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OPTIMALITY CONDITIONS IN INFINITE HORIZON OPTIMIZATION BY CONTINGENT DERIVATIVE

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Abstract: In this paper, the notion of the contingent cone to the set of trajectories in infinite horizon optimization problems is introduced. Some important properties of the contingent cone are investigated. Also the notion of the upper contingent derivative is introduced that is based on the same idea used in the literature when defining with directional derivatives. Then, optimality conditions are derived in terms of the contingent cone and the upper contingent derivative.

Key words: *nfinite horizon optimization, optimal trajectory, contigent cone, upper contingent derivative, optimality condition*

Mathematics Subject Classification: *90C26*

1 Introduction

Infinite horizon optimizations are an important class of optimization problems where the objective function is often defined as a total cost over infinite horizon [1, 2, 9, 13, 16, 18, 17, 19, 20]. This class has many applications in inventory control, production planning, equipment replacement and capacity expansion.

The total cost over an unbounded horizon may be infinite or diverge. Taking this factor into account, different optimality criteria apart from minimal total have been considered [6, 14, 22, 23]. For example, a discounting factor is used to guarantee the convergence of a sum of infinitely many costs over an infinite horizon [1, 2]. Some other examples of such optimality criteria are efficiency or finite optimality [12, 21], the average cost [3, 11, 26], overtaking optimality [6, 10, 15, 27, 28] and 1-optimality [4, 25].

The notion of the tangent cone (contingent cone) plays an important role in driving optimality conditions. The tangent cone gives an approximation of a set around a given point. Different definitions are introduced in the literature for a tangent cone, such as Bouligand tangent cone [5] and Clarke tangent cone [7]. The use of the tangent cones in optimization was initiated by Dubovitskii and Miljutin [8].

In this paper, we consider systems described by the decision network as in [2]. These systems generate trajectories of decisions and there is a cost associated to each decision that could be used to define the functional - total cost for a trajectory. The aim of this paper is to introduce a contingent cone and directional derivative for metric space of trajectories and investigate optimality conditions.

The paper is organized as follows. Notations, problem statement and the new notion “the upper contingent derivative” are presented in the next section. “the Contingent cone” and its some important properties are established in Section 3. Optimality conditions are given in section 4.

2 Notations and Problem Statement

Consider the decision network, (Σ, A, C), where Σ is the set of states (nodes), A is the set of decisions (arcs) and C is a real-valued cost function $C : A \to R^+$. The notation $\mathbb{N}$ is used to denote the set of natural numbers $\{1, 2, \cdots\}$. Throughout the paper we will assume that the following conditions hold:

1. there is a node called single root with the following properties

   there is no incoming arcs to this node,

   every other node can be reached from the single root,

2. the set of decisions available at any node is nonempty and finite,
3. the set of incoming decisions to any node is also finite.

It has been proved that [24, Theorem 1] under these assumptions, the set of nodes can be numbered, say as in the form $\Sigma = \{\sigma_1, \sigma_2, \sigma_3, ...\}$, such that the following holds: if $(\sigma_i, \sigma_j) \in A$ for some nodes $\sigma_i, \sigma_j \in \Sigma$, then $i < j$.

Definition 2.1. A trajectory $\mathbf{s}$ is an infinite sequence of states $(s_1, s_2, s_3, ...)$ where $s_1 = \sigma_1$ is a given fixed root, $s_i \in \Sigma$ and $(s_i, s_{i+1}) \in A$ for all $i = 1, 2, ...$.

The set of all trajectories $\mathbf{s}$ will be denoted by Π. This set can be endowed by a metric. We will use the metric used in [2]; namely, given any two trajectories $\mathbf{s} = (s_1, s_2, s_3, ...)$ and $\mathbf{s}' = (s'_1, s'_2, s'_3, ...)$, the metric ρ is defined as follows:

$$\rho(\mathbf{s}, \mathbf{s}') = \sum_{i=1}^{\infty} \phi_i(\mathbf{s}, \mathbf{s}') 2^{-i}, \tag{2.1}$$

where

$$\phi_i(\mathbf{s}, \mathbf{s}') = \begin{cases} 0 & \text{if} \quad s_i = s'_i \\ 1 & \text{otherwise} \end{cases} .$$

In Lemma 1 in [2], it is proved that the set Π is complete and hence compact in the sense of this metric.

Under this metric, the closeness of trajectories depends on the number of initial nodes over which they agree. For example, given any $i \in \mathbb{N}$, it can easily be verified that the following holds:

$$\rho(\mathbf{s}, \mathbf{s}') \leq \frac{1}{2^{k+1}} \quad \Rightarrow \quad s'_i = s_i, \ \forall i = 1, 2, \cdots, k. \tag{2.2}$$

Optimization problem. Given trajectory $\mathbf{s} = (s_1, s_2, s_3, ...)$, the value $C(s_i, s_{i+1})$ is the cost associated with the decision (s_i, s_{i+1}). Then, the objective function in this problem can be determined as the total cost over the trajectory $\mathbf{s}$ [2]; that is

$$f(\mathbf{s}) = \sum_{i=1}^{\infty} C(s_i, s_{i+1}), \ \forall \mathbf{s} \in \Pi.$$

We assume that f is uniformly convergent over Π; that is, for any $\varepsilon > 0$ there exists n_ε such that for all trajectories $\mathbf{s}$ the relation $\sum_{i=n}^{\infty} C(s_i, s_{i+1}) < \varepsilon$ holds for all $n \geq n_\varepsilon$. In this case f is continuous on Π. Note that this is not a restrictive assumption; it holds if the cost function $C(s_i, s_{i+1})$ is uniformly bounded and also is discounted, for example, by $(1/2)^i$ (see Assumption 1 and Lemma 2 in [2]). Taking this into account, one can define the total cost in the form

$$f(\mathbf{s}) = \sum_{i=1}^{\infty} r^i C(s_i, s_{i+1}); \tag{2.3}$$

where $r \in (0,1)$ is a discount factor.

We consider the following optimization problem

$$\text{Minimize} \quad f(\mathbf{s}), \qquad \text{subject to} \qquad \mathbf{s} \in \Omega; \tag{2.4}$$

where $\Omega \subset \Pi$ is a given closed set. We note that Ω is also a compact set. Since f is continuous and Ω is compact, an optimal solution s^* to problem (2.4) exists.

Given $\varepsilon > 0$, the ε-neighborhood of $\mathbf{s}$ in Π is defined by

$$V_\varepsilon(\mathbf{s}) \doteq \{\mathbf{s}' \in \Pi : \ \rho(\mathbf{s}, \mathbf{s}') < \varepsilon\}.$$

Trajectory $\mathbf{s} \in \Omega$ is called an isolated point of Ω if there is $\varepsilon > 0$ such that $(V_\varepsilon(\mathbf{s}) \setminus \{\mathbf{s}\}) \cap \Omega = \emptyset$. Clearly, if $\mathbf{s} \in \Omega$, $\Omega \setminus \{\mathbf{s}\} \neq \emptyset$ and $\mathbf{s}$ is not an isolated point of Ω then there is a sequence $\mathbf{s}^n \in \Omega$ such that $\mathbf{s}^n \neq \mathbf{s}$, $\forall n$, and $\rho(\mathbf{s}^n, \mathbf{s}) \to 0$ as $n \to \infty$.

We note that the set Π is not a linear space. Below we will define the notions of "direction" and then "directional derivative" by using the main idea behind these notions defined in linear spaces. They will be used to derive optimality conditions.

Definition 2.2. We say the trajectories $\mathbf{s}, \mathbf{h} \in \Omega$ have the same direction if there are $n_s, n_h \in \mathbb{N}$ such that $s_{n_s+i} = h_{n_h+i}$ for all $i = 1, 2, \cdots$. We will use the notation $(\mathbf{s})_\infty = (\mathbf{h})_\infty$ in this case.

According to this definition, having the same direction means that these trajectories coincide/join after some finite steps; i.e. n_s and n_t. Moreover, the "larger" numbers n_s and n_t, assuming the sets $\{s_2, \cdots, s_{n_s}\}$ and $\{t_2, \cdots, t_{n_t}\}$ are disjoint, can be interpreted as $\mathbf{s}$ and $\mathbf{h}$ being "far" from each-other.

Definition 2.3. Assume that $\bar{\mathbf{s}}$ is not an isolated point of Ω; that is, $V_\varepsilon(\bar{\mathbf{s}}) \cap \Omega \neq \emptyset$ for all $\varepsilon > 0$. The *upper contingent derivative* of f at $\bar{\mathbf{s}}$ with respect to the direction $\bar{\mathbf{d}} \in \Pi$ is defined as

$$Uf(\bar{\mathbf{s}}, \bar{\mathbf{d}}) = \limsup_{\substack{\mathbf{s} \to \bar{\mathbf{s}} \\ \mathbf{d} \to \bar{\mathbf{d}} \\ \mathbf{s} \neq \bar{\mathbf{s}} \\ (\mathbf{s})_\infty = (\mathbf{d})_\infty}} \frac{f(\mathbf{s}) - f(\bar{\mathbf{s}})}{\rho(\mathbf{s}, \bar{\mathbf{s}})}.$$

The idea behind this definition is that the limit of the fraction on the right hand side is taken over all sequences $\mathbf{s}^k, \mathbf{d}^k$ such that $\mathbf{s}^k \to \bar{\mathbf{s}}$, $\mathbf{d}^k \to \bar{\mathbf{d}}$ and for each k, the trajectory $\mathbf{s}^k$ has the same direction as $\mathbf{d}^k$. Thus, this idea is similar to the definition of the Clarke's directional derivative of g at $\bar{x}$ in a direction x in linear spaces given below:

$$f^\circ(\bar{x}, x) = \limsup_{\substack{y \to \bar{x} \\ \xi \downarrow 0}} \frac{f(y + \xi x) - f(y)}{\xi}.$$

Since Π is not a linear space and the operation $\lambda \bar{\mathbf{d}}$, (λ is a given number) is not defined, the super-linearity of $Uf(\bar{\mathbf{s}}, \bar{\mathbf{d}})$ with respect to $\bar{\mathbf{d}}$ can not be considered.

We also note that, according to (2.2), for example if $\rho(\mathbf{s}^k, \bar{\mathbf{s}}) \leq \varepsilon$, then the first $[1 + log_2 \frac{1}{\varepsilon}]$ elements of trajectories $\mathbf{s}^k$ and $\bar{\mathbf{s}}$ coincide. Therefore, the statements $\mathbf{s}^k \to \bar{\mathbf{s}}$, $\mathbf{d}^k \to \bar{\mathbf{d}}$ means that trajectories $\mathbf{s}^k$ and $\mathbf{d}^k$ coincide with $\bar{\mathbf{s}}$ and $\bar{\mathbf{d}}$, respectively, at the some "initial stage", and this "initial stage" grows infinitely when $k \to \infty$. Then, the condition $(\mathbf{s}^k)_\infty = (\mathbf{d}^k)_\infty$ states that trajectories $\mathbf{s}^k$, $\mathbf{d}^k$ join after that "initial stage"; that is, have the same direction in terms of Definition 2.2. Therefore, we can interpret this siltation as trajectories $\mathbf{s}^k$ approaching to $\bar{\mathbf{s}}$ in direction $\bar{\mathbf{d}}$.

Clearly, the upper contingent derivative $Uf(\bar{\mathbf{s}}, \bar{\mathbf{d}})$ is not defined for directions (trajectories) $\bar{\mathbf{d}}$ that are not connected with $\bar{\mathbf{s}}$ in the sense of the above interpretation.

3 Contingent Cone

Let $\Omega \subset \Pi$ and $\bar{\mathbf{s}} \in \Omega$ be a non-isolated point of Ω. For the rest of the paper we assume that $\Omega \setminus \{\mathbf{s}\} \neq \emptyset$. We introduce the notion of the contingent cone $T_\Omega(\bar{\mathbf{s}})$.

Definition 3.1. We say $\mathbf{t} \in \Pi$ is an element of the contingent cone $T_\Omega(\bar{\mathbf{s}})$ to the set Ω at $\bar{\mathbf{s}}$ if there exist sequences of trajectories $\mathbf{s}^n \in \Omega, (\mathbf{s}^n \neq \bar{\mathbf{s}}, \forall n)$ and $\mathbf{t}^n \in \Pi$ such that $\mathbf{s}^n \to \bar{\mathbf{s}}$, $\mathbf{t}^n \to \mathbf{t}$ as $n \to \infty$, and $(\mathbf{s}^n)_\infty = (\mathbf{t}^n)_\infty$ for all n :

$$T_\Omega(\bar{\mathbf{s}}) \doteq \{\mathbf{t} \in \Pi : \ \exists \mathbf{s}^n \in \Omega, \mathbf{t}^n \in \Pi; \ \mathbf{s}^n \neq \bar{\mathbf{s}}, \forall n, \ \mathbf{s}^n \to \bar{\mathbf{s}}, \mathbf{t}^n \to \mathbf{t} \text{ and } (\mathbf{s}^n)_\infty = (\mathbf{t}^n)_\infty, \forall n\}.$$

Roughly speaking, the contingent cone $T_\Omega(\bar{\mathbf{s}})$ combines all trajectories in Π having the same direction with some trajectory in Ω being "sufficiently" close to $\bar{\mathbf{s}}$.

Some properties of the contingent cone are presented below.

Lemma 3.2. *Let $\bar{\mathbf{s}}$ be a non-isolated point of Ω. Then $T_\Omega(\bar{\mathbf{s}})$ is not empty and, in particular, $\bar{\mathbf{s}} \in T_\Omega(\bar{\mathbf{s}})$.*

Proof. If $\bar{\mathbf{s}}$ is a non-isolated point of Ω, then there exists a sequence $\mathbf{s}^n \in \Omega$ such that $\mathbf{s}^n \neq \bar{\mathbf{s}}, \forall n$, and $\mathbf{s}^n \to \bar{\mathbf{s}}$ as $n \to \infty$. Then, to prove the relation $\bar{s} \in T_\Omega(\bar{\mathbf{s}})$ it is enough to let $\mathbf{t}^n := \mathbf{s}^n$ in Definition 3.1. □

In the next lemma, we show that the contingent cone in Definition 3.1 is a closed set.

Lemma 3.3. *$T_\Omega(\bar{\mathbf{s}})$ is a closed set.*

Proof. Assume that the sequence $\mathbf{u}^m \in T_\Omega(\bar{\mathbf{s}})$ converges to $\mathbf{u}$. To show $T_\Omega(\bar{\mathbf{s}})$ is closed, it suffices to show that $\mathbf{u} \in T_\Omega(\bar{\mathbf{s}})$; or equivalently, it suffices to show that there exist sequences

$\mathbf{s}^n \in \Omega$ ($\mathbf{s}^n \neq \bar{\mathbf{s}}, \forall n$) and $\mathbf{t}^n \in \Pi$ such that the conditions of Definition 3.1 are satisfied; that is

$$\mathbf{s}^n \to \bar{\mathbf{s}}, \quad \mathbf{t}^n \to \mathbf{u} \text{ as } n \to \infty \quad \text{and } (\mathbf{s}^n)_\infty = (\mathbf{t}^n)_\infty, \forall n. \tag{3.1}$$

Take an arbitrary $\varepsilon > 0$. Let $m(\varepsilon)$ be such that

$$\rho(\mathbf{u}^{m(\varepsilon)}, \mathbf{u}) < \frac{1}{2}\varepsilon.$$

Given $m(\varepsilon)$, by the definition of $\mathbf{u}^{m(\varepsilon)} \in T_\Omega(\bar{\mathbf{s}})$, there are sequences $\mathbf{s}^{m(\varepsilon),n} \in \Omega$, $\mathbf{t}^{m(\varepsilon),n} \in \Pi$ such that $\mathbf{s}^{m(\varepsilon),n} \neq \bar{\mathbf{s}}, \forall n$, and

$$\mathbf{s}^{m(\varepsilon),n} \to \bar{\mathbf{s}}, \quad \mathbf{t}^{m(\varepsilon),n} \to \mathbf{u}^{m(\varepsilon)} \text{ as } n \to \infty \quad \text{and } (\mathbf{s}^{m(\varepsilon),n})_\infty = (\mathbf{t}^{m(\varepsilon),n})_\infty, \forall n.$$

In other words, there is $n(\varepsilon)$ such that

$$\rho(\mathbf{s}^{m(\varepsilon),n(\varepsilon)}, \bar{\mathbf{s}}) \leq \varepsilon \text{ and } \rho(\mathbf{t}^{m(\varepsilon),n(\varepsilon)}, \mathbf{u}^{m(\varepsilon)}) \leq \frac{1}{2}\varepsilon.$$

Then

$$\rho(\mathbf{t}^{m(\varepsilon),n(\varepsilon)}, \mathbf{u}) \leq \rho(\mathbf{t}^{m(\varepsilon),n(\varepsilon)}, \mathbf{u}^{m(\varepsilon)}) + \rho(\mathbf{u}^{m(\varepsilon)}, \mathbf{u}) \leq \varepsilon.$$

This means that

$$\mathbf{s}^{m(\varepsilon),n(\varepsilon)} \to \bar{\mathbf{s}}, \quad \mathbf{t}^{m(\varepsilon),n(\varepsilon)} \to \mathbf{u} \quad \text{as } \varepsilon \to 0.$$

On the other hand $(\mathbf{s}^{m(\varepsilon),n(\varepsilon)})_\infty = (\mathbf{t}^{m(\varepsilon),n(\varepsilon)})_\infty$; that is, the required relation (3.1) is true. □

We note that the sets Ω and $T_\Omega(\bar{\mathbf{s}})$ are generally different. Below we provide an example for which the relation $T_\Omega(\bar{\mathbf{s}}) \nsubseteq \Omega$ holds. A similar example to demonstrate $\Omega \nsubseteq T_\Omega(\bar{\mathbf{s}})$ can be constructed easily.

Example 3.4. Consider the set of nodes $(\delta_1, \delta_2, \delta_3, ...)$, $\delta_i \in \Sigma, \forall i$, and assume that the set

$$\Delta = \{(\delta_{i_1}, \delta_{i_2}, \delta_{i_3}, ...) : \ \{1, 3, 5, \cdots\} \subset \{i_1, i_2, i_3, \cdots\} \text{ and } i_1 < i_2 < \cdots\}$$

belongs to the set of all trajectories Π. Let

$$\bar{\mathbf{s}} = (\delta_1, \delta_3, \delta_5, ...), \quad \mathbf{t} = (\delta_1, \delta_2, \delta_3, \delta_4, ...) \text{ and}$$

$$\Omega = \{\mathbf{s} \in \Delta : \quad (\mathbf{s})_\infty = (\bar{\mathbf{s}})_\infty\}.$$

Clearly $\mathbf{t} \in \Delta \subset \Pi$ and $\mathbf{t} \notin \Omega$ (since each trajectory in Ω contains only odd indices after some finite index). Moreover, $\bar{\mathbf{s}} \in \Omega$ is a non-isolated point of Ω; for example, for the set of trajectories $\mathbf{s}^n \in \Omega$ defined by

$$\mathbf{s}^n = (\delta_1, \delta_3, \delta_5, \cdots, \delta_{2n+1}, \delta_{2n+2}, \delta_{2n+3}, \cdots, \delta_{4n+1}, \delta_{4n+3}, \delta_{4n+5}, ...)$$

we have $\mathbf{s}^n \neq \bar{\mathbf{s}}, \forall n \geq 1$ and $\mathbf{s}^n \to \bar{\mathbf{s}}$ as $n \to \infty$.

We show that $\mathbf{t} \in T_\Omega(\bar{\mathbf{s}})$. Consider the sequence of trajectories $\mathbf{t}^n \in \Delta \subset \Pi$ defined as follows

$$\mathbf{t}^n = (\delta_1, \delta_2, \delta_3, ..., \delta_{2n}, \delta_{2n+1}, \delta_{2n+3}, \delta_{2n+5}, ...).$$

Clearly $\mathbf{t}^n \to \mathbf{t}$ as $n \to \infty$; and moreover, $(\mathbf{t}^n)_\infty = (\mathbf{s}^n)_\infty$ for all n.

Therefore, $\mathbf{t} \in T_\Omega(\bar{\mathbf{s}})$ and $\mathbf{t} \notin \Omega$; that is, $T_\Omega(\bar{\mathbf{s}}) \nsubseteq \Omega$.

In the following, we investigate the contingent cone of intersection and union of sets.

Lemma 3.5. *Let* Ω_1 *and* Ω_2 *be subsets of* Π *and* $\bar{\mathbf{s}} \in \Omega_1 \cap \Omega_2$. *Then,*

(i) $T_{\Omega_1 \cap \Omega_2}(\bar{\mathbf{s}}) \subset T_{\Omega_1}(\bar{\mathbf{s}}) \cap T_{\Omega_2}(\bar{\mathbf{s}})$.

(ii) $T_{\Omega_1 \cup \Omega_2}(\bar{\mathbf{s}}) = T_{\Omega_1}(\bar{\mathbf{s}}) \cup T_{\Omega_2}(\bar{\mathbf{s}})$.

Proof. The proofs of $T_{\Omega_1 \cap \Omega_2}(\bar{\mathbf{s}}) \subset T_{\Omega_1}(\bar{\mathbf{s}}) \cap T_{\Omega_2}(\bar{\mathbf{s}})$ and $T_{\Omega_1}(\bar{\mathbf{s}}) \cup T_{\Omega_2}(\bar{\mathbf{s}}) \subset T_{\Omega_1 \cup \Omega_2}(\bar{\mathbf{s}})$ directly follow from the definition of the contingent cone. Thus, it is enough to show that $T_{\Omega_1 \cup \Omega_2}(\bar{\mathbf{s}}) \subset T_{\Omega_1}(\bar{\mathbf{s}}) \cup T_{\Omega_2}(\bar{\mathbf{s}})$.

Let $\mathbf{t} \in T_{\Omega_1 \cup \Omega_2}(\bar{\mathbf{s}})$. Then, there are sequences $\mathbf{s}^n \in \Omega_1 \cup \Omega_2$ and $\{\mathbf{t}^n\}_{n\in\mathbb{N}}$ such that

$$\rho(\mathbf{s}^n, \bar{\mathbf{s}}) \to 0, \rho(\mathbf{t}^n, \mathbf{t}) \to 0, \text{ and } (\mathbf{s}^n)_\infty = (\mathbf{t}^n)_\infty, \forall n. \tag{3.2}$$

As $\mathbf{s}^n \in \Omega_1 \cup \Omega_2$, for some $j \in \{1, 2\}$ the realtion $\mathbf{s}^n \in \Omega_j$ holds. Then, from (3.2) we conclude that $\mathbf{t} \in T_{\Omega_j}(\bar{\mathbf{s}})$ and hence $\mathbf{t} \in T_{\Omega_1}(\bar{\mathbf{s}}) \cup T_{\Omega_2}(\bar{\mathbf{s}})$. □

In the next example, we show that the inverse of the inclusion (i) in this lemma; that is, the relation

$$T_{\Omega_1 \cap \Omega_2}(\bar{\mathbf{s}}) \supset T_{\Omega_1}(\bar{\mathbf{s}}) \cap T_{\Omega_2}(\bar{\mathbf{s}})$$

may not be true.

Example 3.6. Consider two trajectories $\mathbf{t} = (\bar{s}_1, t_2, t_3, ...)$ and $\bar{\mathbf{s}} = (\bar{s}_1, \bar{s}_2, \bar{s}_3, ...)$ assuming that $t_i \neq \bar{s}_j$ for all $i, j \geq 2$. Define the sets Ω_1 and Ω_2 as follows

$$\Omega_1 = \{\bar{\mathbf{s}}\} \cup \Omega_{1,2} \cup \Omega_1^0, \quad \Omega_2 = \{\bar{\mathbf{s}}\} \cup \Omega_{1,2} \cup \Omega_2^0;$$

where

$$\Omega_1^0 = \{\mathbf{s}^n = (\bar{s}_1, \bar{s}_2, ..., \bar{s}_{2n}, \bar{s}_{2n+2}, \bar{s}_{2n+4}, \bar{s}_{2n+6}, ...), \quad n = 1, 2, \cdots\},$$

$$\Omega_2^0 = \{\mathbf{u}^n = (\bar{s}_1, \bar{s}_2, ..., \bar{s}_{2n}, \bar{s}_{2n+1}, \bar{s}_{2n+3}, \bar{s}_{2n+5}, ...), \quad n = 1, 2, \cdots\},$$

$$\Omega_{1,2} = \{\mathbf{u}^n = (\bar{s}_1, \bar{s}_2, ..., \bar{s}_{2n}, \xi_{2n+1}, \xi_{2n+2}, \xi_{2n+3}, ...), \quad n = 1, 2, \cdots\};$$

and $\xi_i \neq \bar{s}_j$, for $\xi_i \neq t_j$, all $i, j \geq 2$.

Now, let the set of all trajectories Π is given by

$$\Pi = \{\mathbf{t}\} \cup \Omega_1 \cup \Omega_2 \cup T_1 \cup T_2$$

where

$$T_1 = \{\mathbf{t}^n = (t_1, t_2, ..., t_{2n}, \bar{s}_{2n+2}, \bar{s}_{2n+4}, \bar{s}_{2n+6}, ...), \quad n = 1, 2, \cdots\};$$

$$T_2 = \{\mathbf{v}^n = (t_1, t_2, ..., t_{2n}, \bar{s}_{2n+1}, \bar{s}_{2n+3}, \bar{s}_{2n+5}, ...), \quad n = 1, 2, \cdots\}.$$

First we show that $\mathbf{t} \in T_{\Omega_1}(\bar{\mathbf{s}}) \cap T_{\Omega_2}(\bar{\mathbf{s}})$.

Consider the sequences $\mathbf{s}^n \in \Omega_1$ and $\mathbf{t}^n \in T_1$. Clearly

$$\rho(\mathbf{s}^n, \bar{\mathbf{s}}) \to 0, \ \rho(\mathbf{t}^n, \mathbf{t}) \to 0, \text{ and } (\mathbf{s}^n)_\infty = (\mathbf{t}^n)_\infty, \ \forall n.$$

This means that $\mathbf{t} \in T_{\Omega_1}(\bar{\mathbf{s}})$.

In a similar way, for sequences $\mathbf{u}^n \in \Omega_2$ and $\mathbf{v}^n \in T_2$ we have

$$\rho(\mathbf{u}^n, \bar{\mathbf{s}}) \to 0, \ \rho(\mathbf{v}^n, \mathbf{t}) \to 0, \text{ and } (\mathbf{u}^n)_\infty = (\mathbf{v}^n)_\infty, \ \forall n;$$

that leads to $\mathbf{t} \in T_{\Omega_2}(\bar{\mathbf{s}})$. Therefore, $\mathbf{t} \in T_{\Omega_1}(\bar{\mathbf{s}}) \cap T_{\Omega_2}(\bar{\mathbf{s}})$.

Now we show that $\mathbf{t} \notin T_{\Omega_1 \cap \Omega_2}(\bar{\mathbf{s}})$. By contradiction let $\mathbf{t} \in T_{\Omega_1 \cap \Omega_2}(\bar{\mathbf{s}})$; that is, there are sequences $\mathbf{u}^n \in \Omega_1 \cap \Omega_2$, $\mathbf{u}^n \neq \bar{\mathbf{s}}, \forall n$, and $\mathbf{d}^n \in \Pi$ such that

$$\rho(\mathbf{u}^n, \bar{\mathbf{s}}) \to 0, \;\rho(\mathbf{d}^n, \mathbf{t}) \to 0, \text{ and } (\mathbf{u}^n)_\infty = (\mathbf{d}^n)_\infty, \;\forall n. \tag{3.3}$$

As $\Omega_1 \cap \Omega_2 = \{\bar{\mathbf{s}}\} \cup \Omega_{1,2}$, it is not difficult to observe that the relation $\mathbf{u}^n \in \Omega_1 \cap \Omega_2$, $\mathbf{u}^n \neq \bar{\mathbf{s}}$, implies for every n the following holds

$$(\mathbf{u}^n)_\infty \neq (s)_\infty, \quad \forall s \in \{\mathbf{t}\} \cup \Omega_1^0 \cup \Omega_2^0 \cup T_1 \cup T_2.$$

On the other hand, according to the definition of sets Ω_i and T_i, $i = 1, 2$, the convergence $\mathbf{d}^n \to \mathbf{t}$ implies that given any n, one of the following holds: $\mathbf{d}^n = \mathbf{t}$, or $\mathbf{d}^n \in T_1$, or $\mathbf{d}^n \in T_2$. In all of these three cases we have

$$(\mathbf{d}^n)_\infty \neq (\mathbf{u}^n)_\infty.$$

This contradicts (3.3). Thus, $\mathbf{t} \notin T_{\Omega_1 \cap \Omega_2}(\bar{\mathbf{s}})$.

4 Optimality Conditions

Consider the optimization problem (2.4) stated in Section 2. In this section our aim is to investigate optimality conditions for this problem in terms of the contingent cone and the upper contingent derivative.

Assumption (L): The total cost f is locally Lipschitz; that is, for each $\bar{\mathbf{s}} \in \Pi$, there exist a $\delta-$neighborhood $V_\delta(\bar{\mathbf{s}})$ with $\delta > 0$ and a number $L_{\delta,\bar{\mathbf{s}}} < \infty$ such that

$$|f(\mathbf{s}) - f(\mathbf{t})| \leq L_{\delta,\bar{\mathbf{s}}}\, \rho(\mathbf{s}, \mathbf{t}), \;\forall \mathbf{s}, \mathbf{t} \in V_\delta(\bar{\mathbf{s}}). \tag{4.1}$$

The following two propositions provide some sufficient conditions under which Assumption (L) holds.

Proposition 4.1. *Assume that the cost function C is bounded and the total cost f is defined as in (2.3). If $r \leq \frac{1}{2}$ then f is locally Lipschitz at each $\bar{\mathbf{s}} \in \Pi$.*

Proof. Given $\bar{\mathbf{s}} \in \Pi$, we take any $\delta > 0$ and trajectories $\mathbf{s}, \mathbf{t} \in V_\delta(\bar{\mathbf{s}})$. Clearly

$$\rho(\mathbf{s}, \mathbf{t}) \leq 2\delta.$$

According to (2.2) at least the first $[log_2 \frac{1}{2\delta} - 1]$ elements of trajectories $\mathbf{s} = (s_1, s_2, \cdots)$ and $\mathbf{t} = (t_1, t_2, \cdots)$ coincide. Let $N_\delta \geq [log_2 \frac{1}{2\delta} - 1]$ be the first index for which $s_{N_\delta} \neq t_{N_\delta}$ and $s_i = t_i$ for all $i = 1, \cdots, N_\delta - 1$. Then from (2.1) we have

$$\rho(\mathbf{s}, \mathbf{t}) = \sum_{i=1}^{\infty} \phi_i(\mathbf{s}, \mathbf{t}) 2^{-i} \geq 2^{-N_\delta}. \tag{4.2}$$

On the other hand

$$|f(\mathbf{s}) - f(\mathbf{t})| \leq \sum_{i=1}^{\infty} r^i |C(s_i, s_{i+1}) - C(t_i, t_{i+1})| = \sum_{i=N_\delta - 2}^{\infty} r^i |C(s_i, s_{i+1}) - C(t_i, t_{i+1})|.$$

Since cost function C is bounded, there is $M < \infty$ such that $|C(s_i, s_{i+1})| \leq M$ for all i and $\mathbf{s}$. Then

$$|f(\mathbf{s}) - f(\mathbf{t})| \leq \sum_{i=N_\delta-2}^{\infty} 2\, r^i M \leq 2Mr^{N_\delta-2}\frac{1}{1-r}.$$

Taking into account $r \leq \frac{1}{2}$ we have $\frac{1}{1-r} \leq 2$ and $r^{N_\delta-2} \leq 2^{-N_\delta+2}$ and therefore

$$|f(\mathbf{s}) - f(\mathbf{t})| \leq 2M\, 2^{-N_\delta+1}.$$

Thus, denoting $L_{\delta,\bar{\mathbf{s}}} = 4M$ from (4.2) we obtain (4.1). □

Proposition 4.2. *Let $\bar{\mathbf{s}} \in \Pi$ be given and the total cost f be defined as in (2.3). Assume that, there exist a $\delta-$neighborhood $V_\delta(\bar{\mathbf{s}})$ and a number $M < \infty$ such that the cost function C satisfies the following condition*

$$|C(s_i, s_{i+1}) - C(t_i, t_{i+1})| \leq M\,\rho(\mathbf{s}, \mathbf{t}),\ \forall \mathbf{s}, \mathbf{t} \in V_\delta(\bar{\mathbf{s}}) \text{ and } \forall i = 1, 2, 3, \ldots .$$

Then f is locally Lipschitz at $\bar{\mathbf{s}}$.

Proof. Take any $\mathbf{s}, \mathbf{t} \in V_\delta(\bar{\mathbf{s}})$. We have

$$|f(\mathbf{s}) - f(\mathbf{t})| \leq \sum_{i=1}^{\infty} r^i |C(s_i, s_{i+1}) - C(t_i, t_{i+1})|$$

$$\leq \sum_{i=1}^{\infty} r^i\, M\,\rho(\mathbf{s}, \mathbf{t}) = \frac{Mr}{1-r}\,\rho(\mathbf{s}, \mathbf{t}),\ \forall \mathbf{s} \in V_\delta(\mathbf{t}).$$

Thus, (4.1) holds for $L_{\delta,\bar{\mathbf{s}}} = \frac{Mr}{1-r}$. □

The following results is about the existence of the upper contingent derivative.

Lemma 4.3. *Assume that $\bar{\mathbf{s}}$ is a non-isolated point of Ω and Assumption (L) holds. Then the upper contingent derivative $Uf(\bar{\mathbf{s}}, \mathbf{d})$ exists for each direction $\mathbf{d} \in T_\Omega(\bar{\mathbf{s}}) \setminus \{\bar{\mathbf{s}}\}$.*

Proof. Take an arbitrary $\bar{\mathbf{d}} \in T_\Omega(\bar{\mathbf{s}}) \setminus \{\bar{\mathbf{s}}\}$. By Assumption (L) for $\bar{\mathbf{s}} \in \Omega$, there exists a $\delta-$neighborhood $V_\delta(\bar{\mathbf{s}})$ with $\delta > 0$ such that (4.1) holds. Then

$$\frac{|f(\mathbf{s}) - f(\bar{\mathbf{s}})|}{\rho(\mathbf{s}, \bar{\mathbf{s}})} \leq L_{\delta,\bar{\mathbf{s}}},\ \forall \mathbf{s} \in V_\delta(\bar{\mathbf{s}}) \setminus \{\bar{\mathbf{s}}\}. \tag{4.3}$$

Since $\bar{\mathbf{d}} \in T_\Omega(\bar{\mathbf{s}})$, then, it follows from (4.3) that

$$-L_{\delta,\bar{\mathbf{s}}} \leq \sup_{\substack{\rho(\mathbf{s},\bar{\mathbf{s}})<\varepsilon \\ \rho(\mathbf{d},\bar{\mathbf{d}})<\varepsilon \\ \mathbf{s}\neq\bar{\mathbf{s}} \\ (\mathbf{s})_\infty=(\mathbf{d})_\infty}} \frac{f(\mathbf{s}) - f(\bar{\mathbf{s}})}{\rho(\mathbf{s}, \bar{\mathbf{s}})} \leq L_{\delta,\bar{\mathbf{s}}}, \quad \forall \epsilon < \delta. \tag{4.4}$$

Hence

$$Uf(\bar{\mathbf{s}}, \bar{\mathbf{d}}) = \limsup_{\substack{\mathbf{s}\to\bar{\mathbf{s}} \\ \mathbf{d}\to\bar{\mathbf{d}} \\ \mathbf{s}\neq\bar{\mathbf{s}} \\ (\mathbf{s})_\infty=(\mathbf{d})_\infty}} \frac{f(\mathbf{s}) - f(\bar{\mathbf{s}})}{\rho(\mathbf{s}, \bar{\mathbf{s}})}$$

exists, which means, the upper contingent derivative exists for any non-isolated point $\bar{\mathbf{s}} \in \Omega$ and any direction $\bar{\mathbf{d}} \in T_\Omega(\bar{\mathbf{s}}) \setminus \{\bar{\mathbf{s}}\}$. □

Remark. According to Lemma 4.3, as $Uf(\bar{\mathbf{s}}, \mathbf{d})$ exists if $\bar{\mathbf{d}}$ is in the contingent cone, we call it the upper contingent derivative.

We will call d a descent direction of f at $\bar{\mathbf{s}}$ if the upper contingent derivative is negative: $Uf(\bar{\mathbf{s}}, \mathbf{d}) < 0$. The main result of this section is a necessary condition of optimality for a local minimizer of f that are presented in the next theorem and corollary.

Theorem 4.4. *Assume that s^* is a non-isolated point of Ω and Assumption (L) holds at $\mathbf{s}^*$. If $Uf(\mathbf{s}^*, \mathbf{d}) < 0$ for some $\mathbf{d} \in T_\Omega(\mathbf{s}^*) \setminus \{\bar{\mathbf{s}}\}$ then, $\mathbf{s}^*$ is not a local minimizer of the problem (2.4); that is, for every $\varepsilon > 0$, there exist $\mathbf{s}^\varepsilon \in \Omega$ and $\mathbf{d}^\varepsilon \in \Pi$ such that*

$$\rho(\mathbf{s}^\varepsilon, \mathbf{s}^*) < \varepsilon,\ \rho(\mathbf{d}^\varepsilon, \mathbf{d}) < \varepsilon,\ \mathbf{s}^\varepsilon \neq \mathbf{s}^*,\ (\mathbf{s}^\varepsilon)_\infty = (\mathbf{d}^\varepsilon)_\infty \ \text{ and } \ f(\mathbf{s}^\varepsilon) < f(\mathbf{s}^*). \tag{4.5}$$

Proof. By the assumption, there exists $\bar{\mathbf{d}} \in T_\Omega(\mathbf{s}^*) \setminus \{\mathbf{s}^*\}$ such that $Uf(\mathbf{s}^*, \bar{\mathbf{d}}) < 0$. Since $\bar{\mathbf{d}} \in T_\Omega(\bar{\mathbf{s}})$, by the definition of the contingent cone, there exist sequences of trajectories $\mathbf{s}^n \in \Omega$ and $\mathbf{t}^n \in \Pi$ such that

$$\mathbf{s}^n \to \mathbf{s}^*, \mathbf{t}^n \to \bar{\mathbf{d}} \text{ as } n \to \infty;\text{ and } \mathbf{s}^n \neq \mathbf{s}^*,\ (\mathbf{s}^n)_\infty = (\mathbf{d}^n)_\infty, \forall n \geq 1. \tag{4.6}$$

By the definition of the upper contingent derivative we have

$$\limsup_{n\to\infty} \frac{f(\mathbf{s}^n) - f(\mathbf{s}^*)}{\rho(\mathbf{s}^n, \mathbf{s}^*)} \leq Uf(\mathbf{s}^*, \bar{\mathbf{d}}) < 0.$$

Then, given $\epsilon > 0$, there exists a sufficiently large number n_ϵ such that the inequalities $\rho(\mathbf{s}^{n_\epsilon}, \mathbf{s}^*) < \epsilon$, $\rho(\mathbf{t}^{n_\epsilon}, \bar{\mathbf{d}}) < \epsilon$ and $f(\mathbf{s}^{n_\epsilon}) - f(\mathbf{s}^*) < 0$ hold. Therefore, (4.6) yields (4.5). $\square$

From Theorem 4.4 we obtain the following necessary condition of local optimality.

Corollary 4.5 (Necessary condition of optimality)**.** *Assume that s^* is a non-isolated point of Ω and Assumption (L) holds at $\mathbf{s}^*$. If $\mathbf{s}^*$ is a local minimizer of the problem (2.4) then,*

$$Uf(\mathbf{s}^*, \mathbf{d}) \geq 0, \quad \forall \mathbf{d} \in T_\Omega(\mathbf{s}^*) \setminus \{\mathbf{s}^*\}. \tag{4.7}$$

In the following, we present an example that illustrates Theorem 4.4 and Corollary 4.5.

Example 4.6. Let

$$\mathbf{s}^* = (s_1, s_2, s_3, s_4....),\ \bar{\mathbf{d}} = (s_1, \delta_2, \delta_3, \delta_4, ...),\ \text{and}$$

$$\mathbf{s}^n = (s_1, s_2, ..., s_n, \delta_{n+1}, \delta_{n+2}, \delta_{n+3}, ...):\ n = 1, 2, \cdots;$$

where $s_i \neq \delta_i$ for all $i \geq 2$.

The set Ω and the set of all trajectories Π are given by $\Omega = \{\mathbf{s}^*\} \cup \{\mathbf{s}^n :\ n = 1, 2, \cdots\}$ and $\Pi = \{\bar{\mathbf{d}}\} \cup \Omega$.

By the definition of Ω for any given $\mathbf{s}^n \in \Omega$ we have

$$\rho(\mathbf{s}^n, \mathbf{s}^*) = \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots = \frac{1}{2^n}. \tag{4.8}$$

Since $\rho(\mathbf{s}^n, \mathbf{s}^*) \to 0$, the trajectory $\mathbf{s}^*$ is not an isolated point of Ω.

The cost function C is defined as follows: for all $i \in \mathbb{N}$,

$$C(\delta_i, s_{i+1}) = \frac{1-\xi(i)}{2^i}, \text{ and } C(s_i, s_{i+1}) = C(\delta_i, \delta_{i+1}) = \frac{1}{2^i};$$

where $\xi(i)$ satisfies

$$|\xi(i)| \leq M < \infty, \forall i \geq 1, \text{ and } \lim_{i\to\infty} \xi(i) = \xi^*. \tag{4.9}$$

Therefore, we have

$$f(\mathbf{s}^*) = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1 \text{ and } f(\mathbf{s}^n) = 1 - \frac{\xi(n)}{2^n}, \forall n \geq 1. \tag{4.10}$$

Now, we show that Assumption (L) holds at $\mathbf{s}^*$. Let $\mathbf{s}$ and $\mathbf{t}$ in Ω be arbitrary. Then, $\mathbf{s} = \mathbf{s}^n$ and $\mathbf{t} = \mathbf{t}^m$ for some $n, m \in \mathbb{N}$. Assume that $n < m$. We have

$$|f(\mathbf{s}) - f(\mathbf{t})| = \left| -\frac{\xi(n)}{2^n} + \frac{\xi(m)}{2^m} \right| \leq M \left(\frac{1}{2^n} + \frac{1}{2^m} \right).$$

By the fact that

$$\rho(\mathbf{s}, \mathbf{t}) \geq 1/2^n + 1/2^{n+1} + ... + 1/2^m \geq 1/2^n + 1/2^m,$$

we obtain

$$|f(\mathbf{s}) - f(\mathbf{t})| \leq M\, \rho(\mathbf{s}, \mathbf{t}).$$

This implies that Assumption (L) holds at $\mathbf{s}^*$.

Consider arbitrary sequences of trajectories $\mathbf{s}^{n_k} \in \Omega$ and $\mathbf{d}^{n_k} \to \bar{\mathbf{d}}$. First we note that $\mathbf{d}^{n_k} \to \bar{\mathbf{d}}$ implies $\mathbf{d}^{n_k} = \bar{\mathbf{d}}, \forall n_k$. On the other hand, by the definition of Ω it is not not difficult to observe that $\mathbf{s}^{n_k} \to \bar{\mathbf{s}}$ as $n_k \to \infty$. Moreover, $(\mathbf{s}^{n_k})_\infty = (\mathbf{d}^{n_k})_\infty, \forall n_k$. Thus, $\bar{\mathbf{d}} \in T_\Omega(\mathbf{s}^*) \setminus \{\bar{\mathbf{s}}\}$.

By Lemma (4.3) the upper contingent derivative $Uf(\mathbf{s}^*, \bar{\mathbf{d}})$ exists for the direction $\bar{\mathbf{d}} \in T_\Omega(\mathbf{s}^*) \setminus \{\mathbf{s}^*\}$. We have

$$Uf(\mathbf{s}^*, \bar{\mathbf{d}}) = \limsup_{\substack{\mathbf{s}\to\mathbf{s}^* \\ \mathbf{d}\to\bar{\mathbf{d}} \\ \mathbf{s}\neq\mathbf{s}^* \\ (\mathbf{s})_\infty = (\mathbf{d})_\infty}} \frac{f(\mathbf{s}) - f(\mathbf{s}^*)}{\rho(\mathbf{s}, \mathbf{s}^*)} = \limsup_{n_k\to\infty} \frac{f(\mathbf{s}^{n_k}) - f(\mathbf{s}^*)}{\rho(\mathbf{s}^{n_k}, \mathbf{s}^*)}$$

and therefore from (4.8), (4.9) and (4.10) it follows that

$$Uf(\mathbf{s}^*, \bar{\mathbf{d}}) = \limsup_{n_k\to\infty} \frac{1 - \frac{\xi(n_k)}{2^{n_k}} - 1}{\frac{1}{2^{n_k}}} = -\xi^*.$$

If $\mathbf{s}^*$ is a local minimizer in the problem (2.4) then, for all sufficiently large n_k the inequality $f(\mathbf{s}^{n_k}) \geq f(\mathbf{s}^*)$ holds and according to (4.10) we have $\xi(n_k) \leq 0$. Thus in this case $\xi^* \leq 0$ or $Uf(\mathbf{s}^*, \bar{\mathbf{d}}) \geq 0$.

Inversely, if $Uf(\mathbf{s}^*, \bar{\mathbf{d}}) < 0$ for some $\bar{\mathbf{d}}$, then $\xi^* > 0$ and the inequality $\xi(n_k) > 0$ holds for sufficiently large numbers n_k. Therefore in this case, there is n_k such that $f(\mathbf{s}^{n_k}) < f(\mathbf{s}^*)$; that is, $\mathbf{s}^*$ is not not a local minimizer of f.

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