Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 14, No. 4, pp. 659–673, 2018

COMPLETELY POSITIVE CONES AS SUBCONES OF DOUBLY NONNEGATIVE CONES

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Abstract: The convex cone of $n \times n$ completely positive matrices and the convex cone of $n \times n$ copositive matrices are dual to each other. They have attracted interest in mathematical optimization due to the reformulations of some hard problems to copositive optimization problems. These two cones are closely related to the convex cone of $n \times n$ doubly nonnegative matrices and the dual cone of the doubly nonnegative cones. In this paper, we prove a duality result between the set of maximal faces of a proper cone and the set of minimal exposed faces of its dual cone. We also initialize a discussion on which faces of doubly nonnegative cones intersect completely positive cones for more than the origin. This approach is motivated by the idea: using well-studied doubly nonnegative cones to study completely positive cones.

Key words: *completely positive cones, doubly nonnegative cones, maximal faces, minimal exposed Faces*

Mathematics Subject Classification: *15B48; 52A20; 15B99*

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1 Introduction

Like linear semidefinite optimization, copositive optimization is a conic optimization problem. Instead of the cone of semidefinite matrices used in linear semidefinite optimization, copositive cone and its dual cone, namely completely positive cones, are used in the formulation of the primal-dual pair of a copositive optimization problem. Although many hard problems ([4] [7]) can be reformulated as copositive optimization problems, without knowing the structure of the copositive and completely positive cones, the reformulations would not provide much useful information. Studying the structure of the copositive and completely positive cones thus becomes very critical not only in the study of the theory of copositive optimization, but also in the design of algorithms to solve copositive optimization.

The copositive cones and completely positive cones have many other applications in addition to those in optimization, and have been topics of research for many years (see [3] [4]). However, due to their complicated structures, knowledge about the geometric aspects of the copositive cones and completely positive cones are very limited. In [6], a way of representing all the maximal faces of the copositive cones along with a simple equation for the dimension of each one was given because of the known representations of exposed

*This research was supported by Key Project of North Minzu University under Grant No. ZDZX201804; Nonlinear analysis and financial optimization research center of North Minzu University; Natural Science Foundation of China under Grant No. 11861002; Natural Science Foundation of Ningxia under Grant No. NZ17112.

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rays of the completely positive cones. Also in [6] some maximal faces (not all maximal faces) of the completely positive cones and their dimensions were discussed. Those maximal faces of the completely positive cones are determined by the representations of some known exposed rays in the copositive cones. In [5], an algebraic approach was adapted to study the difference between the 5×5 doubly nonnegative cone and 5×5 completely positive cone, more specifically, a representation of extreme rays of the 5×5 doubly nonnegative cone was given and was used to prove the result: if a matrix is doubly nonnegative but not completely positive, then it can be decomposed as the sum of a completely positive matrix and an extremely bad matrix, which is an extreme doubly nonnegative matrix, but not a completely positive matrix, with rank 3.

In this paper, we will prove a duality result between maximal faces of a convex cone $\mathcal{K}$ and minimal exposed faces of $\mathcal{K}^*$, the dual cone of $\mathcal{K}$. This result extends a result in [6], where the result that an exposed ray in $\mathcal{K}$ determines a maximal face in $\mathcal{K}^*$ was proved. To the best of our knowledge, this nice duality result has not appeared in the related literature. In this paper, we also provide a geometric interpretation of a result in [5] by using a different approach other than the one in [5]. Some other geometric properties of the completely positive cones are also presented in this paper.

The paper is organized as follows: in Section 2 we provide some basic definitions and properties associated with a convex cone, for example, we give the definitions of cones, faces, rays, maximal faces, minimal exposed faces, etc., then we prove a duality result between maximal faces and minimal exposed faces in this section. In Section 3, we study the relationship between maximal faces (minimal exposed faces) of a convex cone $\mathcal{K}$ and maximal faces (minimal exposed faces) of a convex subcone of $\mathcal{K}$. In Section 4, we specifically study some geometric properties of the completely positive cones in terms of doubly nonnegative cones. We will have a discussion on which faces of doubly nonnegative cones intersect completely positive cones for more than the origin. A geometric interpretation of a result in [5] is given in this section. Finally in Section 5, we provide some conclusion remarks.

2 Faces of Convex Cones

Throughout this paper, we use $\mathbb{R}^n$ to denote the n-dimensional Euclidean space. $\mathbb{R}_+$ is the set of all nonnegative real numbers. If $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, then $\langle x, y\rangle$ or $x^T y$ is used to represent the inner product of x and y. $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ matrices. For a set $\mathcal{L} \subseteq \mathbb{R}^n$, $cl\,\mathcal{L}$, $int\,\mathcal{L}$, $relint\,\mathcal{L}$, and $span\,\mathcal{L}$ are the closure of $\mathcal{L}$, the set of interior points of $\mathcal{L}$, the set of relative interior points of $\mathcal{L}$, and the space spanned by $\mathcal{L}$, respectively. The reader is referred to [9] for the definitions of these terms.

Let $\mathcal{K}$ be a convex cone in $\mathbb{R}^n$. Then $\mathcal{K}$ can be used to define a partial order in $\mathbb{R}^n$, in other words, $x \succeq_{\mathcal{K}} y$ if and only if $x - y \in \mathcal{K}$. A convex subcone $\mathcal{F} \neq \{0^n\}$ of $\mathcal{K}$ is called a face of $\mathcal{K}$ if $x \in \mathcal{F}$, $x \succeq_{\mathcal{K}} y \succeq_{\mathcal{K}} 0^n$ implies $y \in \mathcal{F}$, where 0^n represents the zero vector in $\mathbb{R}^n$. A face $\mathcal{F}$ is exposed if it is the intersection of $\mathcal{K}$ and a nontrivial supporting hyperplane, in other words, there exists a nonzero $a \in \mathbb{R}^n$ such that $\langle x, a\rangle \geq 0$ for all $x \in \mathcal{K}$ and $\mathcal{F} = \{x \in \mathcal{K} \mid \langle x, a\rangle = 0\}$. We follow the definition in [6] that $\emptyset$ and $\mathcal{K}$ are not exposed faces. For a given face $\mathcal{F}$ of $\mathcal{K}$, the complementary (or conjugate) face of $\mathcal{F}$ is defined to be $\mathcal{F}^c \equiv \{z \in \mathcal{K}^* \mid \langle z, x\rangle = 0 \text{ for all } x \in \mathcal{F}\} = \mathcal{K}^* \cap \mathcal{F}^\perp$, where $\mathcal{K}^*$ is the dual cone of $\mathcal{K}$, that is, $\mathcal{K}^* = \{y \in \mathbb{R}^n \mid \langle x, y\rangle \geq 0 \text{ for all } x \in \mathcal{K}\}$, and $\mathcal{K}^\perp = \mathcal{K}^* \cap (-\mathcal{K}^*)$. The complementary face of a face in $\mathcal{K}^*$ is defined similarly. For $a^* \in \mathcal{K}^*$, we define $\mathcal{F}(\mathcal{K}, a^*) = \{x \in \mathcal{K} \mid \langle a^*, x\rangle = 0\}$. Similarly, we define $\mathcal{F}(\mathcal{K}^*, a) = \{y^* \in \mathcal{K}^* \mid \langle a, y^*\rangle = 0\}$ for $a \in \mathcal{K}$. If a face $\mathcal{F}$ is generated by a nonzero vector, i.e. there exists a nonzero $a \in \mathcal{K}$ such that $\mathcal{F} = \{\alpha a \mid \alpha \in \mathbb{R}_+\}$, then we call $\mathcal{F}$ an extreme ray. If $\mathcal{F}$ is an extreme ray and it is also exposed, then we call $\mathcal{F}$ an exposed ray. We use $\mathrm{Ext}(\mathcal{K})$ and $\mathrm{Exp}(\mathcal{K})$ to represent the sets of extreme rays and exposed rays of $\mathcal{K}$. Similarly, we use $\mathrm{Ext}(\mathcal{K}^*)$ and $\mathrm{Exp}(\mathcal{K}^*)$ to represent the sets of extreme rays and exposed rays of $\mathcal{K}^*$.

Next we list two lemmas that will be used in the proof of our duality theorem.

Lemma 2.1 ([10])**.** *Let $\mathcal{K}$ be a proper convex cone (closed, pointed, and full dimensional) and $\mathcal{K}^*$ be the dual cone of $\mathcal{K}$. Let $\mathcal{F}$ be a face of $\mathcal{K}$ and $a \in relint(\mathcal{F})$. Then $\mathcal{F}^c = \mathcal{F}(\mathcal{K}^*, a)$.*

Lemma 2.2 ([1, Proposition 3.3])**.** *Let $\mathcal{K}$ be a proper convex cone and $\mathcal{K}^*$ the dual cone of $\mathcal{K}$. Let $\mathcal{F}$ be a face of $\mathcal{K}$. Then $\mathcal{F}$ is an exposed face if and only if $\mathcal{F} = (\mathcal{F}^c)^c$, where $(\mathcal{F}^c)^c$ is the conjugate face of the face $\mathcal{F}^c$.*

We now give definitions of maximal faces and minimal exposed faces of a proper convex cone. The characterizations of these faces are very helpful in understanding geometric features of a cone.

Definition 2.3. A face $\mathcal{F}_1$ is a maximal face of a proper convex cone $\mathcal{K}$ if $\mathcal{F}_1 \neq \mathcal{K}$ and there does not exist a face $\mathcal{F}_2 \neq \mathcal{K}$ such that $\mathcal{F}_1 \subset \mathcal{F}_2$.

Definition 2.4. Let $\mathcal{L} \neq \{0\}$ be a subset of $\mathcal{K}$. A minimal exposed face of $\mathcal{K}$ containing $\mathcal{L}$ is an exposed face $\mathcal{F}_1 \supseteq \mathcal{L}$ such that there does not exist an exposed face $\mathcal{F}_2$ with $\mathcal{F}_1 \supset \mathcal{F}_2 \supseteq \mathcal{L}$. An exposed face $\mathcal{F}_1$ is a minimal exposed face of a proper convex cone $\mathcal{K}$ if $\mathcal{F}_1 \neq \{0\}$ and there does not exist an exposed face $\mathcal{F}_2 \neq \{0\}$ such that $\mathcal{F}_1 \supset \mathcal{F}_2$.

Remark 2.5. A maximal face and a minimal exposed face of $\mathcal{K}^*$ can be defined similarly.

The following is the main result in this section.

Theorem 2.6. *Let $\mathcal{K}$ be a proper convex cone and $\mathcal{K}^*$ the dual cone of $\mathcal{K}$. Let $\mathcal{F}$ be an exposed face of $\mathcal{K}$. Then $\mathcal{F}$ is a minimal exposed face of $\mathcal{K}$ if and only if $\mathcal{F}^c$ is a maximal face of $\mathcal{K}^*$.*

Proof. Let $\mathcal{F}$ be a minimal exposed face of $\mathcal{K}$. We now prove that $\mathcal{F}^c$ is a maximal face in $\mathcal{K}^*$. Suppose that $\mathcal{M}$ is a maximal face such that $\mathcal{F}^c \subseteq \mathcal{M}$. Since $\mathcal{M}$ is maximal, by Theorem 2.17 in [6], we know there is an $a \in Ext(\mathcal{K})$ such that $\mathcal{M} = \mathcal{F}(\mathcal{K}^*, a) = \{x^* \in \mathcal{K}^* \mid \langle a, x^* \rangle = 0\}$. Therefore, $\mathcal{M}$ is an exposed face. Since $\mathcal{F}^c \subseteq \mathcal{M}$ and $\mathcal{F}$ is exposed, we obtain that $\mathcal{F} = (\mathcal{F}^c)^c \supseteq \mathcal{M}^c$ by Lemma 2.2. Because $\mathcal{F}$ is a minimal exposed face, we must have $\mathcal{F} = \mathcal{M}^c$. Therefore, $\mathcal{F}^c = (\mathcal{M}^c)^c = \mathcal{M}$, where the last equality is due to the fact that $\mathcal{M}$ is exposed and Lemma 2.2. Hence, $\mathcal{F}^c$ is a maximal face in $\mathcal{K}^*$.

Next we prove that if $\mathcal{F}^c$ is a maximal face in $\mathcal{K}^*$, then $\mathcal{F}$ is a minimal exposed face in $\mathcal{K}$. Suppose that $\mathcal{N} \neq \{0\}$ is an exposed face in $\mathcal{K}$ with $\mathcal{N} \subseteq \mathcal{F}$. Then $\mathcal{N}^c \supseteq \mathcal{F}^c$. The assumption that $\mathcal{F}^c$ is a maximal face gives that $\mathcal{N}^c = \mathcal{F}^c$. Therefore, $\mathcal{N} = (\mathcal{N}^c)^c = (\mathcal{F}^c)^c = \mathcal{F}$ by Lemma 2.2. So $\mathcal{F}$ is a minimal exposed face. $\square$

Remark 2.7. The proof of Theorem 2.6 is easy. However, Theorem 2.6 gives a nice duality result. It provides a one-to-one correspondence between the set of maximal faces (minimal exposed faces) of $\mathcal{K}$ and the set of minimal exposed faces (maximal faces) of $\mathcal{K}^*$. Also the result that the conjugate face of an exposed ray is maximal appearing in [6] becomes a corollary of this theorem.

$\boxed{3}$ Subcones

Let $\mathcal{K}$ be a proper cone in $\mathbb{R}^n$ and $\mathcal{S}$ a convex subcone of $\mathcal{K}$. In this section, we study how maximal faces (minimal exposed faces) of $\mathcal{K}$ and $\mathcal{S}$ are related. Like in [5], we would like to characterize the completely positive cones and copositive cones using the knowledge of some well studied cones, such as semidefinite cones, positive cones, and doubly nonnegative cones. Among the list of these cones, some are convex subcones of the others. Therefore, the results proved in this section can be applied to the next section, which is mainly on completely positive cones.

It is easy to see that the following are true.

1. If $a \in \mathcal{S}$ generates an extreme (exposed) ray of $\mathcal{K}$, then a also generates an extreme (exposed) ray of $\mathcal{S}$. The converse statement may not be true.

2. If $a \in \mathcal{S}$ generates an exposed ray of $\mathcal{K}$, then the conjugate faces of $\{\lambda a \mid \lambda \geq 0\}$ in $\mathcal{K}$ and $\mathcal{S}$ are maximal in $\mathcal{K}^*$ and $\mathcal{S}^*$, respectively. We denote them by $\mathcal{M}_k$ and $\mathcal{M}_s$. Then we have that $\mathcal{M}_k = \mathcal{M}_s \cap \mathcal{K}^*$.

3. If $\mathcal{F}$ is a minimal exposed face of $\mathcal{K}$ and $\mathcal{F} \subseteq \mathcal{S}$, then $\mathcal{F}$ is still an exposed face in $\mathcal{S}$. However, $\mathcal{F}$ may not be minimal exposed in $\mathcal{S}$.

In general, we have the following theorem.

Theorem 3.1. *Let $\mathcal{F}_k$ and $\mathcal{F}_s$ be minimal exposed faces of $\mathcal{K}$ and $\mathcal{S}$, respectively. Let $\mathcal{F}_k^c$ be the conjugate face of $\mathcal{F}_k$ in $\mathcal{K}^*$ and $\mathcal{F}_s^c$ be the conjugate face of $\mathcal{F}_s$ in $\mathcal{S}^*$, that is, $\mathcal{F}_k^c = \mathcal{K}^* \cap \mathcal{F}_k^\perp$ and $\mathcal{F}_s^c = \mathcal{S}^* \cap \mathcal{F}_s^\perp$. Then the following are equivalent.*

(i) $\mathcal{F}_k \cap \mathcal{F}_s \neq \{0\}$

(ii) $\mathcal{F}_s \subseteq \mathcal{F}_k$

(iii) $\mathcal{F}_k^c = \mathcal{F}_s^c \cap \mathcal{K}^*$.

Proof. Since $\mathcal{F}_k$ and $\mathcal{F}_s$ are minimal exposed faces of $\mathcal{K}$ and $\mathcal{S}$, we know there exists $k^* \in \mathcal{K}^*$ and $s^* \in \mathcal{S}^*$ such that $\mathcal{F}_k = \mathcal{F}(\mathcal{K}, k^*)$ and $\mathcal{F}_s = \mathcal{F}(\mathcal{S}, s^*)$. So we have

$$\begin{aligned}\mathcal{F}_k \cap \mathcal{F}_s &= \mathcal{F}(\mathcal{K}, k^*) \cap \mathcal{F}(\mathcal{S}, s^*)\\ &= \{k \in \mathcal{K} \mid \langle k, k^* \rangle = 0\} \cap \{s \in \mathcal{S} \mid \langle s, s^* \rangle = 0\}\\ &= \{k \in \mathcal{S} \mid \langle k, k^* \rangle = 0\} \cap \{s \in \mathcal{S} \mid \langle s, s^* \rangle = 0\}\\ &= \{s \in \mathcal{S} \mid \langle s, k^* + s^* \rangle = 0\},\end{aligned}$$

which shows that if $\mathcal{F}_k \cap \mathcal{F}_s \neq \{0\}$, then $\mathcal{F}_k \cap \mathcal{F}_s$ is also an exposed face in $\mathcal{S}$ due to the fact that $k^* + s^* \in \mathcal{S}^*$. The assumption that $\mathcal{F}_s$ is a minimal exposed faces of $\mathcal{S}$ indicates that $\mathcal{F}_k \cap \mathcal{F}_s = \mathcal{F}_s$. Therefore, (i) implies (ii). Hence, (i) is equivalent to (ii).

Now we show that (ii) and (iii) are equivalent. We first show that (ii) implies (iii). Since $\mathcal{F}_s \subseteq \mathcal{F}_k$, we obtain that $\mathcal{F}_k^c = \mathcal{K}^* \cap \mathcal{F}_k^\perp \subseteq \mathcal{K}^* \cap \mathcal{F}_s^\perp = \mathcal{K}^* \cap \mathcal{S}^* \cap \mathcal{F}_s^\perp = \mathcal{F}_s^c \cap \mathcal{K}^*$. Because $\mathcal{F}_s^c$ is a face of $\mathcal{S}^*$, we know that $\mathcal{F}_s^c \cap \mathcal{K}^*$ is a face of $\mathcal{K}^*$. By Theorem 2.6, we know $\mathcal{F}_k^c$ is a maximal face, therefore, we obtain that $\mathcal{F}_k^c = \mathcal{F}_s^c \cap \mathcal{K}^*$. We next show that (iii) implies (ii). This can be done by $\mathcal{F}_k = (\mathcal{F}_k^c)^c = \mathcal{K} \cap (\mathcal{F}_k^c)^\perp = \mathcal{K} \cap (\mathcal{F}_s^c \cap \mathcal{K}^*)^\perp \supseteq \mathcal{K} \cap (\mathcal{F}_s^c)^\perp \supseteq \mathcal{S} \cap (\mathcal{F}_s^c)^\perp = \mathcal{F}_s$. $\square$

A dual result of Theorem 3.1 by using Theorem 2.6 can be stated as follows. The proof is straightforward, hence omitted.

Theorem 3.2. *Let $\mathcal{F}_k$ and $\mathcal{F}_s$ be maximal faces of $\mathcal{K}$ and $\mathcal{S}$, respectively. Let $\mathcal{F}_k^c$ be the conjugate face of $\mathcal{F}_k$ in $\mathcal{K}^*$ and $\mathcal{F}_s^c$ be the conjugate face of $\mathcal{F}_s$ in $\mathcal{S}^*$, that is, $\mathcal{F}_k^c = \mathcal{K}^* \cap \mathcal{F}_k^\perp$ and $\mathcal{F}_s^c = \mathcal{S}^* \cap \mathcal{F}_s^\perp$. Then the following are equivalent.*

(i) $\mathcal{F}_k^c \cap \mathcal{F}_s^c \neq \{0\}$

(ii) $\mathcal{F}_k^c \subseteq \mathcal{F}_s^c$

(iii) $\mathcal{F}_s = \mathcal{F}_k \cap \mathcal{S}$.

Because a maximal face $\mathcal{F}_k$ of $\mathcal{K}$ should be on the boundary of $\mathcal{K}$, so if Theorem 3.2 (iii) holds, then the maximal face $\mathcal{F}_s$ of $\mathcal{S}$ should also be on the boundary of $\mathcal{K}$. This raises an interesting question: can any maximal face of $\mathcal{S}$ on the boundary of $\mathcal{K}$ be written as the intersection of a maximal face in $\mathcal{K}$ with $\mathcal{S}$? In general, the answer is negative. However, if we assume that $\mathcal{S}$ is proper, then we have the following result.

Proposition 3.3. *Assume further that $\mathcal{S}$ is proper. Consider a face $\mathcal{F}_s$ of $\mathcal{S}$. If $\mathcal{F}_s$ is on the boundary of $\mathcal{K}$ and $\mathcal{F}_s$ is maximal, then $\mathcal{F}_s$ can be written as the intersection of a maximal face $\mathcal{F}_k$ of $\mathcal{K}$ with $\mathcal{S}$.*

Proof. Let $\mathcal{F}_k$ be the maximal face in $\mathcal{K}$ that contains $\mathcal{F}_s$. Then $\mathcal{F}_k \cap \mathcal{S} \neq \mathcal{S}$ due to the assumption that $\mathcal{K}$ and $\mathcal{S}$ are proper. Therefore, we have $\mathcal{F}_k \cap \mathcal{S} \supset \mathcal{F}_s$. Since $\mathcal{F}_s$ is a maximal face of $\mathcal{S}$ and $\mathcal{F}_k \cap \mathcal{S}$ is a face of $\mathcal{S}$, we have $\mathcal{F}_k \cap \mathcal{S} = \mathcal{F}_s$. $\square$

Now let $\mathcal{F}_k$ and $\mathcal{F}_s$ be maximal faces of $\mathcal{K}$ and $\mathcal{S}$ as in Proposition 3.3, respectively. By Theorem 2.6, we know that $\mathcal{F}_k^c$ and $\mathcal{F}_s^c$ are minimal exposed faces of $\mathcal{K}^*$ and $\mathcal{S}^*$. By Theorem 3.2, if $\mathcal{F}_s = \mathcal{F}_k \cap \mathcal{S}$, we have $\mathcal{F}_k^c \subseteq \mathcal{F}_s^c$. On the other hand, if $\mathcal{F}_s$ is not on the boundary of $\mathcal{K}$, even though we assume that $\mathcal{F}_s$ is maximal, we cannot write $\mathcal{F}_s$ as the intersection of a maximal face $\mathcal{F}_k$ of $\mathcal{K}$ with $\mathcal{S}$. Hence, in the case that $\mathcal{F}_s$ is not on the boundary of $\mathcal{K}$, by Theorem 3.2, we know that the intersection of $\mathcal{F}_s^c$ with any minimal exposed faces in $\mathcal{K}^*$ is $\{0\}$. Specifically, if $\mathcal{F}_s^c$ intersects $\mathcal{K}^*$ only at 0, then $\mathcal{F}_s$ is not on the boundary of $\mathcal{K}$. An illustrative example will be given in the next section after various matrix cones are introduced.

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two proper cones in $\mathbb{R}^n$. Then $\mathcal{K}_1 \cap \mathcal{K}_2$ is a convex cone (may not be proper). The next theorem, which is closely related to Proposition 2.1 in [11], characterizes faces in $\mathcal{K}_1 \cap \mathcal{K}_2$ in terms of faces in $\mathcal{K}_1$ and faces in $\mathcal{K}_2$.

Theorem 3.4. *Any face in $\mathcal{K}_1 \cap \mathcal{K}_2$ must be either an intersection of a face in $\mathcal{K}_1$ with a face in $\mathcal{K}_2$, or an intersection of a face in $\mathcal{K}_1$ with $\mathcal{K}_2$, or an intersection of $\mathcal{K}_1$ with a face in $\mathcal{K}_2$.*

Proof. Suppose that $\mathcal{F}$ is a face in $\mathcal{K}_1 \cap \mathcal{K}_2$. Then $\mathcal{F} \subset \mathcal{K}_1$ and $\mathcal{F} \subset \mathcal{K}_2$. Let $\mathcal{F}_{k_1}$ and $\mathcal{F}_{k_2}$ be faces in $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively, which are smallest faces containing $relint(\mathcal{F})$. We should note here that if $relint(\mathcal{F}) \subseteq int(\mathcal{K}_1)$, then $\mathcal{F}_{k_1} = \mathcal{K}_1$. Similarly, if $relint(\mathcal{F}) \subseteq int(\mathcal{K}_2)$, then $\mathcal{F}_{k_2} = \mathcal{K}_2$. It is easy to see that $\mathcal{F}_{k_1} \cap \mathcal{F}_{k_2} \supseteq \mathcal{F}$.

Now we prove that $\mathcal{F}_{k_1} \cap \mathcal{F}_{k_2} \subseteq \mathcal{F}$. Let $x \in \mathcal{F}_{k_1} \cap \mathcal{F}_{k_2}$ and $y \in relint\ (\mathcal{F})$. Since we know that $\mathcal{F}_{k_1}$ is the smallest face in $\mathcal{K}_1$ containing $\mathcal{F}$, we know that $y \in relint(\mathcal{F}_{k_1})$. Therefore, $\lambda_1 x + (1-\lambda_1)z_1 = y$ for some $0 < \lambda_1 < 1$ and $z_1 \in \mathcal{F}_{k_1}$. Similarly, $y \in relint(\mathcal{F}_{k_2})$ implies that $\lambda_2 x + (1-\lambda_2)z_2 = y$ for some $0 < \lambda_2 < 1$ and $z_2 \in \mathcal{F}_{k_2}$. Since both z_1 and z_2 are on the line connecting x and y, we may simply set $z_1 = z_2 = z$. Hence, $z \in \mathcal{F}_{k_1} \cap \mathcal{F}_{k_2} \subseteq \mathcal{K}_1 \cap \mathcal{K}_2$. Because $\mathcal{F}$ is a face in $\mathcal{K}_1 \cap \mathcal{K}_2$, we obtain that $x \in \mathcal{F}$ due to $\lambda_2 x + (1-\lambda_2)z = y \in \mathcal{F}$. Because x is arbitrarily chosen, we have that $\mathcal{F}_{k_1} \cap \mathcal{F}_{k_2} = \mathcal{F}$. $\square$

A facially exposed cone is a cone with the property that all its faces are exposed. The next corollary states that the intersection of two facially exposed cones is still facially exposed, which is a well-known result. The proof is straightforward using Theorem 3.4.

Corollary 3.5. *If $\mathcal{K}_1$ and $\mathcal{K}_2$ are facially exposed, then $\mathcal{K}_1 \cap \mathcal{K}_2$ is also facially exposed. In particular, the doubly nonnegative cone is facially exposed, where the doubly nonnegative cone is the intersection of the positive semidefinite cone and the cone of all nonnegative matrices.*

4 Completely Positive Cones, Copositive Cones, etc.

We first give the definitions of various matrix cones, which will be used in this section.

$\mathcal{P}^n$–the cone of all $n \times n$ positive semidefinite matrices.

$\mathcal{N}^n$–the cone of all $n \times n$ nonnegative symmetric matrices, that is, the cone of all symmetric matrices with nonnegative entries.

$\mathcal{CP}^n$–the cone of all $n \times n$ completely positive matrices. A completely positive matrix is a positive semidefinite matrix, which can be written as XX^T with X being a nonnegative matrix.

$\mathcal{COP}^n$–the cone of all copositive matrices. A matrix A is copositive if $x^TAx \geq 0$ for all $x \in \mathbb{R}^n_+$, the set of all nonnegative vectors.

$\mathcal{P}^n \cap \mathcal{N}^n$–doubly nonnegative cone consisting of all $n \times n$ matrices, which are nonnegative and positive semidefinite.

The following proposition (see [4]) gives the duals of these cones.

Proposition 4.1. *$\mathcal{P}^n$ and $\mathcal{N}^n$ are self-dual. $\mathcal{CP}^n$ and $\mathcal{COP}^n$ are dual to each other. $\mathcal{P}^n \cap \mathcal{N}^n$ and $\mathcal{P}^n + \mathcal{N}^n$ are dual to each other.*

In the proofs of various results in this paper, we need descriptions of faces for the cone $\mathcal{P}^n$ of semidefinite matrices and the cone $\mathcal{N}^n$ of nonnegative symmetric matrices. We list them as lemmas below.

Lemma 4.2. *Let $A = (a_{ij})$ be a $n \times n$ nonnegative symmetric matrix. Then the minimal face of $\mathcal{N}^n$ containing A can be represented by $\mathcal{F}_{\mathcal{N}}(A) = \{(b_{ij}) \in \mathcal{N}^n \mid b_{ij} = 0$ for any $1 \leq i, j \leq n,$ such that $a_{ij} = 0\}$.*

Lemma 4.3. *Let A be positive semidefinite and $\lambda_1 > 0, \lambda_2 > 0, \dots, \lambda_k > 0$ be all positive eigenvalues of A. Let Q be an orthogonal matrix such that $A = Q \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} Q^T$, where $D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_k \end{pmatrix}$. Then the minimal face of $\mathcal{P}^n$ containing A can be represeted by $\mathcal{F}_{\mathcal{P}}(A) = \left\{ Q \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} Q^T \;\middle|\; B \in \mathcal{P}^k \right\}$.*

4.1 Minimal Exposed Faces

There are many geometric properties including characterizations of extreme rays, exposed rays, and maximal faces of these cones that have already been discussed in the literature. But not many discussions on minimal exposed faces of these cones have been given. Because of Theorem 2.6, it is worth studying or characterizing minimal exposed faces of these cones. In this section, we collect and prove some results about minimal exposed faces for $\mathcal{P}^n$, $\mathcal{N}^n$, $\mathcal{P}^n \cap \mathcal{N}^n$, $\mathcal{P}^n + \mathcal{N}^n$, and $\mathcal{CP}^n$. The description of minimal exposed faces of $\mathcal{COP}^n$ seems impossible in the meantime since a complete description of all extreme rays of $\mathcal{COP}^n$ is not available for $n \geq 6$.

The first result we would like to present is the following proposition.

Proposition 4.4. *Facially exposed cones have no other minimal exposed faces except exposed rays.*

Proof. Suppose that $\mathcal{F}$ is a face of a facially exposed cone $\mathcal{K}$. Then $\mathcal{F}$ is the convex hull of some extreme rays. Since an extreme ray is also a face, by the assumption that $\mathcal{K}$ is facially exposed, we know the extreme rays contained in $\mathcal{F}$ must be exposed, which shows that $\mathcal{F}$ must be an exposed ray. $\square$

Since we know that $\mathcal{P}^n$, $\mathcal{N}^n$, and $\mathcal{P}^n \cap \mathcal{N}^n$ are facially exposed, the minimal exposed faces for these cones must be exposed rays. Because whether a completely positive cone is facially exposed is still an open problem ([2]), we cannot apply Proposition 4.4 directly to the completely positive cones. However, we know all extreme rays are exposed in the completely positive cones [6], so we can apply the same argument as in the proof of Proposition 4.4 to show that no other minimal exposed faces exists in the completely positive cones except exposed rays. Now we work with the cone $\mathcal{P}^n + \mathcal{N}^n$. Note that since $\mathcal{P}^n + \mathcal{N}^n$ is not facially exposed (see [6, Figure 1]), we cannot apply Proposition 4.4 to this cone directly. However, we still have the same result, which is stated as a proposition below.

Proposition 4.5. *$\mathcal{P}^n + \mathcal{N}^n$ has no other minimal exposed face except exposed rays.*

Proof. First, we can see that every extreme ray of $\mathcal{P}^n + \mathcal{N}^n$ should be either extreme in $\mathcal{P}^n$ or extreme in $\mathcal{N}^n$. For extreme rays in $\mathcal{P}^n$, we must have the form $\{\alpha xx^T \mid \alpha \in \mathbb{R}_+\}$ with $x \in \mathbb{R}^n$. If $x \in \mathbb{R}^n \backslash (\mathbb{R}^n_+ \cup (-\mathbb{R}^n_+))$, then by Theorem 4.6 in [6] we know that xx^T is exposed in the copositive cone, and hence, it is exposed in $\mathcal{P}^n + \mathcal{N}^n$. If $x \in \mathbb{R}^n_+$ with at least two nonzero entries, then we can easily rewrite it as a sum of two nonnegative matrices. Therefore, xx^T with $x \in \mathbb{R}^n_+$ having at least two nonzero entries does not give an extreme ray in $\mathcal{P}^n + \mathcal{N}^n$. If $x \in \mathbb{R}^n_+$ with one nonzero entry, then we can prove in a similar manner as in the proof of Theorem 4.4 in [6] that xx^T with $x \in \mathbb{R}^n_+$ does not give an exposed ray, only extreme ray.

For extreme rays in $\mathcal{N}^n \backslash \mathcal{P}^n$, it has the form $(e_i e_j^T + e_j e_i^T)$ with $i \neq j$, where e_i is the vector such that the i-th entry is 1 and the other entries are 0. By Theorem 4.6 in [6] again, we know that $(e_i e_j^T + e_j e_i^T)$ with $i \neq j$ is exposed in the copositive cone, and hence, it is exposed in $\mathcal{P}^n + \mathcal{N}^n$. Therefore, the only non-exposed extreme rays in $\mathcal{P}^n + \mathcal{N}^n$ are of the form $\{\alpha e_i e_i^T \mid \alpha \in \mathbb{R}_+\}$.

Now suppose there is an exposed minimal face $\mathcal{F}$ in $\mathcal{P}^n + \mathcal{N}^n$, which is not an exposed ray. Then $\mathcal{F}$ should not contain an exposed ray. Therefore, $\mathcal{F}$ can only be written as the convex hull of non-exposed extreme rays, which are of the form $\{\alpha e_i e_i^T \mid \alpha \in \mathbb{R}_+\}$. Without loss of generality, we may assume

$$\mathcal{F} = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \;\middle|\; A = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_k \end{pmatrix}, a_i \geq 0, \text{ for } i = 1, 2, \dots, k \right\}. \tag{4.1}$$

Since we assume that $\mathcal{F}$ is exposed, we know there is a matrix $M \in (\mathcal{P}^n + \mathcal{N}^n)^* = \mathcal{P}^n \cap \mathcal{N}^n$ such that $\mathcal{F} = \{N \in \mathcal{P}^n + \mathcal{N}^n \mid \langle M, N \rangle = 0\}$. Because of (4.1) and that M is doubly nonnegative, we know that M can be expressed as $\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$, where B is a $(n-k) \times (n-k)$ double nonnegative matrix. Because $\langle M, e_1 e_n^T + e_n e_1^T \rangle = 0$, we know that $\mathcal{F} \neq \{N \in \mathcal{P}^n + \mathcal{N}^n \mid \langle M, N \rangle = 0\}$. Therefore, we prove that $\mathcal{F}$ is not exposed showing that the only minimal exposed faces of $\mathcal{P}^n + \mathcal{N}^n$ are exposed rays. □

$\boxed{4.2}$ Completely Positive Cones as Subcones of Doubly Nonnegative Cones

The cone of completely positive matrices is a convex subcone of the doubly nonnegative cones, that is, $\mathcal{P}^n \cap \mathcal{N}^n \supseteq \mathcal{CP}^n$. What is interesting is the fact that if $A \in \mathcal{CP}^n$, then $\{\alpha A \mid \alpha \in \mathbb{R}_+\} \in \text{Ext}\,(\mathcal{P}^n \cap \mathcal{N}^n)$ if and only if A has rank 1 (Page 1374, [8]). However, when A has rank 1, $\{\alpha A \mid \alpha \in \mathbb{R}_+\}$ is also an extreme ray in $\mathcal{CP}^n$. Actually, $\text{Ext}(\mathcal{CP}^n) \subseteq \text{Ext}(\mathcal{P}^n \cap \mathcal{N}^n)$. Both $\mathcal{CP}^n$ and $\mathcal{P}^n \cap \mathcal{N}^n$ are full dimensional cones, that is, they have interior points. At a first glance, it seems that $\mathcal{CP}^n = \mathcal{P}^n \cap \mathcal{N}^n$. However, some differences between $\mathcal{CP}^n$ and $\mathcal{P}^n \cap \mathcal{N}^n$ have already been discovered in [5, 8] for $n \geq 5$.

Of course, all the results proved in Section 3 can be applied to $\mathcal{P}^n \cap \mathcal{N}^n$ and its subcone $\mathcal{CP}^n$. We start this section by providing an example to illustrate Theorem 3.2 and Proposition 3.3.

Example 4.6. The Horn matrix (see [5]) is in $\mathcal{COP}^5$ but not in $\mathcal{P}^5 + \mathcal{N}^5$. The ray generated by the Horn matrix is exposed in $\mathcal{COP}^5$. Therefore, this exposed ray gives a maximal face in $\mathcal{CP}^5$. Since the exposed ray generated by the Horn matrix intersects $\mathcal{P}^5 + \mathcal{N}^5$ only at the origin, so by Proposition 3.3 and Theorem 3.2, we know the maximal face in $\mathcal{CP}^5$ given by the exposed ray of the Horn matrix is not an intersection of a maximal face in $\mathcal{P}^5 \cap \mathcal{N}^5$ with $\mathcal{CP}^5$. This shows that the maximal face in $\mathcal{CP}^5$ given by the exposed ray of the Horn matrix should pass through the interiors of $\mathcal{P}^5 \cap \mathcal{N}^5$.

Now we turn to the study of some faces of $\mathcal{P}^n \cap \mathcal{N}^n$. Specifically, we are interested in those faces of $\mathcal{P}^n \cap \mathcal{N}^n$ whose intersection with the completely positive cone is more than the origin. Since any face in $\mathcal{P}^n \cap \mathcal{N}^n$ is either an intersection of a face in $\mathcal{P}^n$ with a face in $\mathcal{N}^n$, or an intersection of a face in $\mathcal{P}^n$ with $\mathcal{N}^n$, or an intersection of $\mathcal{P}^n$ with a face in $\mathcal{N}^n$, we consider various cases.

The first result along this line is for a face, which is an intersection of a face in $\mathcal{N}^n$ with $\mathcal{P}^n$.

Theorem 4.7. *Suppose that $\mathcal{F}$ is a face in $\mathcal{P}^n \cap \mathcal{N}^n$, which is an intersection of a face $\mathcal{F}_{\mathcal{N}}$ in $\mathcal{N}^n$ with $\mathcal{P}^n$. Then $\mathcal{F} \cap \mathcal{CP}^n \neq \{0\}$ and is a face of $\mathcal{CP}^n$.*

Proof. If $\mathcal{F} \cap \mathcal{CP}^n \neq \{0\}$, then obviously we know $\mathcal{F} \cap \mathcal{CP}^n$ is a face of $\mathcal{CP}^n$. Therefore, we only need to show that $\mathcal{F} \cap \mathcal{CP}^n \neq \{0\}$. In other words, $\mathcal{F}$ contains a completely positive matrix.

Since $\mathcal{F}$ is a face in $\mathcal{P}^n \cap \mathcal{N}^n$, we know that $\mathcal{F} \neq \{0\}$. This shows there is a matrix $A = (a_{ij})_{n \times n}$ in $\mathcal{F}$ such that at least one entry on the main diagonal should be strictly positive. We may assume that $a_{11} > 0$. We let $Y = (1, 0, \ldots, 0)^T (1, 0, \ldots, 0)$. Because $A \in \mathcal{F}_{\mathcal{N}}$ and $a_{11} > 0$, by Lemma 4.2 we obtain that $Y \in \mathcal{F}_{\mathcal{N}}$. It is also obvious that $Y \in \mathcal{P}^n$. We, therefore, obtain that $Y \in \mathcal{F}$. The fact that $Y \in \mathcal{CP}^n$ is directly from the definition of Y. Therefore, $\mathcal{F} \cap \mathcal{CP}^n \neq \{0\}$ and is a face of $\mathcal{CP}^n$. □

A similar result is also true for a face, which is an intersection of a face in $\mathcal{P}^n$ with $\mathcal{N}^n$. We have the following theorem.

Theorem 4.8. *Suppose that $\mathcal{F}$ is a face in $\mathcal{P}^n \cap \mathcal{N}^n$, which is an intersection of a face $\mathcal{F}_\mathcal{P}$ in $\mathcal{P}^n$ with $\mathcal{N}^n$. Then $\mathcal{F} \cap \mathcal{CP}^n \neq \{0\}$ and is a face of $\mathcal{CP}^n$.*

Proof. Let $A \in relint(\mathcal{F})$. Since A is positive semidefinite, there is an orthogonal matrix $Q = (x_1, x_2, \ldots, x_n)$ and $\lambda_1 > 0, \lambda_2 > 0, \ldots, \lambda_k > 0$ such that $A = Q \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} Q^T$, where $D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_k \end{pmatrix}$, which is equivalent to say that $Ax_i = \lambda_i x_i$ for $1 \leq i \leq k$ and $Ax_i = 0$ for $i \geq k+1$. Because $A \in relint(\mathcal{F})$, we can also set $\mathcal{F}_\mathcal{P} = \left\{ Q \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} Q^T \;\middle|\; B \in \mathcal{P}^k \right\}$. Because $A \neq 0$, there exists a row of A whose entries are not all 0. Without loss of generality, we assume the entries in the first row A_1 are not all zero. We let $Y = A_1^T A_1$. Since $A \in \mathcal{N}^n$, of course, we have $Y \in \mathcal{N}^n$. Now we prove that $Y \in \mathcal{F}_\mathcal{P}$. We know $Q^T Y Q = Q^T A_1^T A_1 Q = (A_1 Q)^T (A_1 Q) = (A_1 x_1, A_1 x_2, \ldots, A_1 x_n)^T (A_1 x_1, A_1 x_2, \ldots, A_1 x_n)$. However, $Ax_i = 0$ for $i \geq k+1$ gives that $A_1 x_i = 0$ for $i \geq k+1$. Therefore,

$$\begin{aligned} Q^T Y Q &= (A_1 x_1, A_1 x_2, \ldots, A_1 x_k, 0 \ldots, 0)^T (A_1 x_1, A_1 x_2, \ldots, A_1 x_k, 0 \ldots, 0) \\ &= \begin{pmatrix} (A_1 x_1, A_1 x_2, \ldots, A_1 x_k)^T (A_1 x_1, A_1 x_2, \ldots, A_1 x_k) & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Now we get $Y = Q \begin{pmatrix} (A_1 x_1, A_1 x_2, \ldots, A_1 x_k)^T (A_1 x_1, A_1 x_2, \ldots, A_1 x_k) & 0 \\ 0 & 0 \end{pmatrix} Q^T$ showing that $Y \in \mathcal{F}_\mathcal{P}$, which implies that $Y \in \mathcal{F}$. Because Y is obviously in $\mathcal{CP}^n$, we obtain that $\mathcal{F} \cap \mathcal{CP}^n \neq \{0\}$ and is a face of $\mathcal{CP}^n$. □

We next consider a face, which is an intersection of a maximal face of $\mathcal{N}^n$ with a face in $\mathcal{P}^n$.

Theorem 4.9. *Suppose that $\mathcal{F}$ is a face in $\mathcal{P}^n \cap \mathcal{N}^n$, which is an intersection of a maximal face of $\mathcal{N}^n$ with a face in $\mathcal{P}^n$. Then $\mathcal{F} \cap \mathcal{CP}^n \neq \{0\}$ and is a face of $\mathcal{CP}^n$.*

Proof. Let $\mathcal{F} = \mathcal{F}_\mathcal{N} \cap \mathcal{F}_\mathcal{P}$, where $\mathcal{F}_\mathcal{N}$ is a maximal face in $\mathcal{N}^n$ and $\mathcal{F}_\mathcal{P}$ is a face in $\mathcal{P}^n$. Since $\mathcal{F}_\mathcal{N}$ is maximal in $\mathcal{N}^n$, we can write

$$\mathcal{F}_\mathcal{N} = \{(a_{ij})_{n \times n} \in \mathcal{S}^n \mid a_{ij} \geq 0, \text{ and for some fixed } i_0 \text{ and } j_0, a_{i_0 j_0} = a_{j_0 i_0} = 0\}.$$

Because $\mathcal{F}_P$ is a face in $\mathcal{P}^n$, there is a $1 \leq k < n$ and an orthogonal matrix Q such that

$$\mathcal{F}_\mathcal{P} = \left\{ Q \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} Q^T \;\middle|\; B \in \mathcal{P}^k \right\}.$$

Since $\mathcal{F}$ is a face of $\mathcal{F}_\mathcal{N} \cap \mathcal{F}_\mathcal{P}$, we can choose $0 \neq A \in relint(\mathcal{F})$. Then $A \in \mathcal{F}_\mathcal{N}$ gives $a_{i_0 j_0} = a_{j_0 i_0} = 0$. Because $\mathcal{F} = \mathcal{F}_\mathcal{N} \cap \mathcal{F}_\mathcal{P}$, we know that there is $B \in \mathcal{P}^k$ such that $A = Q \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} Q^T$. So we have $AQ = Q \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$. If we let $Q = (x_1, x_2, \ldots, x_n)$ with $x_i \in \mathbb{R}^n$ for $i = 1, 2, \ldots, n$, then we have $Ax_p = 0$ for $p = k+1, \ldots, n$. Now we let $A_{i_0} = (a_{i_0 1}, a_{i_0 2}, \ldots, a_{i_0 n})$, that is, the i_0-th row of A. If A_{i_0} is the 0 vector, then all the entries in the i_0-th row and i_0-th column of all the matrices in $\mathcal{F}$ must be zero. So we can delete the i_0-th row and i_0-th column for all the matrices in $\mathcal{F}$, and therefore, $\mathcal{F}$ can be viewed as a face in $\mathcal{P}^{n-1} \cap \mathcal{N}^{n-1}$, which is the intersection of $\mathcal{N}^{n-1}$ with a face in $\mathcal{P}^{n-1}$. By Theorem 4.8, we know that the conclusion is true.

Now we assume that A_{i_0} is not the 0 vector. Then it is easy to see that $Y = A_{i_0}^T A_{i_0} \in \mathcal{F}_{\mathcal{N}}$. Because

$$\begin{aligned} Q^T Y Q &= Q^T A_{i_0}^T A_{i_0} Q \\ &= (A_{i_0}x_1, A_{i_0}x_2, \ldots, A_{i_0}x_n)^T (A_{i_0}x_1, A_{i_0}x_2, \ldots, A_{i_0}x_n) \\ &= (A_{i_0}x_1, A_{i_0}x_2, \ldots, A_{i_0}x_k, 0, \ldots, 0)^T (A_{i_0}x_1, A_{i_0}x_2, \\ &\quad \ldots, A_{i_0}x_k, 0, \ldots, 0) (\text{ because } Ax_p = 0 \text{ for } p = k+1, \ldots, n) \\ &= \begin{pmatrix} B^1 & 0 \\ 0 & 0 \end{pmatrix} (\text{ with } B^1 \in \mathcal{P}^k), \end{aligned}$$

we have that $Y = Q\left[Q^T Y Q\right] Q^T = Q \begin{pmatrix} B^1 & 0 \\ 0 & 0 \end{pmatrix} Q^T$ with B^1 being a $k \times k$ positive semidefinite matrix, which implies that $Y \in \mathcal{F}_{\mathcal{P}}$. Because we already know that $Y = A_{i_0}^T A_{i_0} \in \mathcal{F}_{\mathcal{N}}$, we hence obtain that $Y \in \mathcal{F}$. However, with the definition of Y, we know $Y \in \mathcal{CP}^n$. So we prove that $\mathcal{F} \cap \mathcal{CP}^n \neq \{0\}$. $\square$

The same conclusion can be made for a face, which is an intersection of a face in $\mathcal{N}^n$ with a maximal face of $\mathcal{P}^n$. We have the following theorem.

Theorem 4.10. *Suppose that $\mathcal{F}$ is a face in $\mathcal{P}^n \cap \mathcal{N}^n$, which is an intersection of a face in $\mathcal{N}^n$ with a maximal face of $\mathcal{P}^n$. Then $\mathcal{F} \cap \mathcal{CP}^n \neq \{0\}$ and is a face of $\mathcal{CP}^n$.*

Proof. Suppose that $\mathcal{F}$ is the intersection of a face $\mathcal{F}_{\mathcal{N}}$ in $\mathcal{N}^n$ with a maximal face $\mathcal{F}_{\mathcal{P}}$ in $\mathcal{P}^n$. Since $\mathcal{F}_{\mathcal{P}}$ is a maximal face in $\mathcal{P}^n$, there is an orthogonal matrix Q such that

$$\mathcal{F}_{\mathcal{P}} = \left\{ Q \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} Q^T \;\middle|\; A_{11} \in \mathcal{P}^{n-1} \right\}.$$

We assume $Q = (x_1, x_2, \ldots, x_n)$, $x_i \in \mathbb{R}^n$ for $i = 1, 2, \ldots, n$.

Since $\mathcal{F}_{\mathcal{N}}$ is a face of $\mathcal{N}^n$, without loss of generality, $\mathcal{F}_{\mathcal{N}}$ can be written as follows:

$$\mathcal{F}_{\mathcal{N}} = \{(b_{ij})_{n\times n} \in \mathcal{N}^n \mid b_{ij} = 0 \text{ for } (i,j) \in IJ\},$$

where IJ is a subset of $\{(i,j) \mid 1 \leq i,j \leq n\}$ such that $(i,i) \notin IJ$ and if $(i,j) \in IJ$ then $(j,i) \in IJ$. The reason we assume that $(i,i) \notin IJ$ for all $1 \leq i \leq n$ is that if $(i,i) \in IJ$ for some $1 \leq i \leq n$, then $b_{ii} = 0$, under which when we consider the intersection of the face $\mathcal{F}_{\mathcal{N}}$ with a maximal face of $\mathcal{F}_{\mathcal{P}}$, we must have all the entries in the i-th row and i-th column for all matrices in $\mathcal{F}$ being 0 due to the positive semidefinite requirement of matrices. Hence, we can just delete the i-th row and i-th column and reduce the discussion to the case involving $(n-1)\times(n-1)$ matrices. So the conclusion of the theorem can be proved by using Theorem 4.7.

Let $A = (a_{ij})_{n\times n} \in relint(\mathcal{F})$. By the same argument as above, we can assume that $a_{ii} > 0$ for $1 \leq i \leq n$. $A \in \mathcal{F}_{\mathcal{P}}$ implies that $Ax_n = 0$. Let $x_n = (x_n^1, x_n^2, \ldots, x_n^n)^T$. Since A is not the 0 matrix and x_n is not the 0 vector, with the assumption that $a_{ii} > 0$ for $1 \leq i \leq n$, we conclude that there is i_0 and j_0 with $1 \leq i_0 \leq n$, $1 \leq j_0 \leq n$, and $i_0 \neq j_0$, such that $a_{i_0 j_0} > 0$ and $x_n^{i_0} x_n^{j_0} < 0$. Indeed, $x_n \neq 0$ implies that there is $x_n^{i_0} \neq 0$ for some $1 \leq i_0 \leq n$. By the fact that $Ax_n = 0$, we obtain that $\sum_{j=1}^n a_{i_0 j} x_n^j = a_{i_0 i_0} x_n^{i_0} + \sum_{j=1, j\neq i_0}^n a_{i_0 j} x_n^j = 0$. Since $a_{i_0 i_0} > 0$, $a_{i_0 j} \geq 0$ for $1 \leq j \leq n$, and $x_n^{i_0} \neq 0$, we know that there is j_0 with $1 \leq j_0 \leq n$

and $j_0 \neq i_0$, such that $x_n^{j_0} \neq 0$, $a_{i_0 j_0} > 0$, and $x_n^{i_0} x_n^{j_0} < 0$. We now may assume that $x_n^{i_0} > 0$ and $x_n^{j_0} < 0$. Define $Y = (y_{ij})_{n\times n} = xx^T$, where x is an n dimensional vector with the i_0-th entry being $-x_n^{j_0}$ and j_0-th entry being $x_n^{i_0}$ and all other entries being 0. Y of course is an element of $\mathcal{CP}^n$. Now we prove Y is also an element of $\mathcal{F}$. The conclusion that $Y \in \mathcal{F}_{\mathcal{N}}$ follows from the fact that only nonzero entries of Y are $y_{i_0 i_0}$, $y_{j_0 j_0}$, $y_{i_0 j_0}$ and $y_{j_0 i_0}$, and $(i_0, j_0) \notin IJ$ due to $a_{i_0 i_0} > 0$, $a_{j_0 j_0} > 0$, $a_{i_0 j_0} > 0$ and $a_{j_0 i_0} > 0$. To prove that $Y \in \mathcal{F}_{\mathcal{P}}$, we consider

$$
\begin{aligned}
Q^T Y Q &= Q^T x x^T Q \\
&= (x_1, x_2, \ldots, x_n)^T x x^T (x_1, x_2, \ldots, x_n) \\
&= \begin{pmatrix} -x_1^{i_0} x_n^{j_0} + x_1^{j_0} x_n^{i_0} \\ -x_2^{i_0} x_n^{j_0} + x_2^{j_0} x_n^{i_0} \\ \vdots \\ -x_{n-1}^{i_0} x_n^{j_0} + x_{n-1}^{j_0} x_n^{i_0} \\ 0 \end{pmatrix} \begin{pmatrix} -x_1^{i_0} x_n^{j_0} + x_1^{j_0} x_n^{i_0} \\ -x_2^{i_0} x_n^{j_0} + x_2^{j_0} x_n^{i_0} \\ \vdots \\ -x_{n-1}^{i_0} x_n^{j_0} + x_{n-1}^{j_0} x_n^{i_0} \\ 0 \end{pmatrix}^T \\
&= \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned}
$$

where

$$
B = \begin{pmatrix} -x_1^{i_0} x_n^{j_0} + x_1^{j_0} x_n^{i_0} \\ -x_2^{i_0} x_n^{j_0} + x_2^{j_0} x_n^{i_0} \\ \vdots \\ -x_{n-1}^{i_0} x_n^{j_0} + x_{n-1}^{j_0} x_n^{i_0} \end{pmatrix} \begin{pmatrix} -x_1^{i_0} x_n^{j_0} + x_1^{j_0} x_n^{i_0} \\ -x_2^{i_0} x_n^{j_0} + x_2^{j_0} x_n^{i_0} \\ \vdots \\ -x_{n-1}^{i_0} x_n^{j_0} + x_{n-1}^{j_0} x_n^{i_0} \end{pmatrix}^T \in \mathcal{P}^{n-1}.
$$

Therefore, $Y = QQ^T Y QQ^T = Q \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} Q^T \in \mathcal{F}_{\mathcal{P}}$. Hence, $0 \neq Y \in \mathcal{F} \cap \mathcal{CP}^n$ showing that $\mathcal{F} \cap \mathcal{CP}^n \neq \{0\}$ and is a face of $\mathcal{CP}^n$. $\square$

In [8], it was stated that there does not exist $A \in Ext(\mathcal{P}^n \cap \mathcal{N}^n)$ with $rank(A) = 2$. This statement can be viewed as a corollary of the following theorem.

Theorem 4.11. *Suppose $\mathcal{F}$ is a face of $\mathcal{P}^n \cap \mathcal{N}^n$, which is an intersection of a face $\mathcal{F}_{\mathcal{P}}$ in $\mathcal{P}^n$ of the form $\left\{ Q \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} Q^T \;\middle|\; A_{11} \in \mathcal{P}^2 \right\}$ with a face $\mathcal{F}_{\mathcal{N}}$ in $\mathcal{N}^n$. Here Q is an orthogonal matrix. Then $\mathcal{F} \cap \mathcal{CP}^n \neq \{0\}$ and is a face of $\mathcal{CP}^n$.*

Proof. Let $B = (b_{ij})_{n\times n} \in relint(\mathcal{F})$. Then $rank(B) \leq 2$. If $rank(B) = 1$, then we know that $B \in \mathcal{CP}^n$. Hence, the conclusion holds.

Now consider $rank(B) = 2$. If $b_{ij} > 0$ for all $1 \leq i, j \leq n$, then B is the intersection of a cone in $\mathcal{P}^n$ with $\mathcal{N}^n$. Hence, by Theorem 4.8 we know that the conclusion of the theorem holds.

If for some $b_{ij} = 0$, then by the symmetric property of B we know $b_{ji} = 0$. If all nonzero rows or columns have no zero entry, that is, a zero entry only occurs in a row or a column, whose entries are all zero, then we can delete the rows and columns with all 0 entries and a similar argument as in the previous case shows that the conclusion of the theorem holds.

Now we suppose that there is no row or column with all zero entries. Assume there is a zero entry appearing in a nonzero row and a nonzero column, we assume that row to be the i_0-th row, namely B_{i_0}, and let $J = \{j \in \{1, 2, \ldots, n\} \mid b_{i_0 j} = 0\}$. Then $J \neq \emptyset$. Since

$rank(B) = 2$, there is another row which is linearly independent to B_{i_0}. We can choose any $j \in J$. The rows B_{i_0} and B_j for a fixed $j \in J$ must be linearly independent due to the fact that $b_{i_0 i_0} > 0$, $b_{jj} > 0$, and $b_{i_0 j} = 0$.

Next we will find where zero entries appear in B, so we can construct a nonnegative matrix, which is in $\mathcal{F}$ and eventually will be proved to be a completely positive matrix.

For a fixed $j \in J$, if $b_{i_0 k} > 0$ and $b_{jk} > 0$ for some $1 \leq k \leq n$, then $b_{mk} > 0$ for all $1 \leq m \leq n$ due to the fact that $\alpha B_{i_0} + \beta B_j = B_m$ and $b_{m i_0} = \alpha b_{i_0 i_0}$, $b_{mj} = \beta b_{jj}$, hence, $\alpha \geq 0$ and $\beta \geq 0$. Therefore, when $b_{i_0 k} > 0$ and $b_{jk} > 0$, entries in the k-th row and k-th column are all nonzero. If $b_{i_0 k} > 0$, $b_{jk} = 0$, and $b_{pk} = 0$ for some $p \neq j$, because B_{i_0} and B_j form a basis of the row space of B, we obtain that the p-th row is a multiple of j-th row. Hence, p-th column of B is a multiple of j-th column of B because B is a symmetric matrix. This shows that $b_{i_0 p}$ is a multiple of $b_{i_0 j}$, which is zero. Hence, all zero entries of B should be in the k-th row or k-th column with $k \in J$. Therefore, by the assumption that $B \in relint(\mathcal{F})$ we obtain $B_{i_0}^T B_{i_0} \in \mathcal{F}_{\mathcal{N}}$. However, by the same argument as the one in the proof of the previous theorems, we know $B_{i_0}^T B_{i_0} \in \mathcal{F}_{\mathcal{P}}$. The conclusion that $B_{i_0}^T B_{i_0} \in \mathcal{CP}^n$ is straightforward. Therefore, we prove $\mathcal{F} \cap \mathcal{CP}^n \neq \{0\}$ and is a face of $\mathcal{CP}^n$. $\square$

Corollary 4.12. *Let A be a doubly nonnegative symmetric matrix with rank k. If the ray generated by A is an extreme ray in $\mathcal{N}^n \cap \mathcal{P}^n$, then $k \neq 2$.*

Proof. We prove it by contradiction. Suppose that $k = 2$. Then there is a orthogonal matrix Q such that $A = Q \begin{pmatrix} D_{11} & 0 \\ 0 & 0 \end{pmatrix} Q^T$, where $D_{11} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with $\lambda_1 > 0$ and $\lambda_2 > 0$. Therefore, the ray generated by A can be viewed as an intersection of a face in $\mathcal{P}^n$ of the form $\left\{ Q \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} Q^T \;\middle|\; A_{11} \in \mathcal{P}^2 \right\}$ and a face in $\mathcal{N}^n$. By Theorem 4.11, we know $\mathcal{F} \cap \mathcal{CP}^n \neq \{0\}$ and is a face of $\mathcal{CP}^n$. This shows that $A \in \mathcal{CP}^n$. But a ray in $\mathcal{CP}^n$ must be generated by a matrix with rank 1, which contradicts the assumption that $k = 2$. $\square$

In Theorem 4.11, we worked with a face, which roughly speaking is an intersection of $\mathcal{P}^2$ with a face in $\mathcal{N}^n$. Next theorem studies the face, which is an intersection of $\mathcal{P}^3$ with a face in $\mathcal{N}^n$.

Theorem 4.13. *Suppose $\mathcal{F}$ is a face of $\mathcal{P}^n \cap \mathcal{N}^n$, which is an intersection of a face $\mathcal{F}_{\mathcal{P}}$ in $\mathcal{P}^n$ of the form $\left\{ Q \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} Q^T \;\middle|\; A_{11} \in \mathcal{P}^3 \right\}$ with a face $\mathcal{F}_{\mathcal{N}}$ in $\mathcal{N}^n$. Here Q is an orthogonal matrix. If $\mathcal{F} \cap \mathcal{CP}^n = \{0\}$, then $\mathcal{F}$ is a polyhedral cone.*

Proof. Let $B \in relint(\mathcal{F})$. Then $rank(B) \leq 3$. If $rank(B) = 1$, then $B \in \mathcal{CP}^n$, if $rank(B) = 2$, then we know that $\mathcal{F} \cap \mathcal{CP}^n \neq \{0\}$ by Theorem 4.11. So if $\mathcal{F} \cap \mathcal{CP}^n = \{0\}$, we must have $rank(B) = 3$. Actually for any $C \in \mathcal{F}$, $rank(C) = 3$. This can be proved by the following argument. If $rank(C) = 1$, then $C \in \mathcal{CP}^n$ contradicts the assumption that $\mathcal{F} \cap \mathcal{CP}^n = \{0\}$. If $rank(C) = 2$, then we have a face $\mathcal{F}_C$ of $\mathcal{P}^n$ with C in its relative interior. We have $\mathcal{F}_C \subset \mathcal{F}_{\mathcal{P}}$ by a straightforward argument. $\mathcal{F}_C$ should be of the form described as in Theorem 4.11. By Theorem 4.11, we know that $\mathcal{F}_C \cap \mathcal{F}_{\mathcal{N}} \cap \mathcal{CP}^n \neq \{0\}$. Since $\mathcal{F}_C \subset \mathcal{F}_{\mathcal{P}}$ and $\mathcal{F} = \mathcal{F}_{\mathcal{P}} \cap \mathcal{F}_{\mathcal{N}}$, we have $\mathcal{F} \cap \mathcal{CP}^n \neq \{0\}$. Therefore, $rank(C) = 2$ is impossible. Hence, $rank(C) = 3$ for all $C \in \mathcal{F}$. Therefore, $\mathcal{F}$ can be viewed as an intersection of $\mathcal{F}_{\mathcal{N}}$ with the set of the form $\left\{ Q \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} Q^T \;\middle|\; A_{11} \in \mathcal{P}^3_+ \right\}$, where $\mathcal{P}^3_+$ represents the cone of 3×3 positive definite matrices.

Because $\mathcal{N}^n$ has finitely many faces, the number of faces that is the intersection of a face of $\mathcal{N}^n$ with the set of the form $\left\{ Q \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} Q^T \;\middle|\; A_{11} \in \mathcal{P}^3_+ \right\}$ must be finite. This shows that there are finitely many subfaces of $\mathcal{F}$ showing that $\mathcal{F}$ is a polyhedral cone. □

When $n = 5$, we can prove that $\mathcal{F}$ in Theorem 4.13 becomes an extreme ray. We state this result as a theorem.

Theorem 4.14. *Suppose $\mathcal{F}$ is a face of $\mathcal{P}^5 \cap \mathcal{N}^5$, which is the intersection of a face $\mathcal{F}_\mathcal{P}$ in $\mathcal{P}^5$ of the form $\left\{ Q \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} Q^T \;\middle|\; A_{11} \in \mathcal{P}^3 \right\}$ with a face $\mathcal{F}_\mathcal{N}$ in $\mathcal{N}^5$. Here Q is an orthogonal matrix. If $\mathcal{F} \cap \mathcal{CP}^5 = \{0\}$, then $\mathcal{F}$ is an extreme ray.*

Proof. Since $\mathcal{F}$ can be written as the convex hull of extreme rays, we may assume that A and B generate two extreme rays in $\mathcal{F}$. By Theorem 4.13 we know the rank for both A and B is 3. Since both A and B are in $\mathcal{F}_\mathcal{P}$, we can write A and B as follows:

$$A = Q \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} Q^T \text{ for some } A_{11} \in \mathcal{P}^3_+,$$

and

$$B = Q \begin{pmatrix} B_{11} & 0 \\ 0 & 0 \end{pmatrix} Q^T \text{ for some } B_{11} \in \mathcal{P}^3_+.$$

Let $Q = (x_1, x_2, \ldots, x_5)$, $x_i \in \mathbb{R}^5$ for $i = 1, 2, \ldots, 5$. Then $Ax_i = Bx_i = 0$ for $i = 4, 5$. Since Q is orthogonal, we know that x_4 and x_5 are linearly independent. Hence, the row vectors of A and also the row vectors of B span the space $(span(x_4, x_5))^\perp$. Since $rank(A) = rank(B) = 3$, we know that three linearly independent row vectors of A form a basis of the row space of A. Because we assumed that A and B are both extreme, by a theorem in [8] we know that there are at least 5 zero entries in the upper triangular part of both A and B and also the graphs associated with A and B must be cyclic. Without loss of generality, we may assume that $A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & a_{15} \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ a_{51} & 0 & 0 & a_{54} & a_{55} \end{pmatrix}$. We see that the first three rows are linearly independent, and hence form a basis of the row space of A and also the row space of B. From here, we can prove that $B = \lambda A$ for some $\lambda > 0$. Therefore, all the extreme matrix on the face $\mathcal{F}$ is a multiple of A showing that $\mathcal{F}$ is an extreme ray. □

The following theorem states that all faces in the 5×5 doubly nonnegative cone except certain extreme rays must intersect the 5×5 completely positive cone at more than the origin.

Theorem 4.15. *Let $\mathcal{F}$ be a face of $\mathcal{P}^5 \cap \mathcal{N}^5$, which is not an extreme ray in $(\mathcal{P}^5 \cap \mathcal{N}^5) \backslash \mathcal{CP}^5$. Then $\mathcal{F} \cap \mathcal{CP}^5 \neq \{0\}$ and is a face of $\mathcal{CP}^5$.*

Proof. Since $\mathcal{F}$ is a face of $\mathcal{P}^5 \cap \mathcal{N}^5$, $\mathcal{F}$ can be written as the intersection of a face $\mathcal{F}_\mathcal{N}$ in $\mathcal{N}^5$ with a face $\mathcal{F}_\mathcal{P}$ in $\mathcal{P}^5$. Let $\mathcal{F}_\mathcal{P} = \left\{ Q \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} Q^T \;\middle|\; A_{11} \in \mathcal{P}^k \right\}$ with Q being orthogonal. If $k = 1$, we can easily see that $\mathcal{F}$ is an extremal ray in $\mathcal{CP}^5$. For $k = 2, 3, 4, 5$, we use Theorem 4.11, 4.14, 4.10, and 4.7, respectively, to show that the conclusion holds. □

Now, we conclude this section by providing an alternative proof of a theorem in [5].

Theorem 4.16. *Every doubly nonnegative matrix A, which is not completely positive, can be expressed as $T+E$ with $T \in \mathcal{CP}^5$ and $E \in Ext(\mathcal{P}^5 \cap \mathcal{N}^5)\backslash Ext(\mathcal{CP}^5)$.*

Proof. Let $\mathcal{F}_1$ be the minimal face in $\mathcal{P}^5 \cap \mathcal{N}^5$ which contains A. If $\mathcal{F}_1$ is an extreme ray which is not in $\mathcal{CP}^5$, then we set $T = 0$ and $E = A$.

If $\mathcal{F}_1$ is not an extreme ray, then by Theorem 4.15 we know that $\mathcal{F}_1 \cap \mathcal{CP}^5 \neq \{0\}$. Let $\{0\} \neq T \in \mathcal{F}_1 \cap \mathcal{CP}^5$. Since $\alpha A \in relint(\mathcal{F}_1)$ for any $\alpha > 0$, the ray starting at T and passing through αA for some $\alpha > 0$ should intersect the boundary of $\mathcal{F}_1$, namely B. Otherwise, any point on the ray starting at T and passing through αA must be in $\mathcal{F}_1$ for any $\alpha > 0$, which implies that for any $n \in \mathbb{N}$, we have $n \times \frac{1}{n}A + (1-n)T \in \mathcal{F}_1$. Hence, $\frac{1}{n}A + (\frac{1}{n} - 1)T \in \mathcal{F}_1$, which shows $-T \in \mathcal{F}_1$ contradicting the fact that the doubly nonnegative cone is pointed. Since A is in the line segment with end points T and B, we can write $A = \lambda T + (1-\lambda)B$ for some $0 < \lambda < 1$. Let $T_1 = \lambda T$ and $A_1 = (1-\lambda)B$. Then $T_1 \in \mathcal{CP}^5$ and A_1 is on the boundary of $\mathcal{F}_1$. Now we apply the same argument to A_1, we either have A_1 to be an extreme ray, then we set $T = T_1$ and $E = A_1$, or we continue to have a minimal face $\mathcal{F}_2$ which contains A_1, then a $T_2 \in \mathcal{CP}^5$ and A_2 on the boundary of $\mathcal{F}_2$ are obtained. Since the length of the longest chain of faces must be finite, this process will stop in a finite number of steps k. So we can set $T = T_1 + T_2 + \cdots + T_{k-1}$ and $E = A_k$. □

5 Conclusions

In this paper, we have proved a duality result between maximal faces of a proper cone and minimal exposed faces of its dual cone. We have also presented some geometric properties of completely positive cones in terms of doubly nonnegative cones. As an application of these results, we have provided an alternative proof of a theorem in [5] without using a representation of extreme rays of 5×5 doubly nonnegative cone. We believe these results are new and might be used as a tool to prove other interesting results.

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Manuscript received 28 September 2017
revised 14 February 2018
accepted for publication 8 March 2018

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