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FINDING CONVEX CONTINUOUS FUNCTION FROM AN ACCELERATED SUBGRADIENT PROJECTOR IN EUCLIDEAN SPACE

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Abstract: Yamagishi and Yamada proposed a deep monotone approximation operator based on subgradient projector. Since the subgradient projector is considered to be very important in the field of solving convex feasibility problems, we study the relationship between the two operators. We assume that Yamagishi and Yamada's accelerated operator for a function is a subgradient projector of another convex continuous function in Euclidean Space under some conditions. We derive the necessary condition for finding out the convex continuous function. Then we try to reveal the function from the accelerated operator of a quadratic function and provide some examples of the function.

Key words: *subgradient projector, Yamagishi-Yamada operator, Hessian matrix, convexity*

Mathematics Subject Classification: *47N10,47H05,47H10,47J05*

1 Introduction

The subgradient projector is an algorithmic operator onto a lower level set of a convex function [1]. It is think of as very important from the practical point of view because the subgradient projection iteration is a classical method for solving convex feasibility problem[1, 2, 3]. A series of work have been done to improve the performance of subgradient projection iteration in the field of optimization. In [4], the authors carried out the basic theory of subgradient projector for possibly nonconvex function on a finite dimensional space. In [5], the authors studied the modified cyclic subgradient projection (MCSP) and provided the finite convergence conditions for the iteration. In [6], the author studied the relaxed iteration and got the conclusion that if there exists an interior point in the set of fixed point, the sequence generated by the relaxed iteration converges finitely to a fixed point of the operator. In [7], the authors compared a relaxed cutter method with MCSP [5], Crombez's method [6] and Polyak's method [8], and got some conditions for finite convergence of the cutter method.

Yamagishi and Yamada also provided a deep monotone approximation operator based on subgradient projector in [9]. This accelerated operator was shown that it realized better approximation than subgradient projector. In [10], we have proved that the accelerated operator is actually a subgradient projector of a variant of original continuous convex function on the real line. The aim of this paper is to make a further connection between Yamagishi-Yamada operator and subgradient projector in Euclidean space. We assume that $f(x)$ and

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$g(x)$ are convex continuous functions, $Z(x)$ is an accelerated Yamagishi-Yamada operator of $f(x)$, and $G_g(x)$ is a subgradient projector of $g(x)$, respectively. If $g(x)$ is twice differentiable and $Z(x) = G_g(x)$, then the Hessian matrix $H_f(x)$ and $\nabla f(x) \cdot (\nabla f(x))^{\mathrm{T}}$ must commute with each other. Consequently, we consider the accelerated operator of a quadratic function, and try to derive the expression of another convex continuous function.

The rest of the paper is organized as follows. In Section 2, we recall the definitions of subgradient projector and Yamagishi-Yamada's accelerated operator as well as their properties. In Section 3, we first find the necessary condition for finding a convex continuous function from the accelerated operator. Then we study the relationship between a linear transformation and the subgradient projector. Consequently, we get the expressions of the convex continuous functions from the accelerated operator of a quadratic function. Finally, in Section 4 we conclude the paper.

2 Subgradient Projector and Its Accelerated Operator

Throughout this paper, we consider a subgradient projector in a finite dimensional Euclidean Space. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuous and differentiable function. Suppose that the 0-level set of f satisfies $\mathrm{lev}_0 f \neq \varnothing$. Then the subgradient projector $G_f : \mathbb{R}^m \to \mathbb{R}^m$ defined for f can be written as [1, 10]

$$(\forall x \in \mathbb{R}^m) \quad G_f(x) = \begin{cases} x, & \text{if } f(x) \le 0, \\ x - \frac{f(x)}{\|\nabla f(x)\|^2} \nabla f(x), & \text{if } f(x) > 0, \end{cases} \tag{2.1}$$

where $\nabla f(x)$ is the gradient of f at x and $\| \cdot \|$ is the canonical norm on $\mathbb{R}^m$.

In [9], Yamagishi and Yamada make further assumptions on f: (1) f is Fréchet differentiable on $\mathbb{R}^m$ and ∇f is Lipschitz continuous with constant L, (2) f is bounded below with $\inf f(x) \ge -\rho$. And then they define a function $\theta(x)$ as (See Lemma 1 in [9])

$$(\forall x \in \mathbb{R}^m) \quad \theta(x) = \frac{\|\nabla f(x)\|^2}{2L} - \rho \le f(x). \tag{2.2}$$

Using $\theta(x)$, the authors suggest an accelerated version of subgradient projector as $Z : \mathbb{R}^m \to \mathbb{R}^m$

$$(\forall x \in \mathbb{R}^m)$$
$$Z(x) = \begin{cases} x, & \text{if } f(x) \le 0, \\ x - \frac{\nabla f(x)}{\|\nabla f(x)\|^2} f(x), & \text{if } f(x) > 0 \text{ and } \theta(x) \le 0, \\ x - \frac{\nabla f(x)}{\|\nabla f(x)\|^2}\Big(f(x) + (\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2 \Big), & \text{if } f(x) > 0 \text{ and } \theta(x) > 0. \end{cases} \tag{2.3}$$

3 From Accelerated Operator to Convex Continuous Function

In the following, we study the case that the accelerated operator $Z(x)$ of $f(x)$ equals to the subgradient projector $G_g(x)$ of another convex continuous function $g(x)$, and try to reveal $g(x)$ from the $Z(x)$.

3.1 Necessary Condition

We first provide the necessary condition for finding the convex continuous function. For simplicity, let $D = \{x \in \mathbb{R}^m \mid \theta(x) \le 0\}$.

Theorem 3.1. *Suppose $f : \mathbb{R}^m \to \mathbb{R}$ is twice differentiable and satisfies Yamagishi and Yamada's assumptions. Let $g(x)$ be a continuous, convex, twice differentiable function. If the subgradient projector $G_g = Z$. Then for $x \in \mathbb{R}^m \setminus D$, the Hessian matrix of g is*

$$H_g(x) = \frac{g(x)}{f(x) + (\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2} M(x) H_f(x). \tag{3.1}$$

where $H_f(x)$ is the Hessian matrix of $f(x)$ and

$$M(x) = \mathrm{Id} - \frac{\sqrt{\theta(x)+\rho} - \sqrt{\rho}}{L\sqrt{\theta(x)+\rho}} \frac{\nabla f(x) \cdot (\nabla f(x))^{\mathrm{T}}}{f(x) + (\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2}. \tag{3.2}$$

Furthermore, $\nabla f(x) \cdot (\nabla f(x))^T$ and $H_f(x)$ commute with each other, and $M(x)$ is positive semi-definite.

Proof. (1) We first calculate $\nabla(\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2$. According to the definition of $\theta(x)$, we get

$$\nabla(\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2 = \frac{2(\sqrt{\theta(x)+\rho} - \sqrt{\rho})}{2\sqrt{\theta(x)+\rho}} \nabla\theta(x) = \frac{\sqrt{\theta(x)+\rho} - \sqrt{\rho}}{L\sqrt{\theta(x)+\rho}} H_f^{\mathrm{T}}(x) \cdot \nabla f(x). \tag{3.3}$$

(2) We then find the expression of $H_g(x)$ for $x \in \mathbb{R}^m \setminus D$. Since $G_g(x) = Z(x)$, we have

$$\frac{x - G_g(x)}{\|x - G_g(x)\|^2} = \frac{x - Z(x)}{\|x - Z(x)\|^2}, \tag{3.4}$$

and then

$$\frac{\nabla g(x)}{g(x)} = \frac{\nabla f(x)}{f(x) + (\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2}. \tag{3.5}$$

It is obvious that

$$\frac{\nabla g(x) \cdot (\nabla g(x))^{\mathrm{T}}}{(g(x))^2} = \frac{\nabla f(x) \cdot (\nabla f(x))^{\mathrm{T}}}{\left(f(x) + (\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2\right)^2}. \tag{3.6}$$

Calculate partial derivative with respect to x both sides of Eq. (3.5), we arrive at

$$\begin{aligned} &\frac{H_g(x) \cdot g(x) - \nabla g(x) \cdot (\nabla g(x))^{\mathrm{T}}}{(g(x))^2} \\ &= \frac{H_f(x)\left(f(x) + (\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2\right) - \nabla f(x) \cdot \left(\nabla f(x) + \nabla(\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2\right)^{\mathrm{T}}}{\left(f(x) + (\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2\right)^2}. \end{aligned} \tag{3.7}$$

According to Eqs. (3.3) (3.5) and (3.6), we have

$$\frac{H_g(x)}{g(x)} = \frac{H_f(x)\left(f(x) + (\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2\right) - \nabla f(x) \cdot \left(\nabla(\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2\right)^{\mathrm{T}}}{\left(f(x) + (\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2\right)^2}$$

$$= \frac{\left(f(x) + (\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2\right) \mathrm{Id} - \frac{\sqrt{\theta(x)+\rho} - \sqrt{\rho}}{L\sqrt{\theta(x)+\rho}} \nabla f(x) \cdot (\nabla f(x))^{\mathrm{T}}}{\left(f(x) + (\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2\right)^2} H_f(x). \tag{3.8}$$

According to the definition of $M(x)$ in Eq. (3.2), we obtain the result in Eq. (3.1).

(3) Next, we show that $\nabla f(x)(\nabla f(x))^T$ and $H_f(x)$ commute with each other. According to the definition of $M(x)$, we know that $M(x)$ is symmetric. Since $H_g(x)$ is symmetric, we have $M(x)H_f(x) = H_f(x)M(x)$. Consequently, we find that $\nabla f(x)(\nabla f(x))^T$ and $H_f(x)$ must commute with each other.

(4) Finally, we prove that $M(x)$ is positive semi-definite. The eigenvalues of $\nabla f(x) \cdot (\nabla f(x))^{\mathrm{T}}$ are $\mu_1 = \|\nabla f(x)\|^2$, $\mu_2 = \mu_3 = \cdots = \mu_m = 0$, respectively. Then the eigenvalues of $M(x)$ are

$$\eta_i = 1 - \frac{\sqrt{\theta(x)+\rho} - \sqrt{\rho}}{L\sqrt{\theta(x)+\rho}} \frac{\mu_i}{f(x) + (\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2}, \tag{3.9}$$

for $i = 1, 2, \cdots, m$. Thus

$$\begin{aligned} \eta_1 &= 1 - \frac{\sqrt{\theta(x)+\rho} - \sqrt{\rho}}{L\sqrt{\theta(x)+\rho}} \frac{\|\nabla f(x)\|^2}{f(x) + (\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2} \\ &= \frac{f(x) - \theta(x)}{f(x) + (\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2} \geq 0. \end{aligned} \tag{3.10}$$

And $\eta_2 = \eta_3 = \cdots = \eta_m = 1 \geq 0$. Since all the eigenvalues of $M(x)$ are nonnegative, the symmetric matrix $M(x)$ is positive semi-definite. □

3.2 Linear Transformation $A : \mathbb{R}^n \to \mathbb{R}^m$

In this subsection, we do some calculation on x and G_f.

Theorem 3.2. *Consider a convex continuous differentiable function $f : \mathbb{R}^m \to \mathbb{R}$. Define another function $h : \mathbb{R}^n \to \mathbb{R}$ as $h(\underline{x}) = f(A\underline{x} - b)$, where $b \in \mathbb{R}^m$. Then*

$$(\forall \underline{x} \in \mathbb{R}^n) \quad G_h(\underline{x}) = \left(\mathrm{Id} - \lambda(\underline{x}) A^{\mathrm{T}} A\right) \underline{x} + \lambda(\underline{x}) A^{\mathrm{T}} \left[G_f(A\underline{x} - b) + b\right], \tag{3.11}$$

where

$$\lambda(\underline{x}) = \frac{\|\nabla f(A\underline{x} - b)\|^2}{\|A^{\mathrm{T}} \nabla f(A\underline{x} - b)\|^2}. \tag{3.12}$$

Proof. For $\underline{x} \in \{\underline{x} \in \mathbb{R}^n \mid h(\underline{x}) > 0\}$, we have

$$G_f(A\underline{x} - b) = (A\underline{x} - b) - \frac{f(A\underline{x} - b)}{\|\nabla f(A\underline{x} - b)\|^2} \nabla f(A\underline{x} - b), \tag{3.13}$$

The gradient of $h(\underline{x})$ is $\nabla h(\underline{x}) = A^{\mathrm{T}} \nabla f(A\underline{x} - b)$ and

$$G_h(\underline{x}) = \underline{x} - \frac{h(\underline{x})}{\|\nabla h(\underline{x})\|^2} \nabla h(\underline{x})$$

$$= \underline{x} - \frac{\|\nabla f(A\underline{x}-b)\|^2}{\|A^{\mathrm{T}}\nabla f(A\underline{x}-x)\|^2} A^{\mathrm{T}}[(A\underline{x}-b) - G_f(A\underline{x}-b)] \tag{3.14}$$

$$= \left(\mathrm{Id} - \frac{\|\nabla f(A\underline{x}-b)\|^2}{\|A^{\mathrm{T}}\nabla f(A\underline{x}-b)\|^2} A^{\mathrm{T}}A\right)\underline{x} + \frac{\|\nabla f(A\underline{x}-b)\|^2}{\|A^{\mathrm{T}}\nabla f(A\underline{x}-b)\|^2} A^{\mathrm{T}}\left[G_f(A\underline{x}-b)+b\right]. \tag{3.15}$$

Besides, for $\underline{x} \in \{\underline{x} \in \mathbb{R}^n \mid h(\underline{x}) \le 0\}$, we know that $G_f(A\underline{x}-b) = A\underline{x}-b$. Then the right hand side of above equation becomes $\underline{x}$. Consequently, we can get the conclusion. □

Assume that A is unitary, then $A^{\mathrm{T}}A = \mathrm{Id}$ and $\lambda(\underline{x}) = 1$. Thus we have the following Corollary.

Corollary 3.3 (See Proposition 3.7 in [4])**.** *Assume that $A : \mathbb{R}^m \to \mathbb{R}^m$ is unitary. Then*

$$G_h(\underline{x}) = A^{\mathrm{T}}\left[G_f(A\underline{x}-b)+b\right]. \tag{3.16}$$

3.3 Quadratic Function

Consider a quadratic function $f : \mathbb{R}^m \to \mathbb{R}$

$$f(x) = \|x\|^2 - \alpha \quad (\alpha > 0). \tag{3.17}$$

Let the constants L and ρ satisfy $L \ge 2$ and $\rho \ge \alpha$. Then $\theta(x) = \frac{2}{L}\|x\|^2 - \rho$.

- If $\theta(x) = f(x)$ exactly, define $g : \mathbb{R}^m \to \mathbb{R}$

  $$g(x) = \begin{cases} 0, & \|x\|^2 \le \alpha, \\ \|x\| - \sqrt{\alpha}, & \|x\|^2 > \alpha. \end{cases} \tag{3.18}$$

  It is convex but not differentiable for those x satisfies $\|x\|^2 = \alpha$. However, $Z(x) = G_g(x)$ is still valid.

- If $\theta(x) < f(x)$, define $g : \mathbb{R}^m \to \mathbb{R}$

  $$g(x) = \begin{cases} \|x\|^2 - \alpha, & \|x\|^2 \le \frac{L\rho}{2}, \\ (\frac{L\rho}{2} - \alpha)\exp\left\{q(x) - q(\sqrt{\frac{L\rho}{2}}\frac{x}{\|x\|})\right\}, & \|x\|^2 > \frac{L\rho}{2}, \end{cases} \tag{3.19}$$

  where

  $$\nabla q(x) = \frac{\nabla f(x)}{f(x) + (\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2}. \tag{3.20}$$

  The so-constructed function is convex and twice-differentiable, and satisfies $Z(x) = G_g(x)$.

Next, we provide the details.

$\boxed{3.3.1}$ Different Expressions of $q(x)$

Assume $f(x) > \theta(x)$. Let

$$a \triangleq \frac{L+2}{L}, \qquad b \triangleq \frac{\sqrt{2L\rho}}{L+2}, \qquad \Delta \triangleq \frac{L^2\rho - L(L+2)\alpha}{(L+2)^2}. \tag{3.21}$$

Then

$$f(x) + (\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2 = \|x\|^2 - \alpha + \frac{4\|x\|^2}{2L} - 2\sqrt{\frac{2\rho}{L}}\|x\| + \rho = a[(\|x\| - b)^2 + \Delta]. \tag{3.22}$$

and

$$\nabla q(x) = \frac{\nabla f(x)}{f(x) + (\sqrt{\theta(x)+\rho} - \sqrt{\rho})^2} = \frac{1}{a} \cdot \frac{2x}{(\|x\| - b)^2 + \Delta}. \tag{3.23}$$

For different Δ, we get different expressions of $q(x)$.

1. If $\Delta > 0$, then

$$q(x) = \frac{1}{a}\left[\ln\left((\|x\| - b)^2 + \Delta\right) + \frac{2b}{\sqrt{\Delta}} \arctan \frac{\|x\| - b}{\sqrt{\Delta}}\right] + C. \tag{3.24}$$

2. If $\Delta = 0$, then

$$q(x) = \frac{1}{a}\left[\ln(\|x\| - b)^2 - \frac{2b}{\|x\| - b}\right] + C. \tag{3.25}$$

3. If $\Delta < 0$, then

$$q(x) = \frac{1}{a}\left[\ln|(\|x\| - b)^2 + \Delta| + \frac{b}{\sqrt{-\Delta}} \ln\left|\frac{\|x\| - b - \sqrt{-\Delta}}{\|x\| - b + \sqrt{-\Delta}}\right|\right] + C. \tag{3.26}$$

$\boxed{3.3.2}$ Subgradient Projector and its Accelerated Operator

Next, we prove that $g(x)$ is convex and twice differentiable. Besides, it satisfies $Z(x) = G_g(x)$.

Let $x_0 \in \{x \in \mathbb{R}^m \mid \|x\|^2 = \frac{L\rho}{2}\}$. According to the expression of $q(x)$, we know that $g(x)$ is continuous and $q(\sqrt{\frac{L\rho}{2}}\frac{x}{\|x\|}) = q(x_0)$ is a constant. Consider those $x \in \{x \in \mathbb{R}^m \mid \|x\|^2 > \frac{L\rho}{2}\}$, as x approaches x_0, $\theta(x) \to 0$. According to Eq. (3.20), $\nabla q(x) \to \frac{\nabla f(x_0)}{f(x_0)}$. Consequently,

$$\nabla g(x) = \frac{f(x_0)}{e^{q(x_0)}} e^{q(x)} \nabla q(x) \to \frac{f(x_0)}{e^{q(x_0)}} e^{q(x_0)} \nabla q(x_0) = f(x_0)\nabla q(x_0) = \nabla f(x_0). \tag{3.27}$$

Thus $\nabla g(x)$ is continuous. According to Eqs. (3.1) and (3.2), $H_g(x) = M(x)H_f(x) = 2M(x)$ is positive semi-definite, and again, as x approaches x_0, $H_g(x) \to H_f(x_0)$. It follows that $g(x)$ is convex.

By simple calculation, we get the accelerate version of G_f

$$G_f(x) = \begin{cases} x, & \|x\|^2 \le \alpha, \\ \frac{x}{2\|x\|^2}(\|x\|^2 + \alpha), & \alpha < \|x\|^2 \le \frac{L\rho}{2}, \\ \frac{x}{2\|x\|^2}\left(\frac{L-2}{L}\|x\|^2 + 2\sqrt{\frac{2\rho}{L}}\|x\| + \alpha - \rho\right), & \|x\|^2 > \frac{L\rho}{2}, \end{cases} \tag{3.28}$$

and $Z(x) = G_g(x)$.

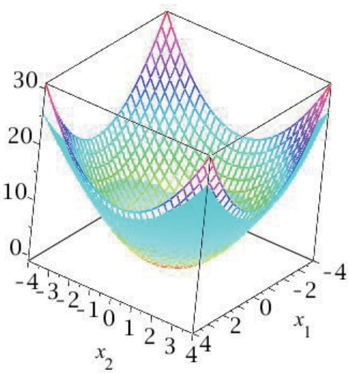

Figure 1: $f(x)$ (colored wireframe patch) and $g(x)$ (surface patch) in Eq. (3.30).

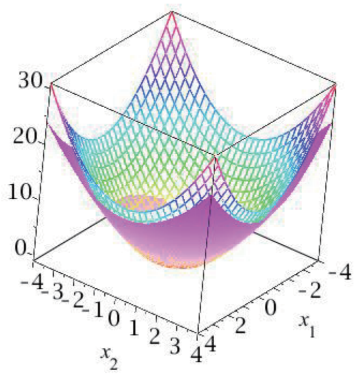

Figure 2: $f(x)$ (colored wireframe patch) and $g(x)$ (surface patch) in Eq. (3.32).

3.3.3 Further Examples

Next, we assume that $f(x) = \|x\|^2 - 1$ is defined on $\mathbb{R}^2$ and the Lipschitz constant $L = 4$. With different ρ, we provide three different examples here.

Example 3.4. Let $\rho = 2$, then $\Delta = \frac{2}{9} > 0$. Afterwards we have

$$q(x) = \frac{2}{3}\ln\left[(\|x\| - \frac{2}{3})^2 + \frac{2}{9}\right] + \frac{4\sqrt{2}}{3}\arctan\frac{3\|x\| - 2}{\sqrt{2}} + C, \tag{3.29}$$

and the function g is given by

$$g(x) = \begin{cases} \|x\|^2 - 1, & \text{if } \|x\|^2 \le 4, \\ 3\left[\frac{1}{2}\left(\|x\| - \frac{2}{3}\right)^2 + \frac{1}{9}\right]^{\frac{2}{3}} \exp\left\{\frac{4\sqrt{2}}{3}(\arctan\frac{3\|x\|-2}{\sqrt{2}} - \arctan 2\sqrt{2})\right\}, & \text{if } \|x\|^2 > 4. \end{cases} \tag{3.30}$$

Example 3.5. Let $\rho = \frac{3}{2}$, then $\Delta = 0$. Afterwards we have

$$q(x) = \frac{2}{3}\left[\ln(\|x\| - \frac{\sqrt{3}}{3})^2 - \frac{2\sqrt{3}}{3\|x\| - \sqrt{3}}\right] + C, \tag{3.31}$$

and the function g is given by

$$g(x) = \begin{cases} \|x\|^2 - 1, & \text{if } \|x\|^2 \le 3, \\ 2\left(\frac{\sqrt{3}}{2}\right)^{\frac{4}{3}}\left(\|x\| - \frac{\sqrt{3}}{3}\right)^{\frac{4}{3}}\exp\left\{\frac{2}{3}\cdot\frac{\sqrt{3}\|x\|-3}{\sqrt{3}\|x\|-1}\right\}, & \text{if } \|x\|^2 > 3. \end{cases} \tag{3.32}$$

Example 3.6. Let $\rho = \frac{5}{4}$, then $\Delta = -\frac{1}{9} < 0$. Afterwards we have

$$q(x) = \frac{2}{3}\ln\left|(\|x\| - \frac{\sqrt{10}}{6})^2 - \frac{1}{9}\right| + \frac{\sqrt{10}}{3}\ln\left|\frac{6\|x\| - \sqrt{10} - 2}{6\|x\| - \sqrt{10} + 2}\right| + C, \tag{3.33}$$

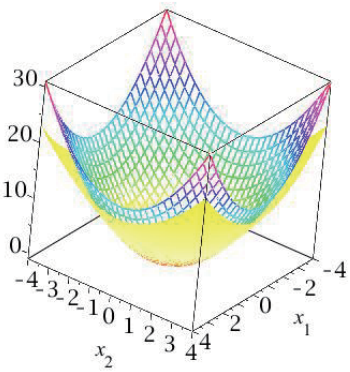

Figure 3: $f(x)$ (colored wireframe patch) and $g(x)$ (surface patch) in Eq. (3.34).

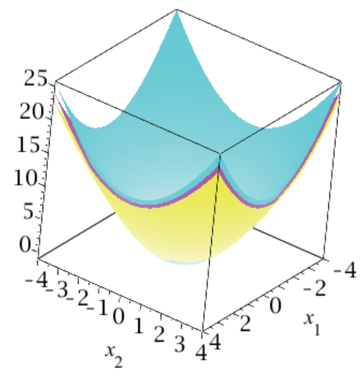

Figure 4: $g(x)$ with different ρ. Upper blue layer: Eq. (3.30), middle magenta layer: Eq. (3.32), and bottom yellow layer: Eq. (3.34).

and the function g is given by

$$g(x)=\begin{cases} \|x\|^2-1, & \text{if } \|x\|^2\le \frac{5}{2},\\ \frac{3}{2}\left[\left(\|x\|-\frac{\sqrt{10}}{6}\right)^2-\frac{1}{9}\right]^{\frac{2}{3}}\left[\frac{\sqrt{10}+1}{\sqrt{10}-1}\cdot\frac{6\|x\|-\sqrt{10}-2}{6\|x\|-\sqrt{10}+2}\right]^{\frac{\sqrt{10}}{3}}, & \text{if } \|x\|^2>\frac{5}{2}.\end{cases} \tag{3.34}$$

The image of the function $g(x)$ and $f(x)$ in Examples 3.4, 3.5 and 3.6 are provided in Figures 1, 2 and 3, respectively. The colored wireframe patch is for $f(x) = \|x\|^2 - 1$, and the surface patch is for $g(x)$ defined in Eqs. (3.30) (3.32) and (3.34), respectively. For comparison, we also display the images of $g(x)$ with different ρ in Figure 4. The upper blue layer is for Example 3.4, the middle magenta layer is for Example 3.5, and the bottom yellow layer is for Example 3.6. As $-\rho$ approaches the lower bound of $f(x)$, the gap between $f(x)$ and $g(x)$ becomes significant.

4 Conclusion

In this paper, we consider the relationship between the Yamagishi-Yamada's accelerated operator and subgradient projector in Euclidean Space. We find the necessary condition of finding a convex continuous function from the accelerated operator if we assume that the accelerated operator of a function is the subgradient projector of another function. We make a connection of the subgradient projectors G_f and G_h when the two function $f(A\underline{x} - b) = h(\underline{x})$. Using a quadratic function as an example, we generate different expressions of the new function with different parameters of L and ρ. Finally, we display the images of different functions so as to catch sight of the difference between those parameters.

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