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TWO INTERIOR POINT CONTINUOUS TRAJECTORY MODELS FOR CONVEX QUADRATIC PROGRAMMING WITH BOUND CONSTRAINTS

Hongwei Yue, Li-Zhi Liao and Xun Qian

Abstract: In this paper, two interior point continuous trajectory models are introduced for solving convex quadratic programming with bound constraints. The main components of our two interior point continuous trajectory models are two dynamical systems, one for each model. These two dynamical systems are very simple, only matrix-vector multiplications are required. Without introducing dual variables or projection, starting from any interior point, all solution trajectories of our two dynamical systems remain in the interior of the feasible regions and will converge to optimal solutions of the underlying optimization problems in the limit. Many theoretical properties for the two dynamical systems are presented. In particular, in our convergence proofs for the solutions of the two dynamical systems, there is no Lyapunov function involved. Furthermore, our preliminary simulation results are very encouraging in obtaining optimal solutions of the corresponding optimization problems.

Key words: *convex quadratic programming , dynamical system , interior point method*

Mathematics Subject Classification: *90C20, 90C51,37C75, 37N40*

1 Introduction

The general bound constrained convex quadratic programming is of the following form

$$\begin{array}{ll} \min & q(x) = \frac{1}{2}x^TQx + c^Tx \\ s.t. & l \le x \le u, \end{array} \tag{P_1}$$

where c, l and u are given vectors in R^n, $Q = (q_{ij})_{n\times n} \in R^{n\times n}$. We assume throughout this paper that Q is symmetric and positive semi-definite.

Problem (P_1) frequently arises in numerical analysis applications, optimal control, and subproblems in general nonlinear optimization algorithms, for instance, in finite-difference discretization of free boundary problems and implementation of robust methods for nonlinear programming [16]. For more details, please see [16, 13, 27] and references therein.

If for any $i \in \{1, \ldots, n\}$, l_i is bounded and $u_i = +\infty$, problem (P_1) can be easily converted into the following nonnegatively constrained convex quadratic programming

$$\begin{array}{ll} \min & q(x) = \frac{1}{2}x^TQx + c^Tx \\ s.t. & x \ge 0. \end{array} \tag{P_2}$$

Nonnegatively constrained convex quadratic programming problems often arise in science, engineering and business, and they may fall into nonnegative least-squares problems.

In support vector machines, computing the maximum margin hyperplane also gives rise to a nonnegatively constrained convex quadratic programming problem [32].

Furthermore, if l_i and u_i are both bounded, and $l_i < u_i$ for any $i \in \{1, \ldots, n\}$, problem (P_1) can be converted into the following special box constrained convex quadratic programming

$$\begin{array}{ll} \min & q(x) = \frac{1}{2}x^T Qx + c^T x \\ s.t. & 0 \le x \le e, \end{array} \qquad (P_3)$$

where e denotes the n-dimensional column vector of all ones.

From a general point of view, there are iterative methods and continuous trajectory methods in the field of computational optimization. Different from the conventional iterative optimization methods, the main feature of the continuous trajectory method is that a continuous trajectory starting from the initial point can be generated. This trajectory eventually will converge to an equilibrium point (or a limit set), which is exactly an optimal solution (or a subset of the optimal solution set) for the underlying optimization problem in the limit [21]. Many iterative schemes can be regarded as the discrete realization of certain continuous trajectories, for example, [5] and [6] for combinatorial optimization, [30] and [31] for constrained minimization problems. Among the methods for solving unconstrained problems, the steepest descend method, the Newton method and the power method, all can be taken as typical discretization examples of the corresponding differential systems [10]. In the literature, the research on continuous trajectory (or dynamical system) methods for linear programming can be found in Adler and Monteiro [2], Anstreicher [3], Bayer and Lagarias [7, 8], Megiddo and Shub [25], Monteiro [26], Chu and Lin [11], and Liao [23].

To solve constrained optimization problems using continuous trajectory methods, there are three ways in constructing a dynamical system:

(i) adopting the gradient of certain penalty function [34, 37];

(ii) employing the Karush-Kuhn-Tucher (KKT) conditions of the underlying optimization problem [19, 36, 46] and;

(iii) utilizing the projection together with variational inequality [9, 18, 40, 41, 42, 44, 47].

Undoubtedly, the three strategies above play important roles in constructing dynamical systems for neural network models in past decades. In the case (i), the objective function will be added by some kind of penalty term, then the penalty parameter must be driven to certain large number for reaching the optimal solution of the original problem. In the case (ii), by using the KKT system, both multiplier and dual variables must be added. Therefore, the variable dimension will be increased. In the case (iii), some projection is needed to ensure that the intermediate solutions remain in the feasible region. Different from above strategies, the construction of our dynamical systems adopts the interior point approach. In this paper, motivated by a first-order interior point method for linearly constrained smooth optimization proposed by Tseng *et al*. [35], we study the corresponding continuous trajectory (or dynamical system, or ordinary differential equation (**ODE**)) model for problem (P_2), and then extend the model to solve problem (P_3). The first-order (iterative) interior point method in [35] unifies and extends the first-order affine scaling method and replicator dynamics method for standard quadratic programming. The direction for no equality constraint case in [35] is as follows:

$$d = -X^{2\gamma} \nabla q(x), \tag{1.1}$$

where $\nabla q(x)$ is the gradient of the objective function $q(x)$ ($q(x)$ is defined in problem (P_1), $\gamma > 0$ is a parameter, and $X^{2\gamma} = diag(x_1^{2\gamma}, \cdots, x_n^{2\gamma}) \in R^{n\times n}$. It should be noticed that based on the relationship between Hopfield networks and interior point algorithms as discussed in [45], the direction of Faybusovich's approach is just (1.1) with $\gamma = \frac{1}{2}$ for no equality constraint case, and the direction of affine scaling approach is just (1.1) with $\gamma = 1$. So our dynamical system model for problem (P_2) in Section 2 can be viewed as the continuous realization of the interior point method. The matrix X (or $X^{2\gamma}$) in direction (1.1) plays the role of a barrier which keeps the continuous trajectory staying in the positive orthant (see [7, 8] for more discussions). This idea can be extended to the box constraint, thus our second dynamical system model in Section 3 can be established for problem (P_3).

Based on above discussions, in this paper, we (a) study a dynamical system model (see system (2.1) in Section 2) by adopting the first-order affine scaling direction d in (1.1) for nonnegatively constrained convex quadratic programming problem (P_2); (b) propose an interior point dynamical system model (see system (3.1) in Section 3) for box constrained convex quadratic programming problem (P_3); (c) prove the convergence of the solutions of the two ODE systems for any interior feasible solution in the limit; and (d) show that the limiting points of the two ODE systems are indeed the optimal solutions for the corresponding optimization problems. It should be mentioned that the two ODE systems in our two dynamical system models only require matrix-vector multiplications, therefore it is expected that the computation for finding the limit points can be achieved very fast.

The rest of this paper is organized as follows. In Section 2, a simple dynamical system model will be constructed for problem (P_2). Then, a thorough study on the continuous trajectory of the ODE resulting from this dynamical system model will be investigated. Various theoretical properties including the strong convergence in the limit will be explored. In Section 3, similar study and investigations for problem (P_3) will be conducted. Some numerical results of the two simple ODE systems will be illustrated in Section 4. Finally, some concluding remarks will be drawn in Section 5.

For simplicity, in what follows, unless otherwise specified, R^n_+ denotes the constrained region $\{x \in R^n \mid x \geq 0\}$, $\|\cdot\|$ denotes the 2-norm, e denotes the n-dimensional vector of all ones, e_i denotes the unity column vector whose ith component is 1. For each index subset $J \subseteq \{1, \ldots, n\}$, we denote by x_J the vector composed of those components of $x \in R^n$ indexed by $j \in J$. I_n stands for the $n \times n$ identity matrix.

2 A Simple Dynamical System for Nonnegatively Constrained Convex Quadratic Programming

Based on direction (1.1) in [35], the following ODE system for problem (P_2) can be constructed.

$$\frac{dx(t)}{dt} = -X^{2\gamma}(Qx + c),\ t \geq t_0,\ x(t_0) = x^0 > 0, \tag{2.1}$$

where $\gamma \geq \frac{1}{2}$ is a fixed constant.

To simplify the following presentation, in the remaining of this paper, $x(t)$ (or $X(t)$) will be replaced by x (or X) whenever no confusion would occur. Throughout this section we make the following assumption.

Assumption 2.1. *The optimal solution set for problem (P_2) is bounded.*

To make our theoretical discussions more readable in this section, we divide the following discussions into 4 subsections.

2.1 Existence and feasibility of trajectory (2.1)

Lemma 2.2 ([14]). *If $g_i : R^n \to R$, $i = 1, \ldots, m$, are concave functions, and if $G = \{x \mid g_i(x) \geq 0,\ i = 1, \ldots, m\}$ is a nonempty bounded set, then for any set of values $\{\epsilon_i\}$, where $\epsilon_i \geq 0$, $i = 1, \ldots, m$, the set*

$$\{x \mid g_i(x) \geq -\epsilon_i, i = 1, \ldots, m\}$$

is bounded.

For any given x^0 in ODE (2.1), let us define the level set

$$L_1(x^0) = \{x \in R^n_+ \mid q(x) \leq q(x^0)\}, \quad \text{where } q(x) \text{ is defined in } (P_2).$$

Following Lemma 2.2, we have

Lemma 2.3. *For problem (P_2), the level set $L_1(x^0)$ is bounded.*

Since $X^{2\gamma}(Qx+c)$ is continuously differentiable on R^n, $X^{2\gamma}(Qx+c)$ is locally Lipschitz continuous on R^n. Thus, there exists a unique solution $x(t)$ for ODE (2.1) on the maximal existence interval $[t_0,\ \beta)$ for some $\beta > 0$.

Theorem 2.4. *Let $x(t)$ be the solution of ODE (2.1) with the maximal existence interval $[t_0,\ \beta)$. Then $x(t) > 0 \in R^n$ for any $t \in [t_0,\ \beta)$.*

Proof. We will prove $x(t) > 0$ for any $t \in [t_0,\ \beta)$ by contradiction. Suppose that there exists a $t^* \in [t_0,\ \beta)$ and an $i \in \{1, \ldots, n\}$ such that $x_i(t^*) = 0$. Since $x_i(t)$ is continuous on t, let t^* be the minimum t such that $x_i(t) = 0$ for some $i \in \{1, \ldots, n\}$, i.e., $x(t) > 0$ for all $0 \leq t < t^*$.

Let

$$M_1 = \max_{t\in[t_0,\ t^*]} \|Qx(t)+c\| + 1,$$

$$M_2 = \max_{t\in[t_0,\ t^*]} \|x(t)\|^{2\gamma-1} + 1,$$

$$t_1 = \max\{t^* - \frac{1}{2M_1M_2}, 0\},$$

and $\bar{t}$ be the time satisfying

$$x_i(\bar{t}) = \max_{t\in[t_1,\ t^*]} x_i(t) > 0.$$

Notice that

$$\frac{dx(t)}{dt} = -X^{2\gamma}(Qx+c),$$

we have

$$x_i(t) = x_i(t^*) + \int_t^{t^*} x_i(\tau)^{2\gamma}(Qx(\tau)+c)_i d\tau.$$

For any $t \in [t_1,\ t^*]$, notice that $x_i(t^*) = 0$ and $x_i(t) \geq 0$, from the above equation, we have

$$\begin{aligned} x_i(t) &\leq M_1(t^*-t) \max_{\tau\in[t,\ t^*]} x_i(\tau)^{2\gamma} \\ &\leq M_1(t^*-t_1) \max_{t\in[t_1,\ t^*]} x_i(t)^{2\gamma} \end{aligned}$$

$$
\begin{aligned}
&= M_1(t^* - t_1)x_i(\bar{t})^{2\gamma} \\
&\leq \frac{1}{2}x_i(\bar{t}).
\end{aligned}
$$

Since $t \in [t_1, t^*]$ is arbitrary, taking $t = \bar{t}$, then

$$x_i(\bar{t}) \leq \frac{1}{2}x_i(\bar{t}).$$

Thus $x_i(\bar{t}) = 0$, which is a contradiction with the definition of $\bar{t}$. □

Suppose $x(t)$ is the solution of ODE (2.1), and $[t_0,\ \beta)$ is the corresponding maximal existence interval. By Theorem 2.4, $x(t) > 0$ for any $t \in [t_0,\ \beta)$. Thus

$$\frac{dq(x(t))}{dt} = -(Qx+c)^T X^{2\gamma}(Qx+c) \leq 0 \quad \forall t \in [t_0,\ \beta),$$

i.e., $q(x)$ is monotonically nonincreasing along the solution trajectory $x(t)$. So $x(t)$ is contained in the compact level set $L_1(x^0)$. Thus $x(t)$ is a global solution, i.e., $\beta = +\infty$.

Corollary 2.5. *There exists a unique solution $x(t)$ for ODE (2.1) on $[t_0,\ +\infty)$, and $x(t) > 0$ for any $t \in [t_0,\ +\infty)$.*

2.2 Weak convergence and infinite stopping of trajectory (2.1)

Lemma 2.6 ([33, Barbalat's Lemma])**.** *If the differentiable function $g(t)$ has a finite limit as $t \to +\infty$, and $\dot{g}$ is uniformly continuous, then $\dot{g} \to 0$ as $t \to +\infty$.*

Now we will show the weak convergence of the solution $x(t)$ of ODE (2.1) as $t \to +\infty$, i.e., the right-hand side of ODE (2.1) approaching to zero as $t \to +\infty$.

Theorem 2.7. *Let $x(t)$ be the solution of ODE (2.1). Then $\lim_{t\to+\infty} X^{2\gamma}(Qx+c) = 0$.*

Proof. From Corollary 2.5, we know that the unique solution $x(t)$ of ODE (2.1) is always positive on $[t_0, +\infty)$. Since $\frac{dq(x)}{dt} = -(Qx+c)^T X^{2\gamma}(Qx+c) \leq 0$ and $q(x)$ is bounded below, $q(x)$ has a finite limit along the trajectory $x(t)$. Obviously, $(Qx+c)^T X^{2\gamma}(Qx+c)$ is continuously differentiable with respect to x, and $x(t)$ is contained in compact level set $L_1(x^0)$. Therefore, there exists a constant $K_1 > 0$ such that

$$
\begin{aligned}
\left|\frac{dq(x)}{dt}\right|_{t=t_1} - \frac{dq(x)}{dt}\Big|_{t=t_2}\Big| &\leq K_1\|x(t_1) - x(t_2)\| \\
&= K_1\left\|\int_{t_1}^{t_2} X^{2\gamma}(Qx+c)dt\right\| \\
&\leq K_2K_1|t_1 - t_2|,
\end{aligned}
$$

where $K_2 = \max_{x\in L_1(x^0)} \|X^{2\gamma}(Qx+c)\|$. Thus $\frac{dq(x)}{dt}$ is uniformly continuous on $[t_0, +\infty)$. Then Lemma 2.6 ensures

$$\lim_{t\to+\infty}(Qx+c)^T X^{2\gamma}(Qx+c) = 0.$$

Since $x(t)$ is bounded and nonnegative, we have

$$\lim_{t\to+\infty} X^{2\gamma}(Qx+c) = 0.$$

□

Theorem 2.7 ensures the weak convergence of $x(t)$ in the limit, i.e. $\frac{dx(t)}{dt} \to 0$ as $t \to +\infty$. The results in the following theorems reveal more properties on the trajectory $x(t)$ of ODE (2.1).

The result in the following theorem indicates that if x^0 is not an optimal solution for problem (P_2), then ODE system (2.1) will never stop, i.e. $\frac{dx}{dt} = 0$, in finite t, and the objective function value will be strictly decreased along the solution of ODE (2.1).

Theorem 2.8. *Let $x(t)$ be the solution of ODE (2.1). If $X^{2\gamma}(Qx+c)|_{t=t_0} \neq 0$, then $X^{2\gamma}(Qx+c) \neq 0$ for any $t \geq t_0$.*

Proof. The proof is similar to the one for Theorem 3 in [21]. Assume that the conclusion is not true. Then there exists a finite time, say $\bar{t} > t_0$, such that $X^{2\gamma}(Qx+c)|_{t=\bar{t}} = 0$. From the continuity of $X^{2\gamma}(Qx+c)$, we can assume that $\bar{t}$ is the minimum t such that $X^{2\gamma}(Qx+c) = 0$. We know that $X^{2\gamma}(Qx+c)$ is Lipschitz continuous in bounded set $L_1(x^0)$, and let $\bar{L}$ be the corresponding Lipschitz constant and $\delta = \min\{\frac{\bar{t}}{2}, \frac{1}{2\bar{L}}\}$. Then for any t_1, $t_2 \in [\bar{t}-\delta, \bar{t}]$, we have

$$\begin{aligned}
\|X^{2\gamma}(Qx+c)|_{t=t_1}\| - \|X^{2\gamma}(Qx+c)|_{t=t_2}\| &\leq \|X^{2\gamma}(Qx+c)|_{t=t_1} - X^{2\gamma}(Qx+c)|_{t=t_2}\| \\
&\leq \bar{L}\|x(t_1) - x(t_2)\| \\
&= \bar{L}\|\int_{t_1}^{t_2} X^{2\gamma}(Qx+c)|_{t=\tau} d\tau\| \\
&\leq \bar{L} \cdot \delta \cdot \max_{\tau \in [\bar{t}-\delta, \bar{t}]} \|X^{2\gamma}(Qx+c)|_{t=\tau}\|.
\end{aligned}$$

Notice that the above inequality is true for any t_1, $t_2 \in [\bar{t}-\delta, \bar{t}]$ and $X^{2\gamma}(Qx+c)|_{t=\bar{t}} = 0$, then we can choose t_1 and t_2 such that

$$\|X^{2\gamma}(Qx+c)|_{t=t_1}\| = \max_{\tau \in [\bar{t}-\delta, \bar{t}]} \|X^{2\gamma}(Qx+c)|_{t=\tau}\|,$$

and

$$\|X^{2\gamma}(Qx+c)|_{t=t_2}\| = \min_{\tau \in [\bar{t}-\delta, \bar{t}]} \|X^{2\gamma}(Qx+c)|_{t=\tau}\|,$$

thus we have

$$\begin{aligned}
0 &= \min_{\tau \in [\bar{t}-\delta, \bar{t}]} \|X^{2\gamma}(Qx+c)|_{t=\tau}\| \\
&\geq (1 - \bar{L}\delta) \max_{\tau \in [\bar{t}-\delta, \bar{t}]} \|X^{2\gamma}(Qx+c)|_{t=\tau}\|.
\end{aligned}$$

This implies that $X^{2\gamma}(Qx+c)|_{t=\tau} = 0$ for any $\tau \in [\bar{t}-\delta, \bar{t}]$ which contradicts with the definition of $\bar{t}$. Thus the proof is complete. □

2.3 Optimality and cluster points of trajectory (2.1)

Next we will show that any cluster point of the solution $x(t)$ of ODE (2.1) as $t \to +\infty$ is an optimal solution for problem (P_2). But first, let us define the limit set

$$\Omega_1(x^0) = \{y \in R^n_+ \mid \ y \text{ is a cluster point of } x(t) \text{ of ODE (2.1) as t} \to +\infty\}. \qquad (2.2)$$

Because of the boundedness of $x(t)$, $\Omega_1(x^0)$ is nonempty, compact, and connected (see Theorem 1.1 on page 390 in [12]).

Following the KKT conditions for problem (P_2) which are:

$$\begin{cases} Qx + c = s, \ s \geq 0, \\ Xs = 0, \ x \geq 0, \end{cases} \tag{2.3}$$

where $X = diag(x) \in R^{n\times n}$ and $s \in R^n$, we can define the dual estimate as

$$s(x) = Qx + c. \tag{2.4}$$

Note: $s(x(t))$ may not be nonnegative for all $t \geq t_0$. Furthermore, we choose an $\bar{x} \in \Omega_1(x^0)$, and define

$$\bar{s} = Q\bar{x} + c.$$

Corollary 2.9. *Let $q(x)$ be defined in problem (P_2). Then (i) $q(x) = q(\bar{x})$ $\forall x \in \Omega_1(x^0)$; (ii) $Xs(x) = 0$ $\forall x \in \Omega_1(x^0)$, where $s(x)$ is defined in (2.4).*

Proof. (i) Since $\frac{dq(x)}{dt} = -(Qx+c)^T X^{2\gamma}(Qx+c) \leq 0$ and $q(x)$ is bounded below, it is easy to see that $q(x)$ equals a constant for any $x \in \Omega_1(x^0)$.
(ii) From Theorem 2.7, this is straightforward. □

For the pair $\bar{x}$ and $\bar{s}$ defined above, we define

$$\bar{J} = \{j|\bar{s}_j = 0, \ j \in \{1, \cdots, n\}\}, \quad \bar{J}^c = \{1, \ldots, n\} \setminus \bar{J},$$

$$\bar{\Lambda}_1 = \{x \in R^n_+ | x_{\bar{J}^c} = 0, \ q(x) = q(\bar{x})\}. \tag{2.5}$$

From Theorem 2.7, we have

$$\bar{X}\bar{s} = 0 \quad \text{or} \quad \bar{x}_i\bar{s}_i = 0, \ i = 1, \cdots, n.$$

This and the definition of $\bar{J}^c$ imply for any $j \in \bar{J}^c$

$$\bar{s}_j \neq 0 \quad \text{and} \quad \bar{x}_j = 0.$$

This and Corollary 2.9 (i) ensure that the set $\bar{\Lambda}_1$ is nonempty since $\bar{x} \in \bar{\Lambda}_1$. In addition, it is easy to see that $\bar{\Lambda}_1$ is closed. Next we will reveal some properties for $\bar{\Lambda}_1$.

Lemma 2.10. *$\bar{\Lambda}_1$ in (2.5) is a convex set.*

Proof. Let x be an arbitrary point in the convex hull $co(\bar{\Lambda}_1)$ of $\bar{\Lambda}_1$, i.e., x is a positive linear convex combination of some points in $\bar{\Lambda}_1$. From the definition of $\bar{\Lambda}_1$ in (2.5), we know that $x_{\bar{J}^c}$=0 and $x \geq 0$. From the convexity of $q(x)$, the following inequality holds

$$q(x) \leq q(\bar{x}).$$

On the other hand, let $\Delta x = x - \bar{x}$. Then $(\Delta x)_{\bar{J}^c} = 0$ and $\bar{s}^T(\Delta x) = 0$. Again by the convexity of $q(x)$, we have

$$\begin{aligned} q(x) &\geq q(\bar{x}) + \nabla q(\bar{x})^T(\Delta x) \\ &= q(\bar{x}) + \bar{s}^T(\Delta x) \\ &= q(\bar{x}). \end{aligned}$$

So $q(x) = q(\bar{x})$ for all $x \in co(\bar{\Lambda}_1)$, thus $x \in \bar{\Lambda}_1$ and $\bar{\Lambda}_1$ is convex. □

Lemma 2.11 ([24, 35]). *Let $f : R^n \to R$ be a convex twice continuously differentiable function. If $f(\cdot)$ is constant on a convex set $\Omega \in R^n$, then $\nabla f(\cdot)$ is constant on Ω.*

Lemma 2.12. *$s(x) = \bar{s}$ for all $x \in \bar{\Lambda}_1$.*

Proof. From the definition of $\bar{\Lambda}_1$, $q(x) = q(\bar{x})$ $\forall x \in \bar{\Lambda}_1$. Then Lemma 2.10 and Lemma 2.11 ensure the result. □

Theorem 2.13. *$\Omega_1(x^0) \subseteq \bar{\Lambda}_1$, where $\Omega_1(x^0)$ is defined in (2.2).*

Proof. Our proof here is similar to the one for Lemma 8 in [35]. If $\bar{J}^c$ is empty, $\bar{\Lambda}_1$ becomes $\{x \in R^n_+ | q(x) = q(\bar{x})\}$. From Corollary 2.9 (i), the result holds clearly. Suppose there exists a point $\hat{x} \in \Omega_1(x^0)$ but $\hat{x} \notin \bar{\Lambda}_1$ with $\hat{x}_{\hat{j}} > 0$ for some $\hat{j} \in \bar{J}^c$, then $q(\hat{x}) = q(\bar{x})$ and $\hat{x} \geq 0$. Clearly $\bar{\Lambda}_1$ lies inside the bounded level set $L_1(x^0)$, this and $\bar{\Lambda}_1$ being closed ensure that $\bar{\Lambda}_1$ is compact. Thus $s(x)$ is uniformly continuous over $\bar{\Lambda}_1$. Lemma 2.12 implies that, for all $\delta > 0$ sufficiently small,

$$|s_j(x)| \geq |\bar{s}_j|/2 \quad \forall j \in \bar{J}^c, \ \forall x \in U(\bar{\Lambda}_1, \delta), \tag{2.6}$$

where $U(\bar{\Lambda}_1, \delta)$ is the δ-neighborhood of set $\bar{\Lambda}_1$. We take δ small enough so that $\delta < \hat{x}_{\hat{j}}$. Then $\hat{x} \notin U(\bar{\Lambda}_1, \delta)$ since $|\hat{x}_{\hat{j}} - x_{\hat{j}}| = \hat{x}_{\hat{j}} > \delta$ for all $x \in \bar{\Lambda}_1$. Notice $\bar{x} \in \Omega_1(x^0) \cap \bar{\Lambda}_1$ and $\hat{x} \in \Omega_1(x^0)$ but $\hat{x} \notin U(\bar{\Lambda}_1, \delta)$, by the connectivity of $\Omega_1(x^0)$, there must exist an $\tilde{x} \in \Omega_1(x^0) \cap U(\bar{\Lambda}_1, \delta)$ but $\tilde{x} \notin \bar{\Lambda}_1$. $\tilde{x} \in \Omega_1(x^0)$ ensures

$$\tilde{x} \geq 0, \ q(\tilde{x}) = q(\bar{x}).$$

$\tilde{x} \notin \bar{\Lambda}_1$ indicates that there must exist some $r \in \bar{J}^c$ such that $\tilde{x}_r \neq 0$. (2.6) and $\tilde{x} \in U(\bar{\Lambda}_1, \delta)$ imply $|s_j(\tilde{x})| \geq |\bar{s}_j|/2$ for all $j \in \bar{J}^c$, thus $\tilde{X} s(\tilde{x}) \neq 0$, which contradicts with the fact $\tilde{X} s(\tilde{x}) = 0$ since $\tilde{x} \in \Omega_1(x^0)$ from Corollary 2.9 (ii). □

Theorem 2.14. *If $x(t)$ is the solution of ODE (2.1), $\lim_{t\to+\infty}(Qx(t) + c) = \bar{s}$ and $\bar{s} \geq 0$.*

Proof. Based on the continuity of $s(x(t))$, compactness of $\Omega_1(x^0)$, Lemma 2.12, and Theorem 2.13, it is easy to have

$$\lim_{t\to+\infty}(Qx(t) + c) = \bar{s}.$$

Suppose there exists some $\bar{j} \in \{1, \ldots, n\}$ such that $\bar{s}_{\bar{j}} < 0$. For any cluster point $\hat{x} \in \Omega_1(x^0)$, from Corollary 2.9 (ii), we have $\hat{X} s(\hat{x}) = 0$. This, Lemma 2.12, and Theorem 2.13 imply $\hat{X}\bar{s} = 0$, thus $\hat{x}_{\bar{j}} = 0$. Since $s(x(t))$ is continuous, there exists some t_K such that $s_{\bar{j}}(x(t)) < 0$ for all $t \geq t_K$, notice that

$$\frac{dx(t)}{dt} = -X^{2\gamma}(Qx + c),$$

and $x(t) > 0$ for all $t \geq 0$. We have $\frac{dx_{\bar{j}}(t)}{dt} \geq 0$ and $x_{\bar{j}}(t) \geq x_{\bar{j}}(t_K) > 0$ for all $t \geq t_K$, which contradicts with $\hat{x}_{\bar{j}} = 0$, thus the proof is complete. □

Theorem 2.15. *Any point $x \in \Omega_1(x^0)$ is an optimal solution of problem (P_2).*

Proof. For any $x \in \Omega_1(x^0)$, by Corollary 2.9 (ii), Lemma 2.12, Theorem 2.13, and Theorem 2.14, the following conditions hold

$$\begin{cases} Qx + c = \bar{s},\ \bar{s} \geq 0, \\ X\bar{s} = 0,\ x \geq 0, \end{cases}$$

which are exactly the KKT conditions (2.3). □

The result in Theorem 2.15 ensures that any limit point of $x(t)$ of ODE (2.1) as $t \to +\infty$ is an optimal solution for problem (P_2).

2.4 Strong convergence of trajectory (2.1)

Lemma 2.16 ([29, Inverse Function Theorem, Theorem 9.24])**.** *Let $D_2 \subset R^n$ be open and $f : D_2 \to R^n$ be a continuously differentiable function on D_2. If $f'(\alpha)$ is invertible for some $\alpha \in D_2$, then, there exists a neighborhood U of α and a neighborhood V of $\eta := f(\alpha)$ such that f is an invertible function on U.*

Theorem 2.17. *$\Omega_1(x^0)$ defined in (2.2) only contains a single point.*

Proof. Our proof strategy is to show that the set $\Omega_1(x^0)$ defined in (2.2) contains only an isolated point.

Since $\Omega_1(x^0)$ is nonempty, assume $x^* \in \Omega_1(x^0)$ and the number of its nonzero components is maximum for all $x \in \Omega_1(x^0)$, the index set $\{1, \ldots, n\}$ can be partitioned into two disjoint sets B and N such that

$$B = \{i | x_i^* > 0,\ i \in \{1, \ldots, n\}\}$$

and

$$N = \{i | x_i^* = 0,\ i \in \{1, \ldots, n\}\}.$$

If $B = \emptyset$, we can conclude there is a single point $x^* = 0$ in $\Omega_1(x^0)$. So we will focus on the case that B is nonempty, without loss of generality, we assume

$$B = \{1, \ldots, k\}\ (k \geq 1) \text{ and } N = \{k+1, \ldots, n\}.$$

Correspondingly, for any $x \in R^n$, it can be denoted by $x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$. Similarly, we can partition $s = \begin{pmatrix} s_B \\ s_N \end{pmatrix}$, $c = \begin{pmatrix} c_B \\ c_N \end{pmatrix}$ respectively, where $x_B,\ s_B,\ c_B \in R^k$, and $x_N,\ s_N,\ c_N \in R^{(n-k)}$.

Let $\delta_1 = \frac{1}{2}\min_{i \in B}\{x_i^*\}$. Together with the definition of x^*, we know

$$x_B > 0 \text{ and } x_N = 0\ \forall x \in \Omega_1(x^0) \cap U(x^*, \delta_1),$$

where $U(x^*, \delta_1)$ is the δ_1-neighborhood of x^*. Next we will prove that x^* is an isolated point of $\Omega_1(x^0)$.

For any point $x \in \Omega_1(x^0)$, from Lemma 2.12 and Theorem 2.13, we have

$$Qx + c = Qx^* + c \doteq s^*. \tag{2.7}$$

For the convenience of discussion, we denote

$$Q = [q_1, q_2, \ldots, q_n] \text{ and } b = s^* - c.$$

If $x \in \Omega_1(x^0) \cap U(x^*, \delta_1)$, we have $x_B > 0$ and $x_N = 0$. Then the first equality in (2.7) can be written as

$$x_1 q_1 + \cdots + x_k q_k = b \quad \forall x \in \Omega_1(x^0) \cap U(x^*, \delta_1). \tag{2.8}$$

Thus, $rank[q_1, q_2, \ldots, q_k] = rank[q_1, q_2, \ldots, q_k, b]$.

If $rank[q_1, q_2, \ldots, q_k] = k$, b can be expressed uniquely as a linear combination of $q_1, q_2, \ldots,\ q_k$, thus except for x^*, there is no x in $\Omega_1(x^0) \cap U(x^*, \delta_1)$ such that (2.8) holds, in other words, x^* is an isolated point of $\Omega_1(x^0)$.

If $rank[q_1, q_2, \ldots, q_k] = r < k$ and $r = 0$, then $q_1 = q_2 = \cdots = q_k = 0$, $\frac{dx_i(t)}{dt}$ in ODE (2.1) will be reduced to

$$\frac{dx_i(t)}{dt} = -c_i {x_i}^{2\gamma}, \quad i = 1, \ldots, k.$$

From Theorem 2.7, $\lim_{t\to+\infty} \frac{dx_i(t)}{dt} = 0$, thus there are two cases

(a) $c_i = 0 \Rightarrow x_i(t) \equiv x_i^0$, since $x_i^0 > 0$ is arbitrary, therefore the optimal solution set is unbounded, which contradicts with the boundedness of the optimal solution set;

(b) $c_i \neq 0 \Rightarrow x_i^* = 0$, this is a contradiction with the assumption that x_i^* is positive.

So we only consider the case that $1 \le r < k$, and assume $\{q_{p_1}, q_{p_2}, \ldots, q_{p_r}\}$ is a maximal linearly independent subset of $\{q_1, q_2, \ldots, q_k\}$, and $\{q_{p_{r+1}}, q_{p_{r+2}}, \ldots, q_{p_k}\} = \{q_1, q_2, \ldots, q_k\} \setminus \{q_{p_1},\ q_{p_2}, \ldots, q_{p_r}\}$. Thus there exists a matrix $W = (w_{ij})_{(k-r)\times r} \in R^{(k-r)\times r}$ such that

$$q_{p_{r+i}} = \sum_{j=1}^{r} w_{ij} q_{p_j},\ i = 1, \ldots, k-r. \tag{2.9}$$

We consider the following sub-system (k rows) of the first equation in (2.7)

$$\begin{cases} q_{p_1}^T x \ = b_{p_1}, \\ \quad \vdots \\ q_{p_r}^T x \ = b_{p_r}, \\ q_{p_{r+1}}^T x = b_{p_{r+1}}, \\ \quad \vdots \\ q_{p_k}^T x \ = b_{p_k}. \end{cases} \tag{2.10}$$

Combining (2.10) with (2.9), we have

$$b_{p_{r+i}} = \sum_{j=1}^{r} w_{ij} b_{p_j},\ i = 1, \ldots, k-r. \tag{2.11}$$

From Corollary 2.9 (ii), we have $X^* s^* = 0$ which implies $s_B^* = 0$. Thus

$$c_B = -b_B,$$

where $b = \begin{pmatrix} b_B \\ b_N \end{pmatrix}$. This and (2.11) indicate

$$c_{p_{r+i}} = \sum_{j=1}^{r} w_{ij} c_{p_j},\ i = 1, \ldots, k-r.$$

Clearly x^* is a solution of (2.10), but linear system (2.10) is degenerate. To overcome the difficulty caused by the degeneracy of linear system (2.10), we define

$$y_i(t) = \begin{cases} \sum_{j=1}^r w_{ij} \ln x_{p_j}(t) - \ln x_{p_{r+i}}(t) & \text{if } \gamma = \frac{1}{2}, \\ \sum_{j=1}^r w_{ij} \frac{x_{p_j}(t)^{1-2\gamma}}{1-2\gamma} - \frac{x_{p_{r+i}}(t)^{1-2\gamma}}{1-2\gamma} & \text{if } \gamma > \frac{1}{2}, \end{cases}$$

where $i = 1, \ldots, k-r$, $t \geq t_0$, and $x(t)$ is the solution of ODE (2.1). From Theorem 2.4, we know $x(t) > 0$ for all $t \geq 0$. Therefore, $y_i(t),\ i = 1, \ldots, k-r$ are well defined for $t \geq t_0$. Notice that

$$\frac{dx(t)}{dt} = -X^{2\gamma}(Qx + c),$$

then we have

$$\begin{aligned} \frac{dy_i(t)}{dt} &= \sum_{j=1}^r w_{ij} \frac{\frac{dx_{p_j}(t)}{dt}}{{x_{p_j}}^{2\gamma}} - \frac{\frac{dx_{p_{r+i}}(t)}{dt}}{{x_{p_{r+i}}}^{2\gamma}} \\ &= (-\sum_{j=1}^r w_{ij} q_{p_j}^T + q_{p_{r+i}}^T)x - \sum_{j=1}^r w_{ij} c_{p_j} + c_{p_{r+i}} \\ &\equiv 0, \quad i = 1, \ldots, k-r,\ t \geq t_0. \end{aligned}$$

Thus there exist $k - r$ constants $\bar{c}_i$ $(i = 1, \ldots, k-r)$ such that

$$y_i(t) \equiv \bar{c}_i, \quad i = 1, \ldots, k-r,\ t \geq t_0.$$

In particular, $\forall x \in \Omega_1(x^0) \cap U(x^*, \delta_1)$

$$\sum_{j=1}^r w_{ij} \ln x_{p_j} - \ln x_{p_{r+i}} \equiv \bar{c}_i, \quad i = 1, \ldots, k-r, \tag{2.12}$$

if $\gamma = \frac{1}{2}$, or

$$\sum_{j=1}^r w_{ij} \frac{x_{p_j}(t)^{1-2\gamma}}{1-2\gamma} - \frac{x_{p_{r+i}}(t)^{1-2\gamma}}{1-2\gamma} \equiv \bar{c}_i, \quad i = 1, \ldots, k-r, \tag{2.13}$$

if $\gamma > \frac{1}{2}$. Let $H \in R^{r \times k}$ be a matrix generated by choosing r linearly independent rows, say $l_1, \ldots, l_r$, from matrix $[q_{p_1}, \ldots, q_{p_r}, q_{p_{r+1}}, \ldots, q_{p_k}]$. Thus H can be written as

$$H = \begin{pmatrix} q_{l_1 p_1} & \cdots & q_{l_1 p_r} & q_{l_1 p_{r+1}} & \cdots & q_{l_1 p_k} \\ q_{l_2 p_1} & \cdots & q_{l_2 p_r} & q_{l_2 p_{r+1}} & \cdots & q_{l_2 p_k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ q_{l_r p_1} & \cdots & q_{l_r p_r} & q_{l_r p_{r+1}} & \cdots & q_{l_r p_k} \end{pmatrix}.$$

Then the following nonlinear system

$$\begin{cases} q_{l_1 p_1} z_1 + q_{l_1 p_2} z_2 + \cdots + q_{l_1 p_k} z_k = b_{l1}, \\ q_{l_2 p_1} z_1 + q_{l_2 p_2} z_2 + \cdots + q_{l_2 p_k} z_k = b_{l2}, \\ \qquad \vdots \\ q_{l_r p_1} z_1 + q_{l_r p_2} z_2 + \cdots + q_{l_r p_k} z_k = b_{lr}, \\ w_{11} \ln z_1 + \cdots + w_{1r} \ln z_r - \ln z_{r+1} = \bar{c}_1, \\ w_{21} \ln z_1 + \cdots + w_{2r} \ln z_r - \ln z_{r+2} = \bar{c}_2, \\ \qquad \vdots \\ w_{(k-r)1} \ln z_1 + \cdots + w_{(k-r)r} \ln z_r - \ln z_k = \bar{c}_{k-r}, \end{cases} \tag{2.14}$$

if $\gamma = \frac{1}{2}$, or

$$\begin{cases} q_{l_1p_1}z_1 + q_{l_1p_2}z_2 + \cdots + q_{l_1p_k}z_k = b_{l1}, \\ q_{l_2p_1}z_1 + q_{l_2p_2}z_2 + \cdots + q_{l_2p_k}z_k = b_{l2}, \\ \quad\vdots \\ q_{l_rp_1}z_1 + q_{l_rp_2}z_2 + \cdots + q_{l_rp_k}z_k = b_{lr}, \\ w_{11}\frac{{z_1}^{1-2\gamma}}{1-2\gamma} + \cdots + w_{1r}\frac{{z_r}^{1-2\gamma}}{1-2\gamma} - \frac{{z_{r+1}}^{1-2\gamma}}{1-2\gamma} = \bar{c}_1, \\ w_{21}r+1 + \cdots + w_{2r}\frac{{z_r}^{1-2\gamma}}{1-2\gamma} - \frac{{z_{r+2}}^{1-2\gamma}}{1-2\gamma} = \bar{c}_2, \\ \quad\vdots \\ w_{(k-r)1}\frac{{z_1}^{1-2\gamma}}{1-2\gamma} + \cdots + w_{(k-r)r}\frac{{z_r}^{1-2\gamma}}{1-2\gamma} - \frac{{z_k}^{1-2\gamma}}{1-2\gamma} = \bar{c}_{k-r}, \end{cases} \tag{2.15}$$

if $\gamma > \frac{1}{2}$, is introduced. From (2.10) , (2.12), and (2.13), we know that for any $x \in \Omega_1(x^0) \cap U(x^*, \delta_1)$, $z = (x_{p_1}, \ldots, x_{p_r}, x_{p_{r+1}}, \ldots, x_{p_k})^T$ is a solution of system (2.14) or (2.15).

The Jacobian matrix of nonlinear system (2.14) or (2.15) is
$J(z) =$

$$\begin{pmatrix} q_{l_1p_1} & \cdots & q_{l_1p_r} & q_{l_1p_{r+1}} & q_{l_1p_{r+2}} & \cdots & q_{l_1p_k} \\ q_{l_2p_1} & \cdots & q_{l_2p_r} & q_{l_2p_{r+1}} & q_{l_2p_{r+2}} & \cdots & q_{l_2p_k} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{l_rp_1} & \cdots & q_{l_rp_r} & q_{l_rp_{r+1}} & q_{l_rp_{r+2}} & \cdots & q_{l_rp_k} \\ w_{11}\frac{1}{{z_1}^{2\gamma}} & \cdots & w_{1r}\frac{1}{{z_r}^{2\gamma}} & -\frac{1}{{z_{r+1}}^{2\gamma}} & 0 & \cdots & 0 \\ w_{21}\frac{1}{{z_1}^{2\gamma}} & \cdots & w_{2r}\frac{1}{{z_r}^{2\gamma}} & 0 & -\frac{1}{{z_{r+2}}^{2\gamma}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{(k-r)1}\frac{1}{{z_1}^{2\gamma}} & \cdots & w_{(k-r)r}\frac{1}{{z_r}^{2\gamma}} & 0 & 0 & \cdots & -\frac{1}{{z_k}^{2\gamma}} \end{pmatrix}.$$

From (2.9), after a series of Gaussian eliminations, it is not hard to verify that $J(z)$ is invertible if $z\ (\in R^k)\ > 0$.

Now let $F(z) = 0$ be system (2.14) or (2.15). From previous discussions, we know (i) $\forall x \in \Omega_1(x^0) \cap U(x^*, \delta_1)$, $z = (x_{p_1}, \ldots, x_{p_r}, x_{p_{r+1}}, \ldots, x_{p_k})^T$ is a solution of $F(z) = 0$, in particular, $z^* = (x^*_{p_1}, \ldots, x^*_{p_r}, x^*_{p_{r+1}}, \ldots, x^*_{p_k})^T$ is also a solution of $F(z) = 0$; (ii) $\frac{\partial F}{\partial z}$ is invertible $\forall z\ (\in R^k)\ > 0$; and (iii) $z^* > 0$. By Lemma 2.16, $z = z^*$ must be an isolated point satisfying $F(z) = 0$. Therefore, there exists a $\delta_2 > 0$ ($\delta_2 \le \delta_1$) such that for any $x \in \Omega_1(x^0) \cap U(x^*, \delta_2)$, $z = (x_{p_1}, \ldots, x_{p_r},\ x_{p_{r+1}}, \ldots, x_{p_k})^T$ is a solution of system (2.14) or (2.15) if and only if $x = x^*$. Thus there is only one point $x^* \in \Omega_1(x^0) \cap U(x^*, \delta_2)$, i.e., x^* is an isolated point of $\Omega_1(x^0)$. But $\Omega_1(x^0)$ is connected, thus there is only one point x^* in $\Omega_1(x^0)$. The proof is complete. □

Theorem 2.17 ensures the strong convergence of the solution $x(t)$ of ODE (2.1) as $t \to +\infty$. This along with Theorem 2.15 guarantee that the limit point is an optimal solution for problem (P_2). It should be mentioned that the limit point depends on the starting point x^0 in general.

3 A Simple Dynamical System for Box Constrained Convex Quadratic Programming

In this section, we focus on solving problem (P_3) by constructing another simple dynamical system model. Based on active set strategies, some algorithms for solving box constrained problems were proposed in [15] and [20]. By solving a series of equality constrained quadratic optimization problems, finally the optimal solution of the original problem is obtained. But for large scale problems, there are two main disadvantages, one is that some constraints are added (dropped) at a time to (from) the working set, which would lead to an excessive number of iterations. The other disadvantage is that the exact minimizer on the current working face is required before adding (dropping) constraints [17]. In order to avoid these disadvantages, some gradient projection based algorithms were proposed in [4, 22, 39, 27, 28]. An algorithm that combines active set strategy with the gradient projection method was discussed in [17]. Xia and Wang presented a projected dynamical system to solve the convex programming with box constraints [43]. As discussed in Section 1, the construction of our dynamical system models is based on the interior point approach. In this section, a dynamical system model will be introduced for the box constrained convex quadratic problem (P_3).

In ODE (2.1), X plays the role of a barrier such that the entire solution trajectory stays in the positive region. Extending the same idea for problem (P_3), we construct the following ODE, which shares the similar properties of ODE (2.1)

$$\frac{dx(t)}{dt} = -X^{2\gamma}(I_n - X)^{2\gamma}(Qx + c),\ x(t_0) = x^0,\ 0 < x^0 < e, \tag{3.1}$$

where $\gamma \geq \frac{1}{2}$ is a fixed constant.

It should be noticed that there are some continuous trajectory models that are very similar to system (3.1). From the form $h(x)$ in Assumptions 1 and 2 in [1], we can see that our ODE in (3.1) shares the same format as the ODE (4) in [1] and ODE (3.1) in [5] (after trivial transformations). But our ODE system (3.1) is not contained in [1] and [5], since the target function in [1] and [5] is multilinear, however our target function is convex quadratic which is not multilinear. So our ODE system (3.1) only coincides with that of [1] and [5] in the linear case. Since the problems concerned in [1] and [5] may not be convex, there is no convergence study there.

In later discussions, the following KKT system of problem (P_3) will be used:

$$\begin{cases} Qx + c = z - y,\ 0 \leq x \leq e, \\ (I_n - X)y = 0,\ y \geq 0, \\ Xz = 0,\ z \geq 0, \end{cases} \tag{3.2}$$

where $y, z \in R^n$. Similar to Section 2, the following discussions will reveal some important theoretical properties for ODE (3.1). The proofs of some results will be provided only if necessary.

Theorem 3.1. *Let $x(t)$ be the solution of ODE (3.1) with the maximal existence interval $[t_0,\ \beta)$. Then $0 < x(t) < e$ for any $t \in [t_0,\ \beta)$.*

Proof. We will prove that $0 < x(t) < e$ for any $t \in [t_0,\ \beta)$ by contradiction. In other words, $rank(X(I_n - X)) \equiv n$ for any $t \in [t_0,\ \beta)$.

Suppose that there exists a $t^* \in [t_0,\ \beta)$ such that $rank(X^*(I_n - X^*)) \leq n - 1$. Since $x_i(t)$ is continuous on t, let t^* be the minimal t such that $rank(X^*(I_n - X^*)) \leq n - 1$, i.e., $0 < x(t) < e$ for all $t_0 \leq t < t^*$. Thus there must exist some $j \in \{1, \ldots, n\}$ such that $x_i(t^*) = 1$ or $x_i(t^*) = 0$. First suppose $x_i(t^*) = 1$, and

$$rank(X(t)(I_n - X(t))) = n\ \forall t \in [t_0, t^*).$$

Let

$$M_1 = \sup\{\|X(Qx+c)\|^{2\gamma} + 1 : 0 \le x \le e\},$$

$$M_2 = \max_{t\in[t_0,\ t^*]} \|1 - x(t)\|^{2\gamma-1} + 1,$$

and

$$t_1 = \max\{t_0, t^* - \frac{1}{2M_1M_2}\}.$$

Further, let $\bar{t}$ be the time satisfying

$$x_i(\bar{t}) = \min_{t\in[t_1,t^*]} x_i(t) < 1.$$

Notice that

$$\frac{dx(t)}{dt} = -(I_n - X)^{2\gamma} X^{2\gamma}(Qx+c),$$

we have

$$x_i(t^*) - x_i(t) = -\int_t^{t^*} (1 - x_i(\tau))^{2\gamma} e_i^T X^{2\gamma}(Qx+c)d\tau.$$

For any $t \in [t_1, t^*]$, $0 \le x_i(t) \le 1$, since $x_i(t^*) = 1$, we have

$$1 - x_i(t) \le M_1|t^* - t|(1 - x_i(\bar{t}))^{2\gamma} \le \frac{1}{2}(1 - x_i(\bar{t})).$$

Since t is arbitrary in $[t_1, t^*]$, taking $t = \bar{t}$ in the above inequality, we have

$$1 - x_i(\bar{t}) \le \frac{1}{2}(1 - x_i(\bar{t})).$$

Thus, $x_i(\bar{t}) = 1$, which is a contradiction with the definition of $x_i(\bar{t})$, $x_i(t^*) = 1$ is rejected. Similarly, $x_i(t^*) = 0$ is rejected for some i and some $t^* \in [t_0,\ \beta)$. □

In (3.1), the matrix $X^{2\gamma}(I_n - X)^{2\gamma}$ plays the role of a box barrier such that the entire solution trajectory of ODE (3.1) will stay in the interior of the box constrained region. Thus $\beta = +\infty$.

Corollary 3.2. *There exists a unique solution $x(t)$ for ODE (3.1) on $[t_0,\ +\infty)$, and $0 < x(t) < e$ for any $t \in [t_0,\ +\infty)$.*

Theorem 3.3. *Let $x(t)$ be the solution of ODE (3.1) on $[t_0,\ +\infty)$. Then $\lim_{t\to+\infty} X^{2\gamma}(I_n - X)^{2\gamma}(Qx+c) = 0$.*

Theorem 3.4. *Let $x(t)$ be the solution of ODE (3.1). If $X^{2\gamma}(I_n - X)^{2\gamma}(Qx+c)|_{t=0} \ne 0$, then $X^{2\gamma}(I_n - X)^{2\gamma}(Qx+c) \ne 0$ for any $t \ge t_0$.*

Similar to (2.4), we define

$$u(x) = Qx + c. \tag{3.3}$$

Let $x(t)$ be the solution of ODE (3.1). Define the limit set

$$\Omega_2(x^0) = \{y \in R^n \mid\ y \text{ is a cluster point of } x(t) \text{ of ODE (3.1) as } t \to +\infty\}. \tag{3.4}$$

Corollary 3.2 implies $0 < x(t) < e$ for any $t \geq t_0$, thus the limit set $\Omega_2(x^0)$ is nonempty, compact, connected [12]. For some given $\bar{x} \in \Omega_2(x^0)$, let

$$\bar{u} = Q\bar{x} + c.$$

By the monotonicity of $q(x(t))$ in terms of t and Theorem 3.3, we have the following corollary.

Corollary 3.5. *Let $q(x)$ be defined in problem (P_3). Then (i) $q(x) = q(\bar{x})$ $\forall x \in \Omega_2(x^0)$; (ii) $X(I_n - X)u(x) = 0$ $\forall x \in \Omega_2(x^0)$, where $u(x)$ is defined in (3.3).*

Let $\bar{J} = \{j | \bar{u}_j = 0,\ j \in \{1, \ldots, n\}\}$ and $\bar{J}^c = \{1, \ldots, n\} \setminus \bar{J}$, then

$$\bar{u}_j \neq 0, \quad \text{for any } j \in \bar{J}^c.$$

Based on this and $\bar{X}(I_n - \bar{X})\bar{u} = 0$, we partition $\bar{J}^c$ by

$$\bar{J}_l^c = \{j | \bar{x}_j = 0,\ j \in \bar{J}^c\} \text{ and } \bar{J}_u^c = \{j | \bar{x}_j = 1,\ j \in \bar{J}^c\},$$

and define set

$$\bar{\Lambda}_2 = \{x \in R^n \mid 0 \leq x \leq e,\ x_{\bar{J}_l^c} = 0,\ x_{\bar{J}_u^c} = 1,\ q(x) = q(\bar{x})\}, \tag{3.5}$$

clearly $\bar{\Lambda}_2$ is closed. $\bar{\Lambda}_2$ is nonempty since $\bar{x} \in \bar{\Lambda}_2$. The following results are similar to those in Section 2.

Theorem 3.6. *$\bar{\Lambda}_2$ in (3.5) is a convex set.*

Proof. Let x be an arbitrary point in the convex hull $co(\bar{\Lambda}_2)$, i.e., x is a positive linear convex combination of some points in $\bar{\Lambda}_2$, thus $x_{\bar{J}_l^c}$=0, $x_{\bar{J}_u^c}$=1, $0 \leq x \leq e$. Based on the convexity of $q(x)$, the following inequality holds

$$q(x) \leq q(\bar{x}). \tag{3.6}$$

On the other hand, let $\Delta x = x - \bar{x}$, then $(\Delta x)_{\bar{J}^c} = 0$ and $\bar{u}^T(\Delta x) = 0$, again by the convexity of $q(x)$, we have

$$\begin{aligned} q(x) &\geq q(\bar{x}) + \nabla q(\bar{x})^T(\Delta x) \\ &= q(\bar{x}) + \bar{u}^T(\Delta x) \\ &= q(\bar{x}), \end{aligned}$$

this and (3.6) imply $q(x) = q(\bar{x})$ for all $x \in co(\bar{\Lambda}_2)$, thus $x \in \bar{\Lambda}_2$, hence $\bar{\Lambda}_2$ in convex. □

By Lemma 2.11 and Theorem 3.6, the following theorem is straightforward.

Theorem 3.7. *$u(x) = \bar{u}$ for all $x \in \bar{\Lambda}_2$, where $u(x)$ is defined in (3.3).*

Theorem 3.8. *$\Omega_2(x^0) \subseteq \bar{\Lambda}_2$, where $\Omega_2(x^0)$ is defined in (3.4).*

Proof. If $\bar{J}_0^c$ is empty, $\bar{\Lambda}_2 = \{x \in R^n \mid 0 \le x \le e,\ q(x) = q(\bar{x})\}$. From Corollary 3.5 (i), the result holds clearly. Now we consider the case that $\bar{J}_0^c$ is nonempty. Suppose there exists a point $\hat{x} \in \Omega_2(x^0)$ but $\hat{x} \notin \bar{\Lambda}_2$. Notice that $\bar{\Lambda}_2$ lies inside the bounded set $\{x \in R^n \mid 0 \le x \le e\}$, so $\bar{\Lambda}_2$ is compact (since $\bar{\Lambda}_2$ is closed). Thus $u(x)$ in (3.3) is uniformly continuous over $\bar{\Lambda}_2$. Theorem 3.7 implies there exists some $\delta_0 > 0$ such that

$$|u_j(x)| \ge |\bar{u}_j|/2 > 0\ \forall\ j \in \bar{J}_0^c,\ \forall\ x \in U(\bar{\Lambda}_2, \delta_0), \tag{3.7}$$

where $U(\bar{\Lambda}_2, \delta_0)$ is the δ_0-neighborhood of set $\bar{\Lambda}_2$. Since $\hat{x} \notin \bar{\Lambda}_2$ and $\bar{\Lambda}_2$ is compact, there exists some $\delta_1 \in (0,\ \delta_0] \cap (0,\ 0.1]$ such that $\hat{x} \notin U(\bar{\Lambda}_2, \delta_1)$. Notice $\bar{x} \in \Omega_2(x^0) \cap \bar{\Lambda}_2$ and $\hat{x} \in \Omega_2(x^0)$ but $\hat{x} \notin U(\bar{\Lambda}_2, \delta_1)$, by the connectivity of $\Omega_2(x^0)$, there must exist some $\tilde{x} \in \Omega_2(x^0) \cap U(\bar{\Lambda}_2, \delta_1)$ but $\tilde{x} \notin \bar{\Lambda}_2$. $\tilde{x} \in \Omega_2(x^0)$ and Corollary 3.5 (i) imply

$$0 \le \tilde{x} \le e,\ q(\tilde{x}) = q(\bar{x}).$$

Since $\tilde{x} \notin \bar{\Lambda}_2$, $\tilde{x} \in \Omega_2(x^0)$, and at least one of the sets $\bar{J}_l^c$ and $\bar{J}_u^c$ is nonempty, then at least one of the following two cases will occur

(a) $\tilde{x}_{\bar{J}_l^c} = 0$ is not true, i.e. there exists some $j_1 \in \bar{J}_l^c$ such that $\tilde{x}_{j_1} > 0$; or

(b) $\tilde{x}_{\bar{J}_u^c} = e_{\bar{J}_u^c}$ is not true, i.e. there exists some $j_2 \in \bar{J}_u^c$ such that $\tilde{x}_{j_2} < 1$.

If case (a) arises, since $\tilde{x} \in \Omega_2(x^0) \cap U(\bar{\Lambda}_2, \delta_1)$ and $\delta_1 \le 0.1$, then $\tilde{x}_{j_1} < 0.1$. Thus, $0 < \tilde{x}_{j_1} < 0.1$.

If case (b) arises, since $\tilde{x} \in \Omega_2(x^0) \cap U(\bar{\Lambda}_2, \delta_1)$ and $\delta_1 \le 0.1$, then $\tilde{x}_{j_2} > 0.9$. Thus, $0.9 < \tilde{x}_{j_2} < 1$.

In either case, there exists a j (j_1 or j_2) $\in \bar{J}_0^c$ such that $\tilde{x}_j(1-\tilde{x}_j) \ne 0$. (3.7) and $\delta_1 \le \delta_0$ ensure $|u_j(\tilde{x})| > 0$, thus $\tilde{x}_j(1-\tilde{x}_j)u_j(\tilde{x}) \ne 0$. This contracts with the fact $\tilde{X}(I_n - \tilde{X})u(\tilde{x}) = 0$ (Corollary 3.5 (ii)) since $\tilde{x} \in \Omega_2(x^0)$, thus the proof is complete. □

Theorem 3.9. *If $x(t)$ is the solution of ODE (3.1), then $\lim_{t\to+\infty}(Qx+c) = \bar{u}$, and $\bar{u}_{\bar{J}_l^c} > 0$ if $\bar{J}_l^c$ is nonempty, $\bar{u}_{\bar{J}_u^c} < 0$ if $\bar{J}_u^c$ is nonempty.*

Proof. By the continuity of $s(x(t))$, compactness of $\bar{\Lambda}_2$, Theorem 3.8, and Theorem 3.9, clearly

$$\lim_{t\to+\infty}(Qx + c) = \bar{u}.$$

If $\bar{J}_l^c$ is nonempty, by the definition of $\bar{J}_l^c$, $\bar{u}_j \ne 0$ for any $j \in \bar{J}_l^c$. Suppose there exists some $\bar{j} \in \bar{J}_l^c$ such that $\bar{u}_{\bar{j}} < 0$. Since $u(x(t))$ is continuous on $[t_0,\ \infty)$, there exists some t_K such that $u_{\bar{j}}(x(t)) < 0$ for all $t \ge t_K$. For any cluster point $\bar{x} \in \Omega_2(x^0)$, $\bar{j} \in \bar{J}_l^c$ implies $\bar{x}_{\bar{j}} = 0$. Notice that

$$\frac{dx(t)}{dt} = -X^{2\gamma}(I_n - X)^{2\gamma}(Qx + c),$$

and $0 < x(t) < e$ for any $t \in [t_0,\ \infty)$. Thus $\frac{dx_{\bar{j}}(t)}{dt} \ge 0$, so $x_{\bar{j}}(t) \ge x_{\bar{j}}(t_K) > 0$ for all $t \ge t_K$, which contradicts with $\bar{x}_{\bar{j}} = 0$.

Similarly, we can prove $\bar{s}_{\bar{J}_u^c} < 0$ if $\bar{J}_u^c$ is nonempty. □

Theorem 3.10. *Any $x \in \Omega_2(x^0)$ is an optimal solution for problem (P_3).*

Proof. For any $x \in \Omega_2(x^0)$, we have

$$Qx + c = Q\bar{x} + c = \bar{u}$$

and

$$X(I_n - X)\bar{u} = 0.$$

By Theorem 3.9, together with the definitions of $\bar{J}$, $\bar{J}_u^c$, and $\bar{J}_l^c$, for any $i \in \{1, \dots, n\}$, we have

$$\begin{array}{ll} \bar{u}_i = 0, & \text{if } 0 < x_i < 1; \\ \bar{u}_i \le 0, & \text{if } x_i = 1; \\ \bar{u}_i \ge 0, & \text{if } x_i = 0. \end{array}$$

Let's define $z,\ y \in R^n$ by

$$\begin{cases} z_i = 0,\ y_i = 0, & \text{if } 0 < x_i < 1; \\ z_i = 0,\ y_i = -\bar{u}_i, & \text{if } x_i = 1; \\ z_i = \bar{u}_i,\ y_i = 0, & \text{if } x_i = 0. \end{cases}$$

Thus the following relations hold

$$\begin{cases} Qx + c = z - y,\ 0 \le x \le e, \\ (I_n - X)y = 0,\ y \ge 0, \\ Xz = 0,\ z \ge 0, \end{cases}$$

which are exactly the KKT conditions (3.2) of problem (P_3). Thus the optimality of x is obvious. □

Theorem 3.11. *The limit set $\Omega_2(x^0)$ only contains a single point.*

Proof. Assume $x^* \in \Omega_2(x^0)$ and the rank of matrix $X^*(I_n - X^*)$ is maximum for all $x \in \Omega_2(x^0)$, then the index set $\{1, 2, \dots, n\}$ can be partitioned into three disjoint sets $\bar{B}$, $\bar{N}_1$, $\bar{N}_2$ such that

$$\begin{cases} 0 < x_i^* < 1,\ i \in \bar{B}, \\ x_i^* = 0,\ i \in \bar{N}_1, \\ x_i^* = 1,\ i \in \bar{N}_2. \end{cases}$$

If $\bar{B} = \emptyset$, we can conclude there exist at most 2^n points in $\Omega_2(x^0)$. Since $\Omega_2(x^0)$ is connected, there is only one point in $\Omega_2(x^0)$. If $\bar{B}$ is nonempty, similar to the lengthy proof of Theorem 2.17, it can be proved that x^* is an isolated point of $\Omega_2(x^0)$. □

For general bound constrained convex quadratic programming (P_1) with $l < u$, if the optimal solution set is bounded, the following ODE system can be used

$$\frac{dx(t)}{dt} = -diag(g(x))(Qx + c),\ x(t_0) = x^0,\ l < x^0 < u, \tag{3.8}$$

where $g : R^n \to R^n$ is defined as follows

$$g_i(x) = \begin{cases} (u_i - x_i)^{2\gamma} & \text{if } l_i = -\infty,\ u_i < +\infty, \\ (u_i - x_i)^{2\gamma}(x_i - l_i)^{2\gamma} & \text{if } -\infty < l_i,\ u_i < +\infty, \\ (x_i - l_i)^{2\gamma} & \text{if } -\infty < l_i,\ u_i = +\infty, \\ 1 & \text{if } l_i = -\infty,\ u_i = +\infty, \end{cases}$$

$\gamma \ge \frac{1}{2}$ is a fixed constant, and $i = 1, 2, \dots, n$. Obviously, ODE systems (2.1) and (3.1) are special forms of ODE system (3.8). Similarly, the previous theoretical results in Section 2 and Section 3 should also hold for ODE (3.8).

4 Numerical Simulation

In this section, two small examples and a set of randomly generated large size (Q, c)s are provided to illustrate the performance of our two interior point continuous trajectory models. The simulation is conducted in Matlab R2015a on a PC. Matlab ODE solvers are not used since these ODE solvers tend to track the trajectory of the ODE rather than computing the limit point (or equilibrium point) of the ODE. Therefore, the following explicit Euler scheme is adopted to solve ODE (3.8). The resulting iterative algorithm is as follows:
Algorithm 1: Explicit Euler scheme for ODE (3.8)

$$x_{k+1} = x_k - h_k diag(g(x_k))(Qx_k + c), \quad k = 0, 1, \ldots$$

where x^0 is chosen as an interior point, $g(x)$ is defined in (3.8) and h_k is chosen so that x_{k+1} stays in the interior region and $q(x_{k+1}) < q(x_k)$ ($q(x)$ is the objective function in problem (P_1)).

In the following computation, $\gamma = 0.5$ is used for Algorithm 1.

Our first small convex quadratic programming with nonnegativity constraint example is constructed to exam the behavior of our dynamical system (2.1) or (3.8). For this example, there are many optimal solutions.

Example 4.1.

$$\min\{\frac{1}{2}x^TQx + c^Tx : x_i \geq 0,\ i = 1, 2, 3\}$$

where matrix Q and vector c are given by

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix} \text{ and } c = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

The optimal solution is $x^* = (\xi^*, 1 - \xi^*, 0)^T$, where $\xi^* \in [0,\ 1]$. The x-space behavior of solution trajectories of ODE (2.1) with 30 random initial points is displayed in Fig. 1.

Fig. 1 illustrate that all solution trajectories of our dynamical system (2.1) converge to some optimal solutions of Example 4.1. In general, the limit point depends on the initial point if there are more than one optimal solution.

Our second example is a box constrained convex quadratic programming problem which is constructed according to Example 1 in [38]:

Example 4.2.

$$\min\{\frac{1}{2}x^TQx + c^Tx : -20 \leq x_i \leq 20,\ i = 1, 2, 3\}$$

where matrix Q and vector c are given by

$$Q = \begin{pmatrix} 0.18 & 0.648 & 0.288 \\ 0.648 & 2.88 & 0.72 \\ 0.288 & 0.72 & 0.72 \end{pmatrix} \text{ and } c = \begin{pmatrix} 0.4 \\ 0.2 \\ 0.3 \end{pmatrix}.$$

In Example 4.2, the optimal solution is $x^* = (-20, 3.38, 4.204)^T$. Starting from 30 random points, the transient behavior of all components of trajectories of ODE (3.1) is displayed in Fig. 2.

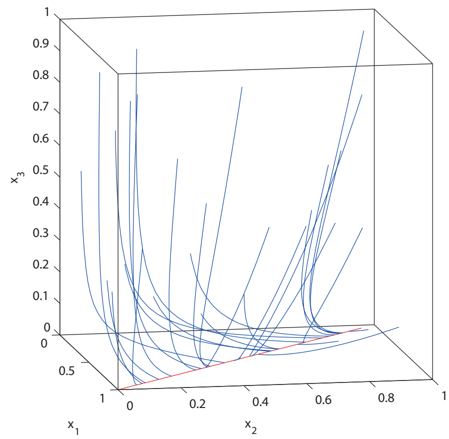

Figure 1: The x-space behavior of solution trajectories of ODE (2.1) with 30 random initial points for Example 4.1.

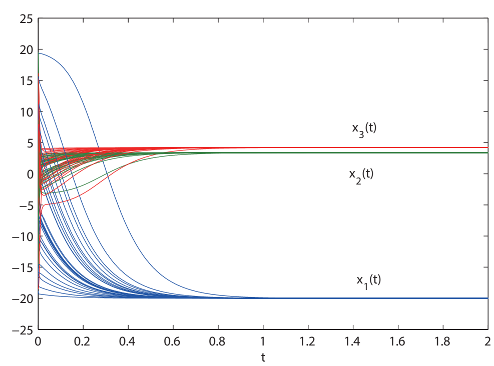

Figure 2: Transient behavior of all components of trajectories of ODE (3.1) with 30 random initial points for Example 4.2.

To better understand the performance of the proposing interior point continuous trajectory models in this paper, a set of large size random problems are tested. For the nonnegatively constrained convex quadratic programming of form (P_2), combining with KKT condition (2.3), Q and c are constructed as follows:

$$\begin{cases} Q = W^T W,\ W = rand(r,n),\ r = rank(Q) < n; \\ c_i = \begin{cases} rand(1,1) - [Qx^*]_i, & \text{if } x_i^* = 0; \\ -[Qx^*]_i, & \text{if } x_i^* \neq 0; \end{cases} \end{cases}$$

where $x^* \in R_+^n$ is given and $rand(\cdot,\cdot)$ is the Matlab random function. In our test, for vector x^*, $\frac{1}{4}n$ components are set to 0, $\frac{1}{4}n$ components equal 1 and the remaining components are assigned to random values between 0 and 1. The above values are distributed randomly among the components of x^*. The initial points are set to $x^0 = rand(n,1)$. The stopping criterion for our Algorithm 1 is set at $\frac{q(x)-q(x^*)}{|q(x^*)|+1} \leq 10^{-6}$. In addition, Matlab function *quadprog*, which is a very reliable and powerful iterative solver for convex quadratic problems, is used to compare with our Algorithm 1. The numbers reported in the following Tables 1 and 2 are the average of 20 runs, where Relative error in q is the final value of $\frac{q(x)-q(x^*)}{|q(x^*)|+1}$.

Similarly, combining with KKT condition (3.2), for the box constrained convex quadratic programming of form (P_3), Q and c are constructed as follows:

$$\begin{cases} Q = W^T W,\ W = rand(r,n),\ r = rank(Q) < n; \\ c_i = \begin{cases} rand(1,1) - [Qx^*]_i, & \text{if } x_i^* = 0; \\ -rand(1,1) - [Qx^*]_i, & \text{if } x_i^* = 1; \\ -[Qx^*]_i, & \text{if } 0 < x_i^* < 1; \end{cases} \end{cases}$$

where $x^* \in \{x \in R^n \mid 0 \leq x \leq e\}$. In our next test, x^* and x^0 are set the same as the ones in the above test. With the same stopping criterion for our Algorithm 1, the simulation results are reported in Tables 3 and 4. Again, the numbers reported in Tables 3 and 4 are the average of 20 runs.

Remarks for Tables 1–4: (i) Matlab function *quadprog* is a second-order algorithm, therefore it requires less number of iterations (normally less than 20) and achieves higher accuracy (see Relative error in q). (ii) Algorithm 1 is a first-order algorithm, therefore it requires more iterations to reach the same accuracy. However, the computation per iteration is much more economic as seen from these tables.

5 Conclusions

Two interior point continuous trajectory models are introduced for solving nonnegatively and box constrained convex quadratic programming problems. Without requiring any projection and introducing any dual variable, the two ODE systems in our interior point continuous trajectory models are sufficiently simple, only requiring matrix-vector multiplication. Furthermore, the two ODE systems have been proved (without using Lyapunov function) to converge to optimal solutions of the respective optimization problems for any interior feasible point in the limit. Our numerical results have shown that the two interior point continuous trajectory models are attractive and efficient in obtaining the optimal solutions. Finally, it should be mentioned that our dynamical system models for problem (P_1) can be extended to include linear constraint $Ax = b$.

Table 1: $Rank(Q) = 0.1n$ for problem (P_2)

	CPU Time (s)		Relative error in q		Iteration No.	
n	Alg. 1	quadprog	Alg. 1	quadprog	Alg. 1	quadprog
2,500	4.0208	95.7349	8.8e-7	1.1e-11	657	9.7
5,000	1.9771	578.4559	9.3e-7	1.3e-12	132	9.9
10,000	5.8089	5250.9351	7.6e-7	5.2e-13	133	10.0
20,000	25.3015	> 8-hour	7.6e-7	–	150	–

where − indicates the unavailable data due to the forced stop.

Table 2: $Rank(Q) = 0.2n$ for problem (P_2)

	CPU Time (s)		Relative error in q		Iteration No.	
n	Alg. 1	quadprog	Alg. 1	quadprog	Alg. 1	quadprog
2,500	3.1860	102.801	8.8e-7	2.7e-12	625	10.3
5,000	3.2265	709.443	8.4e-7	3.0e-12	207	10.2
10,000	6.7420	5757.005	8.1e-7	9.8e-13	182	10.8
20,000	36.5718	> 8-hour	8.0e-7	–	238	–

where − indicates the unavailable data due to the forced stop.

Table 3: $Rank(Q) = 0.1n$ for problem (P_3)

	CPU Time (s)		Relative error in q		Iteration No.	
n	Alg. 1	quadprog	Alg. 1	quadprog	Alg. 1	quadprog
2,500	7.9848	24.1298	1.0e-6	1.2e-8	3,013	9.3
5,000	26.1597	112.3274	1.0e-6	1.5e-8	3,178	10.1
10,000	81.6944	631.8784	1.0e-6	1.6e-8	3,484	11.5
20,000	358.5901	7251.73	9.9e-7	2.3e-8	3,786	12.3

Table 4: $Rank(Q) = 0.2n$ for problem (P_3)

	CPU Time (s)		Relative error in q		Iteration No.	
n	Alg. 1	quadprog	Alg. 1	quadprog	Alg. 1	quadprog
2,500	6.3998	30.9790	9.9e-7	5.7e-9	2,345	10.4
5,000	21.5205	131.717	9.9e-7	5.9e-9	2,636	11.9
10,000	64.6213	681.1962	9.9e-7	1.1e-8	2,772	12.2
20,000	291.0788	7481.171	9.8e-7	7.5e-9	3,087	13.3

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Hongwei Yue
School of Mathematics and Information Science
North China University of Water Resources and Electric Power
Zhengzhou, P. R. China
E-mail address: yuehongwei@ncwu.edu.cn

Li-Zhi Liao
Department of Mathematics, Hong Kong Baptist University
Hong Kong, P. R. China
E-mail address: liliao@hkbu.edu.hk

Xun Qian
Department of Mathematics, Hong Kong Baptist University
Hong Kong, P. R. China
E-mail address:13479784@life.hkbu.edu.hk