Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 14, No. 3, pp. 381–398, 2018

LIPSCHITZ MODULUS OF FULLY PERTURBED LINEAR PROGRAMS*

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Abstract: The present paper deals with the Aubin property of the argmin mapping associated with linear programs under full perturbations (i.e., perturbations of all coefficients). First, we characterize the property itself and, in a second step, we provide a formula for computing the exact Lipschitz modulus. The direct antecedent of this work can be found in [5], which is developed in the framework of canonical perturbations; i.e., keeping fixed the left-hand-side of the constraints. We emphasize the fact that allowing perturbations of the left-hand-side entails notable differences with respect to referred work. In particular, the arguments of this paper are strongly based on the very recent results about calmness moduli of both the feasible set and the argmin mappings.

Key words: *variational analysis, Lipschitz-like property, linear programming, Lipschitz modulus, calmness modulus*

Mathematics Subject Classification: *90C31, 49J53, 49K40, 90C05.*

1 Introduction

The present work is focused on the computation of the exact *Lipschitz modulus* for the optimal set mapping in the context of *fully* perturbed linear programs. Paper [5] constitutes our direct antecedent, as far as it provides the counterpart of the current analysis when confined to the framework of *canonical* perturbations (where perturbations fall on the right hand side of the constraints and on the coefficient vector of the objective function). To this respect, paper [5] solved an open problem previously posed by Li [28]. A key starting point in our analysis is traced out from [29]; where under appropriate assumptions (held in our framework) it is established that any *uniform calmness constant* becomes a Lipschitz constant. In this way, we take advantage of recent studies about calmness of the optimal set mapping to derive a formula for computing the aimed Lipschitz modulus; specifically, an upper bound on the calmness modulus of this mapping given in [7] (and further developed in [9]) constitutes a key ingredient in our analysis.

The structure of the paper is as follows. Section 1 is devoted to integrating the present paper in the existing literature about Lipschitz and calmness type properties, specially in relation to linear programming. Section 2 contains the model presentation as well as the necessary notation, stability notions of multifunctions, and some preliminary results about Lipschitz and calmness properties in our linear setting which are needed later on. Section

*This research has been partially supported by Grant MTM2014-59179-C2-2-P from MINECO, Spain, and FEDER, "Una manera de hacer Europa", European Union.

3 establishes various characterizations of the Aubin property of the argmin mapping $\mathcal{S}$ (see Theorem 3.5). At this moment we advance that this property holds under full perturbations if and only if it holds under canonical perturbations, which at a first glance seems to be a weaker property. Finally, Section 4 provides in Theorem 4.1 the aimed formula for the Lipschitz modulus of $\mathcal{S}$, which is expressed exclusively in terms of the nominal data. Both results, Theorems 3.5 and 4.1, gather the main original contributions of this paper. In comparison with its counterpart in the case of canonical perturbations (gathered in [5]), we point out the notable differences arising in the proofs when we allow perturbations of all data. In particular, a crucial ingredient in the new development is the recent upper bound on the calmness modulus of $\mathcal{S}$ given in Theorem 2.9. We finish Section 4 with an illustrative example intended to show some of the geometrical ideas underlying the perturbations strategies followed in the proof of Theorem 4.1.

It is well-known that linear optimization, besides its own interest, constitutes a key tool for approaching broader global optimization models as, for instance, the standard quadratic optimization problem (see, e.g., [2, 15]), as well as mathematical programs with complementarity constraints (MPCCs). As an application of the referred upper bound on the calmness modulus, [7, Subsection 5.2] deals with a concrete regularization scheme for MPCCs introduced in [23], applied in the context of linear MPCCs. Complementarity constraints naturally appear in numerous applications in economics and engineering; see [31] (and references therein) for details on theory and applications of MPCCs. The application of the results provided in the present paper about Lispchitz modulus to linear MPCCs requires a specific study of constraint systems containing equations (to be perturbed as equations, not split into two inequalities), and constitutes an open problem at the moment.

Lipschitz and calmness properties play an important role in optimization and variational analysis, and one can find in the literature deep contributions to the study of these properties; besides the monographs [17, 24, 32, 35], see for instance [16], [22] and [25] in relation to metric regularity of generic multifunctions (the last one analyzes its relationship with the behavior of methods for solving a *generalized equation*), and [30] and [10] with respect to metric regularity for convex systems and problems, respectively. In the context of linear systems, the Aubin property and the associated modulus are analyzed in [3] and [4] (see also [12] for extensions to an infinite dimensional setting). With respect to calmness, when confined to linear programming, we cite the pioneer works of Robinson [33] and [34]. In a more general framework we refer to [8, 18, 20, 25, 26, 27], which are developed in the setting of nonlinear constraint systems under canonical perturbations (where calmness translates into the existence of a *local error bound* for a certain related supremum function).

2 The Model and Preliminaries

This section is divided into three subsections. The first one introduces our parametric model. The second recalls some continuity and Lipschitz type properties for generic multifunctions between metric spaces and gathers some results dealing with the relationships among them. The third subsection presents some backgrounds about the Aubin and calmness properties in our linear programming setting.

2.1 The model

We consider a parameterized family of linear optimization problems in $\mathbb{R}^n$ given by:

$$P(c,a,b): \quad \begin{array}{ll} \text{minimize} & c'x \\ \text{subject to} & a_t'x \leq b_t, \ \ t \in T := \{1,2,...,m\}, \end{array} \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the vector of variables, and $c \in \mathbb{R}^n$, $a \equiv (a_t)_{t\in T} \in (\mathbb{R}^n)^T$, and $b \equiv (b_t)_{t\in T} \in \mathbb{R}^T$ are the problem's data, which are considered as parameters to be perturbed. All elements in $\mathbb{R}^n$ are regarded as column-vectors and $u'v$ denotes the usual inner product of u and v in $\mathbb{R}^n$.

Associated with (2.1), we consider the *optimal set mapping* (also called *argmin mapping*), $\mathcal{S} : \mathbb{R}^n \times (\mathbb{R}^n)^T \times \mathbb{R}^T \rightrightarrows \mathbb{R}^n$, given by

$$\mathcal{S}(c,a,b) := \{x \in \mathbb{R}^n \mid x \text{ is an optimal solution of } P(c,a,b)\}.$$

The parameter space $\mathbb{R}^n \times (\mathbb{R}^n)^T \times \mathbb{R}^T$ is endowed with the norm

$$\|(c,a,b)\| := \max\{\|c\|_*, \|(a,b)\|_\infty\}, \tag{2.2}$$

where $\mathbb{R}^n$ is equipped with an arbitrary norm $\|\cdot\|$, with *dual norm* given by $\|u\|_* = \max_{\|x\|\le 1} |u'x|$, and $\|(a,b)\|_\infty := \max_{t\in T} \left\| \binom{a_t}{b_t} \right\|$, where

$$\left\| \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\| = \max\{\|a_t\|_*, |b_t|\}. \tag{2.3}$$

We also deal with the optimal set mapping in the context of canonical perturbations (i.e., where the left-hand-side of the constraints is fixed), $\mathcal{S}_{\overline{a}} : \mathbb{R}^n \times \mathbb{R}^T \rightrightarrows \mathbb{R}^n$, which is given by

$$\mathcal{S}_{\overline{a}}(c,b) = \mathcal{S}(c,\overline{a},b), \text{ for all } (c,b) \in \mathbb{R}^n \times \mathbb{R}^T.$$

In relation to the constraint system of problem (2.1), we consider the *feasible set mappings* $\mathcal{F}_{\overline{a}}$ and $\mathcal{F}$ corresponding, respectively, to the settings of canonically and fully perturbed systems. Formally, $\mathcal{F} : (\mathbb{R}^n)^T \times \mathbb{R}^T \rightrightarrows \mathbb{R}^n$ is given by

$$\mathcal{F}(a,b) := \{x \in \mathbb{R}^n \mid a_t'x \le b_t,\ t \in T\},\ (a,b) \in (\mathbb{R}^n)^T \times \mathbb{R}^T,$$

and $\mathcal{F}_{\overline{a}}(b) := \mathcal{F}(\overline{a},b)$, $b \in \mathbb{R}^T$. We also appeal to the *set of active indices* at $x \in \mathcal{F}(a,b)$, denoted by $T_{a,b}(x)$, which is defined as

$$T_{a,b}(x) := \{t \in T \mid a_t'x = b_t\}.$$

$\boxed{2.2}$ Aubin property, calmness, and some continuity properties for generic multifunctions

At this moment we recall some variational concepts for a generic multifunction between metric spaces Y and X (with distances denoted indistinctly by d), $\mathcal{M} : Y \rightrightarrows X$. The reader is addressed to the monographs [17, 24, 32, 35] for a comprehensive analysis of these notions. Multifunction $\mathcal{M}$ is said to satisfy –or have– the *Aubin* property (also called *pseudo-Lipschitz*, cf. [24], or *Lipschitz-like*, cf. [32]) at $(\overline{y},\overline{x}) \in \operatorname{gph}\mathcal{M}$ (the graph of $\mathcal{M}$) if there exist a constant $\kappa \ge 0$ and neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that

$$d(x',\mathcal{M}(y)) \le \kappa d(y',y) \tag{2.4}$$

whenever $x' \in \mathcal{M}(y') \cap U$ and $y', y \in V$; here $d(x',\mathcal{M}(y))$ denotes the usual distance from point x' to set $\mathcal{M}(y)$, defined as $+\infty$ when $\mathcal{M}(y) = \emptyset$. It is well-known that the Aubin property of $\mathcal{M}$ at $(\overline{y},\overline{x})$ is equivalent to the *metric regularity* of $\mathcal{M}^{-1}$ at $(\overline{x},\overline{y})$, i.e., to the existence of a constant $\kappa \geq 0$ and (possibly smaller) neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that

$$d\left(x, \mathcal{M}\left(y\right)\right) \leq \kappa d\left(y, \mathcal{M}^{-1}\left(x\right)\right), \text{ for all } x \in U \text{ and all } y \in V. \tag{2.5}$$

Recall that $\mathcal{M}^{-1} : X \rightrightarrows Y$ is defined by $y \in \mathcal{M}^{-1}\left(x\right) \Leftrightarrow x \in \mathcal{M}\left(y\right)$. The *Lipschitz modulus* of $\mathcal{M}$ at $\left(\overline{y}, \overline{x}\right)$, $\text{lip}\mathcal{M}\left(\overline{y}, \overline{x}\right)$, is the infimum of those $\kappa \geq 0$ for which (2.4) –or (2.5)– holds (for some associated neighborhoods). The case when $\mathcal{M}$ does not satisfy the Aubin property at $\left(\overline{y}, \overline{x}\right)$ corresponds to $\text{lip}\mathcal{M}\left(\overline{y}, \overline{x}\right) = +\infty$.

$\mathcal{M}$ is said to be *calm* at $\left(\overline{y}, \overline{x}\right)$ when (2.4) is valid when replacing y with the nominal element $\overline{y}$. It is also known that calmness of $\mathcal{M}$ at $\left(\overline{y}, \overline{x}\right)$ is equivalent to the *metric subregularity* of $\mathcal{M}^{-1}$ at $\left(\overline{x}, \overline{y}\right)$, which is defined by fixing $y = \overline{y}$ in inequality (2.5). The corresponding infimum of all κ's is then called *calmness modulus* of $\mathcal{M}$ at $\left(\overline{y}, \overline{x}\right)$ and denoted by $\text{clm}\mathcal{M}\left(\overline{y}, \overline{x}\right)$. Obviously,

$$\text{clm}\mathcal{M}\left(\overline{y}, \overline{x}\right) \leq \text{lip}\mathcal{M}\left(\overline{y}, \overline{x}\right). \tag{2.6}$$

Taking the previous notation into account, the main goal of the present work is to compute $\text{lip}\mathcal{S}(\overline{p}, \overline{x})$ at a nominal pair $(\overline{p}, \overline{x}) \in \text{gph}\mathcal{S}$, with $\overline{p} = \left(\overline{c}, \overline{a}, \overline{b}\right)$ and, as mentioned above, the recent results about $\text{clm}\mathcal{S}(\overline{p}, \overline{x})$ gathered in [7] constitute crucial tools.

In order to apply (in Sections 3 and 4) some results traced out from [29], we need to introduce additional stability properties. We start by recalling two continuity properties for a multifunction between metric spaces $\mathcal{M} : Y \rightrightarrows X$ at $\overline{y} \in Y$. For avoiding trivial situations, we assume $\mathcal{M}\left(\overline{y}\right) \neq \emptyset$ throughout this subsection. We say that:

- $\mathcal{M}$ is *Berge lower semicontinuous* (*B-lsc*) at $\overline{y}$ if for each open set $U \subset X$ such that $\mathcal{M}\left(\overline{y}\right) \cap U \neq \emptyset$ there exists a neighborhood V of $\overline{y}$ such that $\mathcal{M}\left(y\right) \cap U \neq \emptyset$ for all $y \in V$.
- $\mathcal{M}$ is *Hausdorff lower semicontinuous* (*H-lsc*) at $\overline{y}$ if
$$\lim_{y \to \overline{y}} e\left(\mathcal{M}\left(\overline{y}\right), \mathcal{M}\left(y\right)\right) = 0,$$
where
$$e\left(\mathcal{M}\left(\overline{y}\right), \mathcal{M}\left(y\right)\right) := \sup_{x \in \mathcal{M}(\overline{y})} d\left(x, \mathcal{M}\left(y\right)\right)$$
stands for the Hausdorff excess of $\mathcal{M}\left(\overline{y}\right)$ over $\mathcal{M}\left(y\right)$.

The statements in the following lemma can be found, e.g., in [1, Section 2.2].

Lemma 2.1. *The following implications hold:*
(i) *If* $\mathcal{M}$ *is* H-lsc *at* $\overline{y}$, *then* $\mathcal{M}$ *is* B-lsc *at* $\overline{y}$;
(ii) *If* $\mathcal{M}$ *is* B-lsc *at* $\overline{y}$ *and* $\mathcal{M}(\overline{y})$ *is a compact set, then* $\mathcal{M}$ *is* H-lsc *at* $\overline{y}$.

The next definitions (see [29]) and the subsequent lemma deal with an open and convex subset $V_0 \subset Y$ (with Y assumed to be a normed space). First, recall that $\mathcal{M}$ is *H-lsc* (resp. *B-lsc)* on V_0 if it is *H-lsc* (resp. *B-lsc*) at any $y \in V_0$.

- $\mathcal{M}$ (assumed to be closed-valued) is said to be λ-*Lipschitz continuous* on V_0 if
$$H\left(\mathcal{M}\left(y\right), \mathcal{M}\left(y'\right)\right) \leq \lambda d\left(y, y'\right) \text{ for all } y, y' \in V_0,$$
where $H\left(\mathcal{M}\left(y\right), \mathcal{M}\left(y'\right)\right)$ is the (extended) Hausdorff distance between $\mathcal{M}\left(y\right)$ and $\mathcal{M}\left(y'\right)$, given by
$$H\left(\mathcal{M}\left(y\right), \mathcal{M}\left(y'\right)\right) = \max\left\{e\left(\mathcal{M}\left(y\right), \mathcal{M}\left(y'\right)\right), e\left(\mathcal{M}\left(y'\right), \mathcal{M}\left(y\right)\right)\right\}.$$

- $\mathcal{M}$ is said to be *locally upper Lipschitz continuous with modulo* λ on V_0 if for each $y \in V_0$ there exists a neighborhood V_y of y such that

$$e\left(\mathcal{M}\left(y'\right), \mathcal{M}\left(y\right)\right) \leq \lambda d(y', y), \text{ for all } y' \in V_y.$$

Remark 2.2. *As a consequence of the definitions, one has:*

(*i*) *If* $\mathcal{M}$ *is* λ-Lipschitz continuous *on* V_0, *then* $\mathcal{M}$ *has the Aubin property at any* $(y, x) \in$ gph$\mathcal{M}$ *such that* $y \in V_0$.

(*ii*) *If* $\mathcal{M}$ *is locally upper Lipschitz continuous with modulo* λ *on* V_0, *then* $\mathcal{M}$ *is calm at any* $(y, x) \in$ gph$\mathcal{M}$ *such that* $y \in V_0$.

One can easily see that the converse implications in the previous remark are not true in general. Besides the fact that the Aubin and calmness properties at $(y, x) \in$ gph$\mathcal{M}$ are local properties (only involve elements around (y, x)), the λ-Lipschitz continuity and local upper Lipschitz continuity are stated in terms of a *uniform* constant λ (the same constant for the whole neighborhood V_0 of $\overline{y}$).

Here we recall the above mentioned result by Li [29] which constitutes the key starting point in our analysis of the Aubin property of $\mathcal{S}$. Let us comment that Theorem 2.1 in [29] is stated for $V_0 = \mathbb{R}^m$ in the statement of the following lemma, but the proof also works in the current setting.

Lemma 2.3. [29, Thm. 2.1] *Let* $\mathcal{M} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *and let* $V_0 \neq \emptyset$ *be an open convex subset of* $\mathbb{R}^m$. *If* $\mathcal{M}$ *is* H-lsc *on* V_0 *and locally upper Lipschitz continuous with modulo* λ *on* V_0, *then* $\mathcal{M}$ *is* λ*-Lipschitz continuous on* V_0.

2.3 Background on the Aubin and calmness properties in linear optimization

First, we introduce some basic notation. Given $X \subset \mathbb{R}^k$, $k \in \mathbb{N}$, we denote by conv$X$ and cone$X$ the *convex hull* and the *conical convex hull* of X, respectively. It is assumed that cone$X$ always contains the zero-vector 0_k, in particular cone$(\emptyset) = \{0_k\}$. If X is a subset of any topological space, int$X$, cl$X$ and bd$X$ stand, respectively, for the interior, the closure, and the boundary of X.

The following proposition provides the calmness modulus of the feasible set mapping $\mathcal{F}$ in our case of interest (see the subsequent remark).

Proposition 2.4. [14, Thm. 5] and [6, Prop. 4.1] *Let* $((\overline{a}, \overline{b}), \overline{x}) \in$ gph$\mathcal{F}$. *Then*

$$\text{clm}\mathcal{F}((\overline{a}, \overline{b}), \overline{x}) = (\|\overline{x}\| + 1)\, \text{clm}\mathcal{F}_{\overline{a}}(\overline{b}, \overline{x}).$$

In the particular case when $\mathcal{F}(\overline{a}, \overline{b}) = \{\overline{x}\}$, *we have*

$$\text{clm}\mathcal{F}((\overline{a}, \overline{b}), \overline{x}) = \frac{\|\overline{x}\| + 1}{d_*(0_n, \text{bd}\,\text{conv}\{\overline{a}_t,\ t \in T_{\overline{a}, \overline{b}}\,(\overline{x})\})},$$

where d_* *stands for the distance associated with the dual norm* $\|\cdot\|_*$.

Remark 2.5. *A more general expression for* clm$\mathcal{F}_{\overline{a}}(\overline{b}, \overline{x})$ *can be found in [14, Thm. 4] without requiring* $\mathcal{F}(\overline{a}, \overline{b}) = \{\overline{x}\}$. *We do not include it here in order to avoid introducing additional notation. Besides being used in the proof of Theorem 4.1, Proposition 2.4, applied to suitably enlarged systems, underlies Theorem 2.9 below.*

For the sake of brevity, from now on we use the notation

$$p = (c, a, b) \in \mathbb{R}^n \times (\mathbb{R}^n)^T \times \mathbb{R}^T.$$

We consider the following family of index subsets closely related to the Karush-Kuhn-Tucker (KKT) optimality conditions at $(p, x) \in \mathrm{gph}\mathcal{S}$:

$$\mathcal{T}_p(x) = \{D \subset T_{a,b}(x) \mid \ |D| = n, \ -c \in \mathrm{cone}\{a_t, \ t \in D\}\}.$$

The following theorem comes from particularizing conditions (i),(ii),(iv) and (v) in [10, Thm. 16] (developed in a semi-infinite framework) to our finite linear setting. In it we appeal to the well-known *Slater constraint qualification* (SCQ) which is satisfied at (a, b) if there exists $\widehat{x} \in \mathbb{R}^n$ (called a *Slater point*) such that $a_t'\widehat{x} < b_t$ for all $t \in T$.

Theorem 2.6. [10, Thm. 16] *Let $((\overline{c}, \overline{b}), \overline{x}) \in \mathrm{gph}\mathcal{S}_{\overline{a}}$. The following conditions are equivalent:*

(i) $\mathcal{S}_{\overline{a}}$ has the Aubin property at $((\overline{c}, \overline{b}), \overline{x})$;

(ii) $\mathcal{S}_{\overline{a}}$ is strongly Lipschitz stable *at $((\overline{c}, \overline{b}), \overline{x})$ (single valued and Lipschitz continuous in a neighborhood of $(\overline{c}, \overline{b})$);*

(iii) $\mathcal{S}_{\overline{a}}$ is single valued in some neighborhood of $(\overline{c}, \overline{b})$;

(iv) The Nürnberger condition (NC) *holds at $(\overline{p}, \overline{x})$; i.e., SCQ is satisfied at $(\overline{a}, \overline{b})$ and there is no $D \subset T_{\overline{a},\overline{b}}(\overline{x})$ with $|D| < n$ such that $-\overline{c} \in \mathrm{cone}\{\overline{a}_t, \ t \in D\}$.*

Remark 2.7. *According to the previous theorem, under the Aubin property of $\mathcal{S}_{\overline{a}}$ at $((\overline{c}, \overline{b}), \overline{x})$, we may simplify the notation by writing $\mathrm{lip}\mathcal{S}_{\overline{a}}(\overline{c}, \overline{b})$ instead of $\mathrm{lip}\mathcal{S}_{\overline{a}}((\overline{c}, \overline{b}), \overline{x})$, due to the fact that $\mathcal{S}_{\overline{a}}(\overline{c}, \overline{b})$ is a singleton. The same convention will be applied to $\mathcal{S}$ in Section 4. Moreover:*

(i) $\mathcal{T}_{\overline{p}}(\overline{x}) \neq \emptyset$, as far as $-\overline{c} \in \mathrm{int\,cone}\{\overline{a}_t, \ t \in T_{\overline{a},\overline{b}}(\overline{x})\}$ (as a consequence of $\mathcal{S}_{\overline{a}}(\overline{c}, \overline{b}) = \{\overline{x}\}$).

(ii) As a consequence of NC, for any $D \in \mathcal{T}_{\overline{p}}(\overline{x})$, $\{\overline{a}_t, \ t \in D\}$ is linearly independent and, so, the matrix A_D whose rows are the transposes of $\overline{a}_t, t \in D$ (given in some prefixed order), is non-singular and

$$\left\|A_D^{-1}\right\| = (d_*(0_n, \mathrm{bd\,conv}\{\pm\overline{a}_t, t \in D\}))^{-1}.$$

(See, e.g., [7, Rem. 3.3]). Observe that, according to our choice of norms, A_D acts from $(\mathbb{R}^n, \|\cdot\|)$ to $(\mathbb{R}^D, \|\cdot\|_\infty)$.

Theorem 2.8. [5, Cor. 2] and [7, Rem. 3.3] *If $\mathcal{S}_{\overline{a}}$ has the Aubin property at $((\overline{c}, \overline{b}), \overline{x}) \in \mathrm{gph}\mathcal{S}_{\overline{a}}$, then*

$$\mathrm{lip}\mathcal{S}_{\overline{a}}(\overline{c}, \overline{b}) = \max_{D \in \mathcal{T}_{\overline{p}}(\overline{x})} \left\|A_D^{-1}\right\| = \max_{D \in \mathcal{T}_{\overline{p}}(\overline{x})} \frac{1}{d_*(0_n, \mathrm{bd\,conv}\{\pm\overline{a}_t, t \in D\})}.$$

The following theorem provides a formula for the calmness modulus of $\mathcal{S}_{\overline{a}}$ (which is always calm as a consequence of a classical result by Robinson [34] about calmness of polyhedral mappings), as well as an upper bound on the calmness modulus of $\mathcal{S}$ under SCQ. From now on, we use the notation

$$\mathcal{K}_p(x) = \{D \subset T_{a,b}(x) \mid |D| \leq n \text{ and } -c \in \mathrm{cone}\{a_t, \ t \in D\}\},$$

(note that condition $|D| \leq n$ comes from Caratheodory's Theorem). Observe that the only difference with the definition of $\mathcal{T}_p(x)$ is that condition '$|D| = n$' is replaced with '$|D| \leq n$'. So, in general $\mathcal{T}_p(x) \subset \mathcal{K}_p(x)$; and NC at $(\overline{p}, \overline{x})$ turns out to be equivalent to condition '$\mathcal{K}_{\overline{p}}(\overline{x}) = \mathcal{T}_{\overline{p}}(\overline{x})$'.

Theorem 2.9. *Assume $\mathcal{S}(\overline{p}) = \{\overline{x}\}$. Then, we have:*

(*i*) [7, Thm. 3.1]

$$\operatorname{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c},\overline{b}\right),\overline{x}\right) = \max_{D\in\mathcal{K}_{\overline{p}}(\overline{x})} \frac{1}{d_*\left(0_n, \operatorname{bd}\operatorname{conv}\left\{\overline{a}_t,\ t\in T_{\overline{a},\overline{b}}\left(\overline{x}\right); -\overline{a}_t,\ t\in D\right\}\right)};$$

(*ii*) [7, Thm. 4.2(*i*)] *If SCQ holds at* $\left(\overline{a},\overline{b}\right)$, *then*

$$\operatorname{clm}\mathcal{S}\left(\overline{p},\overline{x}\right) \leq \left(\|\overline{x}\|+1\right)\operatorname{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c},\overline{b}\right),\overline{x}\right).$$

3 Characterization of the Aubin Property for Fully Perturbed Problems

This section is devoted to characterizing the Aubin property of $\mathcal{S}$. At this moment we advance that this property turns out to be equivalent to the Aubin property of $\mathcal{S}_{\overline{a}}$, which at a first glance seems to be less restrictive as far as it involves less parameters to be perturbed. In this way, we are able to apply Theorem 2.6 for deriving additional equivalent conditions to the Aubin property of $\mathcal{S}$. From the practical point of view, we pay special attention to NC (Theorem 2.6 (iv)).

Now, we introduce two technical lemmas. The first one constitutes the counterpart of [13, Lem. 6] for convex hulls, instead of certain convex sets called there *hypographical* sets.

Lemma 3.1. *Let B be the closed unit ball associated with an arbitrary norm in $\mathbb{R}^k$. Let us consider $\{u_i,\ i=1,...,p\}\subset\mathbb{R}^k$ such that $\delta B\subset \operatorname{conv}\{u_i,\ i=1,...,p\}$, for some $\delta>0$. Then, for any $0<\varepsilon<\delta$ we have that*

$$\left(\delta-\varepsilon\right)B\subset\operatorname{conv}\{\widetilde{u}_i, i=1,...,p\},\ \textit{whenever}\ \|u_i-\widetilde{u}_i\|<\varepsilon,\ i=1,...,p.$$

Proof. Reasoning by contradiction, assume the existence of $0<\varepsilon_0<\delta$ and $\{u_i^0, i=1,...,p\}$, such that $\left\|u_i-u_i^0\right\|<\varepsilon_0$, for all $i=1,...,p$, and

$$\left(\delta-\varepsilon_0\right)B\not\subset\operatorname{conv}\{u_i^0, i=1,...,p\},$$

which entails the existence of $\widehat{u}\in\left(\delta-\varepsilon_0\right)B\setminus\operatorname{conv}\{u_i^0, i=1,...,p\}$. Then, by strong separation, there exists $(v,w)\in\mathbb{R}^k\times\mathbb{R}$ such that $v'\widehat{u}<w$, while $v'u_i^0\geq w$, for all $i=1,...,p$. Take any $z\in\mathbb{R}^k$ such that $\|z\|=\varepsilon_0$ and $v'z=\|v\|_*\|z\|$ (see [21, V.3.2]). Then $\widehat{u}-z\in\operatorname{conv}\{u_i,\ i=1,...,p\}$, since $\|\widehat{u}-z\|\leq\delta$; so, we can write

$$\widehat{u}-z:=\sum_{i=1}^p\lambda_i u_i=\sum_{i=1}^p\lambda_i u_i^0+\sum_{i=1}^p\lambda_i\left(u_i-u_i^0\right),$$

for certain $\lambda_i\geq 0$, $i=1,...,p$ and $\sum_{i=1}^p\lambda_i=1$. This implies

$$\begin{aligned} v'\left(\widehat{u}-z\right) &= \sum_{i=1}^p\lambda_i v'u_i^0+\sum_{i=1}^p\lambda_i v'\left(u_i-u_i^0\right) \\ &\geq w-\sum_{i=1}^p\lambda_i\|v\|_*\left\|u_i-u_i^0\right\|\geq w-\|v\|_*\varepsilon_0. \end{aligned}$$

On the other hand, the choice of z yields the contradiction

$$v'(\widehat{u}-z) < w - \|v\|_* \|z\| = w - \|v\|_* \varepsilon_0.$$

□

Condition (i) in the following lemma has its counterpart for canonical perturbations in [10, Prop. 9 (i)]. We emphasize the fact that condition (iii) constitutes a key result in the framework of the current paper. From now on we use the notation:

$$\overline{\delta} := \min_{D \in \mathcal{T}_{\overline{p}}(\overline{x})} d_*(0_n, \operatorname{bd}\operatorname{conv}\{\pm \overline{a}_t,\ t \in D\}). \tag{3.1}$$

Remark 3.2. *If NC holds at* $(\overline{p}, \overline{x}) \in \operatorname{gph}\mathcal{S}$*, then,* $\overline{\delta} > 0$ *(see, Remark 2.7(ii)).*

Lemma 3.3. *Assume that NC holds at* $(\overline{p}, \overline{x}) \in \operatorname{gph}\mathcal{S}$*. Then, there exists* $0 < \varepsilon < \overline{\delta}$ *such that whenever* $\|p - \overline{p}\| < \varepsilon$ *one has:*

(i) *If* $x \in \mathcal{S}(p)$*, then NC holds at* (p, x)*. In particular,* $\mathcal{S}(p)$ *is a singleton and we can write*

$$\mathcal{S}(p) = \{x(p)\}.$$

(ii) $\mathcal{T}_p(x(p)) \subset \mathcal{T}_{\overline{p}}(\overline{x})$.

(iii) $(\overline{\delta} - \varepsilon) B_* \subset \operatorname{conv}\{\pm a_t,\ t \in D\}$, *for all* $D \in \mathcal{T}_p(x)$.

Here B_* *represents the closed unit ball in* $\mathbb{R}^n$ *associated with the dual norm* $\|\cdot\|_*$.

Proof. Assume that NC is satisfied at $(\overline{p}, \overline{x}) \in \operatorname{gph}\mathcal{S}$. Observe that the NC assumption at $(\overline{p}, \overline{x})$ easily entails $\mathcal{S}(\overline{p}) = \{\overline{x}\}$. Then, taking SCQ into account and applying [19, Thms. 6.1 and 10.4] we have that $\mathcal{S}$ is both lower and upper semicontinuous in the sense of Berge at $\overline{p}$. This entails that, given any $p^r \to \overline{p}$, we have $\mathcal{S}(p^r) \neq \emptyset$ for r large enough (without loss of generality, w.l.o.g. in brief, for all r), and for any sequence $\{z^r\}$ with $z^r \in \mathcal{S}(p^r)$ we have that $\{z^r\}$ converges to $\overline{x}$.

Reasoning by contradiction, assume the existence of a sequence $\{p^r\}$ converging to $\overline{p}$, with $\|p^r - \overline{p}\| < \overline{\delta}$ for all r, and such that (i) fails at all p^r. More specifically, assume the existence of $\{x^r\}$ such that $x^r \in \mathcal{S}(p^r)$ and NC fails at (p^r, x^r), for all $r = 1, 2, \ldots$ According to the previous comment $\{x^r\}$ converges to $\overline{x}$. Moreover, it is well-known that SCQ is a stable property (see, e.g., [19, Thm. 6.1]) and, so, SCQ holds at (a^r, b^r) for r large enough (w.l.o.g. for all r). Then, the unfulfillment of NC at (p^r, x^r) yields

$$-c^r \in \operatorname{cone}\{a_t^r,\ t \in D^r\},\ \text{for some } D^r \subset T_{a^r, b^r}(x^r),\ \text{with } |D^r| < n.$$

The finiteness of T allows us to assume (taking an appropriate subsequence if necessary) that $\{D^r\}$ is constant; i.e., $D^r = D$ for all r. One immediately has that $D \subset T_{\overline{a}, \overline{b}}(\overline{x})$. Now, let us write

$$-c^r = \sum_{t \in D} \lambda_t^r a_t^r, \tag{3.2}$$

for some scalars $\lambda_t^r \geq 0$, for all $t \in D$ and $r = 1, 2, \ldots$ Observe that the sequence $\left\{\sum_{t \in D} \lambda_t^r\right\}_r$ must be bounded since otherwise, dividing both sides of (3.2) by $\gamma_r := \sum_{t \in D} \lambda_t^r$ and letting $r \to +\infty$, we would attain the contradiction with SCQ (see, e.g. [19, Thm. 6.1])

$$0_n \in \operatorname{conv}\{\overline{a}_t,\ t \in D\}.$$

Finally, since $\{c^r\}$ converges to $\overline{c}$, $\{a_t^r\}_r$ converges to $\overline{a}_t$, for each $t \in D$, and we may assume (again by taking a subsequence if necessary) that $\{\lambda_t^r\}_r$ converges to some $\lambda_t \geq 0$ for each $t \in D$, we conclude

$$-\overline{c} = \sum_{t \in D} \lambda_t \overline{a}_t,$$

which contradicts NC at $(\overline{p}, \overline{x})$.

Thus we have proved the existence of $0 < \varepsilon_1 < \overline{\delta}$ verifying that condition (i) holds at all p such that $\|p - \overline{p}\| < \varepsilon_1$. Now, let us establish the existence of $0 < \varepsilon_2 < \varepsilon_1$ such that condition (ii) is satisfied at all p such that $\|p - \overline{p}\| < \varepsilon_2$. Again, reasoning by contradiction, we assume the existence of a sequence $\{p^r\}$ converging to $\overline{p}$ such that $\|p^r - \overline{p}\| < \varepsilon_1$ and $\mathcal{T}_{p^r}(x(p^r)) \not\subset \mathcal{T}_{\overline{p}}(\overline{x})$, $r = 1, 2, \ldots$ So, for each r, let $D^r \in \mathcal{T}_{p^r}(x(p^r)) \setminus \mathcal{T}_{\overline{p}}(\overline{x})$. Again we may assume that $\{D^r\}$ is constant, say $D^r = D$, for all r (obviously $|D| = n$). Since $\{p^r\}$ converges to $\overline{p}$ and $\{x(p^r)\}$ converges to $\overline{x}$ (see the proof of condition(i)), we have that $D \subset T_{\overline{a},\overline{b}}(\overline{x})$. So, $D \notin \mathcal{T}_{\overline{p}}(\overline{x})$ implies that $-\overline{c} \notin \operatorname{cone}\{\overline{a}_t,\ t \in D\}$. Write again $-c^r$ as in (3.2) for some scalars $\lambda_t^r \geq 0$, for all $t \in D$ and $r = 1, 2, \ldots$ Again the sequence $\left\{\sum_{t \in D} \lambda_t^r\right\}_r$ must be bounded since otherwise we come to a contradiction with SCQ at $\overline{p}$; then letting $r \to +\infty$, we attain the contradiction

$$-\overline{c} = \sum_{t \in D} \lambda_t \overline{a}_t$$

for certain $\lambda_t \geq 0$, $t \in D$.

Fix now ε_2 such that $0 < \varepsilon_2 < \varepsilon_1$ and condition (ii) is satisfied whenever $\|p - \overline{p}\| < \varepsilon_2$.

Finally, let us prove the existence of $0 < \varepsilon < \varepsilon_2$ such that

$$\left(\overline{\delta} - \varepsilon\right) B_* \subset \operatorname{conv}\{\pm a_t,\ t \in D\} \text{ for all } D \in \mathcal{T}_p(x(p)), \text{ if } \|p - \overline{p}\| < \varepsilon. \tag{3.3}$$

NC at $(\overline{p}, \overline{x})$ implies that, for each $D \in \mathcal{T}_{\overline{p}}(\overline{x})$, $\delta_D := d_*(0_n, \operatorname{bd}\operatorname{conv}\{\pm\overline{a}_t, t \in D\}) > 0$ (see the Remark 3.2). Then, since $0_n \in \operatorname{conv}\{\pm\overline{a}_t,\ t \in D\}$, it is clear that $0_n \in \operatorname{int}\operatorname{conv}\{\pm\overline{a}_t,\ t \in D\}$. Consequently, it is immediate that

$$\delta_D B_* \subset \operatorname{conv}\{\pm\overline{a}_t,\ t \in D\}, \text{ for } D \in \mathcal{T}_{\overline{p}}(\overline{x}).$$

Then, since $\overline{\delta} = \min\{\delta_D,\ D \in \mathcal{T}_{\overline{p}}(\overline{x})\}$, we have

$$\overline{\delta} B_* \subset \operatorname{conv}\{\pm\overline{a}_t,\ t \in D\}, \text{ for all } D \in \mathcal{T}_{\overline{p}}(\overline{x}).$$

Take any $0 < \varepsilon < \varepsilon_2$ $(< \overline{\delta})$ and let $p = (c, a, b)$ be such that $\|p - \overline{p}\| < \varepsilon$ and let $D \in \mathcal{T}_p(x(p))$. By condition (ii), $D \in \mathcal{T}_{\overline{p}}(\overline{x})$. In our choice of norms, $\|p - \overline{p}\| < \varepsilon$, implies $\|a_t - \overline{a}_t\|_* < \varepsilon$, for all $t \in D$. Then, applying Lemma 3.1 we have

$$\left(\overline{\delta} - \varepsilon\right) B_* \subset \operatorname{conv}\{\pm a_t,\ t \in D\}, \tag{3.4}$$

as we aimed to prove. □

Condition (ii) in the following theorem constitutes a key result in the computation of upper bounds for the Lipschitz modulus of $\mathcal{S}$ (by appealing to Lemma 2.3) under NC. Roughly speaking, it provides ‘uniform’ calmness constants for $\mathcal{S}$, around $\overline{p}$.

Theorem 3.4. *Assume that NC holds at $(\overline{p}, \overline{x}) \in \operatorname{gph}\mathcal{S}$. Then,*

(i) $\mathcal{S}$ is single valued and continuous on some neighborhood of $\overline{p}$.

(ii) For any $0 < \alpha < \overline{\delta}$, there exists $\varepsilon > 0$ such that

$$\mathrm{clm}\mathcal{S}\left(p, x\left(p\right)\right) < \frac{\left\Vert \overline{x}\right\Vert + 1 + \alpha}{\overline{\delta} - \alpha}, \tag{3.5}$$

whenever $\left\Vert p - \overline{p}\right\Vert < \varepsilon$, with $x\left(p\right)$ and $\overline{\delta}$ being defined in Lemma 3.3(i) and (3.1), respectively.

Proof. (i) Let us take $0 < \varepsilon < \overline{\delta}$ verifying all conditions of Lemma 3.3. In particular, we have that

$$\mathcal{S}\left(p\right) = \left\{x\left(p\right)\right\} \text{ and } NC \text{ holds at } \left(p, x\left(p\right)\right), \text{ whenever } \left\Vert p - \overline{p}\right\Vert < \varepsilon,$$

and in particular SCQ is satisfied at $\left(a, b\right)$. Then, $\left\Vert p - \overline{p}\right\Vert < \varepsilon$ implies that $\mathcal{S}$ is *B-lsc* at p (see [19, Thm. 10.4 (ii)]), which entails, together with the fact that $\mathcal{S}\left(p\right)$ is a singleton, the continuity of $\mathcal{S}$ at p.

(ii) Let $0 < \varepsilon_0 < \overline{\delta}$ verifying Lemma 3.3 and, according to condition (i), such that

$$\left\Vert x\left(p\right)\right\Vert < \left\Vert \overline{x}\right\Vert + \alpha, \text{ whenever } \left\Vert p - \overline{p}\right\Vert < \varepsilon_0. \tag{3.6}$$

Consider any p verifying $\left\Vert p - \overline{p}\right\Vert < \varepsilon_0$. Applying Theorem 2.9 (ii), we have:

$$\mathrm{clm}\mathcal{S}\left(p, x\left(p\right)\right) \leq \max_{D \in \mathcal{T}_p(x(p))} \frac{\left\Vert x\left(p\right)\right\Vert + 1}{d_*\left(0_n, \mathrm{bd}\,\mathrm{conv}\left\{a_t,\ t \in T_{a,b}\left(x\left(p\right)\right); -a_t,\ t \in D\right\}\right)};$$

(recall that $\mathcal{T}_p\left(x\right) = \mathcal{K}_p\left(x\right)$ under NC). Since (again by NC) we have

$$0_n \in \mathrm{int}\,\mathrm{conv}\left\{\pm a_t,\ t \in D\right\} \subset \mathrm{int}\,\mathrm{conv}\left\{a_t,\ t \in T_{a,b}\left(x\right); -a_t,\ t \in D\right\},$$

then

$$\mathrm{clm}\mathcal{S}\left(p, x\left(p\right)\right) \leq \max_{D \in \mathcal{T}_p(x(p))} \frac{\left\Vert x\left(p\right)\right\Vert + 1}{d_*\left(0_n, \mathrm{bd}\,\mathrm{conv}\left\{\pm a_t,\ t \in D\right\}\right)}. \tag{3.7}$$

Moreover, applying condition (iii) in Lemma 3.3, we have that

$$\min_{D \in \mathcal{T}_p(x(p))} d_*\left(0_n, \mathrm{bd}\,\mathrm{conv}\left\{\pm a_t,\ t \in D\right\}\right) \geq \overline{\delta} - \alpha > 0, \text{ if } \left\Vert p - \overline{p}\right\Vert < \alpha. \tag{3.8}$$

Consequently, from (3.6), (3.7) and (3.8) we have that

$$\mathrm{clm}\mathcal{S}\left(p, x\left(p\right)\right) < \frac{\left\Vert \overline{x}\right\Vert + 1 + \alpha}{\overline{\delta} - \alpha},$$

whenever $\left\Vert p - \overline{p}\right\Vert < \varepsilon := \min\{\varepsilon_0, \alpha\}$. □

Now, by gathering all previous results, we establish the aimed characterization result. In it, $B\left(\overline{p}, \varepsilon\right)$ represents the open ball centered at $\overline{p}$ and with radius ε (associated with the metric given by (2.2) and (2.3)).

Theorem 3.5. *Let $\left(\overline{p}, \overline{x}\right) \in \mathrm{gph}\mathcal{S}$. The following conditions are equivalent:*

(i) $\mathcal{S}$ has the Aubin property at $\left(\overline{p}, \overline{x}\right)$;

(ii) $\mathcal{S}$ is strongly Lipschitz stable at $\left(\overline{p}, \overline{x}\right)$;

(iii) $\mathcal{S}_{\overline{a}}$ has the Aubin property at $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right)$;

(iv) NC holds at $\left(\overline{p}, \overline{x}\right)$;

(v) For any $0 < \alpha < \overline{\delta}$, there exists $\varepsilon > 0$ such that $\mathcal{S}$ is H-lsc *on $B(\overline{p}, \varepsilon)$ and locally upper Lipschitz continuous with modulo*

$$\lambda_\alpha := \frac{\|\overline{x}\| + 1 + \alpha}{\overline{\delta} - \alpha}$$

on $B(\overline{p}, \varepsilon)$;

(vi) For any $\alpha > 0$, there exists $\varepsilon > 0$ such that $\mathcal{S}$ is λ_α-Lipschitz continuous on $B(\overline{p}, \varepsilon)$, where λ_α is defined as in the previous condition.

Proof. $(i) \Rightarrow (ii)$ comes from [24, Cor. 4.7]; $(ii) \Rightarrow (iii)$ is a trivial consequence of the definitions; $(iii) \Leftrightarrow (iv)$ was already established in Theorem 2.6.

$(iv) \Rightarrow (v)$. Let $0 < \alpha < \overline{\delta}$. By applying the previous theorem, we find $\varepsilon > 0$ such that $\mathcal{S}$ is single valued and continuous on $B(\overline{p}, \varepsilon)$, which entails that $\mathcal{S}$ is *H-lsc* on $B(\overline{p}, \varepsilon)$, and

$$\mathrm{clm}\mathcal{S}(p, x(p)) < \lambda_\alpha, \text{ if } \|p - \overline{p}\| < \varepsilon.$$

The definition of calmness modulus yields, for each $p \in B(\overline{p}, \varepsilon)$, the existence of a neighborhood $V_p \times W_{x(p)}$ of $(p, x(p))$ (we assume that $V_p \subset B(\overline{p}, \varepsilon)$) such that

$$d(x(p'), x(p)) \leq \lambda_\alpha d(p', p), \text{ whenever } p' \in V_p,\ x(p') \in W_{x(p)}. \tag{3.9}$$

In other words, $\mathcal{S}$ is calm at any $(p, x(p))$, $p \in B(\overline{p}, \varepsilon)$, with the same calmness constant λ_α. In order to establish that $\mathcal{S}$ is locally upper Lipschitz continuous with modulo λ_α on $B(\overline{p}, \varepsilon)$, we should be able to remove condition '$x(p') \in W_{x(p)}$' in (3.9). In fact, due to the continuity of $\mathcal{S}$ on $B(\overline{p}, \varepsilon)$, for each $p \in B(\overline{p}, \varepsilon)$, we can take $\widetilde{V}_p \subset V_p$ such that $x(p') \in W_{x(p)}$ for all $p' \in \widetilde{V}_p$. Consequently, we have

$$d(x(p'), x(p)) \leq \lambda_\alpha d(p', p), \text{ whenever } p' \in \widetilde{V}_p.$$

Finally, $(v) \Rightarrow (vi)$ comes from Lemma 2.3 and $(vi) \Rightarrow (i)$ is valid in general (see Remark 2.2). □

4 Lipschitz Modulus

The following theorem constitutes, together with Theorem 3.5, the main contribution of the paper. The proof is quite technical, and the subsequent Example 4.2 is intended to illustrate the underlying geometrical ideas. Recalling Remark 2.7, we write $\mathrm{lip}\mathcal{S}(\overline{p})$ instead of $\mathrm{lip}\mathcal{S}(\overline{p}, \overline{x})$.

Theorem 4.1. *If $\mathcal{S}$ has the Aubin property at $(\overline{p}, \overline{x}) \in \mathrm{gph}\mathcal{S}$, then,*

$$\begin{aligned} \mathrm{lip}\mathcal{S}(\overline{p}) &= (\|\overline{x}\| + 1)\, \mathrm{lip}\mathcal{S}_{\overline{a}}\left(\overline{c}, \overline{b}\right) \\ &= \max_{D \in \mathcal{T}_{\overline{p}}(\overline{x})} \frac{\|\overline{x}\| + 1}{d_*\left(0_n, \mathrm{bd}\,\mathrm{conv}\left\{\pm \overline{a}_t, t \in D\right\}\right)}. \end{aligned}$$

Proof. Observe that the last equality comes from Theorem 2.8. For the sake of simplicity, we keep the notation (3.1); i.e., we are going to establish

$$\mathrm{lip}\mathcal{S}(\overline{p}) = \frac{\|\overline{x}\| + 1}{\overline{\delta}}.$$

The '$\leq$' inequality comes from Theorem 3.5 (vi). Specifically, for any $0 < \alpha < \overline{\delta}$ we can find $\varepsilon > 0$, assumed to be sufficiently small to guarantee that $\mathcal{S}(p) = \{x(p)\}$ whenever $\|p - \overline{p}\| < \varepsilon$, such that

$$d(x(p), x(p')) \leq \lambda_\alpha d(p, p'), \text{ for all } p, p' \in B(\overline{p}, \varepsilon).$$

Then, by definition we obtain our aimed inequality

$$\mathrm{lip}\mathcal{S}(\overline{p}) \leq \liminf_{\alpha \to 0} \lambda_\alpha = \frac{\|\overline{x}\| + 1}{\overline{\delta}}.$$

In order to establish the '$\geq$' inequality, we are going to prove the existence of two sequences of parameters $\{p^r\}$ and $\{\widetilde{p}^r\}$ converging to $\overline{p}$, such that

$$\lim_r \frac{\|x(p^r) - x(\widetilde{p}^r)\|}{\|p^r - \widetilde{p}^r\|} \geq \frac{\|\overline{x}\| + 1}{\overline{\delta}}, \tag{4.1}$$

where $\{x(p^r)\} = \mathcal{S}(p^r)$ and $\{x(\widetilde{p}^r)\} = \mathcal{S}(\widetilde{p}^r)$, which immediately entails

$$\mathrm{lip}\mathcal{S}(\overline{p}) \geq \frac{\|\overline{x}\| + 1}{\overline{\delta}}.$$

Take a $D \in \mathcal{T}_{\overline{p}}(\overline{x})$ with

$$\overline{\delta} = d_*(0_n, \mathrm{bd}\,\mathrm{conv}\{\pm\overline{a}_t,\ t \in D\}),$$

i.e., where the minimum in (3.1) is attained. Consider the index set $T_D := (\{1\} \times D) \cup (\{2\} \times D)$ and the associated multifunction $\mathcal{L}_D : (\mathbb{R}^n)^{T_D} \times (\mathbb{R})^{T_D} \rightrightarrows \mathbb{R}^n$, given by

$$\mathcal{L}_D(u, v) := \left\{x \in \mathbb{R}^n \mid (u_{1,t})' x \leq v_{1,t},\ t \in D;\ \ (u_{2,t})' x \leq v_{2,t},\ t \in D\right\}.$$

Let $(\overline{u}, \overline{v}) \in (\mathbb{R}^n)^{T_D} \times (\mathbb{R})^{T_D}$ be defined by

$$\begin{cases} (\overline{u}_{1,t}, \overline{v}_{1,t}) := \left(\overline{a}_t, \overline{b}_t\right),\ t \in D, \\ (\overline{u}_{2,t}, \overline{v}_{2,t}) := -\left(\overline{a}_t, \overline{b}_t\right),\ t \in D. \end{cases}$$

Note that, by the definition of $\mathcal{T}_{\overline{p}}(\overline{x})$, we have

$$\{\overline{x}\} = \mathcal{S}(\overline{p}) \subset \mathcal{L}_D(\overline{u}, \overline{v}) = \left\{x \in \mathbb{R}^n \mid \overline{a}_t' x \leq \overline{b}_t,\ t \in D;\ -\overline{a}_t' x \leq -\overline{b}_t,\ t \in D\right\}.$$

Since, under NC (see Theorem 3.5), $\{\overline{a}_t,\ t \in D\}$ is a basis of $\mathbb{R}^n$, $\mathcal{L}_D(\overline{u}, \overline{v})$ is a singleton. So, indeed we have $\mathcal{L}_D(\overline{u}, \overline{v}) = \{\overline{x}\}$. In this way, $\mathcal{L}_D$ is nothing else but a feasible set mapping with uniqueness of feasible solution at $(\overline{u}, \overline{v})$. Then, applying Proposition 2.4, we have that

$$\mathrm{clm}\mathcal{L}_D((\overline{u}, \overline{v}), \overline{x}) = \frac{\|\overline{x}\| + 1}{\overline{\delta}}. \tag{4.2}$$

Here $(\mathbb{R}^n)^{T_D} \times (\mathbb{R})^{T_D}$ is assumed to be endowed with the supremum norm

$$\|(u, v)\| := \max_{d \in T_D} \left\| \begin{pmatrix} u_d \\ v_d \end{pmatrix} \right\|,$$

where $\left\|\begin{pmatrix} u_d \\ v_d \end{pmatrix}\right\|$ is considered as in (2.3). Recalling the equivalence between the calmness of a multifunction and metric subregularity of its inverse, (4.2) entails the existence of a sequence $\{x^r\} \subset \mathbb{R}^n$ converging to $\overline{x}$ such that

$$\lim_r \frac{\|x^r - \overline{x}\|}{d\left((\overline{u}, \overline{v}), \mathcal{L}_D^{-1}(x^r)\right)} = \frac{\|\overline{x}\| + 1}{\overline{\delta}}. \tag{4.3}$$

Applying [13, Lem. 10] we can write

$$\begin{aligned} d\left((\overline{u}, \overline{v}), \mathcal{L}_D^{-1}(x^r)\right) &= \frac{\max\{[\overline{a}_t' x^r - \overline{b}_t]_+; [-(\overline{a}_t' x^r - \overline{b}_t)]_+,\ t \in D\}}{\|x^r\| + 1} \\ &= \frac{\max_{t\in D} \left|\overline{a}_t' x^r - \overline{b}_t\right|}{\|x^r\| + 1}, \end{aligned} \tag{4.4}$$

where $[\alpha]_+ := \max\{\alpha, 0\}$ denotes the positive part of $\alpha \in \mathbb{R}$.

Now, we define two sequences $\{(a^r, b^r)\}, \{(\widetilde{a}^r, \widetilde{b}^r)\} \subset (\mathbb{R}^n)^T \times (\mathbb{R})^T$ in the following form:

$$\begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} := \begin{cases} \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix} - \dfrac{\overline{a}_t' x^r - \overline{b}_t}{\|x^r\| + 1} \begin{pmatrix} u^r \\ -1 \end{pmatrix}, & t \in D, \\ \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix} + [\overline{a}_t' x^r - \overline{b}_t]_+ \begin{pmatrix} 0_n \\ 1 \end{pmatrix}, & t \in T \diagdown D, \end{cases}$$

where $u^r \in \mathbb{R}^n$ is such that $\|u^r\|_* = 1$ and $(u^r)' x^r = \|x^r\|$;

$$\begin{pmatrix} \widetilde{a}_t^r \\ \widetilde{b}_t^r \end{pmatrix} := \begin{cases} \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix}, & t \in D, \\ \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix}, & t \in T \diagdown D. \end{cases}$$

For each $r \in \mathbb{N}$, define

$$p^r := (\overline{c}, a^r, b^r) \text{ and } \widetilde{p}^r := \left(\overline{c}, \widetilde{a}^r, \widetilde{b}^r\right).$$

Let us show that $\{p^r\}$ and $\{\widetilde{p}^r\}$ verify our aimed statement (4.1). First, from the definition one easily checks that $\{p^r\}$ and $\{\widetilde{p}^r\}$ converge to $\overline{p}$ (since $x^r \to \overline{x}$). Then, the Aubin property assumption for $\mathcal{S}$ at $(\overline{p}, \overline{x})$ together with Lemma 3.3(i) yield the uniqueness of optimal solution at p^r and $\widetilde{p}^r$ for r large enough (w.l.o.g., for all r). Let us check that

$$\mathcal{S}(p^r) = \{x^r\} \text{ and } \mathcal{S}(\widetilde{p}^r) = \{\overline{x}\}, \text{ for all } r. \tag{4.5}$$

The uniqueness condition yields that it is sufficient to prove that $\mathcal{S}(p^r) \ni x^r$ and $\mathcal{S}(\widetilde{p}^r) \ni \overline{x}$ for all r. For each $t \in D$ we have

$$\begin{aligned} \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix}' \begin{pmatrix} x^r \\ -1 \end{pmatrix} &= \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix}' \begin{pmatrix} x^r \\ -1 \end{pmatrix} - \frac{\overline{a}_t' x^r - \overline{b}_t}{\|x^r\| + 1} \begin{pmatrix} u^r \\ -1 \end{pmatrix}' \begin{pmatrix} x^r \\ -1 \end{pmatrix} \\ &= \overline{a}_t' x^r - \overline{b}_t - \frac{\overline{a}_t' x^r - \overline{b}_t}{\|x^r\| + 1} (\|x^r\| + 1) = 0, \end{aligned} \tag{4.6}$$

and

$$\begin{pmatrix} \widetilde{a}_t^r \\ \widetilde{b}_t^r \end{pmatrix}' \begin{pmatrix} \overline{x} \\ -1 \end{pmatrix} = \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix}' \begin{pmatrix} \overline{x} \\ -1 \end{pmatrix} = 0. \tag{4.7}$$

If $t \in T \diagdown D$ we have

$$\begin{aligned} \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix}' \begin{pmatrix} x^r \\ -1 \end{pmatrix} &= \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix}' \begin{pmatrix} x^r \\ -1 \end{pmatrix} + [\overline{a}_t' x^r - \overline{b}_t]_+ \begin{pmatrix} 0_n \\ 1 \end{pmatrix}' \begin{pmatrix} x^r \\ -1 \end{pmatrix} \\ &= \overline{a}_t' x^r - \overline{b}_t - [\overline{a}_t' x^r - \overline{b}_t]_+ \leq 0, \end{aligned}$$

and

$$\begin{aligned} \begin{pmatrix} \widetilde{a}_t^r \\ \widetilde{b}_t^r \end{pmatrix}' \begin{pmatrix} \overline{x} \\ -1 \end{pmatrix} &= \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix}' \begin{pmatrix} \overline{x} \\ -1 \end{pmatrix} = \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix}' \begin{pmatrix} \overline{x} \\ -1 \end{pmatrix} + [\overline{a}_t' x^r - \overline{b}_t]_+ \begin{pmatrix} 0_n \\ 1 \end{pmatrix}' \begin{pmatrix} \overline{x} \\ -1 \end{pmatrix} \\ &= -[\overline{a}_t' x^r - \overline{b}_t]_+ \leq 0. \end{aligned}$$

So,

$$x^r \in \mathcal{F}(a^r, b^r), \ \overline{x} \in \mathcal{F}(\widetilde{a}^r, \widetilde{b}^r), \text{ for all } r.$$

Now, let us see that x^r and $\overline{x}$ satisfy the KKT conditions for p^r and $\widetilde{p}^r$, respectively. From (4.6) and (4.7) we have that

$$D \subset T_{a^r, b^r}(x^r) \cap T_{\widetilde{a}^r, \widetilde{b}^r}(\overline{x}), \text{ for all } r.$$

Moreover, since both $\{(a^r, b^r)\}$ and $\{(\widetilde{a}^r, \widetilde{b}^r)\}$ converge to $(\overline{a}, \overline{b})$ and $-\overline{c} \in \operatorname{int}\operatorname{cone}\{\overline{a}_t, t \in D\}$ (because of NC; see Theorem 3.5), appealing e.g. to [19, Exercise 6.12] we obtain

$$-\overline{c} \in \operatorname{int}\operatorname{cone}\{(a_t^r, t \in D)\} \cap \operatorname{int}\operatorname{cone}\{(\widetilde{a}_t^r, t \in D)\}, \text{ for } r \text{ large enough}$$

(we may assume for all r). In this way, we have proved (4.5).

Finally, observe that, from the corresponding definitions, we have

$$\begin{aligned} \|p^r - \widetilde{p}^r\| &= \max_{t \in T} \left\| \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} - \begin{pmatrix} \widetilde{a}_t^r \\ \widetilde{b}_t^r \end{pmatrix} \right\| \\ &= \max_{t \in D} \frac{|\overline{a}_t' x^r - \overline{b}_t|}{\|x^r\| + 1} \left\| \begin{pmatrix} u^r \\ -1 \end{pmatrix} \right\| = \max_{t \in D} \frac{|\overline{a}_t' x^r - \overline{b}_t|}{\|x^r\| + 1}. \end{aligned} \tag{4.8}$$

Consequently,

$$\lim_r \frac{\|x(p^r) - x(\widetilde{p}^r)\|}{\|p^r - \widetilde{p}^r\|} = \lim_r \frac{\|x^r - \overline{x}\|}{\|p^r - \widetilde{p}^r\|} = \frac{\|\overline{x}\| + 1}{\overline{\delta}},$$

where for the last equality we have applied (4.3), (4.4), and (4.8). □

We finish the paper with an example illustrating the ingredients p^r, $\widetilde{p}^r$, and $\widetilde{x}^r$ in the proof of Theorem 4.1. This example goes back to [11, Exa. 2], which was revisited in [7, Exa. 3.1]. As the only difference with those examples, our current one considers a nonzero optimal solution, what entails a nonzero nominal right hand side $\overline{b}$.

Example 4.2. *Consider the linear optimization problem $P\left(\overline{c}, \overline{a}, \overline{b}\right)$ in $\mathbb{R}^2$*

$$\begin{array}{lll} \textit{minimize} & x_1 + \frac{1}{3}x_2 & \\ \textit{subject to} & -x_1 \leq -2, & (t=1) \\ & -x_1 - \frac{1}{2}x_2 \leq -\frac{5}{2}, & (t=2) \\ & -x_1 - x_2 \leq -3, & (t=3) \\ & -x_1 + x_2 \leq -1, & (t=4) \end{array}$$

whose unique optimal solution is $\bar{x} = \binom{2}{1}$, and where

$$\mathcal{K}_{\overline{p}}(\bar{x}) = \mathcal{T}_{\overline{p}}(\bar{x}) = \{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}.$$

First we comment some geometrical ideas for canonical perturbations and later we will consider full perturbations, because the choice of indices where perturbations are performed is the same. For simplicity, computations are referred to the Euclidean norm in $\mathbb{R}^2$. The reader can easily check that the corresponding maximum over $D \in \mathcal{K}_{\overline{p}}(\bar{x})$ providing $\mathrm{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)$ in Theorem 2.9(i) is attained at both $D = \{1,2\}$ and $D = \{1,3\}$, and therefore $\mathrm{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c},\overline{b}\right),\overline{x}\right) = \sqrt{5}$ (see [7, Exa. 3.1]). Indeed, [11, Exa. 2] provides an *ad hoc* argument which could be reproduced here for the current $\overline{b}$ in order to show that, for any given $\varepsilon > 0$, the furthest (with respect to $\overline{x}$) possible optimal solution $x \in \mathcal{S}_{\overline{a}}(\overline{c}, b)$ corresponding to some $b \in \mathbb{R}^T$ with $\left\|b-\overline{b}\right\|_\infty \leq \varepsilon$ is $\widehat{x}^\varepsilon := \binom{2+\varepsilon}{1-2\varepsilon}$. Nevertheless, in this example

$$\mathrm{lip}\mathcal{S}_{\overline{a}}\left(\left(\overline{c},\overline{b}\right),\overline{x}\right) = \sqrt{17} = \left\|A_{\{1,2\}}^{-1}\right\|.$$

More in detail (see Theorem 2.8), $d_*\left(0_2, \mathrm{bd}\,\mathrm{conv}\left\{\pm\overline{a}_t, t \in \{1,2\}\right\}\right)$ is attained, for instance, at $\binom{1/17}{-4/17} \in \mathrm{conv}\left\{-\overline{a}_1, +\overline{a}_2\right\}$, which suggests the idea of, on the one hand, reducing the feasible set by replacing $\overline{b}_1$ with $\overline{b}_1 - \varepsilon$ and, on the other hand, enlarging the feasible set by replacing $\overline{b}_2$ with $\overline{b}_2 + \varepsilon$, leading to $x^\varepsilon := \binom{2+\varepsilon}{1-4\varepsilon}$ as the only solution of $\left\{\overline{a}_1'x = \overline{b}_1 - \varepsilon,\ \overline{a}_2'x = \overline{b}_2 + \varepsilon\right\}$. The problem (if we are thinking in terms of calmness) is that x^ε is not optimal (even not feasible) for any $(\overline{c}, \overline{a}, b)$ corresponding to some $b \in \mathbb{R}^T$ with $\left\|b-\overline{b}\right\|_\infty \leq \varepsilon$; in fact, for making x^ε feasible we have to replace $\overline{b}_3$ with $\overline{b}_3 + 3\varepsilon$. This is not a problem in terms of the Aubin property, since we can consider a ratio $\left\|x^\varepsilon - \overline{x}\right\| / \left\|b^\varepsilon - \widetilde{b}^\varepsilon\right\|_\infty = \sqrt{17}$ with $b^\varepsilon := \overline{b} + (-\varepsilon, \varepsilon, 3\varepsilon, 0)'$ and $\widetilde{b}^\varepsilon := \overline{b} + (0, 0, 3\varepsilon, 0)'$, keeping fixed $\overline{c}$ and $\overline{a}$. Of course, we can replace ε with a sequence $\varepsilon_r \downarrow 0$ as $r \to \infty$.

Now, let us consider full perturbations in the current example. Perturbations of $\overline{c}$ are negligible in our analysis because of NC. Given the inequality $\overline{a}_t'\overline{x} \leq \overline{b}_t$, the way of perturbing $\left(\overline{a}_t, \overline{b}_t\right)$ with perturbation size, using our norm (2.3), less than or equal to ε in order to get the maximum slack increase and the maximum decrease is, respectively $\left(\overline{a}_t \mp \varepsilon\overline{u}\right)'\overline{x} \leq \overline{b}_t \pm \varepsilon$, where $\overline{u} \in \mathbb{R}^n$ is any vector satisfying $\left\|\overline{u}\right\|_* = 1$ and $\overline{u}'\overline{x} = \left\|\overline{x}\right\|$. In the particular case when $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$, we have $\overline{u} = \overline{x}/\left\|\overline{x}\right\|$. With this idea in mind, taking into account the notation of the previous paragraph, we have two options. The first one is to define z^ε as the solution of system

$$\left\{\left(\overline{a}_1 + \varepsilon\overline{x}/\left\|\overline{x}\right\|\right)'x = \overline{b}_1 - \varepsilon,\ \left(\overline{a}_2 - \varepsilon\overline{x}/\left\|\overline{x}\right\|\right)'x = \overline{b}_2 + \varepsilon\right\},$$

i.e., $z^\varepsilon := \left(\sqrt{5} + 2\varepsilon\right)^{-1}\left(2\sqrt{5} + \left(9 + \sqrt{5}\right)\varepsilon, \sqrt{5} - 2\left(9 + 2\sqrt{5}\right)\varepsilon\right)'$. Write as a_1^ε and a_2^ε the left-hand-side coefficients above, while $\overline{a}_3$ and $\overline{a}_4$ remain unchanged. In this way $z^\varepsilon \in \mathcal{S}\left(\overline{c}, a^\varepsilon, b^\varepsilon\right)$, $\overline{x} \in \mathcal{S}\left(\overline{c}, \overline{a}, \widetilde{b}^\varepsilon\right)$, and

$$\lim_{\varepsilon \to 0^+} \frac{\left\|z^\varepsilon - \overline{x}\right\|}{\left\|\left(a^\varepsilon, b^\varepsilon\right) - \left(\overline{a}, \widetilde{b}^\varepsilon\right)\right\|_\infty} = \lim_{\varepsilon \to 0^+} \frac{\left\|z^\varepsilon - \overline{x}\right\|}{\varepsilon} = \sqrt{17}\left(1 + \sqrt{5}\right).$$

The second option is the one followed in the proof of Theorem 4.1, consisting in keeping the same x^ε defined above (motivated by canonical perturbations) and perturbing all coefficients in order to make x^ε and $\overline{x}$ optimal for the respective perturbed parameter in such a way that the distance between the perturbed parameters is minimum.

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Manuscript received 28 September 2015
revised 21 March 2016, 5 May 2016
accepted for publication 14 May 2016

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