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# ON THE SOLUTION EXISTANCE OF CAUCHY TENSOR VARIATIONAL INEQUALITY PROBLEMS*

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**Abstract:** In this paper, we consider the solution existence of Cauchy tensor variational inequality problem. With the help of the homotopy mapping and degree theory, we establish the solution existence of the problem, and based on the tensor positive (semi-)definiteness, we explore the solution set structure.

**Key words:** *Cauchy tensor; tensor variational inequality; positive definiteness; necessary and sufficient condition*

**Mathematics Subject Classification:** *15A18, 15A69, 65F15, 65F10*

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## 1 Introduction

Tensor is a higher-order extension of matrix which finds wide applications in higher-order statistics [2], signal and image processing[15, 19], continuum phiysics[23, 17], blind source separation[12, 27], and it receives much attention of researchers in recent decade [1, 3, 4, 5, 6, 8, 10, 11, 14, 13, 16, 18, 20, 21, 22, 24, 25, 26]. As the research continues, tensor analysis and computing are now developed into a new research branch in mathematics named multilinear algebra.

Let $\mathcal{A} = (a_{i_1\ldots i_m})$ be an $m$-th order $n$-dimensional tensor, $q \in R^n$, and $\Omega$ be a bounded closed convex set in $R^n$. Consider the tensor variational inequality problem (TVIP) of finding a vector $x \in \Omega$ such that

$$(y - x)^T F(x) \geq 0, \quad \forall\, y \in \Omega, \tag{1.1}$$

where $F(x) = \mathcal{A}x^{m-1} + q$ and $\mathcal{A}x^{m-1} \in R^n$ with $i$-th element being that

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}, \quad i = 1, 2, \cdots, n.$$

For convenience, we denote the set of the $m$-th order $n$-dimensional tensor by $T_{m,n}$ for abbreviation.

Obviously, the TVIP is a new type of tensor computing problem. In a certain sense, the TVIP is an optimal condition of a constrained tensor optimization problem, and on the other

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hand, it is a special type of the classical variational inequality problem (VIP) introduced by Hartman and Stampacchia [9]. The VIP was widely and well studied around 2000, see [7] and references therein.

It is well known that for a mathematical problem, the solution existence is an important issue. For the VIP, there are many theoretical results on the solution existence obtained by an analytic approach, e.g., the existence and uniqueness of a solution to the VIP is established under the strong monotonicity of the underlying mapping, by virtue of the degree theory and set-valued analysis. Further, under the pseudo-monotonicity of the underlying function, the convexity of the solution set of VIP is established [7]. Since the TVIP is special type of the VIP, it is natural to ask whether we can establish the solution existence of the problem based on the tensor structure? and if it has a solution, what is its structure? This constitutes the motivation of the paper.

The main contribution of this paper is as follows. First, we show that if the underlying tensor is a positive definite Cauchy tensor, then the corresponding TVIP$(\mathcal{A}, q)$ has at most one solution for any nonempty set $\Omega$ in $R^n$ and any $q \in R^n$. Second, if the underlying tensor is a positive semi-definite Cauchy tensor, then the solution set of the corresponding TVIP$(\mathcal{A}, q)$ is convex. Finally, for the second case, we establish some equivalent conditions for the TVIP$(\mathcal{A}, q)$ to have a solution.

The remainder of this paper is organized as follows. In Section 2, we give some basic definitions and introduce a specially structured tensor. We also explore the monotonicity of function $F(x) = \mathcal{A}x^{m-1} + q$ in virtue of the positive (semi-)definiteness. In Section 3, we first investigate the existence of solution of the tensor variational inequality problems with some milder conditions, then we discuss the solution set structure in virtue of the positive (semi-)definiteness of a Cauchy tensor.

To end this section, we give some notations used in this paper. Throughout the paper, we use Greek letters $\alpha, \beta, \cdots$ for scalars, use lower case letters $x, y, \cdots$ for vectors, use capital letters $A, B, \cdots$ for matrices, and use bold and calligraphic letters $\mathcal{A}, \mathcal{B}, \cdots$ for tensors. Further, we use $[n]$ to denote index set $\{1, 2, \cdots, n\}$, and write $x \geq 0$ $(x > 0)$ to mean that every entry of vector $x$ is nonnegative (positive). The topological boundary and the topological closure of a set $\Omega$ is denoted by $\mathrm{bd}(\Omega)$ and $\mathrm{cl}(\Omega)$, respectively.

## $\boxed{2}$ Preliminaries

In this section, we first present some definitions developed in tensor analysis, introduce some specially structured tensors, and then establish some monotonicity properties of function $F(x) = \mathcal{A}x^{m-1} + q$ via the positive (semi-)definiteness of tensor $\mathcal{A}$.

**Definition 2.1.** [5] For $c = (c_1, c_2, \cdots, c_n) \in R^n$, tensor $C = (c_{i_1 \cdots i_m})$ with entries

$$c_{i_1 \cdots i_m} = \frac{1}{c_{i_1} + c_{i_2} + \cdots + c_{i_m}}, \quad j \in [m], i_j \in [n],$$

is said to be a Cauchy tensor and vector $c \in R^n$ is said to be the generating vector of the tensor.

In the following, we use $\mathcal{C}_{m,n}$ to denote the Cauchy tensor set of order $m$ dimensional $n$. It is easy to see that any Cauchy tensor is symmetric in the sense that its any entries $c_{ii_2 \cdots i_m}$ are invariant under any permutation of their indices [11].

To discuss the monotonicity of the underlying function in the TVIP, we recall the definition of positive (semi-)definiteness of a tensor [16] and the monotonicity of vector-mapping [7].

**Definition 2.2.** Let $\mathcal{A} = (a_{i_1\dots i_m}) \in T_{m,n}$. $\mathcal{A}$ is said to be
(i) positive semi-definite if $\mathcal{A}x^m \geq 0$ for any vector $x \in R^n$;
(ii) positive definite if $\mathcal{A}x^m > 0$ for any nonzero vector $x \in R^n$.

**Definition 2.3.** Let $\Omega$ be a nonempty set in $R^n$. Then mapping $f : \Omega \to R^n$ is said to be
(i) pseudo monotone on $\Omega$ if

$$(x-y)^T f(y) \geq 0 \Longrightarrow (x-y)^T f(x) \geq 0, \quad \forall\ x, y \in \Omega;$$

(ii) monotone on $\Omega$ if

$$(x-y)^T (f(x) - f(y)) \geq 0, \quad \forall x, y \in \Omega;$$

(iii) strictly monotone on $\Omega$ if

$$(x-y)^T (f(x) - f(y)) > 0, \quad \forall x, y \in \Omega \text{ with } x \neq y.$$

To establish the monotonicity of the underlying function in the TVIP via the (semi-) positiveness of Cauchy tensor, we need the following conclusions.

**Lemma 2.4.** *If an even order Cauchy tensor $\mathcal{A}$ is positive semi-definite, then function $F(x) = \mathcal{A}x^{m-1} + q$ is monotone on $R^n$ for any $q \in R^n$.*

*Proof.* To show the conclusion, it suffices to show that for any $x, y \in R^n$, it holds that $F_i(x) - F_i(y) \geq 0$ if $x_i \geq y_i$ for any $i \in [n]$.

In fact, for any $i \in [n]$ such that $x_i \geq y_i$, from the positive semi-definiteness of Cauchy tensor $\mathcal{A} \in \mathcal{C}_{m,n}$, we conclude that for any $t \in (0,1)$,

$$t^{c_j + \frac{c_i - 1}{m-1}} x_j \geq t^{c_j + \frac{c_i - 1}{m-1}} y_j,$$

which implies that

$$\Big( \sum_{j \in [n]} t^{c_j + \frac{c_i - 1}{m-1}} x_j \Big)^{m-1} \geq \Big( \sum_{j \in [n]} t^{c_j + \frac{c_i - 1}{m-1}} y_j \Big)^{m-1}.$$

Hence,

$$\int_0^1 \Big( \sum_{j \in [n]} t^{c_j + \frac{c_i - 1}{m-1}} x_j \Big)^{m-1} \mathrm{d}t \geq \int_0^1 \Big( \sum_{j \in [n]} t^{c_j + \frac{c_i - 1}{m-1}} y_j \Big)^{m-1} \mathrm{d}t.$$

Thus,

$$\begin{aligned}
F(x) - F(y))_i =& (\mathcal{A}x^{m-1} + q - \mathcal{A}y^{m-1} - q)_i \\
=& (\mathcal{A}x^{m-1} - \mathcal{A}y^{m-1})_i \\
=& \sum_{i_2, \cdots, i_m \in [n]} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} - \sum_{i_2, \cdots, i_m \in [n]} a_{ii_2\cdots i_m} y_{i_2} \cdots y_{i_m} \\
=& \sum_{i_2, \cdots, i_m \in [n]} \frac{x_{i_2} \cdots x_{i_m}}{c_i + c_{i_2} + \cdots + c_{i_m}} - \sum_{i_2, \cdots, i_m \in [n]} \frac{y_{i_2} \cdots y_{i_m}}{c_i + c_{i_2} + \cdots + c_{i_m}} \\
=& \sum_{i_2, \cdots, i_m \in [n]} \int_0^1 t^{c_i + c_{i_2} + \cdots + c_{i_m} - 1} x_{i_2} \cdots x_{i_m} \,\mathrm{d}t - \sum_{i_2, \cdots, i_m \in [n]} \int_0^1 t^{c_i + c_{i_2} + \cdots + c_{i_m} - 1} y_{i_2} \cdots y_{i_m} \,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \sum_{i_2,\cdots,i_m \in [n]} t^{c_i + c_{i_2} + \cdots + c_{i_m} - 1} x_{i_2} \cdots x_{i_m} \, \mathrm{d}t \\
&\quad - \int_0^1 \sum_{i_2,\cdots,i_m \in [n]} t^{c_i + c_{i_2} + \cdots + c_{i_m} - 1} y_{i_2} \cdots y_{i_m} \, \mathrm{d}t \\
&= \int_0^1 (\sum_{j \in [n]} t^{c_j + \frac{c_i - 1}{m-1}} x_j)^{m-1} \, \mathrm{d}t - \int_0^1 (\sum_{j \in [n]} t^{c_j + \frac{c_i - 1}{m-1}} y_j)^{m-1} \, \mathrm{d}t \\
&\geq 0.
\end{aligned}
$$

The desired result follows. □

Similar with the proof of the Lemma 2.4, we can easily obtain the following conclusion.

**Lemma 2.5.** *If an even order tensor $\mathcal{A} \in \mathcal{C}_{m,n}$ is positive definite, then $F(x)$ is strictly monotone on $R^n$.*

## 3 Solution Existence of the TVIP

In this section, we first establish the solution existence of the TVIP, and then explore the solution set structure. To this end, we need the following definition [7].

**Definition 3.1.** Let an integer $\deg(\Phi, \Omega, p)$ be associated with each triple $(\Phi, \Omega, p)$ in the domain of $\Phi$. The function deg is called a (topological) degree if the following three axioms are satisfied

$(A1)$ $\deg(\Phi, \Omega, p) = 1$ *if* $p \in \Omega$;

$(A2)$ $\deg(\Phi, \Omega, p) = \deg(\Phi, \Omega_1, p) + \deg(\Phi, \Omega_2, p)$ if $\Omega_1$ and $\Omega_2$ are two disjoint open subsets of $\Omega$ and $p \notin \Phi((cl\Omega) \backslash (\Omega_1 \cup \Omega_2))$;

$(A3)$ $\deg(H(\cdot, t), \Omega, p(t))$ is independent of $t \in [0, 1]$ for any two continuous functions $H : cl\Omega \times [0, 1] \to R^n$ and $p : [0, 1] \to R^n$ such that

$$p(t) \notin H(bd\Omega, t), \quad \forall t \in [0, 1].$$

We call $\deg(\Phi, \Omega, p)$ the degree of $\Phi$ at $p$ relative to $\Omega$. If $p = 0$, we simply write $\deg(\Phi, \Omega)$ for $\deg(\Phi, \Omega, p)$.

Based on the definition given above, the solution existence of the VIP is derived [7].

**Lemma 3.2.** *Let $\Omega \subset R^n$ be closed convex, $F : D \supseteq \Omega \to R^n$ be continuous on an open set $D$, and let $G(v) = v - P_\Omega(v - F(v))$. Then, if there exists a bounded open set $U$ such that $cl(U) \supseteq D$ and $\deg(G, U)$ is well defined and nonzero, then the VI$(\mathcal{A}, q)$ has a solution in $U$, where $P_\Omega(x) = \arg\min_{y \in \Omega} \|y - x\|$ for any $x \in R^n$.*

Using the monotonicity and the degree theory, we establish an equivalent condition under which the TVIP has a solution.

**Theorem 3.3.** *If an even order tensor $\mathcal{A} \in \mathcal{C}_{m,n}$ is positive semi-definite, then the following statements are equivalent.*

(i) *There exists a vector $x^* \in \Omega$ such that*

$$L_< = \{x \in \Omega \mid (x - x^*)^T (\mathcal{A}x^{m-1} + q) < 0\}$$

*is bounded (possibly empty).*

(ii) *There exist a bounded open set* $U \subset R^n$ *and a vector* $x^* \in \Omega \bigcap U$*, such that*

$$(x - x^*)^T(\mathcal{A}x^{m-1} + q) \geq 0, \forall x \in \Omega \cap bdU.$$

(iii) *The TVIP has a solution.*

*Proof.* $(i) \Rightarrow (ii)$ Let $U \subset R^n$ be a bounded open set, such that $x^* \in U$ and $L_< \subseteq U$. As $U$ is an open set, one has

$$L_< \bigcap \mathrm{bd}(U) = \emptyset.$$

Thus,

$$(x - x^*)^T(\mathcal{A}x^{m-1} + q) \geq 0, \quad \forall x \in \Omega \cap \mathrm{bd}(U),$$

and (ii) follows.

$(ii) \Rightarrow (iii)$ We will prove this by reductio ad absurdum. For the sake of contradiction, suppose that the solution set of TVIP$(\mathcal{A}, q)$ is empty. Let

$$G(x) = x - P_\Omega(x - (\mathcal{A}x^{m-1} + q)).$$

Then $G^{-1}(0) \bigcap \mathrm{bd}(U) = \emptyset$, and hence $\deg(G, U)$ is well defined. In the following, we will show that $\deg(G, U) \neq 0$.

In fact, for the homotopy mapping

$$H(x, t) = x - P_\Omega(t(x - (\mathcal{A}x^{m-1} + q)) + (1 - t)x^*),$$

it is easy to see that $H(x, 0) = x - x^*$. Since $x^* \in U$, it holds that $\deg(H(x, 0), U) = 1$ and $H(x, 1) = x - P_\Omega(x - (\mathcal{A}x^{m-1} + q))$.

Now, we show that if $H(x, t) = 0$ for some $(x, t) \in \mathrm{cl}(U) \times (0, 1)$, then $x \notin \mathrm{bd}(U)$.

Assume that $H(x, t) = 0$ for some $t \in (0, 1)$. Without loss of generality, we assume that $x \neq x^*$. Since $H(x, t) = 0$, we conclude that $x \in \Omega$ and

$$(y - x)^T[x - t(x - (\mathcal{A}x^{m-1} + q)) - (1 - t)x^*] \geq 0.$$

In particular, for $y = x^*$, one has

$$(x^* - x)^T[x - t(x - (\mathcal{A}x^{m-1} + q)) - (1 - t)x^*] \geq 0.$$

Then

$$(x^* - x)^T[t(\mathcal{A}x^{m-1} + q) + (1 - t)(x - x^*)] \geq 0,$$

which implies that

$$(x^* - x)^T(\mathcal{A}x^{m-1} + q) \geq \frac{1 - t}{t}\|x - x^*\|^2 > 0,$$

where the last inequality uses the facts that $t \in (0, 1)$ and $x \neq x^*$. Thus $x \notin \mathrm{bd}(U)$. Consequently, by the homotopy invariance property of the degree, one has

$$\deg(G, U) = \deg(H(x, 0), U) = \deg(H(x, 1), U) = 1.$$

By Lemma 3.2, we know that $S(\mathcal{A}, q)$ is nonempty. A contradiction is obtained and (iii) follows.

$(iii) \Rightarrow (i)$ Let $x^*$ be the solution of TVIP. Then for all $y \in \Omega$,

$$(y - x^*)^T(\mathcal{A}(x^*)^{m-1} + q) \geq 0. \tag{3.1}$$

By Lemma 2.4, it holds that

$$(y-x^*)^T(\mathcal{A}y^{m-1}-\mathcal{A}(x^*)^{m-1})\geq 0.$$

Connecting this with (3.1) yields that

$$(y-x^*)^T(\mathcal{A}y^{m-1}+q)\geq 0,$$

which implies that $L_<$ is empty and (i) follows. □

Now, we explore the structure of the solution set of the TVIP by virtue of the positive definiteness of the underlying tensor in the TVIP.

**Theorem 3.4.** *If the even order tensor $\mathcal{A}\in\mathcal{C}_{m,n}$ is positive definite, then the VIP$(\mathcal{A},q)$ has at most one solution on $\Omega$ for any $q\in R^n$.*

*Proof.* Our proof is based on contraposition. Suppose $x_1,x_2\in\Omega$ are two distinct solutions of the TVIP. Then

$$(y-x_1)^T(\mathcal{A}x_1^{m-1}+q)\geq 0,\quad (y-x_2)^T(\mathcal{A}x_2^{m-1}+q)\geq 0,\quad \forall y\in\Omega.$$

Taking $y=x_2$ in the first inequality and $y=x_1$ in the second inequality gives

$$(x_2-x_1)^T(\mathcal{A}x_1^{m-1}+q)\geq 0,\quad (x_1-x_2)^T(\mathcal{A}x_2^{m-1}+q)\geq 0,$$

which implies that

$$(x_1-x_2)^T(\mathcal{A}x_1^{m-1}-\mathcal{A}x_2^{m-1})\leq 0. \tag{3.2}$$

By Lemma 2.5, we have

$$(x_1-x_2)^T(\mathcal{A}x_1^{m-1}-\mathcal{A}x_2^{m-1})>0.$$

Obviously, this contradicts (3.4), and the desired conclusions follows. □

The next result shows that the convexity of the solution set of the TVIP provided that the underlying tensor $\mathcal{A}\in\mathcal{C}_{m,n}$ is positive semi-definite.

**Theorem 3.5.** *Let $\mathcal{A}$ be an even order positive semi-definite Cauchy tensor. Then the solution set of TVIP is convex.*

*Proof.* First, we claim that the solution set of the TVIP$(\mathcal{A},q)$ satisfies that

$$S(\mathcal{A},q)=\bigcap_{y\in\Omega}\{x\in\Omega\mid (y-x)^T(\mathcal{A}y^{m-1}+q)\geq 0\}.$$

In fact, if $x\in S(\mathcal{A},q)$, then

$$(y-x)^T(\mathcal{A}x^{m-1}+q)\geq 0. \tag{3.3}$$

Combining inequality 3.3, Definition 2.3 with with Lemma 2.4 yields that

$$(y-x)^T(\mathcal{A}y^{m-1}+q)\geq 0,\quad \forall\ y\in\Omega. \tag{3.4}$$

Thus,

$$x\in\bigcap_{y\in\Omega}\{x\in\Omega\mid (y-x)^T(\mathcal{A}y^{m-1}+q)\geq 0\}.$$

This means that

$$S(\mathcal{A}, q) \subseteq \bigcap_{y \in \Omega} \{x \in \Omega \mid (y - x)^T (\mathcal{A} y^{m-1} + q) \geq 0\}. \tag{3.5}$$

On the other hand, since $\Omega$ is convex, for any $z \in \Omega$ and $\tau \in (0, 1)$, it holds that

$$y = \tau x + (1 - \tau) z \in \Omega.$$

Then it follows from (3.4) that

$$(z - x)(\mathcal{A}(\tau x + (1 - \tau) z)^{m-1} + q) \geq 0.$$

Letting $\tau \to 1$ yields that

$$(z - x)(\mathcal{A} x^{m-1} + q) \geq 0.$$

From the arbitrariness of $z \in \Omega$, we conclude that $x \in S(\mathcal{A}, q)$ which implies that

$$S(\mathcal{A}, q) \supseteq \bigcap_{y \in \Omega} \{x \in \Omega \mid (y - x)^T (\mathcal{A} y^{m-1} + q) \geq 0\}.$$

Recalling (3.5), we can obtain the desired result, i.e.,

$$S(\mathcal{A}, q) = \bigcap_{y \in \Omega} \{x \in \Omega \mid (y - x)^T (\mathcal{A} y^{m-1} + q) \geq 0\}.$$

Now, we show that the solution set $S(\mathcal{A}, q)$ of the TVIP$(\mathcal{A}, q)$ is convex based on the above claim.

In fact, since for any fixed $y \in \Omega$, the set

$$\{x \in \Omega \mid (y - x)^T (\mathcal{A} y^{m-1} + q) \geq 0\}$$

is convex. Using the fact that the intersection of any number of convex set is convex, we conclude that the solution set $S(\mathcal{A}, q)$ of the TVIP$(\mathcal{A}, q)$ is convex. The desired result follows. $\square$

## 4 Conclusions

In this paper, by exploring structure properties of the Cauchy tensor and by virtue of of the homotopy mapping and degree theory, we established the solution existence of the TVIP and explored the solution set structure of the TVIP which are useful in TVIP analysis. It is well known that there are many efficient solution methods for the VIP, and they all apply to the TVIP. However, due to the involvement of the tensor in the TVIP, establishing an efficient solution method based on the tensor structure is a significant research work, and this will be discussed in the future.

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