Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 14, No. 3, pp. 463–477, 2018

SENSITIVITY FOR SECOND-ORDER COMPOSED CONTINGENT DERIVATIVES UNDER BENSON PROPER EFFICIENCY

Hongwei Liang, Zhongping Wan, Qilong He

Abstract: In this paper, we analyze the quantitative behavior of the perturbation map by employing second-order composed contingent derivative for the set-valued maps under Benson proper efficiency. By using a standard separation theorem for convex sets, we provide relationships between second-order composed contingent derivative of a given feasible set map and that of the proper perturbation map.

Key words: *second-order composed contingent derivatives, sensitivity analysis, proper perturbation map, Benson proper efficiency*

Mathematics Subject Classification: *49J53, 46G05, 90C31*

1 Introduction

Sensitivity analysis usually means the quantitative analysis. It studys the derivatives of the perturbation map when an optimization problem is perturbed. Some helpful results have been proved in scalar optimization (see [5, 20]). In the case, the optimal solution reached is a minimal solution, and hence unique. Lots of researchers have discussed the sensitivity of vector optimization in [13, 14, 15, 16, 19, 21, 22, 24, 25]. For this case, the optimal solutions most of times are multiple ones and the efficient set, in general, is not a singleton, and therefore, sensitivity analysis may consider a set-valued perturbation map more than a function. In general vector optimization, Tanino [24] obtained the first results about the efficient set map called the perturbation map by making use of contingent derivatives introduced in [2]. Several quantitative results in this direction can be found in Klose [13], Kuk et al.[15], Li [16], Shi [21]. By virtue of convexity assumptions in vector optimization, the papers [14, 22, 25] studied quantitative properties of perturbation maps in nonsmooth convex vector optimization.

The second-order sensitivity analysis was considered by virtue of the second-order contingent derivatives in [27, 28, 29]. Higher-order adjacent derivatives in [23] and higher-order contingent derivatives in [26] were applied to higher-order sensitivity analysis in vector optimization. Using the higher-order contingent derivatives and a separation theorem for convex sets, Xu [30] also obtained some results about higher-order sensitivity analysis.

It is well knonwn that, unlike the contingent cone, the second-order (resp.higher-order) contingent set is not necessarily a cone and even not a convex set in general. Hence, second-order (resp.higher-order) contingent derivatives that introduced by the second-order (resp.higher-order) contingent set lack some inherent properties the contingent derivatives own. Then, it is more difficult to study the set-valued optimization by using the second-order (resp.higher-order) contingent derivatives. Note that second-order composed contingent cone is always closed and convex when the set we disscuss is convex. Therefore, by using the second-order composed contingent derivatives presented in Khan [11], Zhu [31] proposed Karush-Kuhn-Tucker sufficient and necessary optimality conditions for set-valued optimization in second-order case extended the results of Götz in [6].

In vector optimization problem, minimal or weakly minimal points of a subset of a partially ordered linear space are mainly considered. But as we konw, the range of the set of (weakly) minimal points is often too large, then, investigating some variants of these concepts makes more meaningful. For example, various notions of proper minimality have been introduced in [3, 4, 9, 12, 17], and Benson proper efficiency given in [3, 9, 17] plays a major role in set-valued optimization.

Inspired by the preceding work [23, 26, 27, 28, 29, 30, 31], we investigate the qualitative analysis for second-order composed contingent derivative under Benson proper perturbation maps in set-valued optimization. The remaining of this article is organized as follows. Section 2 provides some basic definitions we need in the paper. The main results in establishing relationships between the second-order composed contingent derivative of the perturbation map and the set of Benson proper minimal points of the second-order composed contingent derivative of the set-valued map are given in Section 3.

2 Preliminaries

Throughout the paper, let X, Y be two Banach spaces, $F : X \to 2^Y$ be a nonempty set-valued map, $C \subseteq Y$ be a closed convex pointed cone with nonempty interior $\text{int}C \neq \emptyset$. where C is pointed if $C \cap (-C) = \{0\}$. Assume that 0_X and 0_Y denote the origins of X and Y, respectively, X^* and Y^* are the topological dual spaces of X and Y. Let C^* be the negative dual cone of cone C, defined by

$$C^* := \{\varphi \in Y^* : \varphi(c) \leq 0, \ \forall c \in C\}.$$

Let Q be a nonempty subset of Y, denote the closure of Q by $\text{cl}Q$ and the interior of Q by $\text{int}Q$. The cone hull of Q is defined by

$$\text{cone}Q := \{tq : t \geq 0, \ q \in Q\}.$$

The domain, graph and epigraph of a given set-valued map $F : X \to 2^Y$ are defined by

$$\text{dom}F := \{x \in X \mid F(x) \neq \emptyset\},$$

$$\text{gph}F := \{(x, y) \in X \times Y \mid y \in F(x), \ x \in X\},$$

$$\text{epi}F := \{(x, y) \in X \times Y \mid y \in F(x) + C, \ x \in X\},$$

respectively. Obviously, the $\text{epi}F$ is the graph of $F + C$, i.e., $\text{epi}F = \text{gph}(F + C)$.

The profile map F_+ of F is denoted by $F_+(x) := F(x) + C$, for every $x \in \text{dom}F$.

The Painleve-Kuratowski (sequential) outer (or upper) limit is defined by

$$\limsup_{x \to \hat{x}} F(x) := \{\hat{y} \in Y \mid \exists x_n \to \hat{x}, y_n \to \hat{y} \ \text{ s.t. } y_n \in F(x_n), \ \forall n \in \mathbb{N}\}.$$

In this paper, we let $\mathbb{R}$ is the set of real numbers and $\mathbb{R}_+ = \{r : r \geq 0\}$.

Definition 2.1 ([9])**.** A nonempty convex subset B of a convex cone $C \neq \{0_X\}$ is called a base for C, if $0_X \notin B$ and every $x \in C \setminus \{0_X\}$ has a unique representation of the form

$$x = \lambda b \; for \; some \; \lambda > 0 \; and \; some \; b \in B.$$

Definition 2.2 ([9])**.** Let S be a nonempty convex subset of X. A set-valued map $F : S \to 2^Y$ is said to be C-convex if and only if for all $x_1, x_2 \in S$ and $\lambda \in [0, 1]$, we have

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subset F(\lambda x_1 + (1 - \lambda)x_2) + C.$$

It is well known that if F is C-convex on S, then gph$(F + C)$ is a convex subset in $X \times Y$.

Definition 2.3 ([2])**.** Let S be a nonempty subset of X, $\hat{x} \in \text{cl}S$. The contingent cone $T(S, \hat{x})$ of S at $\hat{x}$, is given by $T(S, \hat{x}) := \{v \in X \mid \exists t_n \downarrow 0, \exists v_n \to v, \text{ s.t. } \hat{x} + t_n v_n \in S, \ \forall n \in N\}$. Or equivalently, $T(S, \hat{x}) := \{v \in X \mid \exists \lambda_n \to +\infty, \exists x_n \in S, \ \text{ s.t. } x_n \to \hat{x} \textit{ and } \lambda_n(x_n - \hat{x}) \to v\}$.

Definition 2.4 ([2])**.** Let S be a nonempty subset of X, $\hat{x} \in \text{cl}S$. The second-order contingent set $T^2(S, \hat{x}, \omega)$ of S at $\hat{x}$ in the direction $\omega \in X$, is given by $T^2(S, \hat{x}, \omega) := \limsup\limits_{t \downarrow 0} \frac{S - \hat{x} - t\omega}{\frac{1}{2}t^2}$. Or equivalently, $T^2(S, \hat{x}, \omega) := \{v \in X \mid \exists t_n \downarrow 0, \exists v_n \to v, \text{ such that } \hat{x} + t_n\omega + \frac{1}{2}t_n^2 v_n \in S, \ \forall n \in N\}$.

Proposition 2.5 ([10, 12])**.** *Let S be a convex subset of X, $\hat{x} \in clS$ and $\omega \in T(S, \hat{x})$. Then*

$$T(T(S, \hat{x}), \omega) = \text{cl}(\text{cone}(\text{cone}(S - \hat{x}) - \omega)),$$

and

$$T^2(S, \hat{x}, \omega) \subset T(T(S, \hat{x}), \omega).$$

Additionally, if $0_X \in T^2(S, \hat{x}, \omega)$, then

$$T^2(S, \hat{x}, \omega) = T(T(S, \hat{x}), \omega).$$

Where $T(T(S, \hat{x}), \omega)$ is called second-order composed contingent cone.

Definition 2.6 ([2])**.** Let $F : X \to 2^Y$ be a set-valued map. The contingent derivative $DF(\hat{x}, \hat{y})$ of F at $(\hat{x}, \hat{y}) \in \text{gph}F$ is the set-valued map from X to Y defined by

$$\text{gph}DF(\hat{x}, \hat{y}) = T(\text{gph}F, (\hat{x}, \hat{y})).$$

Definition 2.7 ([2])**.** Let $F : X \to 2^Y$ be a set-valued map. The second-order contingent derivative $D^2F(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ of F at $(\hat{x}, \hat{y}) \in \text{gph}F$ in the direction $(\hat{u}, \hat{v}) \in X \times Y$ is the set-valued map X to Y defined by

$$\text{gph}D^2F(\hat{x}, \hat{y}, \hat{u}, \hat{v}) = T^2(\text{gph}F, (\hat{x}, \hat{y}), (\hat{u}, \hat{v})).$$

Definition 2.4 and Proposition 2.5 show that the second order contingent set $T^2(S, \hat{x}, \omega)$ is not necessarily a cone and even is not convex although S is convex . However, contingent cone $T(S, \hat{x})$ and second-order composed contingent cone $T(T(S, \hat{x}), \omega)$ are always closed cones, and especially, are convex when S is convex. So Khan [11] and Zhu [31] proposed a new notion of second-order composed contingent derivative for set-valued maps by terms of the second-order composed contingent cone.

Definition 2.8 ([11, 31])**.** Let $F : X \to 2^Y$ be a set-valued map. The second-order composed contingent derivative $D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ of F at $(\hat{x}, \hat{y}) \in gphF$ in the direction $(\hat{u}, \hat{v}) \in X \times Y$ is the set-valued map X to Y defined by

$$\mathrm{gph}D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v}) = T(T(\mathrm{gph}F, (\hat{x}, \hat{y})), (\hat{u}, \hat{v})).$$

Lemma 2.9. *If* $(\hat{x}, \hat{y}) \in \mathrm{gph}F$ *and* $(\hat{u}, \hat{v}) \in X \times Y$*, then* $D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x)$ *is closed for every* $x \in \mathrm{dom}D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})$.

Proof. It follows dircctly from Proposition 2.5 and Definition 2.8.

Definition 2.10 ([1])**.** A set-valued map $F : X \to 2^Y$ is called lower semicontinuous (l.s.c) at $\hat{x} \in X$ if for any sequence $x_n \in X$ satisfying $x_n \to \hat{x}$ and any $\hat{y} \in F(\hat{x})$, there exists a sequence $y_n \in F(x_n)$ such that $y_n \to \hat{y}$. We say F is said to be lower semicontinuous on $S \subseteq X$ if F is l.s.c. at every point $x \in S$.

Lemma 2.11 ([25])**.** *If* $F : X \to 2^Y$ *is a* C*-convex set-valued map and* $x \in \mathrm{int}X$*, then* $F + C$ *is lower semicontinuous at* x*.*

Definition 2.12 ([2])**.** Let A be a nonempty subset of Y. A point $\bar{x} \in A$ is called a minimal point of A if $A \cap (\bar{x} - C) = \{\bar{x}\}$. The set of all minimal points of A is denoted by $\mathrm{Min_C}A$.

Definition 2.13 ([2])**.** Let A be a nonempty subset of Y. A point $\bar{x} \in A$ is called a weakly minimal point of A if $A \cap (\bar{x} - \mathrm{int}C) = \emptyset$. The set of all weakly minimal points of A is denoted by $\mathrm{WMin_C}A$.

Definition 2.14 ([3, 12])**.** Let A be a nonempty subset of Y. $\bar{y} \in A$ is called a Benson proper minimal point of A, written as $\bar{y} \in \mathrm{PrMin}(A, C)$, if

$$\mathrm{clcone}(A + C - \bar{y}) \cap (-C) = \{0\}.$$

Remark 2.15. $\mathrm{PrMin}(A, C) \subseteq \mathrm{Min_C}A \subseteq \mathrm{WMin_C}A$. However, neither of the inverse inclusions is true, as is shown in the following example.

Example 2.16. Consider the set $A := \{(y_1, y_2) \in [0, 2] \times [0, 2] \mid y_2 \geq 1 - \sqrt{1 - (y_1 - 1)^2}$ *for* $y_1 \in [0, 1]\}$ in $Y := \mathbb{R}^2$ with the natural ording cone $C := \mathbb{R}^2_+$. A direct calculation gives

$$\mathrm{Min_C}A := M = \{(y_1, 1 - \sqrt{1 - (y_1 - 1)^2}) \mid y_1 \in [0, 1]\},$$
$$\mathrm{WMin_C}A = M \cup \{(0, y_2) \in \mathbb{R}^2 \mid y_2 \in (1, 2]\} \cup \{(y_1, 0) \in \mathbb{R}^2 \mid y_1 \in (1, 2]\},$$
$$\mathrm{PrMin}(A, C) = M \setminus \{(0, 1), (1, 0)\}.$$

Consequently, we have

$$\mathrm{WMin_C}A \nsubseteq \mathrm{Min_C}A \nsubseteq \mathrm{PrMin}(A, C).$$

Definition 2.17 ([25])**.** Let A be a nonempty subset of X, $\hat{x} \in X$. The normal cone $N_A(\hat{x})$ to A at $\hat{x}$ is the negative polar cone of the tangent cone $T(A, \hat{x})$, i.e.,

$$N_A(\hat{x}) = [T(A, \hat{x})]^0 = \{\varphi \in X^* : \varphi(x) \leq 0,\ \forall x \in T(A, \hat{x})\}.$$

When A is convex and $\hat{x} \in A$, we have $N_A(\hat{x}) = \{\varphi \in X^* : \varphi(\hat{x}) \geq \varphi(x),\ \forall x \in A\}$.

In the following, we introduce a concept that similar to Definition 13.2.4 in [12] (p616).

Definition 2.18. Let $A \subset Y$ be a nonempty subset of Y and $A + C$ be convex. A point $\hat{y} \in \mathrm{PrMin}(A, C)$ is called a normally Benson proper minimal point of A, if

$$N_{A+C}(\hat{y}) \subset \mathrm{int}(C^*) \cup \{0_{Y^*}\},$$

where C^* is the negative dual cone of cone C.

3 Second-order Composed Contingent Derivative of the Perturbation Map

Let F is considered to be a feasible set map from X to Y, where X is the Banach space of perturbation parameter vectors and Y is the objective space. Another set-valued map G from X to Y is denoted by

$$G(x) = \text{PrMin}(F(x), C), \text{ for any } x \in X, \tag{3.1}$$

and call it the (proper) perturbation map. The aim of this section is to discuss the relationship between the second-order composed contingent derivative of G and that of F.

Definition 3.1. F is said to be C-minicomplete by G near $\bar{x}$, if

$$F(x) \subset G(x) + C, \ \forall x \in V(\bar{x}),$$

where $V(\bar{x})$ is a neighborhood of $\bar{x}$.

Definition 3.2 ([18])**.** Let H be a nonempty subset of Y, H is said to hold the C-domination property iff $H \subset \text{Min}_{\text{C}} H + C$.

Lemma 3.3 ([8])**.** *For a cone $K \subset Y$ and its negative dual cone $K^* = \{\varphi \in Y^* : \varphi(k) \leq 0, \forall k \in K\}$, we have $\varphi(k) < 0$ for $\varphi \in Y^* \setminus \{0_{Y^*}\}, \ k \in \text{int}K$, or $\varphi \in \text{int}K^*, \ k \in K \setminus \{0_Y\}$.*

Lemma 3.4. *Let $(\hat{x}, \hat{y}) \in \text{gph}F$ and $(\hat{u}, \hat{v}) \in X \times Y$. Suppose that C has a compact base B. Then*

$$\text{Min}_{\text{C}} D^{''} F_+(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) \subset D^{''} F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x),$$

for any $x \in X$.

Proof. Let

$$y \in \text{Min}_{\text{C}} D^{''} F_+(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x), \tag{3.2}$$

then $y \in D^{''} F_+(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x)$. It follows from Definition 2.8 that $(x, y) \in T(T(\text{gph}(F + C), (\hat{x}, \hat{y})), (\hat{u}, \hat{v}))$. By Definition 2.3, there exist sequences $t_n \to 0^+$ and $(x_n, y_n) \to (x, y)$, such that for $\forall n \in N$,

$$(\hat{u}, \hat{v}) + t_n(x_n, y_n) \in T(\text{gph}(F + C), (\hat{x}, \hat{y})).$$

Moreover, $\forall n \in N$, there exist sequences $t_n^k \to 0^+$ and $(x_n^k, y_n^k) \to (\hat{u}, \hat{v}) + t_n(x_n, y_n)$, such that

$$(\hat{x}, \hat{y}) + t_n^k(x_n^k, y_n^k) \in \text{gph}(F + C), \forall k \in N,$$

that is

$$\hat{y} + t_n^k y_n^k \in F(\hat{x} + t_n^k x_n^k) + C, \forall n, k \in N.$$

Since C has a compact base B, there exist $\alpha_n^k \geq 0$ and $b_n^k \in B$ such that

$$\hat{y} + t_n^k(y_n^k - \frac{\alpha_n^k}{t_n^k} b_n^k) \in F(\hat{x} + t_n^k x_n^k), \tag{3.3}$$

where $b_n^k \to b_n$ and $b_n \to b$. Since B is compact, $b \in B$.

We now show $\frac{\alpha_n^k}{t_n^k} \to 0$. Suppose to the contrary that for some $\varepsilon > 0$, we may assume without loss of generality that $\frac{\alpha_n^k}{t_n^k} \geq \varepsilon$, by taking a subsequence if necessary. Then

$$\begin{aligned}\hat{y} + t_n^k(y_n^k - t_n\varepsilon b_n^k) =& \hat{y} + t_n^k(y_n^k - \frac{\alpha_n^k}{t_n^k} b_n^k) + \alpha_n^k b_n^k - \varepsilon t_n^k t_n b_n^k \\ \in& F(\hat{x} + t_n^k x_n^k) + C.\end{aligned}$$

Since $y_n^k - t_n\varepsilon b_n^k \to \hat{v} + t_n y_n - t_n \varepsilon b_n$, we get

$$(\hat{u}, \hat{v}) + t_n(x_n, (y_n - \varepsilon b_n)) \in T(\mathrm{gph}(F + C), (\hat{x}, \hat{y})).$$

Because $x_n \to x$, $y_n - \varepsilon b_n \to y - \varepsilon b$, one obtains

$$(x, y - \varepsilon b) \in T(T(\mathrm{gph}(F + C), (\hat{x}, \hat{y}), (\hat{u}, \hat{v})),$$

which together with Definition 2.8 gives

$$y - \varepsilon b \in D^{''} F_+(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x).$$

This is a contradiction to the assumption (3.2). Therefore, $\frac{\alpha_n^k}{t_n^k} \to 0$. Then, from (3.3), it follows that

$$y \in D^{''} F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x).$$

Consequently,

$$\mathrm{Min_C} D^{''} F_+(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) \subset D^{''} F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x).$$

The proof is complete. □

Lemma 3.5 ([31])**.** *Let $F : X \to 2^Y$ be a set-valued map. $(\hat{x}, \hat{y}) \in \mathrm{gph}F$ and $(\hat{u}, \hat{v}) \in X \times Y$. Then, for every $x \in \mathrm{dom}D^{''} F(\hat{x}, \hat{y}, \hat{u}, \hat{v})$. we have*

$$D^{''} F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) + C \subset D^{''} F_+(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x).$$

Theorem 3.6. *Let $(\hat{x}, \hat{y}) \in \mathrm{gph}F$ and $(\hat{u}, \hat{v}) \in X \times Y$. Suppose that C has a compact base B and $D^{''} F_+(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x)$ fulfills the C-domination property for all $x \in \Xi := \mathrm{dom}D^{''} F(\hat{x}, \hat{y}, \hat{u}, \hat{v})$. Then, for any $x \in \Xi$,*

$$D^{''} F_+(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) = D^{''} F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) + C. \tag{3.4}$$

Proof. By Lemma 3.5, the conclusion

$$D^{''} F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) + C \subset D^{''} F_+(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x).$$

is provided. We only prove that the inclusion $D^{''} F_+(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) \subset D^{''} F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) + C$. It follows from the C-domination property $D^{''} F_+(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x)$ and Lemma 3.4 that

$$D^{''} F_+(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) \subset \mathrm{Min_C} D^{''} F_+(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) + C \subset D^{''} F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) + C.$$

Thus, this completes the proof. □

Remark 3.7. If the C-domination property of $D^{''} F_+(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x)$ in Theorem 3.6 is not fulfilled, then Theorem 3.6 may not hold. The following example illustrates the case.

Example 3.8. Let $X = \mathbb{R}$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$. A set-valued map $F : X \to 2^Y$ is defined by

$$F(x) = \begin{cases} \{x, -2\}, & if\ x \geq 0, \\ \{-1\}, & if\ x < 0. \end{cases}$$

Let $(\hat{x}, \hat{y}) = (0, 0) \in \text{gph}F$ and $(\hat{u}, \hat{v}) = (1, 1)$. Then, for any $x \geq 0$,

$$D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) = \{y \mid y = x\},\ D^{''}F_+(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) = \mathbb{R}.$$

Clearly, $D^{''}F_+(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x)$ does not satisfy the C-domination property and

$$D^{''}F_+(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) \neq D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) + C.$$

Lemma 3.9. *Let A be a nonempty subset of Y. Then*

$$\text{PrMin}(A, C) = \text{PrMin}(A + C, C)$$

Proof. Let $\bar{y} \in \text{PrMin}(A, C)$, by the definition 2.14, then

$$\text{clcone}(A + C - \bar{y}) \cap (-C) = \{0\}.$$

Since C is a convex cone, then $C + C = C$, one obtains

$$\text{clcone}(A + C + C - \bar{y}) \cap (-C) = \{0\}.$$

Because of $\bar{y} \in A \subset A + C$,

$$\bar{y} \in \text{PrMin}(A + C, C).$$

On the other hand, let $\bar{y} \in \text{PrMin}(A + C, C)$, then $\bar{y} \in A + C$ and

$$\text{clcone}(A + C + C - \bar{y}) \cap (-C) = \{0\}. \tag{3.5}$$

It follows from $0 \in C$ that

$$\text{clcone}(A + C - \bar{y}) \cap (-C) = \{0\}.$$

In the following, we prove $\bar{y} \in A$. If we assume that $\bar{y} \notin A$, then there exists $\hat{y} \in A$ with $\hat{y} \neq \bar{y}$ such that $\bar{y} \in \{\hat{y}\} + C$. Then,

$$0 \neq \hat{y} - \bar{y} \in \text{clcone}(A + C + C - \bar{y}) \cap (-C),$$

which contradicts (3.5). Therefore,

$$\bar{y} \in \text{PrMin}(A, C).$$

□

Theorem 3.10. *Let $(\hat{x}, \hat{y}) \in \text{gph}F$ and $(\hat{u}, \hat{v}) \in X \times Y$. Suppose that F is C-minicomplete by G near $\hat{x}$, C has a compact base B and $D^{''}F_+(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x)$ fulfills the C-domination property for all $x \in \Xi := \text{dom}D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})$. Then, for any $x \in \Xi$,*

$$\text{PrMin}(D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x), C) \subset D^{''}G(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x).$$

Proof. Firstly, we prove that

$$D^{''}F_+(\hat{x},\hat{y},\hat{u},\hat{v})(x) = D^{''}G_+(\hat{x},\hat{y},\hat{u},\hat{v})(x).$$

Since $G(x) \subset F(x)$ and F is C-minicompleteness by G near $\hat{x}$, there exists a neighborhood $V(\hat{x})$ of $\hat{x}$ such that

$$G(x) + C = F(x) + C, \quad \forall x \in V(\hat{x}).$$

Hence, for any $\hat{y} \in G(\hat{x}),\ x \in V(\hat{x})$,

$$D^{''}F_+(\hat{x},\hat{y},\hat{u},\hat{v})(x) = D^{''}G_+(\hat{x},\hat{y},\hat{u},\hat{v})(x), \tag{3.6}$$

which implies that, for any $\hat{y} \in G(\hat{x}),\ x \in V(\hat{x})$,

$$\mathrm{Min_C} D^{''}F_+(\hat{x},\hat{y},\hat{u},\hat{v})(x) = \mathrm{Min_C} D^{''}G_+(\hat{x},\hat{y},\hat{u},\hat{v})(x). \tag{3.7}$$

In what follows, we prove

$$D^{''}G_+(\hat{x},\hat{y},\hat{u},\hat{v})(x) = D^{''}G(\hat{x},\hat{y},\hat{u},\hat{v})(x) + C. \tag{3.8}$$

Lemma 3.5 applies to G, we conclude that

$$D^{''}G(\hat{x},\hat{y},\hat{u},\hat{v})(x) + C \subset D^{''}G_+(\hat{x},\hat{y},\hat{u},\hat{v})(x). \tag{3.9}$$

The proof of the inverse inclusion $D^{''}G_+(\hat{x},\hat{y},\hat{u},\hat{v})(x) \subset D^{''}G(\hat{x},\hat{y},\hat{u},\hat{v})(x) + C$ is given in the following. By Lemma 3.4, one obtains

$$\mathrm{Min_C} D^{''}G_+(\hat{x},\hat{y},\hat{u},\hat{v})(x) \subset D^{''}G(\hat{x},\hat{y},\hat{u},\hat{v})(x). \tag{3.10}$$

Noticing that the C-domination property of $D^{''}F_+(\hat{x},\hat{y},\hat{u},\hat{v})(x)$ and (3.6), (3.7), (3.10), we obtians

$$\begin{aligned}
&D^{''}G_+(\hat{x},\hat{y},\hat{u},\hat{v})(x)\\
=&D^{''}F_+(\hat{x},\hat{y},\hat{u},\hat{v})(x)\\
\subset&\mathrm{Min_C} D^{''}F_+(\hat{x},\hat{y},\hat{u},\hat{v})(x) + C\\
=&\mathrm{Min_C} D^{''}G_+(\hat{x},\hat{y},\hat{u},\hat{v})(x) + C\\
\subset&D^{''}G(\hat{x},\hat{y},\hat{u},\hat{v})(x) + C,
\end{aligned}$$

which together with (3.9), (3.8) holds.

From (3.6), (3.8), Theorem 3.6 and Lemma 3.9, it follows that

$$\begin{aligned}
&\mathrm{PrMin}(D^{''}F(\hat{x},\hat{y},\hat{u},\hat{v})(x), C)\\
=&\mathrm{PrMin}(D^{''}F(\hat{x},\hat{y},\hat{u},\hat{v})(x) + C, C)\\
=&\mathrm{PrMin}(D^{''}F_+(\hat{x},\hat{y},\hat{u},\hat{v})(x), C)\\
=&\mathrm{PrMin}(D^{''}G_+(\hat{x},\hat{y},\hat{u},\hat{v})(x), C)\\
=&\mathrm{PrMin}(D^{''}G(\hat{x},\hat{y},\hat{u},\hat{v})(x) + C, C)\\
=&\mathrm{PrMin}(D^{''}G(\hat{x},\hat{y},\hat{u},\hat{v})(x), C)\\
&\subset D^{''}G(\hat{x},\hat{y},\hat{u},\hat{v})(x).
\end{aligned}$$

□

The following example explains C-minicompleteness of F in Theorem 3.10 cannot be omitted.

Example 3.11. Let $X = \mathbb{R}$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$. Consider the set-valued map $F : X \to 2^Y$ defined by

$$F(x) = \{y \mid y > |x|\} \cup \{0\}, \forall x \in X.$$

Let $(\hat{x}, \hat{y}) = (0, 0) \in gphF$ and $(\hat{u}, \hat{v}) = (-1, 1)$, a direct calculation gives

$$G(x) = \{(0, 0)\},\ \forall x \in X.$$

Since

$$T(\text{gph}F, (\hat{x}, \hat{y})) = \{(x, y) \mid y \geq |x|\},$$

and

$$T(T(\text{gph}F, (\hat{x}, \hat{y})), (\hat{u}, \hat{v})) = \{(x, y) \mid y \geq -x\},$$

$$D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) = \{y \mid y \geq -x\},\ \forall x \in X,$$

$$D^{''}G(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) = \emptyset,\ \forall x \in X,$$

$$\text{PrMin}(D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x), C) = \{y \mid y = -x\},\ \forall x \in X.$$

Obviously, F is not C-minicomplete by G near $\hat{x}$ and for any $x \in X$,

$$\text{PrMin}(D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x), C) \not\subset D^{''}G(\hat{x}, \hat{y}, \hat{u}, \hat{v}(x).$$

Now, we provide an example to explain Theorem 3.10.

Example 3.12. Let $X = \mathbb{R}$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$ and $F : X \to 2^Y$ be a set-valued map with

$$F(x) = \{y \mid y \geq |x|\}, \forall x \in X.$$

Then

$$G(x) = \{y \mid y = |x|\},\ \forall x \in X.$$

Take $(\hat{x}, \hat{y}) = (1, 1) \in gphF$ and $(\hat{u}, \hat{v}) = (0, 0)$. By directly calculating, one has

$$T(\text{gph}F, (\hat{x}, \hat{y})) = \{(x, y) \mid y \geq x\},$$

$$T(T(\text{gph}F, (\hat{x}, \hat{y})), (\hat{u}, \hat{v})) = \{(x, y) \mid y \geq x\},$$

$$D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) = \{y \mid y \geq x\},\ \forall x \in X,$$

$$D^{''}G(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) = \{y \mid y = x\},\ \forall x \in X,$$

$$\text{PrMin}(D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x), C) = \{y \mid y = x\},\ \forall x \in X.$$

Consequently,

$$\text{PrMin}(D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x), C) \subset D^{''}G(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x).$$

Theorem 3.13. *Let $F : X \to 2^Y$ be a set-valued map, $(\hat{x}, \hat{y}) \in gphF$ and $(\hat{u}, \hat{v}) \in X \times Y$. C has a compact base B, X^* and Y^* are * weak compact. Suppose that the following assumptions hold:*

(i) *F is C-convex and $\hat{x} \in \text{int}(\text{dom}F)$.*

(ii) *$\hat{y}$ is a normally Benson proper efficient point of $F(\hat{x})$.*

Then, for every $x \in \mathrm{dom} D^{''}F(\hat{x},\hat{y},\hat{u},\hat{v})$*, we have*

$$D^{''}G(\hat{x},\hat{y},\hat{u},\hat{v})(x) \subset \mathrm{PrMin}(D^{''}F(\hat{x},\hat{y},\hat{u},\hat{v})(x), C).$$

Proof. Let $y \in D^{''}G(\hat{x},\hat{y},\hat{u},\hat{v})(x)$ which implies that $y \in D^{''}F(\hat{x},\hat{y},\hat{u},\hat{v})(x)$. Now suppose that $y \notin \mathrm{PMin}(D^{''}F(\hat{x},\hat{y},\hat{u},\hat{v})(x), C)$. Then

$$\mathrm{clcone}(D^{''}F(\hat{x},\hat{y},\hat{u},\hat{v})(x) + C - y) \cap (-C \setminus \{0_Y\}) \neq \emptyset. \tag{3.11}$$

Since $y \in D^{''}G(\hat{x},\hat{y},\hat{u},\hat{v})(x)$, there exist sequences $t_n \to 0^+$ and $(x_n, y_n) \to (x, y)$, such that

$$(\hat{u},\hat{v}) + t_n(x_n, y_n) \in T(\mathrm{gph}G, (\hat{x},\hat{y})), \ \forall n \in N.$$

Moreover, $\forall n \in N$, there exist sequences $t_n^k \to 0^+$ and $(x_n^k, y_n^k) \to (\hat{u},\hat{v}) + t_n(x_n, y_n)$, such that

$$(\hat{x},\hat{y}) + t_n^k(x_n^k, y_n^k) \in \mathrm{gph}G, \ \forall k \in N,$$

implying

$$\hat{y} + t_n^k y_n^k \in G(\hat{x} + t_n^k x_n^k) = \mathrm{PrMin}(F(\hat{x} + t_n^k x_n^k), C), \quad \forall n, k \in N.$$

Therefore, $(\hat{x} + t_n^k x_n^k, \hat{y} + t_n^k y_n^k)$ is a boundary point of the convex set $\mathrm{gph}(F + C)$. By a separation theorem for convex sets, there exists $(\varphi_n^k, \phi_n^k) \in X^* \times Y^* \setminus \{(0_{X^*}, 0_{Y^*})\}$, such that

$$\varphi_n^k(\hat{x} + t_n^k x_n^k) + \phi_n^k(\hat{y} + t_n^k y_n^k) \geq \varphi_n^k(x') + \phi_n^k(y'), \ \forall (x', y') \in \mathrm{gph}(F + C). \tag{3.12}$$

We normalize these vectors without loss of generality so that $\|(\varphi_n^k, \phi_n^k)\| = 1$. By the assumption that X^* and Y^* are * weak compact, we may assume without loss of generality that $(\varphi_n^k, \phi_n^k) \to^{*w} (\varphi_n, \phi_n), (k \to \infty)$. Passing to the limit as $k \to \infty$ in (3.12), we obtain

$$\varphi_n(\hat{x}) + \phi_n(\hat{y}) \geq \varphi_n(x') + \phi_n(y'), \ \forall (x', y') \in \mathrm{gph}(F + C). \tag{3.13}$$

Because X^* and Y^* are * weak compact, in the similar way, we may assume that $(\varphi_n, \phi_n) \to^{*w} (\varphi, \phi) \neq (0_{X^*}, 0_{Y^*}), (n \to \infty)$. By taking the limit as $n \to \infty$ in (3.13), one has

$$\varphi(\hat{x}) + \phi(\hat{y}) \geq \varphi(x') + \phi(y'), \ \forall (x', y') \in \mathrm{gph}(F + C).$$

Let $\tilde{y} \in F(\hat{x}) + C$. From the assumption (i) and Lemma 2.11, it follows that $F + C$ is lower semicontinuous at $\hat{x}$. Therefore, there exists a sequence $\tilde{y}_n^k \in F(\hat{x} + t_n^k x_n^k) + C$, such that $\tilde{y}_n^k \to \tilde{y}$. By setting $(x', y') = (\hat{x} + t_n^k x_n^k, \tilde{y}_n^k)$ in (3.12), we get that

$$\varphi_n^k(\hat{x} + t_n^k x_n^k) + \phi_n^k(\hat{y} + t_n^k y_n^k) \geq \varphi_n^k(\hat{x} + t_n^k x_n^k) + \phi_n^k(\tilde{y}_n^k).$$

Let $k \to \infty$, we give

$$\phi_n(\hat{y}) \geq \phi_n(\tilde{y}).$$

By taking the limit as $n \to \infty$, one obtains

$$\phi(\hat{y}) \geq \phi(\tilde{y}).$$

In view of the Definition 2.17, we have

$$\phi \in N_{F(\hat{x})+C}(\hat{y}).$$

Moreover, by the assumption (ii) and Definition 2.18, we deduce that

$$\phi \in \text{int}C^* \cup \{0_{Y^*}\}.$$

However, because of the assumption that $\hat{x} \in \text{int}(\text{dom}F)$, we conclude $\phi \neq 0_{Y^*}$, assuring that

$$\phi \in \text{int}C^*. \tag{3.14}$$

According to (3.11), let $\gamma \in \text{clcone}(D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) + C - y)$, then there exists $\bar{y}_n \in D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x)$ and $c_n \in C, \gamma_n > 0$, such that

$$\gamma = \lim_{n\to\infty} \gamma_n(\bar{y}_n + c_n - y) \in -C \setminus \{0_Y\}. \tag{3.15}$$

It follows from (3.14), (3.15) and Lemma 3.5 that

$$\phi(\gamma) = \lim_{n\to\infty} \gamma_n(\phi(\bar{y}_n) + \phi(c_n) - \phi(y)) > 0,$$

which together with Lemma 2.9, there exists $\bar{y} \in D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x)$, $\bar{y}_n \longrightarrow \bar{y}$, such that $n \longrightarrow \infty$,

$$\phi(\bar{y}) - \phi(y) > 0 \tag{3.16}$$

holds. Analogously, because of $\bar{y} \in D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x)$, this means that there are sequences $\bar{t}_n \to 0^+$ and $(\bar{x}_n, \bar{y}_n) \to (x, \bar{y})$, such that

$$(\hat{u}, \hat{v}) + \bar{t}_n(\bar{x}_n, \bar{y}_n) \in T(\text{gph}F, (\hat{x}, \hat{y})),\ \forall n \in N.$$

Moreover, $\forall n \in N$, there exist sequences $\bar{t}_n^k \to 0^+$ and $(\bar{x}_n^k, \bar{y}_n^k) \to (\hat{u}, \hat{v}) + \bar{t}_n(\bar{x}_n, \bar{y}_n)$, such that

$$(\hat{x}, \hat{y}) + \bar{t}_n^k(\bar{x}_n^k, \bar{y}_n^k) \in \text{gph}F,\ \forall k \in N,$$

that is

$$\hat{y} + \bar{t}_n^k \bar{y}_n^k \in F(\hat{x} + \bar{t}_n^k \bar{x}_n^k),\ \forall n, k \in N.$$

Since $t_n \to 0^+$, $t_n^k \to 0^+$, we may assume that $t_n \leq \bar{t}_n$, $t_n^k \leq \bar{t}_n^k$ by taking a subsequence if necessary.

Because $\hat{y} + t_n^k y_n^k \in F(\hat{x} + t_n^k x_n^k), \hat{y} + \bar{t}_n^k \bar{y}_n^k \in F(\hat{x} + \bar{t}_n^k \bar{x}_n^k)$, and C-convexity of F at $\hat{x}$, one has

$$\theta_n^k(\hat{y} + t_n^k y_n^k) + (1 - \theta_n^k)(\hat{y} + \bar{t}_n^k \bar{y}_n^k) \in F(\theta_n^k(\hat{x} + t_n^k x_n^k) + (1 - \theta_n^k)(\hat{x} + \bar{t}_n^k \bar{x}_n^k)) + C,$$

where $\theta_n^k = \frac{t_n^k}{t_n^k + \bar{t}_n^k} \in (0, 1)$, for sufficiently large k. From (3.12), we get

$$\begin{aligned}\varphi_n^k(\hat{x} + t_n^k x_n^k) + \phi_n^k(\hat{y} + t_n^k y_n^k) \geq &\varphi_n^k(\theta_n^k(\hat{x} + t_n^k x_n^k) + (1 - \theta_n^k)(\hat{x} + \bar{t}_n^k \bar{x}_n^k)) \\ &+\phi_n^k(\theta_n^k(\hat{y} + t_n^k y_n^k) + (1 - \theta_n^k)(\hat{y} + \bar{t}_n^k \bar{y}_n^k)),\end{aligned}$$

hence

$$t_n^k(\varphi_n^k(x_n^k) + \phi_n^k(y_n^k)) \geq \bar{t}_n^k(\varphi_n^k(\bar{x}_n^k) + \phi_n^k(\bar{y}_n^k)). \tag{3.17}$$

On the other hand, since $(\hat{x}, \hat{y}) \in gph(F + C)$, it follows from (3.12) that

$$\varphi_n^k(x_n^k) + \phi_n^k(y_n^k) \geq 0.$$

Since $\bar{t}_n^k \geq t_n^k$, one has

$$\bar{t}_n^k(\varphi_n^k(x_n^k) + \phi_n^k(y_n^k)) \geq t_n^k(\varphi_n^k(x_n^k) + \phi_n^k(y_n^k)). \quad (3.18)$$

From (3.17) and (3.18), it follows that

$$\varphi_n^k(x_n^k) + \phi_n^k(y_n^k) \geq \varphi_n^k(\bar{x}_n^k) + \phi_n^k(\bar{y}_n^k).$$

By taking the limit as $k \to \infty$ and $t_n \leq \bar{t}_n$, one obtians

$$\varphi_n(x_n) + \phi_n(y_n) \geq \varphi(\bar{x}_n) + \phi_n(\bar{y}_n).$$

And let $n \to \infty$, we have

$$\phi(y) \geq \phi(\bar{y}),$$

which contradicts (3.16). Consequently

$$y \in \mathrm{PrMin}(D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x), C).$$

The proof is complete. □

To illustrate Theorem 3.13, we give the following example.

Example 3.14. Let $X = \mathbb{R}$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$ and $F : X \to 2^Y$ be defined by

$$F(x) = \{y \mid y \geq x^2\}, \forall x \in X.$$

Then

$$G(x) = \{y \mid y = x^2\},\ \forall x \in X.$$

Take $(\hat{x}, \hat{y}) = (0, 0) \in gphF$ and $(\hat{u}, \hat{v}) = (0, 0)$. By calculating, we get

$$T(\mathrm{gph}F, (\hat{x}, \hat{y})) = \{(x, y) \mid y \geq 0\},$$
$$T(T(\mathrm{gph}F, (\hat{x}, \hat{y})), (\hat{u}, \hat{v})) = \{(x, y) \mid y \geq 0\},$$
$$D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) = \{y \mid y \geq 0\},\ \forall x \in X,$$
$$D^{''}G(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) = \{y \mid y = 0\},\ \forall x \in X,$$
$$\mathrm{PrMin}(D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x), C) = \{y \mid y = 0\},\ \forall x \in X.$$

Obviously,

$$D^{''}G(\hat{x}, \hat{y}, \hat{u}, \hat{v}(x)) \subset \mathrm{PrMin}(D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x), C).$$

Remark 3.15. Theorem 3.13 doesn't contain the assumption of gph$F$-derivability. therefore, the proof of the Theorem is different from Theorem 3.3 in [30].

4 Conclusions and Perspectives

In this paper, we studied the sensitivity of second-order composed contingent derivative for Benson proper perturbation maps. Relationships between the second-order composed contingent derivative of proper perturbation maps G and the set of Benson proper minimal points of $D^{''}F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x)$ were discussed under some assumptions. It is well known that the range of the set of Benson proper minimal points is smaller than (weakly) minimal points, therefore, the study of Benson proper efficient points makes more sense. Second-order composed contingent derivatives that introduced by the second-order composed contingent cone has some special properties that second-order contingent derivative doesn't own. Hence, we can explore the weaker condition to study the sensitivity for second-order composed contingent derivative for further works. Moreover, stability results of second-order composed contingent derivative for Benson perturbation maps may be a great of interests.

References

[1] J.P. Aubin and I. Ekeland, *Applied Nonlinear Analysis,* John Wiley and Sons, New York, 1984.

[2] J.P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.

[3] H.P. Benson, An improved definition of proper efficiency for vector maximization with respect to cones, *J. Math. Anal. Appl.* 71 (1979) 232–241.

[4] J.M. Borwein and D. Zhuang, Super efficiency in vector optimization, *Trans. Amer. Math. Soc.* 338 (1993) 105–122.

[5] A.V. Fiacco, *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming*, Academic Press, New York, 1983.

[6] A. Götz and J. Jahn, The Lagrange multiplier rule in set-valued optimization, *SIAM J. Optim.* 10 (1999) 331–344.

[7] M.I. Henig, Proper efficiency with respect to cones, *J. Optim. Theory Appl.* 36 (1982) 387–407.

[8] Y.D. Hu and C. Ling, Connectedness of cone superefficient point sets in locally convex topological vector spaces, *J. Optim. Theory Appl.* 107 (2000) 433–446.

[9] J. Jahn, *Vector Optimization: Theory, Applications, and Extensions*, Springer-Verlag, Berlin Heidelberg, 2004.

[10] B. Jim └ nez and V. Novo, Optimality conditions in differentiable vector optimization via second-order tangent sets, *Appl. Math. Optim.* 49 (2004) 123–144.

[11] A.A. Khan and G. Isac, Second-order optimality conditions in set-valued optimization by a new tangential derivative, *Acta Math. Vietnam.* 34 (2009) 81-90.

[12] A.A. Khan, C. Tammer and C. Zalinescu, *Set-valued optimization: An introduction with applications,* Springer-Verlag, Berlin Heidelberg, 2015.

[13] J. Klose, Sensitivity analysis using the tangent derivative, *Numer. Funct. Anal. Optim.* 13 (1992) 143–153.

[14] H. Kuk, T. Tanino and M. Tanaka, Sensitivity analysis in parametrized convex vector optimization, *J. Math. Anal. Appl.* 202 (1996) 511–522.

[15] H. Kuk, T. Tanino and M. Tanaka, Sensitivity analysis in vector optimization, *J. Optim. Theory Appl.* 89 (1996) 713–730.

[16] S.J. Li, Sensitivity and stability for contingent derivative in multiobjective optimization, *Math. Appl.* 11 (1998) 49–53.

[17] Z.F. Li, Benson Proper Efficiency in the Vector Optimization of Set-Valued Maps, *J. Optim. Theory Appl.* 98 (1998) 623–649.

[18] D.T. Luc, Theory of vector optimization. Springer-Verlag, Berlin Heidelberg, 1989.

[19] Z.H. Peng, Z.P. Wan and W.Z. Xiong, Sensitivity analysis in set-valued optimization under strictly minimal efficiency, *Evol. Equ. Control Theory.* 6 (2017) 427–436.

[20] R.T. Rockafellar, Lagrange multipliers and subderivatives of optimal value functions in nonlinear programming, *Math. Programming Study.* 17 (1982) 28–66.

[21] D.S. Shi, Contingent derivative of the perturbation map in multiobjective optimization, *J. Optim. Theory Appl.* 70 (1991) 385–396.

[22] D.S. Shi, Sensitivity analysis in convex vector optimization, *J. Optim. Theory Appl.* 77 (1993) 145–159.

[23] X.K. Sun and S.J. Li, Stability analysis for higher-order adjacent derivative in parametrized vector optimization, *J. Inequal. Appl.* 2010 (2010) Article ID 510838 .

[24] T. Tanino, Sensitivity analysis in multiobjective optimization, *J. Optim. Theory Appl.* 56 (1988) 479–499.

[25] T. Tanino, Stability and sensitivity analysis in convex vector optimization, *SIAM J. Control Optim.* 26 (1988) 521–536.

[26] L.T. Tung, Higher-order contingent derivatives of perturbation maps in multiobjective optimization, *J. Nonlinear Funct. Anal.* 2015 (2015) Article ID 19 .

[27] Q.L. Wang, Continuity of second-order contingent derivative of the efficient set map for parametrized multiobjective optimization, *Positivity.* 17 (2013) 415–429.

[28] Q.L. Wang and S.J. Li, Second-order contingent derivative of the pertubation map in multiobjective optimization, *Fixed Point Theory Appl.* 2011 (2011) Article ID 857520.

[29] Q.L. Wang and S.J. Li, Sensitivity and stability for the second-order contingent derivative of the proper perturbation map in vector optimization, *Optim. Lett.* 6 (2012) 731–748.

[30] Y.H. Xu and Z.H. Peng, Higher-order sensitivity analysis in set-valued optimization under henig efficiency, *J. Ind. Manag. Optim.* 13 (2017) 313-327.

[31] S.K. Zhu, S.J. Li and K.L. Teo, Second-order Karush-Kuhn-Tucker optimality conditions for set-valued optimization, *J. Global Optim.* 58 (2014) 673–679.

Manuscript received 17 November 2016
revised 7 August 2017
accepted for publication October 17 2017

Hongwei Liang
School of Mathematics and Statistics
Wuhan University, Wuhan 430072, P.R.China
E-mail address: hongweiliang@whu.edu.cn

Zhongping Wan
School of Mathematics and Statistics
Computational Science Hubei Key Laboratory
Wuhan University, Wuhan 430072, P.R.China
E-mail address: zpwan-whu@126.com

Qilong He
Economic and Management school of Wuhan University
Wuhan 430072, P.R.China
E-mail address: 2014101050076@whu.edu.cn