows from (3.1) that there exists a sequence $v_n' \to v$ such that

$$(\hat{x} + t_n u_1 + \cdots + t_n^{m-1} u_{m-1} + t_n^m u_n', \hat{y} + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m v_n') \in \mathrm{epi}F,$$

which implies

$$\hat{y} + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m v_n' \in F(\hat{x} + t_n u_1 + \cdots + t_n^{m-1} u_{m-1} + t_n^m u_n') + C.$$

Consequently,

$$\begin{aligned}
v_n' &\in \frac{1}{t_n^m}\left(F(\hat{x} + t_n u_1 + \cdots + t_n^{m-1} u_{m-1} + t_n^m u_n') + C - \hat{y} - t_n v_1 - \cdots - t_n^{m-1} v_{m-1}\right) \\
&\subset \frac{1}{t_n^m}\left(F(E) + C - \hat{y} - t_n v_1 - \cdots - t_n^{m-1} v_{m-1}\right) \\
&= \frac{1}{t_n}\left(\frac{1}{t_n}\left(\cdots \frac{1}{t_n}\left(\frac{1}{t_n}\left(\frac{1}{t_n}\left(F(E) + C - \hat{y}\right) - v_1\right) - v_2\right) - \cdots - v_{m-2}\right) - v_{m-1}\right) \\
&\subset \mathrm{cone}\left(\mathrm{cone}(\cdots \mathrm{cone}(\mathrm{cone}(F(E) + C - \hat{y}) - v_1) - \cdots - v_{m-2}) - v_{m-1}\right).
\end{aligned}$$

Taking $n \to \infty$, one gets

$$v \in \mathrm{clcone}\left(\mathrm{cone}(\cdots \mathrm{cone}(\mathrm{cone}(F(E) + C - \hat{y}) - v_1) - \cdots - v_{m-2}) - v_{m-1}\right).$$

$\square$

Remark 3.5. If we substitute $D^{(m)}F(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})$ for $M^{(m)}F(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})$ in Proposition 3.4, the following inclusion

$$\begin{aligned}
&D^{(m)}F_+(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})\left(T^{(m)}(E, \hat{x}, u_1, \ldots, u_{m-1})\right) \\
&\subset \mathrm{clcone}\left(\mathrm{cone}(\cdots \mathrm{cone}(\mathrm{cone}(F(E) + C - \hat{y}) - v_1) - \cdots - v_{m-2}) - v_{m-1}\right).
\end{aligned}$$

is not necessarily true, as is shown in the following example.

Example 3.6. Let R be the set of real numbers, $X = R$, $Y = R^2$, $E = \{x : x \geq 0\}$, $C = \{(t_1, t_2) \in R^2 : t_1 \geq 0, t_2 \geq 0\}$. A set-valued map $F : X \to 2^Y$ is defined by

$$F(x) = \begin{cases} \{(y_1, y_2) \in R^2 : y_1 \geq 0, y_2 \geq 0\}, & \text{if } x \geq 0, \\ \{(y_1, y_2) \in R^2 : y_1 \geq -\sqrt{-x}, y_2 \geq \sqrt[3]{x}\}, & \text{otherwise.} \end{cases}$$

Let $(\hat{x}, \hat{y}) = (0, 0)$, a direct calculation gives

$$T^{(1)}(E, 0) = [0, +\infty),$$

$$\begin{aligned}
T^{(1)}(\mathrm{graph}F, (0,0)) = T^{(1)}(\mathrm{epi}F, (0,0)) &= \{(x, y) \in R \times R^2 : x > 0, y_1 \geq 0, y_2 \geq 0\} \\
&\quad \cup \{(x, y) \in R \times R^2 : x \leq 0, y_1 \in R, y_2 \in R\},
\end{aligned}$$

$$D^{(1)}F_+(0,0)(x) = \begin{cases} R \times R, & x \leq 0, \\ \{(y_1, y_2) \in R^2 : y_1 \geq 0, y_2 \geq 0\}, & x > 0, \end{cases}$$

$$\begin{aligned} M^{(1)}(\text{epi}F,(0,0)) &= \{(x,y) \in R \times R^2 : x \geq 0, y_1 \geq 0, y_2 \geq 0\} \\ &\cup \{(x,y) \in R \times R^2 : x < 0, y_1 \in R, y_2 \in R\}, \end{aligned}$$

$$M^{(1)}F(0,0)(x) = \begin{cases} R \times R, & x < 0, \\ \{(y_1,y_2) \in R^2 : y_1 \geq 0, y_2 \geq 0\}, & x \geq 0. \end{cases}$$

Consequently,

$$D^{(1)}F_+(0,0)(T^{(1)}(E,0)) = R \times R,$$

$$M^{(1)}F(0,0)(T^{(1)}(E,0)) = \{(y_1,y_2) \in R^2 : y_1 \geq 0, y_2 \geq 0\},$$

$$\text{clcone}(F(E) + C - \hat{y}) = \{(y_1,y_2) \in R^2 : y_1 \geq 0, y_2 \geq 0\}.$$

Thus, the conclusion of Proposition 3.4 is true. However,

$$D^{(1)}F_+(\hat{x},\hat{y})\left(T^{(1)}(E,\hat{x})\right) \not\subset \text{clcone}(F(E) + C - \hat{y}).$$

Proposition 3.7. *Suppose that $Q \subset X \times Y$ is convex, $(\hat{x},\hat{y}) \in \text{cl}Q$, then $M^{(m)}(Q,\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$ is convex.*

Proof. Let $(u',v'),(u'',v'') \in M^{(m)}(Q,\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$. For any $\lambda \in (0,1)$, we shall prove

$$\lambda(u',v') + (1-\lambda)(u'',v'') \in M^{(m)}(Q,\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1}),$$

that is,

$$(\lambda u' + (1-\lambda)u'', \lambda v' + (1-\lambda)v'') \in M^{(m)}(Q,\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1}).$$

In fact, for any $t_n \to 0^+$, $p_n \to \lambda u' + (1-\lambda)u''$, let $q_n := p_n - \lambda u' - (1-\lambda)u''$, then $q_n \to 0$. Thus

$$\frac{1}{\lambda}\Big(p_n - (1-\lambda)u''\Big) - q_n \to u'.$$

Since $(u',v') \in M^{(m)}(Q,\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$, there exists a sequence $v'_n \to v'$ such that

$$\begin{aligned} \Big(\hat{x} + t_n u_1 + \cdots + t_n^{m-1}u_{m-1} + t_n^m\left(\tfrac{p_n-(1-\lambda)u''}{\lambda} - q_n\right), \\ \hat{y} + t_n v_1 + \cdots + t_n^{m-1}v_{m-1} + t_n^m v'_n\Big) \in Q. \end{aligned} \tag{3.2}$$

In the similar way, $\frac{1}{1-\lambda}(p_n - \lambda u') - q_n \to u''$, thus there exists a sequence $v''_n \to v''$ such that

$$\begin{aligned} \Big(\hat{x} + t_n u_1 + \cdots + t_n^{m-1}u_{m-1} + t_n^m\left(\tfrac{p_n-\lambda u'}{1-\lambda} - q_n\right), \\ \hat{y} + t_n v_1 + \cdots + t_n^{m-1}v_{m-1} + t_n^m v''_n\Big) \in Q. \end{aligned} \tag{3.3}$$

Since Q is convex, multiplying (3.2) and (3.3) by λ and $1-\lambda$, respectively, one obtains

$$\begin{aligned} \Big(\hat{x} + t_n u_1 + \cdots + t_n^{m-1}u_{m-1} + t_n^m\left(p_n - (1-\lambda)u'' + p_n - \lambda u' - q_n\right), \\ \hat{y} + t_n v_1 + \cdots + t_n^{m-1}v_{m-1} + t_n^m\left(\lambda v'_n + (1-\lambda)v''_n\right)\Big) \in Q. \end{aligned}$$

From $q_n = p_n - \lambda u' - (1-\lambda)u''$, it follows that

$$\Big(\hat{x} + t_n u_1 + \cdots + t_n^{m-1} u_{m-1} + t_n^m p_n,$$
$$\hat{y} + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m \left(\lambda v_n' + (1-\lambda) v_n''\right)\Big) \in Q.$$

Let $\bar{v}_n := \lambda v_n' + (1-\lambda)v_n''$, then $\bar{v}_n \to \lambda v' + (1-\lambda)v''$. Consequently

$$\left(\hat{x} + t_n u_1 + \cdots + t_n^{m-1} u_{m-1} + t_n^m p_n, \hat{y} + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m \bar{v}_n\right) \in Q.$$

By the definition of $M^{(m)}(Q, \hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})$, we have

$$(\lambda u' + (1-\lambda)u'', \lambda v' + (1-\lambda)v'') \in M^{(m)}(Q, \hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1}).$$

□

Proposition 3.8. *Suppose that $S \subset X$ is convex, $F : S \to 2^Y$ is $C-$convex and lower semicontinuous on S, the higher-order $M-$tangent derivative of F at $(\hat{x}, \hat{y})$ for vectors $(u_1 - \hat{x}, v_1 - \hat{y}), \ldots, (u_{m-1} - \hat{x}, v_{m-1} - \hat{y})$ exists, then*

$$F(x) - \{\hat{y}\} \subset M^{(m)} F(\hat{x}, \hat{y}, u_1 - \hat{x}, v_1 - \hat{y}, \ldots, u_{m-1} - \hat{x}, v_{m-1} - \hat{y})(x - \hat{x}), \forall x \in S,$$

where $u_1, u_2, \ldots, u_{m-1} \in S$ and $(u_i, v_i) \in \mathrm{epi}F, i = 1, 2, \ldots, m-1$.

Proof. Let $x \in S$, $y \in F(x)$, we shall prove

$$y - \hat{y} \in M^{(m)} F(\hat{x}, \hat{y}, u_1 - \hat{x}, v_1 - \hat{y}, \ldots, u_{m-1} - \hat{x}, v_{m-1} - \hat{y})(x - \hat{x}).$$

It suffices to show

$$(x - \hat{x}, y - \hat{y}) \in \mathrm{graph} M^{(m)} F(\hat{x}, \hat{y}, u_1 - \hat{x}, v_1 - \hat{y}, \ldots, u_{m-1} - \hat{x}, v_{m-1} - \hat{y}).$$

By Definition 3.1, it suffices to demonstrate

$$(x - \hat{x}, y - \hat{y}) \in M^{(m)} \left(\mathrm{epi}F, \hat{x}, \hat{y}, u_1 - \hat{x}, v_1 - \hat{y}, \ldots, u_{m-1} - \hat{x}, v_{m-1} - \hat{y}\right).$$

In fact, let $t_n \to 0^+$, $x_n \to x - \hat{x}$, then $x_n + \hat{x} \to x$. From F is lower semicontinuous at x, it follows that there exists a sequence $y_n \in F(x_n + \hat{x})$ such that $y_n \to y$. Then $y_n - \hat{y} \to y - \hat{y}$. Since S is convex, one obtains

$$\hat{x} + 2t_n^m x_n = 2t_n^m (x_n + \hat{x}) + (1 - 2t_n^m)\hat{x} \in S$$

and

$$\begin{aligned} &\hat{x} + 2\left(t_n(u_1 - \hat{x}) + \cdots + t_n^{m-1}(u_{m-1} - \hat{x})\right) \\ &= 2t_n u_1 + \cdots + 2t_n^{m-1} u_{m-1} + (1 - 2t_n - \cdots - 2t_n^{m-1})\hat{x} \\ &\in S. \end{aligned}$$

Then

$$\begin{aligned} &\hat{x} + t_n(u_1 - \hat{x}) + \cdots + t_n^{m-1}(u_{m-1} - \hat{x}) + t_n^m x_n \\ &= \tfrac{1}{2}\left(\hat{x} + 2t_n^m x_n\right) + \tfrac{1}{2}\left(\hat{x} + 2\left(t_n(u_1 - \hat{x}) + \cdots + t_n^{m-1}(u_{m-1} - \hat{x})\right)\right) \\ &\in S. \end{aligned}$$

It follows from F is $C-$convex on S that

$$\hat{y} + 2t_n^m(y_n - \hat{y}) = 2t_n^m y_n + (1 - 2t_n^m)\hat{y} \quad \in F(2t_n^m(x_n + \hat{x}) + (1 - 2t_n^m)\hat{x}) + C$$

$$= F(\hat{x} + 2t_n^m x_n) + C. \tag{3.4}$$

Since $(u_i, v_i) \in \text{epi}F, i = 1, 2, \ldots, m-1$ and F is $C-$convex on S, we have

$$\begin{aligned}
&\hat{y} + 2\left(t_n(v_1 - \hat{y}) + \cdots + t_n^{m-1}(v_{m-1} - \hat{y})\right) \\
&= 2t_n v_1 + \cdots + 2t_n^{m-1} v_{m-1} + (1 - 2t_n - \cdots - 2t_n^{m-1})\hat{y} \\
&\in F(2t_n u_1 + \cdots + 2t_n^{m-1} u_{m-1} + (1 - 2t_n - \cdots - 2t_n^{m-1})\hat{x}) + C \\
&= F\left(\hat{x} + 2\left(t_n(u_1 - \hat{x}) + \cdots + t_n^{m-1}(u_{m-1} - \hat{x})\right)\right) + C,
\end{aligned}$$

which together with (3.4) gives

$$\begin{aligned}
&\hat{y} + t_n(v_1 - \hat{y}) + \cdots + t_n^{m-1}(v_{m-1} - \hat{y}) + t_n^m(y_n - \hat{y}) \\
&= \frac{1}{2}\left(\hat{y} + 2t_n^m(y_n - \hat{y})\right) + \frac{1}{2}\left(\hat{y} + 2\left(t_n(v_1 - \hat{y}) + \cdots + t_n^{m-1}(v_{m-1} - \hat{y})\right)\right) \\
&\in \frac{1}{2}F(\hat{x} + 2t_n^m x_n) + \frac{1}{2}F\left(\hat{x} + 2\left(t_n(u_1 - \hat{x}) + \cdots + t_n^{m-1}(u_{m-1} - \hat{x})\right)\right) + C \\
&\subset F(\hat{x} + t_n(u_1 - \hat{x}) + \cdots + t_n^{m-1}(u_{m-1} - \hat{x}) + t_n^m x_n) + C.
\end{aligned}$$

Hence

$$(\hat{x}, \hat{y}) + t_n(u_1 - \hat{x}, v_1 - \hat{y}) + \cdots + t_n^{m-1}(u_{m-1} - \hat{x}, v_{m-1} - \hat{y}) + t_n^m(x_n, y_n - \hat{y}) \in \text{epi}F.$$

Consequently,

$$(x - \hat{x}, y - \hat{y}) \in \text{graph}M^{(m)}F(\hat{x}, \hat{y}, u_1 - \hat{x}, v_1 - \hat{y}, \ldots, u_{m-1} - \hat{x}, v_{m-1} - \hat{y}).$$

$\square$

Proposition 3.9. *Let $F_i : X \to 2^Y$, $\hat{x} \in \text{dom}F_1 \cap \text{dom}F_2$, $\hat{y}_i \in F_i(\hat{x}), i = 1, 2$. The higher-order M-tangent derivative of F_1 at $(\hat{x}, \hat{y}_1)$ for vectors $(u_1, v_{1,1}), \ldots, (u_{m-1}, v_{1,m-1})$ and the higher-order M-tangent derivative of F_2 at $(\hat{x}, \hat{y}_2)$ for vectors $(u_1, v_{2,1}), \ldots, (u_{m-1}, v_{2,m-1})$ exist, then for any $x \in \text{dom}F_1 \cap \text{dom}F_2$,*

$$\begin{aligned}
&M^{(m)}F_1(\hat{x}, \hat{y}_1, u_1, v_{1,1}, \ldots, u_{m-1}, v_{1,m-1})(x) \\
&+ M^{(m)}F_2(\hat{x}, \hat{y}_2, u_1, v_{2,1}, \ldots, u_{m-1}, v_{2,m-1})(x) \\
&\subset M^{(m)}(F_1 + F_2)(\hat{x}, \hat{y}_1 + \hat{y}_2, u_1, v_{1,1} + v_{2,1}, \ldots, u_{m-1}, v_{1,m-1} + v_{2,m-1})(x).
\end{aligned}$$

Proof. For any $t_n \to 0^+$, $x_n \to x$, let $y_1 \in M^{(m)}F_1(\hat{x}, \hat{y}_1, u_1, v_{1,1}, \ldots, u_{m-1}, v_{1,m-1})(x)$ and $y_2 \in M^{(m)}F_2(\hat{x}, \hat{y}_2, u_1, v_{2,1}, \ldots, u_{m-1}, v_{2,m-1})(x)$, then for above t_n and x_n, "is replaced by" Let $y_1 \in M^{(m)}F_1(\hat{x}, \hat{y}_1, u_1, v_{1,1}, \ldots, u_{m-1}, v_{1,m-1})(x)$, $y_2 \in M^{(m)}F_2(\hat{x}, \hat{y}_2, u_1, v_{2,1}, \ldots, u_{m-1}, v_{2,m-1})(x)$, $t_n \to 0^+$ and $x_n \to x$, then there exist $y_n^1 \to y_1$ and $y_n^2 \to y_2$ such that

$$\hat{y}_1 + t_n v_{1,1} + \cdots + t_n^{m-1} v_{1,m-1} + t_n^m y_n^1 \in F_1(\hat{x} + t_n u_1 + \cdots + t_n^{m-1} u_{m-1} + t_n^m x_n) + C$$

and

$$\hat{y}_2 + t_n v_{2,1} + \cdots + t_n^{m-1} v_{2,m-1} + t_n^m y_n^2 \in F_2(\hat{x} + t_n u_1 + \cdots + t_n^{m-1} u_{m-1} + t_n^m x_n) + C.$$

Thus

$$\begin{aligned}
&(\hat{y}_1 + \hat{y}_2) + t_n(v_{1,1} + v_{2,1}) + \cdots + t_n^{m-1}(v_{1,m-1} + v_{2,m-1}) + t_n^m(y_n^1 + y_n^2) \\
&\in (F_1 + F_2)(\hat{x} + t_n u_1 + \cdots + t_n^{m-1} u_{m-1} + t_n^m x_n) + C + C \\
&\subset (F_1 + F_2)(\hat{x} + t_n u_1 + \cdots + t_n^{m-1} u_{m-1} + t_n^m x_n) + C.
\end{aligned}$$

Hence,

$$y_1 + y_2 \in M^{(m)}(F_1 + F_2)(\hat{x}, \hat{y}_1 + \hat{y}_2, u_1, v_{1,1} + v_{2,1}, \dots, u_{m-1}, v_{1,m-1} + v_{2,m-1})(x).$$

□

Proposition 3.10. *Let $G : X \to 2^Y$, $F : Y \to 2^Z$ with* $\mathrm{Im}G \subset \mathrm{dom}F$, $(\hat{x}, \hat{y}) \in \mathrm{graph}G$, $(\hat{y}, \hat{z}) \in \mathrm{graph}F$ *and* $(u_i, v_i, w_i) \in X \times Y \times Z, i = 1, 2, \dots, m-1$. *The higher-order M-tangent derivative of F at $(\hat{y}, \hat{z})$ for vectors $(v_1, w_1), \dots, (v_{m-1}, w_{m-1})$ and the higher-order M-tangent derivative of G at $(\hat{x}, \hat{y})$ for vectors $(u_1, v_1), \dots, (u_{m-1}, v_{m-1})$ exist, then*

$$\begin{aligned} &M^{(m)}F(\hat{y}, \hat{z}, v_1, w_1, \dots, v_{m-1}, w_{m-1})\left(M_c^{(m)}G(\hat{x}, \hat{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x)\right) \\ &\subset M^{(m)}(F \circ G)(\hat{x}, \hat{z}, u_1, w_1, \dots, u_{m-1}, w_{m-1})(x), \forall x \in \mathrm{dom}G, \end{aligned}$$

where $F \circ G(x) = F(G(x))$ *and*

$$\mathrm{graph}M_c^{(m)}G(\hat{x}, \hat{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1}) = M^{(m)}(\mathrm{graph}G, \hat{x}, \hat{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1}).$$

Proof. Let

$$z \in M^{(m)}F(\hat{y}, \hat{z}, v_1, w_1, \dots, v_{m-1}, w_{m-1})\left(M_c^{(m)}G(\hat{x}, \hat{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x)\right),$$

then there exists $y \in M_c^{(m)}G(\hat{x}, \hat{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x)$ such that

$$z \in M^{(m)}F(\hat{y}, \hat{z}, v_1, w_1, \dots, v_{m-1}, w_{m-1})(y).$$

For any $t_n \to 0^+$, $x_n \to x$, it follows from $y \in M_c^{(m)}G(\hat{x}, \hat{y}, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x)$ that there exists $y_n \to y$ such that

$$\hat{y} + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m y_n \in G(\hat{x} + t_n u_1 + \cdots + t_n^{m-1} u_{m-1} + t_n^m x_n). \tag{3.5}$$

For above t_n and y_n, it follows from $z \in M^{(m)}F(\hat{y}, \hat{z}, v_1, w_1, \dots, v_{m-1}, w_{m-1})(y)$ that there exists $z_n \to z$ such that

$$\hat{z} + t_n w_1 + \cdots + t_n^{m-1} w_{m-1} + t_n^m z_n \in F(\hat{y} + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m y_n) + C,$$

which together with (3.5) gives

$$\hat{z} + t_n w_1 + \cdots + t_n^{m-1} w_{m-1} + t_n^m z_n \in F(G(\hat{x} + t_n u_1 + \cdots + t_n^{m-1} u_{m-1} + t_n^m x_n)) + C.$$

Then

$$\hat{z} + t_n w_1 + \cdots + t_n^{m-1} w_{m-1} + t_n^m z_n \in F \circ G(\hat{x} + t_n u_1 + \cdots + t_n^{m-1} u_{m-1} + t_n^m x_n) + C.$$

Therefore,

$$z \in M^{(m)}(F \circ G)(\hat{x}, \hat{z}, u_1, w_1, \dots, u_{m-1}, w_{m-1})(x).$$

□

$\boxed{4}$ Higher-Order Kuhn-Tucker Optimality Conditions

Let $F : X \to 2^Y$, $G : X \to 2^Z$, $(F, G) : X \to 2^{Y \times Z}$ be defined by $(F, G)(x) = F(x) \times G(x)$.

Consider the following optimization problem with set-valued maps:

$$\text{(VP)} \quad \min \ F(x), \qquad \text{s.t.} \ G(x) \cap (-D) \neq \emptyset, x \in X.$$

The feasible set of (VP) is denoted by $\hat{E}$, i.e., $\hat{E} = \{x \in X : G(x) \cap (-D) \neq \emptyset\}$.

Definition 4.1. Suppose $\hat{x} \in \hat{E}$, $\hat{y} \in F(\hat{x})$. A pair $(\hat{x}, \hat{y})$ is called a weak minimizer of (VP), if

$$(F(\hat{E}) - \hat{y}) \cap (-\mathrm{int}C) = \emptyset.$$

Theorem 4.2 (Fritz John necessary optimality condition)**.** *Suppose that $(\hat{x}, \hat{y})$ is a weak minimizer of (VP), $\hat{z} \in G(\hat{x}) \cap (-D)$, $(u_i, v_i, w_i) \in X \times (-C) \times (-D), i = 1, 2, \ldots, m-1$, F is higher-order M-derivable at $(\hat{x}, \hat{y})$ for vectors $(u_1, v_1), \ldots, (u_{m-1}, v_{m-1})$, G is higher-order M-derivable at $(\hat{x}, \hat{z})$ for vectors $(u_1, w_1), \ldots, (u_{m-1}, w_{m-1})$, and*

$$\mathrm{dom}M^{(m)}F(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1}) = \mathrm{dom}M^{(m)}G(\hat{x}, \hat{z}, u_1, w_1, \ldots, u_{m-1}, w_{m-1}) = X.$$

The set-valued map

$$\varphi^*(x) = \Big(M^{(m)}F(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x), \\ M^{(m)}G(\hat{x}, \hat{z}, u_1, w_1, \ldots, u_{m-1}, w_{m-1})(x)\Big) : X \to 2^{Y \times Z}$$

is nearly $C \times D-$subconvexlike on X, then there exist $s^* \in C^*$, $k^* \in D^*$, $(s^*, k^*) \neq (0_{Y^*}, 0_{Z^*})$ such that

$$\inf_{x \in X} \Big(s^*\left(M^{(m)}F(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)\right) + \\ k^*\left(M^{(m)}G(\hat{x}, \hat{z}, u_1, w_1, \ldots, u_{m-1}, w_{m-1})(x)\right)\Big) \geq 0, \tag{4.1}$$

where

$$s^*\Big(M^{(m)}F(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)\Big) = \bigcup_{y \in M^{(m)}F(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)} s^*(y)$$

and

$$k^*\Big(M^{(m)}G(\hat{x}, \hat{z}, u_1, w_1, \ldots, u_{m-1}, w_{m-1})(x)\Big) = \bigcup_{z \in M^{(m)}G(\hat{x}, \hat{z}, u_1, w_1, \ldots, u_{m-1}, w_{m-1})(x)} k^*(z).$$

Proof. Since $(\hat{x}, \hat{y})$ is a weak minimizer of (VP),

$$(F(\hat{E}) - \hat{y}) \cap (-\mathrm{int}C) = \emptyset.$$

In what follows, we prove

$$\mathrm{cone}\,(\varphi^*(X) + C \times D) \cap ((-\mathrm{int}C) \times (-\mathrm{int}D)) = \emptyset, \tag{4.2}$$

where

$$\varphi^*(X) = \bigcup_{x\in X} \varphi^*(x) = \bigcup_{x\in X} \Big(M^{(m)}F(\hat{x},\hat{y},u_1,v_1,\dots,u_{m-1},v_{m-1})(x), M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\dots,u_{m-1},w_{m-1})(x) \Big).$$

On the contrary suppose that (4.2) does not hold, from $(0_Y, 0_Z) \notin (-\text{int}C) \times (-\text{int}D)$, it follows that there exist $\bar{\lambda} > 0$, $\bar{x} \in X$, such that

$$\Big(\bar{\lambda}\Big(M^{(m)}F(\hat{x},\hat{y},u_1,v_1,\dots,u_{m-1},v_{m-1})(\bar{x}), M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\dots,u_{m-1},w_{m-1})(\bar{x})\Big)\Big) \cap ((-\text{int}C) \times (-\text{int}D)) \neq \emptyset.$$

Hence there exist $\bar{y} \in M^{(m)}F(\hat{x},\hat{y},u_1,v_1,\dots,u_{m-1},v_{m-1})(\bar{x})$, $\bar{z} \in M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\dots,u_{m-1},w_{m-1})(\bar{x})$, such that

$$\bar{\lambda}(\bar{y},\bar{z}) \in (-\text{int}C) \times (-\text{int}D).$$

Since $-\text{int}C$ and $-\text{int}D$ are cones, one obtains

$$\bar{y} \in -\text{int}C \tag{4.3}$$

and

$$\bar{z} \in -\text{int}D. \tag{4.4}$$

From $\bar{z} \in M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\dots,u_{m-1},w_{m-1})(\bar{x})$, it follows that

$$\begin{aligned}(\bar{x},\bar{z}) &\in \text{graph}M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\dots,u_{m-1},w_{m-1}) \\ &= M^{(m)}\left(\text{epi}G,\hat{x},\hat{z},u_1,w_1,\dots,u_{m-1},w_{m-1}\right) \\ &\subset T^{(m)}\left(\text{epi}G,\hat{x},\hat{z},u_1,w_1,\dots,u_{m-1},w_{m-1}\right).\end{aligned}$$

Hence there exist $t_n \to 0^+$, $x_n \to \bar{x}$, $z_n \to \bar{z}$ such that

$$(\hat{x} + t_n u_1 + \cdots + t_n^{m-1}u_{m-1} + t_n^m x_n, \hat{z} + t_n w_1 + \cdots + t_n^{m-1} w_{m-1} + t_n^m z_n) \in \text{epi}G.$$

Thus

$$\hat{z} + t_n w_1 + \cdots + t_n^{m-1} w_{m-1} + t_n^m z_n \in G(\hat{x} + t_n u_1 + \cdots + t_n^{m-1}u_{m-1} + t_n^m x_n) + D. \tag{4.5}$$

A set of positive integers is denoted by N. From (4.4) and $z_n \to \bar{z}$, it follows that there exists $N_1 \in N$ such that

$$z_n \in -\text{int}D, \forall n > N_1.$$

Since $-\text{int}D$ and $-D$ are convex cones, one obtains

$$\hat{z} + t_n w_1 + \cdots + t_n^{m-1} w_{m-1} + t_n^m z_n \in -D - \cdots - D - \text{int}D = -\text{int}D, \forall n > N_1. \tag{4.6}$$

It follows from (4.5) that there exists $\tilde{z}_n \in G(\hat{x} + t_n u_1 + \cdots + t_n^{m-1}u_{m-1} + t_n^m x_n)$ such that

$$\hat{z} + t_n w_1 + \cdots + t_n^{m-1} w_{m-1} + t_n^m z_n \in \tilde{z}_n + D.$$

Since D is a convex cone, from (4.6) one dedues

$$\tilde{z}_n \in \hat{z} + t_n w_1 + \cdots + t_n^{m-1} w_{m-1} + t_n^m z_n - D \subset -\text{int}D - D = -\text{int}D \subset -D,$$

which gives

$$G(\hat{x} + t_n u_1 + \cdots + t_n^{m-1} u_{m-1} + t_n^m x_n) \cap (-D) \neq \emptyset,$$

that is

$$\hat{x} + t_n u_1 + \cdots + t_n^{m-1} u_{m-1} + t_n^m x_n \in \hat{E}.$$

From $t_n \to 0^+$, $x_n \to \bar{x}$, it follows that $\bar{x} \in T^{(m)}(\hat{E}, \hat{x}, u_1, \ldots, u_{m-1})$, which along with Proposition 3.4 and $\bar{y} \in M^{(m)} F(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(\bar{x})$ leads to

$$\begin{aligned} \bar{y} \quad &\in M^{(m)} F(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1}) \left(T^{(m)}(\hat{E}, \hat{x}, u_1, \ldots, u_{m-1}) \right) \\ &\subset \text{clcone} \left(\text{cone}(\cdots \text{cone}(\text{cone}(F(\hat{E}) + C - \hat{y}) - v_1) - \cdots - v_{m-2}) - v_{m-1} \right). \end{aligned}$$

It follows from (4.3) that

$$\text{clcone} \left(\text{cone}(\cdots \text{cone}(\text{cone}(F(\hat{E}) + C - \hat{y}) - v_1) - \cdots - v_{m-2}) - v_{m-1} \right) \cap (-\text{int}C) \neq \emptyset.$$

Since $-\text{int}C$ is open, one obtains

$$\text{cone} \left(\text{cone}(\cdots \text{cone}(\text{cone}(F(\hat{E}) + C - \hat{y}) - v_1) - \cdots - v_{m-2}) - v_{m-1} \right) \cap (-\text{int}C) \neq \emptyset.$$

Consequently, there exists a sequence $\{\lambda_i : \lambda_i \geq 0, i = 0, 1, 2, \ldots, m-1\}$ such that

$$\lambda_{m-1} \left(\lambda_{m-2}(\cdots \lambda_1(\lambda_0(F(\hat{E}) + C - \hat{y}) - v_1) - \cdots - v_{m-2}) - v_{m-1} \right) \cap (-\text{int}C) \neq \emptyset,$$

then

$$\Pi_{i=0}^{m-1} \lambda_i (F(\hat{E}) + C - \hat{y}) \cap (\lambda_{m-1} v_{m-1} + \lambda_{m-1}\lambda_{m-2} v_{m-2} + \cdots + \Pi_{i=1}^{m-1} \lambda_i v_1 - \text{int}C) \neq \emptyset.$$

It follows from $v_i \in -C, i = 1, 2, \ldots, m-1$ and $-C$ is a convex cone that

$$\lambda_{m-1} v_{m-1} + \lambda_{m-1}\lambda_{m-2} v_{m-2} + \cdots + Pi_{i=1}^{m-1} \lambda_i v_1 - \text{int}C \subset -C - \cdots - C - \text{int}C = -\text{int}C.$$

Consequently,

$$\Pi_{i=0}^{m-1} \lambda_i (F(\hat{E}) + C - \hat{y}) \cap (-\text{int}C) \neq \emptyset.$$

From $0 \notin -\text{int}C$, it follows that $\Pi_{i=0}^{m-1} \lambda_i > 0$. Together with C is a cone gives

$$(F(\hat{E}) + C - \hat{y}) \cap (-\text{int}C) \neq \emptyset.$$

Since C is a pointed cone, one obtains

$$(F(\hat{E}) - \hat{y}) \cap (-\text{int}C) \neq \emptyset.$$

This is a contradiction to the assumption that $(\hat{x}, \hat{y})$ is a weak minimizer of (VP). Hence,

$$\text{cone}\,(\varphi^*(X) + C \times D) \cap ((-\text{int}C) \times (-\text{int}D)) = \emptyset.$$

Since $-\text{int}C$ and $-\text{int}D$ are open, one deduces

$$\text{clcone}\,(\varphi^*(X) + C \times D) \cap ((-\text{int}C) \times (-\text{int}D)) = \emptyset.$$

By the assumption that $\varphi^*(x)$ is nearly $C\times D-$subconvexlike on X we know clcone($\varphi^*(X)$ $+C\times D$) is convex. By a separation theorem for convex sets, there exists $(s^*,k^*)\in Y^*\times Z^*\setminus\{(0_{Y^*},0_{Z^*})\}$, such that

$$(s^*,k^*)\left(\text{clcone}\left(\varphi^*(X)+C\times D\right)\right)\geq s^*(-\text{int}C)+k^*(-\text{int}D). \tag{4.7}$$

Since clcone($\varphi^*(X)+C\times D$) is a cone and on which (s^*,k^*) has a lower bound, we conclude

$$(s^*,k^*)(\text{clcone}(\varphi^*(X)+C\times D))\geq 0. \tag{4.8}$$

In view of $(0_Y,0_Z)\in$ clcone($\varphi^*(X)+C\times D$) and (4.7),

$$s^*(-\text{int}C)+k^*(-\text{int}D)\leq 0. \tag{4.9}$$

It follows from $(0_Y,0_Z)\in C\times D$ and (4.8) that $(s^*,k^*)(\varphi^*(X))\geq 0$. Hence $(s^*,k^*)\left(\varphi^*(x)\right)$ ≥ 0, $\forall x\in X$. In other words,

$$s^*\left(M^{(m)}F(\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x)\right)+$$
$$k^*\left(M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\ldots,u_{m-1},w_{m-1})(x)\right)\geq 0,$$

Therefore,

$$\inf_{x\in X}\Big(s^*\left(M^{(m)}F(\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x)\right)+$$
$$k^*\left(M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\ldots,u_{m-1},w_{m-1})(x)\right)\Big)\geq 0,$$

Then, we prove $k^*\in D^*$ and $s^*\in C^*$.

In view of (4.9),

$$k^*(\text{int}D)\geq s^*(-\text{int}C).$$

Since int$D$ is a cone and on which k^* is bounded blow, we derive

$$k^*(\text{int}D)\geq 0.$$

Since D is a closed convex cone, we get $D=\text{cl}(\text{int}D)$. For any $d\in D$, there exists a net $\{d_\alpha\}\subset\text{int}D$ such that $d=\lim d_\alpha$. Hence,

$$k^*(d)=\lim k^*(d_\alpha)\geq 0,$$

which implies $k^*(D)\geq 0$, thus $k^*\in D^*$. In the similar way, we conclude $s^*\in C^*$. The proof is completed. □

Corollary 4.3. *Suppose that $(\hat{x},\hat{y})$ is a weak minimizer of (VP), $\hat{z}\in G(\hat{x})\cap(-D)$, $(u_i,v_i,w_i)\in X\times(-C)\times(-D), i=1,2,\ldots,m-1$, F is higher-order M-derivable at $(\hat{x},\hat{y})$ for vectors $(u_1,v_1),\ldots,(u_{m-1},v_{m-1})$, G is higher-order M-derivable at $(\hat{x},\hat{z})$ for vectors $(u_1,w_1),\ldots,(u_{m-1},w_{m-1})$, F and G are $C-$convex and $D-$convex on X, respectively,*

$$\text{dom}M^{(m)}F(\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})=\text{dom}M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\ldots,u_{m-1},w_{m-1})=X.$$

Then there exist $s^*\in C^*$, $k^*\in D^*$, $(s^*,k^*)\neq(0_{Y^*},0_{Z^*})$ such that

$$\inf_{x\in X}\Big(s^*\left(M^{(m)}F(\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x)\right)+$$

$$k^*\left(M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\ldots,u_{m-1},w_{m-1})(x)\right)\Big)\geq 0,$$

where

$$s^*\Big(M^{(m)}F(\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x)\Big)=\bigcup_{y\in M^{(m)}F(\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x)}s^*(y)$$

and

$$k^*\Big(M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\ldots,u_{m-1},w_{m-1})(x)\Big)=\bigcup_{z\in M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\ldots,u_{m-1},w_{m-1})(x)}k^*(z).$$

Proof. Since F and G are $C-$convex and $D-$convex, respectively, we conclude epi$F$ and epi$G$ are convex. From Proposition 3.7, one decluces that $M^{(m)}\Big(\text{epi}F,\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1}\Big)$ and $M^{(m)}\Big(\text{epi}G,\hat{x},\hat{z},u_1,w_1,\ldots,u_{m-1},w_{m-1}\Big)$ are convex. Consequently, graph$M^{(m)}F(\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$ and graph$M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\ldots,u_{m-1},w_{m-1})$ are convex. Then, $M^{(m)}F(\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})$ and $M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\ldots,u_{m-1},w_{m-1})$ are $C-$convex and $D-$convex, respectively. Since

$$\begin{aligned}\varphi^*(x)= &\Big(M^{(m)}F(\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x),\\ &M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\ldots,u_{m-1},w_{m-1})(x)\Big),\end{aligned}$$

we get φ^* is $C\times D-$convex. Thus, φ^* is nearly $C\times D-$subconvexlike on X. Therefore, the conditions of Theorem 4.2 are satisfied, we complete the proof. □

Theorem 4.4 (Kuhn-Tucker necessary optimality condition)**.** *Suppose that $(\hat{x},\hat{y})$ is a weak minimizer of (VP), $\hat{z}\in G(\hat{x})\cap(-D)$, $(u_i,v_i,w_i)\in X\times(-C)\times(-D), i=1,2,\ldots,m-1$, F is higher-order M-derivable at $(\hat{x},\hat{y})$ for vectors $(u_1,v_1),\ldots,(u_{m-1},v_{m-1})$, G is higher-order M-derivable at $(\hat{x},\hat{z})$ for vectors $(u_1,w_1),\ldots,(u_{m-1},w_{m-1})$, and*

$$\text{dom}M^{(m)}F(\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})=\text{dom}M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\ldots,u_{m-1},w_{m-1})=X.$$

The set-valued map

$$\begin{aligned}\varphi^*(x)=\Big(&M^{(m)}F(\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x),\\ &M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\ldots,u_{m-1},w_{m-1})(x)\Big):X\to 2^{Y\times Z}\end{aligned}$$

is nearly $C\times D-$subconvexlike on X. If there exists an $x''\in X$ such that $M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\ldots,u_{m-1},w_{m-1})(x'')\cap(-\text{int}D)\neq\emptyset$, then there exist $s^*\in C^*\setminus\{0_{Y^*}\}$ and $k^*\in D^*$ such that

$$\begin{aligned}\inf_{x\in X}\Big(&s^*\left(M^{(m)}F(\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x)\right)+\\ &k^*\left(M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\ldots,u_{m-1},w_{m-1})(x)\right)\Big)\geq 0,\end{aligned}$$

where

$$s^*\Big(M^{(m)}F(\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x)\Big)=\bigcup_{y\in M^{(m)}F(\hat{x},\hat{y},u_1,v_1,\ldots,u_{m-1},v_{m-1})(x)}s^*(y)$$

and

$$k^*\left(M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\dots,u_{m-1},w_{m-1})(x)\right)=\bigcup_{z\in M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\dots,u_{m-1},w_{m-1})(x)} k^*(z).$$

Proof. From Theorem 4.2, it suffices to show $s^*\neq 0_{Y^*}$. Suppose to the contrary that $s^*=0_{Y^*}$, then from $(s^*,k^*)\neq(0_{Y^*},0_{Z^*})$ it follows that $k^*\neq 0_{Z^*}$. By the assumption of the theorem, there exists $z''\in M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\dots,u_{m-1},w_{m-1})(x'')$ such that $z''\in -\text{int}D$, which together with $k^*\in D^*\backslash\{0_{Z^*}\}$ gives $k^*(z'')<0$. On the other hand, by the assumption of the theorem and $s^*=0_{Y^*}$, we get

$$k^*\left(M^{(m)}G(\hat{x},\hat{z},u_1,w_1,\dots,u_{m-1},w_{m-1})(x'')\right)\geq 0.$$

This is a contradiction.The proof is completed. □

We give the following example to illustrate Theorems 4.2 and 4.4.

Example 4.5. Suppose R is the set of real numbers, $X=Y=Z=R$, $C=D=\{t:t\geq 0\}$, set-valued maps $F:X\to 2^Y$ and $G:X\to 2^Z$ are defined respectively by

$$F(x)=\{y:y\geq x^4\},x\in R,$$

$$G(x)=\begin{cases}\{y:y\geq x^2\}, & \text{if } \ x\geq 0,\\ \{y:y\geq x\}, & \text{otherwise.}\end{cases}$$

Let $(\hat{x},\hat{y})=(0,0)$, $(u_i,v_i,w_i)=(-1,0,-1), i=1,2,3$, $x''=-1$, a direct calculation gives

$$\hat{z}\in G(0)\cap(-D)=\{0\},$$

$$M^{(4)}(\text{epi}F,0,0,-1,0,-1,0,-1,0)=\{(x,y):x\in R,y\geq 1\},$$

$$M^{(4)}(\text{epi}G,0,0,-1,-1,-1,-1,-1,-1)=\{(x,y):y\geq x,x\in R\},$$

$$M^{(4)}F(0,0,-1,0,-1,0,-1,0)(x)=[1,+\infty),x\in R,$$

$$M^{(4)}G(0,0,-1,-1,-1,-1,-1,-1)(x)=\{y:y\geq x\},x\in R,$$

$$M^{(4)}G(0,0,-1,-1,-1,-1,-1,-1)(-1)\cap(-\text{int}D)=[-1,0).$$

Thus, $\varphi^*(x)=\left(M^{(4)}F(0,0,-1,0,-1,0,-1,0)(x),M^{(4)}G(0,0,-1,-1,-1,-1,-1,-1)(x)\right)$ is nearly $C\times D-$subconvexlike on X. Consequently, the conditions of Theorems 4.1 and 4.2 are satisfied. Taking $s^*=1$ and $k^*=0$, one has

$$\begin{aligned}&\inf_{x\in X}\Big(s^*\left(M^{(4)}F(0,0,-1,0,-1,0,-1,0)(x)\right)+\\ &\quad k^*\left(M^{(4)}G(0,0,-1,-1,-1,-1,-1,-1)(x)\right)\Big)=1\geq 0.\end{aligned}$$

Then, the conclusions of Theorems 4.2 and 4.4 are true.

Theorem 4.6 (Kuhn-Tucker sufficient optimality condition)**.** *Suppose that* $F:\hat{E}\to 2^Y$ *and* $G:\hat{E}\to 2^Z$ *are* $C-$*convex and* $D-$*convex on the convex set* $\hat{E}$*, respectively.* F *and* G *are lower semicontinuous on* $\hat{E}$*.* $\hat{z}\in G(\hat{x})\cap(-D)$*,* $(\hat{x},\hat{y})\in\text{graph}F$*,* $(u_i,v_i)\in\text{epi}F, i=1,2,\dots,m-1$*,* $(u_i,w_i)\in\text{epi}G, i=1,2,\dots,m-1$*. If there exist* $s^*\in C^*\setminus\{0_{Y^*}\}$ *and* $k^*\in D^*$ *such that* $k^*(\hat{z})=0$ *and*

$$\begin{aligned}&\inf_{x\in X}\Big(s^*\left(M^{(m)}F(\hat{x},\hat{y},u_1-\hat{x},v_1-\hat{y},\dots,u_{m-1}-\hat{x},v_{m-1}-\hat{y})(x-\hat{x})\right)\\ &+k^*\left(M^{(m)}G(\hat{x},\hat{z},u_1-\hat{x},w_1-\hat{z},\dots,u_{m-1}-\hat{x},w_{m-1}-\hat{z})(x-\hat{x})\right)\Big)\geq 0,\end{aligned}\tag{4.10}$$

then $(\hat{x},\hat{y})$ *is a weak minimizer of (*VP*).*

Proof. On the contrary suppose that $(\hat{x}, \hat{y})$ is not a weak minimizer of (VP), then

$$(F(\hat{E}) - \hat{y}) \cap (-\text{int}C) \neq \emptyset,$$

Hence, there exist $\tilde{x} \in \hat{E}$, $\tilde{y} \in F(\tilde{x})$ such that

$$\tilde{y} - \hat{y} \in -\text{int}C.$$

It follows from $\tilde{x} \in \hat{E}$ that we can find a $\tilde{z} \in G(\tilde{x}) \cap (-D)$. Along with Proposition 3.8 gives

$$\tilde{y} - \hat{y} \in M^{(m)}F(\hat{x}, \hat{y}, u_1 - \hat{x}, v_1 - \hat{y}, \ldots, u_{m-1} - \hat{x}, v_{m-1} - \hat{y})(\tilde{x} - \hat{x})$$

and

$$\tilde{z} - \hat{z} \in M^{(m)}G(\hat{x}, \hat{z}, u_1 - \hat{x}, w_1 - \hat{z}, \ldots, u_{m-1} - \hat{x}, w_{m-1} - \hat{z})(\tilde{x} - \hat{x}).$$

From (4.10), one gets

$$s^*(\tilde{y} - \hat{y}) + k^*(\tilde{z} - \hat{z}) \geq 0. \tag{4.11}$$

On the other hand, from $s^* \in C^* \setminus \{0_{Y^*}\}$ and $\tilde{y} - \hat{y} \in -\text{int}C$, it follows that

$$s^*(\tilde{y} - \hat{y}) < 0.$$

Since $k^*(\hat{z}) = 0$, $\tilde{z} \in G(\tilde{x}) \cap (-D)$ and $k^* \in D^*$, one obtains

$$k^*(\tilde{z} - \hat{z}) = k^*(\tilde{z}) - k^*(\hat{z}) = k^*(\tilde{z}) \leq 0.$$

So,

$$s^*(\tilde{y} - \hat{y}) + k^*(\tilde{z} - \hat{z}) < 0,$$

which contradicts (4.11). The proof is completed. □

We end this section by presenting an example to illustrate the higher-order sufficient optimality condition established above.

Example 4.7. Suppose that R is the set of real numbers, $X = Y = Z = R$, $C = D = \{t : t \geq 0\}$, set-valued maps $F : X \to 2^Y$ and $G : X \to 2^Z$ are defined respectively by

$$F(x) = \{y : y \geq |x|\}, x \in R,$$

$$G(x) = \{y : y \geq x\}, x \in R.$$

Then F is $C-$convex and lower semicontinuous on R and G is $D-$convex and lower semicontinuous on R. Let $(\hat{x}, \hat{y}) = (0, 0)$, $(u_i, v_i, w_i) = (-1, 1, -1), i = 1, 2, \ldots, m-1$, a direct calculation gives

$$\hat{z} \in G(0) \cap (-D) = \{0\},$$

$$M^{(m)}(\text{epi}F, 0, 0, -1, 1, \ldots, -1, 1) = \{(x, y) : y \geq -x, x \in R\},$$

$$M^{(m)}(\text{epi}G, 0, 0, -1, -1, \ldots, -1, -1) = \{(x, y) : y \geq x, x \in R\},$$

$$M^{(m)}F(0, 0, -1, 1, \ldots, -1, 1)(x) = \{y : y \geq -x\}, x \in R,$$

$$M^{(m)}G(0, 0, -1, -1, \ldots, -1, -1)(x) = \{y : y \geq x\}, x \in R.$$

Taking $s^* = k^* = 1$, one has

$$\inf_{x \in X} \left(s^* \left(M^{(m)}F(0, 0, -1, 1, \ldots, -1, 1)(x) \right) + k^* \left(M^{(m)}G(0, 0, -1, -1, \ldots, -1, -1)(x) \right) \right) = 0$$

and

$$k^*(\hat{z}) = k^*(0) = 0.$$

Consequently, the conditions of Theorem 4.6 are satisfied and $(0, 0)$ is a weak minimizer of (VP). Then, the conclusion of Theorem 4.6 holds.

$\boxed{5}$ Conclusions

In this paper, we introduced a new kind of higher-order M-tangent set and with which, higher-order M-tangent derivative for a set-valued function was introduced. Compared with the higher-order contingent derivative $D^{(m)}F(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})$, higher-order $M-$tangent derivative $M^{(m)}F(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})$ has a nice property:

$$\begin{aligned}&M^{(m)}F(\hat{x}, \hat{y}, u_1, v_1, \ldots, u_{m-1}, v_{m-1})\left(T^{(m)}(E, \hat{x}, u_1, \ldots, u_{m-1})\right)\\ \subset\ &\mathrm{clcone}\left(\mathrm{cone}(\cdots \mathrm{cone}(\mathrm{cone}(F(E) + C - \hat{y}) - v_1) - \cdots - v_{m-2}) - v_{m-1}\right).\end{aligned}$$

which is demonstrated by Proposition 3.4, while the higher-order contingent derivative has not the property, see Remark 3.5. Just applying the property, we established higher-order Fritz John and Kuhn-Tucker necessary optimality conditions for a point pair to be a weak minimizer of set-valued optimization problem, where the higher-order tangent derivatives of multiobjective function and constraint function are separated.

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