Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL
Pacific Journal of Optimization
Vol. 16, No. 1, pp. 1–18, 2020

SADDLE POINT CRITERIA AND WOLFE DUALITY FOR CONVEX NONSMOOTH INTERVAL-VALUED VECTOR OPTIMIZATION PROBLEMS

Tadeusz Antczak

Abstract: In this paper, a convex nonsmooth vector optimization problem with multiple interval objective function is considered. The vector-valued Lagrange function and its saddle point are defined for such a vector interval-valued optimization problem. Then, the saddle point criteria are established for the considered nonsmooth interval-valued vector optimization problem under assumption that the functions constituting it are nondifferentiable convex. Further, its vector Wolfe dual problem with multiple interval-valued objective function is defined and several duality results in the sense of Wolfe are proved between vector interval-valued optimization problems also under convexity assumptions.

Key words: *nonsmooth vector optimization problem with multiple interval-valued objective function, vector-valued Lagrange function, saddle point criteria, vector Wolfe dual problem with multiple interval-valued objective function, nondifferentiable interval-valued convex function*

Mathematics Subject Classification: *90C25, 90C29, 90C30, 90C46*

1 Introduction

In mathematical programming problems, the coefficients of optimization problems are, in general, considered as deterministic values. This assumption is not satisfied by great majority of real-life engineering and economical problems. The introduction of imprecision and uncertainty in the modeling process is an important issue of the approaching real practical problems. Uncertainty can be handled in various manners, namely by a stochastic process and fuzzy numbers. However, sometimes it is hard to find an appropriate membership function or probability distribution with insufficiency of data. Therefore, interval-valued optimization problems may provide an alternative choice for considering the uncertainty into optimization problems. Namely, the coefficients in interval-valued optimization problems are assumed to be closed intervals. Although the specifications of closed intervals may still be judged as subjective viewpoint, however, it seems that the bounds of uncertain data (i.e., determining the closed intervals to bound the possible observed data) are easier to be handled than specifying the distributions and membership functions in stochastic optimization and fuzzy optimization problems, respectively. In recent years, therefore, optimality conditions and duality results for scalar interval-valued optimization problems have recently attracted the attention of many researchers (see, for instance, Jayswal et al. (2011), Jayswal et al. (2016), Jiang et al. (2008), Osuna-Gómez et al. (2015), Rohn (1980), Sun and Wang

(2013), Wu (2007), (2008), (2009), (2010), Zhang et al. (2014), Zhou and Wang (2009), and others).

Although nonlinear multiobjective programming problems with the coefficients considered as deterministic values occur in various fields of applications in O.R., however, this assumption is not satisfied by great majority of real-life engineering and economical problems. In recent years, therefore, attempts were made by several authors to prove optimality conditions and duality results for interval-valued vector optimization problems (see, for instance, Ahmad et al. (2015), Bhurjee and Panda (2012), Bitran (1980), Chanas and Kuchta (1996), Hosseinzade and Hassanpour (2011), Ishihuchi and Tanaka (1990), Jana and Panda (2014), Oliveira and Antunes (2007), Wu (2009), and others). Whereas many optimality and duality results have been explored for interval-valued optimization problems if the involved functions are smooth, while only few papers studied optimality conditions and duality results for scalar nonsmooth interval-valued optimization problems. For the considered nonsmooth scalar optimization problem with interval-valued objective function, Sun and Wang (2013) proved optimality conditions and several duality results under convexity assumption. Recently, Antczak (2017) derived the Fritz John and the Karush-Kuhn-Tucker type necessary optimality conditions for (weakly LU-efficiency) LU-efficiency of a feasible solution for a nondifferentiable interval-valued vector optimization problem. Further, he also proved the sufficiency of the Karush-Kuhn-Tucker type necessary optimality conditions and Mond-Weir duality results for convex nonsmooth interval-valued vector optimization problems.

Recent years have seen an increasing interest amongst researchers to explore saddle point criteria for scalar interval-valued optimization problems. Namely, Sun et al. (2014) derived saddle point optimality conditions and established a relation between an optimal solution of the considered interval-valued optimization problem and a saddle point of the Lagrangian function. Recently, Jayswal et al. (2016) proved saddle-point optimality conditions under invexity assumption. Hence, they found a relation between a LU-optimal solution of the considered nonsmooth scalar optimization problem with interval-valued objective function and a saddle-point of the Lagrangian function.

However, to the author's knowledge, there are not any result on saddle point criteria for nondifferentiable vector optimization problems with multiple interval-valued objective functions. The purpose of this paper is, therefore, to study saddle point criteria for nondifferentiable interval-valued multiobjective programming problems, that is, for nonsmooth vector optimization problems with interval-valued objective functions and inequality constraints. We define the vector-valued Lagrange function and its saddle point for the aforesaid interval-valued multiobjective programming problem. Then we derive the saddle point criteria for the considered nonsmooth vector optimization problem with multiple interval-valued objective function. In other words, we prove the equivalence between a saddle point of the vector-valued Lagrange function and a weak LU-Pareto solution (a LU-Pareto solution) of the considered nonsmooth interval-valued multiobjective programming problem under the assumption that the involved functions are convex. Further, we define vector Wolfe dual problem with multiple interval-valued objective function for the aforesaid interval-valued multiobjective programming problem. We establish several duality theorems between these nonsmooth multiobjective programming problems with interval-valued multiple objective functions also under convexity hypotheses.

2 Notations and Preliminaries

Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant. The following convention for equalities and inequalities will be used in the paper.

For any vectors $x = (x_1, x_2, \ldots, x_n)^T$ and $y = (y_1, y_2, \ldots, y_n)^T$ in R^n, we define:
(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \ldots, n$;
(ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, \ldots, n$;
(iii) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, \ldots, n$;
(iv) $x \geq y$ if and only if $x \geqq y$ and $x \neq y$.

Let $I(R)$ be a class of all closed and bounded intervals in R. Throughout this paper, when we say that A is a closed interval, we mean that A is also bounded in R. If A is a closed interval, we use the notation $A = [a^L, a^U]$, where a^L and a^U mean the lower and upper bounds of A, respectively. In other words, if $A = [a^L, a^U] \in I(R)$, then $A = [a^L, a^U] = \{x \in R : a^L \leqq x \leqq a^U\}$. If $a^L = a^U = a$, then $A = [a, a] = a$ is a real number.

Let $A = [a^L, a^U]$, $B = [b^L, b^U]$, then, by definition, we have:

i) $A + B = \{a + b : a \in A \text{ and } b \in B\} = [a^L + b^L, a^U + b^U]$,

ii) $-A = \{-a : a \in A\} = [-a^U, -a^L]$,

iii) $A - B = A + (-B) = \{a - b : a \in A \text{ and } b \in B\} = [a^L - b^U, a^U - b^L]$,

iv) $k + A = \{k + a : a \in A\} = [k + a^L, k + a^U]$, where k is a real number,

v) $kA = \begin{cases} [ka^L, ka^U] & \text{if} \quad k > 0, \\ [ka^U, ka^L] & \text{if} \quad k \leqq 0 \end{cases}$ where k is a real number.

In interval mathematics, an order relation is often used to rank interval numbers and it implies that an interval number is better than another but not that one is larger than another. For $A = [a^L, a^U]$ and $B = [b^L, b^U]$, we write $A \leqq_{LU} B$ if and only if $a^L \leqq b^L$ and $a^U \leqq b^U$. It is easy to see that $\leqq_{LU}$ is a partial ordering on $I(R)$. Also, we can write $A <_{LU} B$ if and only if $A \leqq_{LU} B$ and $A \neq B$. Equivalently, $A <_{LU} B$ if and only if $(a^L < b^L,\ a^U \leqq b^U)$ or $(a^L \leqq b^L,\ a^U < b^U)$ or $(a^L < b^L,\ a^U < b^U)$.

Throughout this section, let X be a nonempty subset of R^n. Further, $\psi : X \to I(R)$ is called an interval-valued function if $\psi(x) = [\psi^L(x), \psi^U(x)]$ with ψ^L, $\psi^U : X \to R$ such that $\psi^L(x) \leqq \psi^U(x)$ for each $x \in X$.

It is well-known that a function $f : X \to R$ defined on a convex set $X \subset R^n$ is said to be (strictly) convex provided that, for all $x \in X$, $(x \neq u)$ and any $\alpha \in [0, 1]$, one has

$$f(u + \alpha(x - u)) \leqq \alpha f(x) + (1 - \alpha) f(u). \quad (<)$$

Definition 2.1 (Rockafellar (1970))**.** Let $f : X \to R$ be a convex function defined on a convex set $X \subset R^n$. The subdifferential of a convex function f at $u \in X$, denoted by $\partial f(u)$, is defined as follows

$$\partial f(u) = \{\xi \in R^n : f(x) - f(u) \geqq \xi^T (x - u) \quad \forall x \in X\}.$$

Let $f : X \to R^p$ be a (strictly) convex vector-valued function defined on a nonempty convex set $X \subset R^n$. Then the inequalities

$$f_i(x) - f_i(u) \geqq \xi_i^T (x - u), \; i = 1, \ldots, p \qquad (>) \tag{2.1}$$

hold for all $x \in X$, $(x \neq u)$ and each $\xi_i \in \partial f(u)$, where $\partial f_i(u)$ denotes the subdifferential of f_i at u.

Similar to the definition of convexity for a real-valued function, the notion of convexity for an interval-valued function is defined as follows:

Definition 2.2 (Wu (2007))**.** Let X be a nonempty convex subset of R^n and $f : X \to I(R)$ be an interval-valued function defined on X. It is said that f is convex on X if the inequality

$$f(u + \alpha(x - u)) \leqq_{LU} \alpha f(x) + (1 - \alpha) f(u)$$

holds for all $x, u \in X$ and any $\alpha \in [0, 1]$.

Proposition 2.3 (Wu (2007))**.** *Let X be a nonempty convex subset of R^n and $f : X \to I(R)$ be an interval-valued function defined on X. The interval-valued function f is convex at $u \in X$ if and only if the real-valued functions f^L and f^U are convex at u.*

Remark 2.4. If $f : X \to I(R)$ is convex at u on X, then the inequalities

$$f^L(x) - f^L(u) \geqq \left(\xi^L\right)^T (x - u), \quad \forall \xi^L \in \partial f^L(u), \tag{2.2}$$

$$f^U(x) - f^U(u) \geqq \left(\xi^U\right)^T (x - u), \quad \forall \xi^U \in \partial f^U(u) \tag{2.3}$$

hold for all $x \in X$, where $\partial f^L(u)$ and $\partial f^U(u)$ denote the subdifferentials of f^L and f^U at u, respectively. If inequalities (2.2) and (2.3) are satisfied at every $u \in X$, then f is convex on X.

Remark 2.5. If $f : X \to I(R)$ is strictly convex at $u \in X$ on X, then both the functions f^L and f^U are strictly convex at $u \in X$ on X, that is, inequalities (2.2) and (2.3) are strict for all $x \in X$, $x \neq u$.

In this paper, we consider the following vector optimization problem with interval-valued multiple objective function:

$$\begin{aligned} f(x) = (f_1(x), \ldots, f_p(x)) &\to \min \\ g(x) = (g_1(x), \ldots, g_m(x)) &\leqq 0, \qquad \text{(IVP)} \\ x &\in R^n, \end{aligned}$$

where each $f_k : R^n \to I(R)$, $k \in K = \{1, \ldots, p\}$ is an interval-valued function, that is,

$$f_k(x) = \left[f_k^L(x), f_k^U(x)\right], \; i \in I,$$

and, moreover, $g : X \to R^m$. We shall assume, moreover, that f_k^L, $f_k^U : R^n \to R$, $k \in K$ and $g_j : R^n \to R$, $j \in J$, are locally Lipschitz functions on R^n. For the purpose of simplifying our presentation, we introduce the following notations $f^L = \left(f_1^L, \ldots, f_p^L\right)^T$, $f^U = \left(f_1^U, \ldots, f_p^U\right)^T$. Further, let us denote by D the set of all feasible solutions in the considered interval-valued multiobjective optimization problem (IVP), that is, the set $D = \{x \in R^n : g(x) \leqq 0\}$ and, moreover, by $J(x)$ the set of constraint indices that are active at a feasible solution x, that is, $J(x) = \{j \in J : g_j(x) = 0\}$.

Since each of objective values f_i is a closed interval, we need to provide an ordering relation between any two closed intervals. The most direct way is to invoke the ordering relation $\leqq_{LU}$ that was defined above. However, $\leqq_{LU}$ is a partial ordering, not a total ordering, on $I(R)$. Therefore, we shall follow the similar concept of nondominated solution used in multiobjective programming problem to investigate the solution concepts.

For such an interval-valued multicriterion optimization problem, its optimal solution is defined in terms of a weak LU-Pareto (LU-Pareto) solution in the following sense (see, for instance, Ahmad et al. (2015), Wu (2009)):

Definition 2.6. A feasible point $\overline{x}$ is said to be a weak LU-Pareto solution (a weakly LU-efficient solution) of (IVP) if and only if there is no another feasible solution x such that, for each $k \in K$,

$$f_k(x) <_{LU} f_k(\overline{x}).$$

Definition 2.7. A feasible point $\overline{x}$ is said to be a LU-Pareto solution (a LU-efficient solution) of (IVP) if and only if there is no another feasible solution x such that

$$f(x) <_{LU} f(\overline{x}).$$

Recently, Antczak (2017) established the following Karush-Kuhn-Tucker necessary optimality conditions for a nonsmooth multiobjective programming problem with multiple interval-valued objective function involving inequality constraints under the generalized Slater constraint qualification.

Theorem 2.8 (Karush-Kuhn-Tucker necessary optimality conditions)**.** *Let $\overline{x} \in D$ be a weak LU-Pareto solution of the interval-valued multiobjective optimization problem (IVP) and the constraint qualification of Slater's type be satisfied at $\overline{x}$. Then there exist $\overline{\lambda}^L \in R^p$, $\overline{\lambda}^U \in R^p$ and $\overline{\mu} \in R^m$ such that*

$$0 \in \sum_{k=1}^{p} \overline{\lambda}_k^L \partial f_k^L(\overline{x}) + \sum_{k=1}^{p} \overline{\lambda}_k^U \partial f_k^U(\overline{x}) + \sum_{j=1}^{m} \overline{\mu}_j \partial g_j(\overline{x}), \tag{2.4}$$

$$\overline{\mu}_j g_j(\overline{x}) = 0, \quad j \in J, \tag{2.5}$$

$$\overline{\lambda}^L \geq 0, \, \overline{\lambda}^U \geq 0, \quad \overline{\mu} \geqq 0. \tag{2.6}$$

Definition 2.9. $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right) \in D \times R_+^{2p} \times R_+^m$ is said to be a Karush-Kuhn-Tucker point of the considered interval-valued vector optimization problem (IVP) if the Karush-Kuhn-Tucker necessary optimality conditions (2.4)-(2.6) are satisfied at $\overline{x}$ with Lagrange multipliers $\overline{\lambda}^L \in R^p$, $\overline{\lambda}^U \in R^p$ and $\overline{\mu} \in R^m$.

3 Vector Saddle Point Criteria

In this section, for the constrained nonsmooth multiobjective programming problem (IVP) with multiple interval-valued objective function, we define the vector-valued Lagrange function L_p and a saddle point of L_p. Then, we prove the saddle point criteria for the problem (IVP) under assumption that the involved functions are convex.

The vector-valued Lagrange function L_p of the considered interval-valued vector optimization problem (IVP) is the function $L_p : D \times R_+^{2p} \times R_+^m \to R^p$ defined by

$$L_p\left(x, \lambda, \mu\right) := diag\lambda^L \, f^L(x) + diag\lambda^U \, f^U(x) + \frac{1}{p} \sum_{j=1}^{m} \overline{\mu}_j g_j(x) e, \tag{3.1}$$

where $e = [1, \ldots, 1] \in R^p$, $\lambda = \left[\lambda^L, \lambda^U\right] \in R_+^{2p}$ and, for any $\tau \in R^p$, the symbol $diag\,\tau$ is defined as follows

$$diag\,\tau = \begin{bmatrix} \tau_1 & 0 & \ldots & 0 \\ 0 & \tau_2 & \ldots & 0 \\ & & \ddots & 0 \\ 0 & \ldots & 0 & \tau_p \end{bmatrix}. \tag{3.2}$$

Now, we give the definition of a saddle point of the vector-valued Lagrange function L_p defined for the considered nonsmooth interval-valued vector optimization problem (IVP).

Definition 3.1. A point $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in D \times R_+^{2p} \times R_+^m$ is said to be a saddle point of the vector-valued Lagrange function L_p defined for the considered multiobjective programming problem (IVP) with multiple interval-valued objective function if,

i) $L_p(\overline{x}, \overline{\lambda}, \mu) \leqq L_p(\overline{x}, \overline{\lambda}, \overline{\mu}) \quad \forall \mu \in R_+^m$,

ii) $L_p(x, \overline{\lambda}, \overline{\mu}) \nleq L_p(\overline{x}, \overline{\lambda}, \overline{\mu}) \quad \forall x \in D$.

Theorem 3.2. *Let $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in D \times R_+^{2p} \times R_+^m$ be a saddle point of the vector-valued Lagrange function L_p defined for the considered interval-valued vector optimization problem (IVP). Then $\overline{x}$ is a LU-Pareto solution of the problem (IVP). Further, if Lagrange multipliers $\overline{\lambda}^L$ and $\overline{\lambda}^U$ are assumed to satisfy $\left(\overline{\lambda}_k^L, \overline{\lambda}_k^U\right) > 0$ for some $k \in K$, then $\overline{x}$ is a weak LU-Pareto solution of the problem (IVP).*

Proof. Since $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in D \times R_+^{2p} \times R_+^m$ is a saddle point of the vector-valued Lagrange function L_p defined for the considered interval-valued vector optimization problem (IVP), by Definition 3.1, the conditions i) and ii) are fulfilled. Thus, by the condition i) and the definition of the vector-valued Lagrange function L_p, the inequalities

$$\overline{\lambda}_k^L f_k^L(\overline{x}) + \overline{\lambda}_k^U f_k^U(\overline{x}) + \frac{1}{p}\sum_{j=1}^m \mu_j g_j(\overline{x}) \leqq \overline{\lambda}_k^L f_k^L(\overline{x}) + \overline{\lambda}_k^U f_k^U(\overline{x}) + \frac{1}{p}\sum_{j=1}^m \overline{\mu}_j g_j(\overline{x}), \ \ k \in K$$

hold for all $\mu \in R_+^m$. From the feasibility of $\overline{x}$ in (IVP), it follows that the inequality

$$\sum_{j=1}^m \mu_j g_j(\overline{x}) \leqq \sum_{j=1}^m \overline{\mu}_j g_j(\overline{x})$$

holds for all $\lambda \in R_+^m$. Therefore, for $\mu = 0$, we have

$$\sum_{j=1}^m \overline{\mu}_j g_j(\overline{x}) \geqq 0. \tag{3.3}$$

Then, by $\overline{x} \in D$ and $\overline{\mu} \in R_+^m$, it follows that

$$\sum_{j=1}^m \overline{\mu}_j g_j(\overline{x}) \leqq 0. \tag{3.4}$$

Thus, (3.3) and (3.4) yield

$$\sum_{j=1}^m \overline{\mu}_j g_j(\overline{x}) = 0. \tag{3.5}$$

We proceed by contradiction. Suppose, contrary to the result, that $\overline{x} \in D$ is not a *LU*-Pareto solution of the problem (IVP). Then, by Definition 2.7, there exists $\widetilde{x} \in D$ such that

$$f(\widetilde{x}) <_{LU} f(\overline{x}).$$

Therefore, by definition of the relation $<_{LU}$, it follows that

$$\left(f^L(\widetilde{x}) < f^L(\overline{x}) \ \wedge \ f^U(\widetilde{x}) \leqq f^U(\overline{x})\right)$$
$$\text{or } \left(f^L(\widetilde{x}) \leqq f^L(\overline{x}) \ \wedge \ f^U(\widetilde{x}) < f^U(\overline{x})\right)$$
$$\text{or } \left(f^L(\widetilde{x}) < f^L(\overline{x}) \ \wedge \ f^U(\widetilde{x}) < f^U(\overline{x})\right).$$

Hence, by the Karush-Kuhn-Tucker necessary optimality condition (2.6), the above inequalities imply

$$\overline{\lambda}_k^L f_k^L(\widetilde{x}) + \overline{\lambda}_k^U f^U(\widetilde{x}) \leqq \overline{\lambda}_k^L f_k^L(\overline{x}) + \overline{\lambda}_k^U f^U(\overline{x}),\ k \in K, \tag{3.6}$$

$$\overline{\lambda}_k^L f_k^L(\widetilde{x}) + \overline{\lambda}_k^U f^U(\widetilde{x}) < \overline{\lambda}_k^L f_k^L(\overline{x}) + \overline{\lambda}_k^U f^U(\overline{x}) \text{ for at least one } k \in K. \tag{3.7}$$

Since $\widetilde{x} \in D$, $\overline{x} \in D$ and $\overline{\mu} \in R_+^m$, by (3.5), inequalities (3.6) and (3.7) yield, respectively,

$$\begin{gathered}\overline{\lambda}_k^L f_k^L(\widetilde{x}) + \overline{\lambda}_k^U f_k^U(\widetilde{x}) + \frac{1}{p}\sum_{j=1}^m \overline{\mu}_j g_j(\widetilde{x}) \leqq \\ \overline{\lambda}_k^L f_k^L(\overline{x}) + \overline{\lambda}_k^U f_k^U(\overline{x}) + \frac{1}{p}\sum_{j=1}^m \overline{\mu}_j g_j(\overline{x}),\ k \in K,\end{gathered} \tag{3.8}$$

$$\begin{gathered}\overline{\lambda}_k^L f_k^L(\widetilde{x}) + \overline{\lambda}_k^U f_k^U(\widetilde{x}) + \frac{1}{p}\sum_{j=1}^m \overline{\mu}_j g_j(\widetilde{x}) < \\ \overline{\lambda}_k^L f_k^L(\overline{x}) + \overline{\lambda}_k^U f_k^U(\overline{x}) + \frac{1}{p}\sum_{j=1}^m \overline{\mu}_j g_j(\overline{x}) \text{ for at least one } k \in K.\end{gathered} \tag{3.9}$$

Thus, by (3.2), (3.8) and (3.9) imply

$$\begin{gathered}diag\overline{\lambda}^L f^L(\widetilde{x}) + diag\overline{\lambda}^U f^U(\widetilde{x}) + \frac{1}{p}\sum_{j=1}^m \overline{\mu}_j g_j(\widetilde{x})e \leq \\ diag\overline{\lambda}^L f^L(\overline{x}) + diag\overline{\lambda}^U f^U(\overline{x}) + \frac{1}{p}\sum_{j=1}^m \overline{\mu}_j g_j(\overline{x})e.\end{gathered}$$

By definition of the vector-valued Lagrange function L_p, we have that the inequality

$$L_p\left(\widetilde{x}, \overline{\lambda}, \overline{\mu}\right) \leq L_p\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$$

holds, which contradicting the inequality ii) in Definition 3.1.

Now, we prove that $\overline{x} \in D$ is a weak LU-Pareto solution of the problem (IVP). By means of contradiction, suppose that $\overline{x} \in D$ is not a weak LU-Pareto solution of the problem (IVP). Then, by Definition 2.6, there exists $\widetilde{x} \in D$ such that

$$f_k(\widetilde{x}) <_{LU} f_k(\overline{x}),\ k \in K.$$

Therefore, by definition of the relation $<_{LU}$, we have for any $k \in K$,

$$\left(f_k^L(\widetilde{x}) < f_k^L(\overline{x}) \ \wedge \ f_k^U(\widetilde{x}) \leqq f_k^U(\overline{x})\right)$$
$$\text{or } \left(f_k^L(\widetilde{x}) \leqq f_k^L(\overline{x}) \ \wedge \ f_k^U(\widetilde{x}) < f_k^U(\overline{x})\right)$$

$$\text{or } \left(f_k^L(\widetilde{x}) < f_k^L(\overline{x}) \ \wedge \ f_k^U(\widetilde{x}) < f_k^U(\overline{x})\right).$$

Since $(\overline{\lambda}_k^L, \overline{\lambda}_k^U) > 0$ for some $k \in K$, multiplying each inequality above by the corresponding Lagrange multiplier, we obtain that inequalities (3.6) and (3.7) are satisfied. The rest of the proof is the same as in the case of LU-efficiency and, therefore, it is omitted in the paper. □

Now, under stronger hypotheses, we prove the converse result.

Theorem 3.3. *Let $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in D \times R_+^{2p} \times R_+^m$ be a Karush-Kuhn-Tucker point of the considered nonsmooth vector optimization problem (IVP) with multiple interval-valued objective function. Furthermore, assume that the objective functions f_k^L, $k \in K$, f_k^U, $k \in K$, and the constraint functions g_j, $j \in J$, are convex on D. Then $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is a saddle point of the vector-valued Lagrange function L_p defined for the problem (IVP).*

Proof. First, we prove the inequality i) in Definition 3.1. By assumption, $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in D \times R_+^{2p} \times R_+^m$ is a Karush-Kuhn-Tucker point of the considered interval-valued vector optimization problem (IVP). Using the feasibility of $\overline{x}$ for the problem (IVP) together with the Karush-Kuhn-Tucker necessary optimality condition (2.5), we get that the following inequalities

$$\mu_j g_j(\overline{x}) \leqq \overline{\mu}_j g_j(\overline{x}), \ j \in J \tag{3.10}$$

hold for all $\mu = (\mu_1, \ldots, \mu_m) \in R_+^m$. By (3.10), it follows that the inequalities

$$\begin{aligned} \overline{\lambda}_k^L f_k^L(\overline{x}) + \overline{\lambda}_k^U f_k^U(\overline{x}) + \frac{1}{p}\sum_{j=1}^m \mu_j g_j(\overline{x}) \leqq \\ \overline{\lambda}_k^L f_k^L(\overline{x}) + \overline{\lambda}_k^U f_k^U(\overline{x}) + \frac{1}{p}\sum_{j=1}^m \overline{\mu}_j g_j(\overline{x}), \quad k \in K \end{aligned} \tag{3.11}$$

hold for any $\mu \in R_+^m$. Thus, by (3.2), (3.11) implies that the inequality

$$\begin{aligned} diag\overline{\lambda}^L f^L(\overline{x}) + diag\overline{\lambda}^U f^U(\overline{x}) + \frac{1}{p}\sum_{j=1}^m \mu_j g_j(\overline{x}) e \leqq \\ diag\overline{\lambda}^L f^L(\overline{x}) + diag\overline{\lambda}^U f^U(\overline{x}) + \frac{1}{p}\sum_{j=1}^m \overline{\mu}_j g_j(\overline{x}) e \end{aligned}$$

holds for any $\mu \in R_+^m$. Hence, by the definition of the vector-valued Lagrange function (3.1), the inequality

$$L_p(\overline{x}, \overline{\lambda}, \mu) \leqq L_p(\overline{x}, \overline{\lambda}, \overline{\mu}) \tag{3.12}$$

holds for any $\mu \in R_+^m$.

Now, we prove the second inequality in Definition 3.1. We proceed by contradiction. Suppose, contrary to the result, that there exists $\widetilde{x} \in D$ such that $L_p(\widetilde{x}, \overline{\lambda}, \overline{\mu}) \leq L_p(\overline{x}, \overline{\lambda}, \overline{\mu})$. Then, by the definition of the vector-valued Lagrange function, it follows that

$$\begin{aligned} diag\overline{\lambda}^L f^L(\widetilde{x}) + diag\overline{\lambda}^U f^U(\widetilde{x}) + \frac{1}{p}\sum_{j=1}^m \overline{\mu}_j g_j(\widetilde{x}) e \leq \\ diag\overline{\lambda}^L f^L(\overline{x}) + diag\overline{\lambda}^U f^U(\overline{x}) + \frac{1}{p}\sum_{j=1}^m \overline{\mu}_j g_j(\overline{x}) e. \end{aligned}$$

Thus, by (3.2), it follows that

$$\overline{\lambda}_k^L f_k^L(\widetilde{x}) + \overline{\lambda}_k^U f_k^U(\widetilde{x}) + \frac{1}{p}\sum_{j=1}^m \overline{\mu}_j g_j(\widetilde{x}) \leqq \overline{\lambda}_k^L f_k^L(\overline{x}) + \overline{\lambda}_k^U f_k^U(\overline{x}) + \frac{1}{p}\sum_{j=1}^m \overline{\mu}_j g_j(\overline{x}),\ k \in K.$$

$$\overline{\lambda}_k^L f_k^L(\widetilde{x}) + \overline{\lambda}_k^U f_k^U(\widetilde{x}) + \frac{1}{p}\sum_{j=1}^m \overline{\mu}_j g_j(\widetilde{x}) < \overline{\lambda}_k^L f_k^L(\overline{x}) + \overline{\lambda}_k^U f_k^U(\overline{x}) + \frac{1}{p}\sum_{j=1}^m \overline{\mu}_j g_j(\overline{x}) \text{ for at least one } k \in K.$$

Adding both sides of the inequalities above, we get

$$\sum_{k=1}^p \overline{\lambda}_k^L f_k^L(\widetilde{x}) + \sum_{k=1}^p \overline{\lambda}_k^U f_k^U(\widetilde{x}) + \sum_{j=1}^m \overline{\mu}_j g_j(\widetilde{x}) < \sum_{k=1}^p \overline{\lambda}_k^L f_k^L(\overline{x}) + \sum_{k=1}^p \overline{\lambda}_k^U f_k^U(\overline{x}) + \sum_{j=1}^m \overline{\mu}_j g_j(\overline{x}). \tag{3.13}$$

By assumption, f_k^L, $k \in K$, f_k^U, $k \in K$, g_j, $j \in J$, are convex functions on D. Hence, the inequalities

$$f_k^L(\widetilde{x}) - f_k^L(\overline{x}) \geqq \xi_k^L\left(\widetilde{x} - \overline{x}\right), \quad k \in K, \tag{3.14}$$

$$f_k^U(\widetilde{x}) - f_k^U(\overline{x}) \geqq \xi_k^U\left(\widetilde{x} - \overline{x}\right), \quad k \in K, \tag{3.15}$$

$$g_j(\widetilde{x}) - g_j(\overline{x}) \geqq \zeta_j\left(\widetilde{x} - \overline{x}\right), \quad j \in J \tag{3.16}$$

hold for any $\xi_k^L \in \partial f_k^L(\overline{x})$, $\xi_k^U \in \partial f_k^U(\overline{x})$, $k \in K$, $\zeta_j \in \partial g_j(\overline{x})$, $j \in J$. Multiplying inequalities (3.14)-(3.16) by the associated Lagrange multiplier and then adding both sides of the obtained inequalities, respectively, we get

$$\sum_{k=1}^p \overline{\lambda}_k^L f_k^L(\widetilde{x}) - \sum_{k=1}^p \overline{\lambda}_k^L f_k^L(\overline{x}) \geqq \sum_{k=1}^p \overline{\lambda}_k^L \xi_k^L\left(\widetilde{x} - \overline{x}\right), \tag{3.17}$$

$$\sum_{k=1}^p \overline{\lambda}_k^U f_k^U(\widetilde{x}) - \sum_{k=1}^p \overline{\lambda}_k^U f_k^U(\overline{x}) \geqq \sum_{k=1}^p \overline{\lambda}_k^U \xi_k^U\left(\widetilde{x} - \overline{x}\right), \tag{3.18}$$

$$\sum_{j=1}^m \overline{\mu}_j g_j(\widetilde{x}) - \sum_{j=1}^m \overline{\mu}_j g_j(\overline{x}) \geqq \sum_{j=1}^m \overline{\mu}_j \zeta_j\left(\widetilde{x} - \overline{x}\right). \tag{3.19}$$

Combining inequalities (3.17)-(3.19), we obtain

$$\sum_{k=1}^p \overline{\lambda}_k^L f_k^L(\widetilde{x}) + \sum_{k=1}^p \overline{\lambda}_k^U f_k^U(\widetilde{x}) + \sum_{j=1}^m \overline{\mu}_j g_j(\widetilde{x}) - \sum_{k=1}^p \overline{\lambda}_k^L f_k^L(\overline{x}) - \sum_{k=1}^p \overline{\lambda}_k^U f_k^U(\overline{x}) - \sum_{j=1}^m \overline{\mu}_j g_j(\overline{x}) \geqq \sum_{k=1}^p \overline{\lambda}_k^L \xi_k^L\left(\widetilde{x} - \overline{x}\right) + \sum_{k=1}^p \overline{\lambda}_k^U \xi_k^U\left(\widetilde{x} - \overline{x}\right) + \sum_{j=1}^m \overline{\mu}_j \zeta_j\left(\widetilde{x} - \overline{x}\right). \tag{3.20}$$

Thus, by (3.13) and (3.20), the inequality

$$\left[\sum_{k=1}^{p}\overline{\lambda}_k^L\xi_k^L+\sum_{k=1}^{p}\overline{\lambda}_k^U\xi_k^U+\sum_{j=1}^{m}\overline{\mu}_j\zeta_j\right](\widetilde{x}-\overline{x})<0 \tag{3.21}$$

holds, contradicting the Karush-Kuhn-Tucker necessary optimality condition (2.4). This completes the proof of this theorem. □

The following result follows directly from the above theorem.

Corollary 3.4. *Let $\overline{x}\in D$ be a weak LU-Pareto solution of the considered nonsmooth vector optimization problem (IVP) with multiple interval-valued objective function and the Karush-Kuhn-Tucker necessary optimality conditions (2.4)-(2.6) be satisfied at $\overline{x}$ with Lagrange multipliers $\overline{\lambda}=\left[\overline{\lambda}^L,\overline{\lambda}^U\right]\in R_+^{2p}$ and $\overline{\mu}\in R_+^m$. Further, assume that all hypotheses of Theorem 3.3 are fulfilled. Then $\left(\overline{x},\overline{\lambda},\overline{\mu}\right)$ is a saddle point of the vector-valued Lagrange function L_p defined for the problem (IVP).*

In order to illustrate the results established in the paper, we consider an example of a convex nondifferentiable multiobjective programming problem with interval-valued objective functions in which all involved functions are convex.

Example 3.1. Consider the family of convex nondifferentiable optimization problem with interval-valued objective functions:

$$\begin{array}{c} f(x)=\left([1,2]\left(|x_1|+|x_2|\right),\ldots,[p,p+1]\left(|x_1|+|x_2|\right)\right)\rightarrow\min \\ g_1(x)=x_1^2-x_1\leqq 0, \\ g_1(x)=x_2^2-x_2\leqq 0, \end{array} \tag{IVP1}$$

where $p\geq 1$ is a finite integer number. Note that $D=\{(x_1,x_2)\in R^2: x_1^2-x_1\leqq 0,\ x_2^2-x_2\leqq 0\}$ and $\overline{x}=(0,0)$ is a feasible point of the problem (IVP1). It can be shown that there exist $\overline{\lambda}^L\in R^p$, $\overline{\lambda}^U\in R^p$ and $\overline{\mu}\in R^2$ (for example, $\overline{\lambda}^L=(1,\ldots,1)\in R^p$, $\overline{\lambda}^U=(1,\ldots,1)\in R^p$, $\overline{\mu}=\left(\frac{1}{2},\frac{1}{2}\right)\in R^2$) such that the Karush-Kuhn-Tucker necessary optimality conditions (2.4)-(2.6) are satisfied at $\overline{x}=(0,0)$ with these Lagrange multipliers. Also it is not difficult to show that f_k^L, $k\in K$, f_k^U, $k\in K$, g_j, $j\in J\left(\overline{x}\right)$, are convex functions on D. Since all hypotheses of Theorem 3.3 are satisfied, $\left(\overline{x},\overline{\lambda},\overline{\mu}\right)$ is a saddle point of the vector-valued Lagrange function L_p defined for the considered nonsmooth vector optimization problem (IVP1) with interval-valued objective functions. Further, by Theorem 3.2, it follows that $\overline{x}=(0,0)$ is a *LU*-Pareto solution of the problem (IVP1).

4 Wolfe Duality

In this section, for the considered nonsmooth multiobjective programming problem (IVP) with multiple interval-valued objective function, its vector Wolfe dual problem with multiple interval-valued objective function is defined as follows:

$$\vartheta\left(y,\mu\right)=\left(\left[f_1^L\left(y\right)+\frac{1}{2}\sum_{j=1}^{m}\mu_jg_j(y),f_1^U\left(y\right)+\frac{1}{2}\sum_{j=1}^{m}\mu_jg_j(y)\right],\ldots,\right. \tag{4.1}$$

$$\left[f_p^L(y)+\frac{1}{2}\sum_{j=1}^m \mu_j g_j(y), f_p^U(y)+\frac{1}{2}\sum_{j=1}^m \mu_j g_j(y)\right]\Bigg) \to \max$$

$$\text{such that } \ 0 \in \sum_{k=1}^p \lambda_k^L \partial f_k^L(y) + \sum_{k=1}^p \lambda_k^U \partial f_k^U(y) + \sum_{j=1}^m \mu_j \partial g_j(y), \qquad \text{(IVWD)}$$
$$y \in X,\ \lambda^L \geq 0,\ \lambda^L e = 1,\ \lambda^U \geq 0,\ \lambda^U e = 1,\ \mu \geqq 0,$$

where $e = (1, \ldots, 1)^T \in R^p$.

Let

$$W = \Bigg\{\left(y, \lambda^L, \lambda^U, \mu\right) \in R^n \times R^p \times R^p \times R^m : 0 \in \sum_{k=1}^p \lambda_k^L \partial f_k^L(y) + \sum_{k=1}^p \lambda_k^U \partial f_k^U(y) + \sum_{i=1}^m \mu_i \partial g_i(y),\ \ \lambda^L \geq 0,\ \lambda^L e = 1,\ \lambda^U \geq 0,\ \lambda^U e = 1,\ \mu \geqq 0\Bigg\}$$

be the set of all feasible solutions of the problem (IVWD). Further, let us denote by Y the projection of W on X, that is, $Y = \left\{y \in R^n : \left(y, \lambda^L, \lambda^U, \mu\right) \in W\right\}$. We now derive duality relations between vector optimization problems (IVP) and (IVWD) with multiple interval-valued objective functions.

Theorem 4.1 (Weak duality)**.** *Let x and $\left(y, \lambda^L, \lambda^U, \mu\right)$ be any feasible solutions for the problems (IVP) and (IVWD), respectively. Further, assume that f_k^L, $k \in K$, f_k^U, $k \in K$, g_j, $j \in J$,, are convex functions on $D \cup Y$. Then,*

$$f(x) \nless_{LU} \vartheta(y, \mu).$$

Proof. Suppose, contrary to the result, that

$$f(x) <_{LU} \vartheta(y, \mu).$$

Hence, by definition of the objective function ϑ, we have

$$f(x) <_{LU} f(y) + \frac{1}{2}\sum_{j=1}^m \mu_j g_j(y) e.$$

Thus, by definition of the relation $<_{LU}$, we have

$$\begin{cases} f^L(x) < f^L(y) + \frac{1}{2}\sum_{j=1}^m \mu_j g_j(y) e, \\ f^U(x) \leqq f^U(y) + \frac{1}{2}\sum_{j=1}^m \mu_j g_j(y) e, \end{cases}$$

$$\text{or } \begin{cases} f^L(x) \leqq f^L(y) + \frac{1}{2}\sum_{j=1}^m \mu_j g_j(y) e, \\ f^U(x) < f^U(y) + \frac{1}{2}\sum_{j=1}^m \mu_j g_j(y) e, \end{cases}$$

$$\text{or }\begin{cases} f^L(x) < f^L(y) + \frac{1}{2}\sum_{j=1}^m \mu_j g_j(y)e, \\ f^U(x) < f^U(y) + \frac{1}{2}\sum_{j=1}^m \mu_j g_j(y)e. \end{cases}$$

By $(y, \lambda^L, \lambda^U, \mu) \in W$, the above relations give, respectively,

$$\begin{cases} \sum_{k=1}^p \lambda_k^L f_k^L(x) < \sum_{k=1}^p \lambda_k^L f_k^L(y) + \frac{1}{2}\sum_{j=1}^m \mu_j g_j(y) \sum_{k=1}^p \lambda_k^L, \\ \sum_{k=1}^p \lambda_k^U f_k^U(x) \leqq \sum_{k=1}^p \lambda_k^U f_k^U(y) + \frac{1}{2}\sum_{j=1}^m \mu_j g_j(y) \sum_{k=1}^p \lambda_k^U, \end{cases}$$

$$\text{or }\begin{cases} \sum_{k=1}^p \lambda_k^L f_k^L(x) \leqq \sum_{k=1}^p \lambda_k^L f_k^L(y) + \frac{1}{2}\sum_{j=1}^m \mu_j g_j(y) \sum_{k=1}^p \lambda_k^L, \\ \sum_{k=1}^p \lambda_k^U f_k^U(x) < \sum_{k=1}^p \lambda_k^U f_k^U(y) + \frac{1}{2}\sum_{j=1}^m \mu_j g_j(y) \sum_{k=1}^p \lambda_k^U, \end{cases}$$

$$\text{or }\begin{cases} \sum_{k=1}^p \lambda_k^L f_k^L(x) < \sum_{k=1}^p \lambda_k^L f_k^L(y) + \frac{1}{2}\sum_{j=1}^m \mu_j g_j(y) \sum_{k=1}^p \lambda_k^L, \\ \sum_{k=1}^p \lambda_k^U f_k^U(x) < \sum_{k=1}^p \lambda_k^U f_k^U(y) + \frac{1}{2}\sum_{j=1}^m \mu_j g_j(y) \sum_{k=1}^p \lambda_k^U. \end{cases}$$

Hence, by $\sum_{k=1}^p \lambda_k^L = 1$ and $\sum_{k=1}^p \lambda_k^U = 1$, the above relations imply

$$\sum_{k=1}^p \lambda_k^L f_k^L(x) + \sum_{k=1}^p \lambda_k^U f_k^U(x) < \sum_{k=1}^p \lambda_k^L f_k^L(y) + \sum_{k=1}^p \lambda_k^U f_k^U(y) + \sum_{j=1}^m \mu_j g_j(y).$$

By $x \in D$ and $(y, \lambda^L, \lambda^U, \mu) \in W$, the above inequality gives

$$\sum_{k=1}^p \lambda_k^L f_k^L(x) + \sum_{k=1}^p \lambda_k^U f_k^U(x) + \sum_{j=1}^m \mu_j g_j(x) < \sum_{k=1}^p \lambda_k^L f_k^L(y) + \sum_{k=1}^p \lambda_k^U f_k^U(y) + \sum_{j=1}^m \mu_j g_j(y). \tag{4.2}$$

By assumption, f_k^L, $k \in K$, f_k^U, $k \in K$, g_j, $j \in J$, are convex functions on $D \cup Y$. Hence, the inequalities

$$f_k^L(x) - f_k^L(y) \geqq \xi_k^L(x-y) \quad k \in K, \tag{4.3}$$

$$f_k^U(x) - f_k^U(y) \geqq \xi_k^U(x-y) \quad k \in K, \tag{4.4}$$

$$g_j(x) - g_j(y) \geqq \zeta_j(x-y), \quad j \in J \tag{4.5}$$

hold for any $\xi_k^L \in \partial f_k^L(y)$, $\xi_k^U \in \partial f_k^U(y)$, $k \in K$, $\zeta_j \in \partial g_j(y)$, $j \in J$. By the last constraint of the problem (IVWD), we obtain from (4.3)-(4.5), respectively,

$$\sum_{k=1}^{p} \lambda_k^L f_k^L(x) - \sum_{k=1}^{p} \lambda_k^L f_k^L(y) \geqq \sum_{k=1}^{p} \lambda_k^L \xi_k^L (x-y), \tag{4.6}$$

$$\sum_{k=1}^{p} \lambda_k^U f_k^U(x) - \sum_{k=1}^{p} \lambda_k^U f_k^U(y) \geqq \sum_{k=1}^{p} \lambda_k^U \xi_k^U (x-y), \tag{4.7}$$

$$\sum_{j=1}^{m} \mu_j g_j(x) - \sum_{j=1}^{m} \mu_j g_j(y) \geqq \sum_{j=1}^{m} \mu_j \zeta_j (x-y). \tag{4.8}$$

Thus, (4.6)-(4.8) yield

$$\begin{gathered}\sum_{k=1}^{p} \lambda_k^L f_k^L(x) + \sum_{k=1}^{p} \lambda_k^U f_k^U(x) + \sum_{j=1}^{m} \mu_j g_j(x) + \\ - \sum_{k=1}^{p} \lambda_k^L f_k^L(y) - \sum_{k=1}^{p} \lambda_k^U f_k^U(y) - \sum_{j=1}^{m} \mu_j g_j(y) \geqq \\ \sum_{k=1}^{p} \lambda_k^L \xi_k^L (x-y) + \sum_{k=1}^{p} \lambda_k^U \xi_k^U (x-y) + \sum_{j=1}^{m} \mu_j \zeta_j (x-y).\end{gathered} \tag{4.9}$$

By (4.2) and (4.9), it follows that the inequality

$$\left[\sum_{k=1}^{p} \lambda_k^L \xi_k^L + \sum_{k=1}^{p} \lambda_k^U \xi_k^U + \sum_{j=1}^{m} \mu_j \zeta_j\right](x-y) < 0 \tag{4.10}$$

holds, which is a contradiction to the first constraint of the interval-valued vector Wolfe dual problem (IVWD). Thus, the proof of this theorem is completed. □

If the stronger assumption of convexity is imposed on the objective functions, then the following result is true:

Theorem 4.2 (Weak duality)**.** *Let x and $(y, \lambda^L, \lambda^U, \mu)$ be feasible solutions for the problems (IVP) and (IVWD), respectively. Further, assume that the objective functions f_k^L, $k \in K$, f_k^U, $k \in K$, are strictly convex functions on $D \cup Y$, g_j, $j \in J(y)$, are convex functions on $D \cup Y$. Then,*

$$f_k(x) \nless_{LU} \vartheta_k(y, \mu) \text{ for each } k \in K.$$

Theorem 4.3 (Strong duality)**.** *Let $\overline{x}$ be a LU-Pareto solution (a weak LU-Pareto solution) of the considered interval-valued vector optimization problem (IVP) and the constraint qualification of Slater's type be satisfied at $\overline{x}$. Then there exist $\overline{\lambda}^L \in R^p$, $\overline{\lambda}^U \in R^p$ and $\overline{\mu} \in R^m$ such that $(\overline{x}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu})$ is feasible for the vector Wolfe dual problem (IVWD) with multiple interval-valued objective function and the objective functions of (IVP) and (IVWD) are equal at these points. If also all hypotheses of the weak duality theorem (Theorem 4.1 or Theorem 4.2) are satisfied, then $(\overline{x}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu})$ is a LU-efficient solution (a weakly LU-efficient solution) of maximum type for the problem (IVWD).*

Proof. By assumption, $\overline{x}$ is a LU-Pareto optimal solution of the problem (IVP) and the constraint qualification of Slater's type is satisfied at $\overline{x}$. Then, the necessary optimality conditions (2.4)-(2.6) are satisfied with Lagrange multipliers $\overline{\lambda}^L \in R^p$, $\overline{\lambda}^U \in R^p$ and $\overline{\mu} \in R^m$. Thus, the feasibility of $(\overline{x}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu})$ for the problem (IVWD) follows directly from these conditions. Hence, the objective functions in (IVP) and (IVWD) are equal at $\overline{x}$ and $(\overline{x}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu})$, respectively.

Suppose that $(\overline{x}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu})$ is not a LU-efficient solution of a maximum type for the problem (IVWD). Then, by definition, there exists $(y, \lambda^L, \lambda^U, \mu) \in W$ such that

$$\vartheta(\overline{x}, \overline{\mu}) <_{LU} \vartheta\left(y, \mu\right).$$

Thus,

$$f\left(\overline{x}\right) + \frac{1}{2}\sum_{i=1}^{m} \overline{\mu}_i g_i(\overline{x})e <_{LU} f(y) + \frac{1}{2}\sum_{i=1}^{m} \mu_i g_i(y)e.$$

Using $\overline{x} \in D$ together with the necessary optimality condition (2.6), we have that the following inequality

$$f\left(\overline{x}\right) <_{LU} f(y) + \frac{1}{2}\sum_{i=1}^{m} \mu_i g_i(y)e$$

holds, contradicting the weak duality theorem (Theorem 4.1). Hence, $(\overline{x}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu})$ is a LU-efficient solution of a maximum type for the problem (IVWD). In order to prove that $(\overline{x}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu})$ is a weakly LU-efficient solution of maximum type for the problem (IVWD), hypotheses of the weak duality theorem (Theorem 4.2) should be assumed. □

Theorem 4.4 (Converse duality)**.** *Let* $(\overline{y}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu})$ *be a weakly LU-efficient solution (a LU-efficient solution) of a maximum type in the problem (IVWD) such that* $\overline{y} \in D$*. Furthermore, assume that* f_k^L*,* $k \in K$*,* f_k^U*,* $k \in K$*, are strictly convex functions (convex functions) on* $D \cup Y$ *and* g_j*,* $j \in J\left(\overline{y}\right)$*, are convex functions on* $D \cup Y$*. Then* $\overline{y}$ *is a weak LU-Pareto solution (a LU-Pareto solution) of the problem (IVP).*

Proof. Proof of this theorem follows directly from weak duality (Theorem 4.1 or Theorem 4.2). □

A restricted version of converse duality for interval-valued vector optimization problems (IVP) and (IVWD) is presented in two next theorems.

Theorem 4.5 (Restricted converse duality)**.** *Let* $(\overline{y}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu})$ *be feasible of the vector Wolfe dual problem (IVWD) with multiple interval-objective function. Further, assume that functions* f_k^L*,* f_k^U*,* $k \in K$*,* g_j*,* $j \in J$*, are convex on* $D \cup Y$*. If there exists* $\overline{x} \in D$ *such that* $f(\overline{x}) = \vartheta\left(\overline{y}, \overline{\mu}\right)$*, then* $\overline{x}$ *is a LU-Pareto solution of the interval-valued vector optimization problem (IVP).*

Proof. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not a LU-Pareto solution of the problem (IVWP). This means, by Definition 2.6, that there exists $\widetilde{x} \in D$ such that

$$f(\widetilde{x}) <_{LU} f(\overline{x}). \tag{4.11}$$

Thus, by assumption $f(\overline{x}) = \vartheta\left(\overline{y}, \overline{\mu}\right)$, (4.11) gives

$$f(\widetilde{x}) <_{LU} \vartheta\left(\overline{y}, \overline{\mu}\right). \tag{4.12}$$

Hence, by the definition of the relation $<_{LU}$ and the definition of the objective function ϑ in (IVDW), (4.12) implies

$$\left(f^L(\widetilde{x}) < f^L(\overline{y}) + \frac{1}{2}\sum_{j=1}^{m} \overline{\mu}_j g_j(\overline{y})e \ \wedge \ f^U(\widetilde{x}) \leqq f^U(\overline{y}) + \frac{1}{2}\sum_{j=1}^{m} \overline{\mu}_j g_j(\overline{y})e \right) \tag{4.13}$$

$$\text{or} \ \left(f^L(\widetilde{x}) \leqq f^L(\overline{y}) + \frac{1}{2}\sum_{j=1}^{m} \overline{\mu}_j g_j(\overline{y})e \ \wedge \ f^U(\widetilde{x}) < f^U(\overline{y}) + \frac{1}{2}\sum_{j=1}^{m} \overline{\mu}_j g_j(\overline{y})e \right) \tag{4.14}$$

$$\text{or} \ \left(f^L(\widetilde{x}) < f^L(\overline{y}) + \frac{1}{2}\sum_{j=1}^{m} \overline{\mu}_j g_j(\overline{y})e \ \wedge \ f^U(\widetilde{x}) < f^U(\overline{y}) + \frac{1}{2}\sum_{j=1}^{m} \overline{\mu}_j g_j(\overline{y})e \right). \tag{4.15}$$

By $\left(\overline{y}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu}\right) \in W$, it follows that $\overline{\lambda}^L \geq 0$, $\lambda^L e = 1$, $\lambda^U \geq 0$, $\lambda^U e = 1$ Hence, (4.13)-(4.15) yield

$$\sum_{k=1}^{p} \overline{\lambda}_k^L f_k^L(\widetilde{x}) + \sum_{k=1}^{p} \overline{\lambda}_k^U f_k^U(\widetilde{x}) < \sum_{k=1}^{p} \overline{\lambda}_k^L f_k^L(\overline{y}) + \sum_{k=1}^{p} \overline{\lambda}_k^U f_k^U(\overline{y}) + \sum_{j=1}^{m} \overline{\mu}_j g_j(\overline{y}). \tag{4.16}$$

Using $\widetilde{x} \in D$ and $(\overline{y}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu}) \in W$, we have

$$\begin{aligned} \sum_{k=1}^{p} \overline{\lambda}_k^L f_k^L(\widetilde{x}) + \sum_{k=1}^{p} \overline{\lambda}_k^U f_k^U(\widetilde{x}) + \sum_{j=1}^{m} \overline{\mu}_j g_j(\widetilde{x}) < \\ \sum_{k=1}^{p} \overline{\lambda}_k^L f_k^L(\overline{y}) + \sum_{k=1}^{p} \overline{\lambda}_k^U f_k^U(\overline{y}) + \sum_{j=1}^{m} \overline{\mu}_j g_j(\overline{y}). \end{aligned} \tag{4.17}$$

Since f_k^L, f_k^U, $k \in K$, g_j, $j \in J$, are convex on $D \cup Y$, by (2.1), (2.2) and (2.3), the inequalities

$$f_k^L(\widetilde{x}) - f_k^L(\overline{y}) \geqq \left(\xi_k^L\right)^T (\widetilde{x} - \overline{y}), \ k \in K, \tag{4.18}$$

$$f_k^U(\widetilde{x}) - f_k^U(\overline{y}) \geqq \left(\xi_k^U\right)^T (\widetilde{x} - \overline{y}), \ k \in K, \tag{4.19}$$

$$g_j(\widetilde{x}) - g_j(\overline{y}) \geqq \zeta_j^T (\widetilde{x} - \overline{y}), \quad j \in J \tag{4.20}$$

hold for each $\xi^L \in \partial f_k^L(\overline{y})$, $\xi^U \in \partial f_k^U(\overline{y})$, $k \in K$, $\zeta_j \in \partial g_j(\overline{y})$, $j \in J$, respectively. Hence, by $(\overline{y}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu}) \in W$, (4.18)-(4.20) yield, respectively,

$$\overline{\lambda}_k^L f_k^L(\widetilde{x}) - \overline{\lambda}_k^L f_k^L(\overline{y}) \geqq \overline{\lambda}_k^L \left(\xi_k^L\right)^T (\widetilde{x} - \overline{y}), \ \ k \in K, \tag{4.21}$$

$$\overline{\lambda}_k^U f_k^U(\widetilde{x}) - \overline{\lambda}_k^U f_k^U(\overline{y}) \geqq \overline{\lambda}_k^U \left(\xi_k^U\right)^T (\widetilde{x} - \overline{y}), \ \ k \in K, \tag{4.22}$$

$$\overline{\mu}_j g_j(\widetilde{x}) - \overline{\mu}_j g_j(\overline{y}) \geqq \overline{\mu}_j \zeta_j^T (\widetilde{x} - \overline{y}), \quad j \in J. \tag{4.23}$$

Thus, (4.21)-(4.23) imply

$$\sum_{k=1}^{p} \overline{\lambda}_k^L f_k^L(\widetilde{x}) - \sum_{k=1}^{p} \overline{\lambda}_k^L f_k^L(\overline{y}) + \sum_{k=1}^{p} \overline{\lambda}_k^U f_k^U(\widetilde{x}) - \sum_{k=1}^{p} \overline{\lambda}_k^U f_k^U(\overline{y}) + \tag{4.24}$$

$$\sum_{j=1}^{m}\overline{\mu}_j g_j(\widetilde{x}) - \sum_{j=1}^{m}\overline{\mu}_j g_j(\overline{y}) \geqq \sum_{k=1}^{p}\overline{\lambda}_k^L \left(\xi_k^L\right)^T (\widetilde{x}-\overline{y}) +$$
$$\sum_{k=1}^{p}\overline{\lambda}_k^U \left(\xi_k^U\right)^T (\widetilde{x}-\overline{y}) + \sum_{j=1}^{m}\overline{\mu}_j \zeta_j^T (\widetilde{x}-\overline{y}).$$

Combining (4.17) and (4.24), we get that the inequality

$$\left[\sum_{k=1}^{p}\overline{\lambda}_k^L \left(\xi_k^L\right)^T + \sum_{k=1}^{p}\overline{\lambda}_k^U \left(\xi_k^U\right)^T + \sum_{j=1}^{m}\overline{\mu}_j \zeta_j^T\right](\widetilde{x}-\overline{y}) < 0$$

holds for each $\xi_k^L \in \partial f_k^L(\overline{y})$, $\xi_k^U \in \partial f_k^U(\overline{y})$, $k \in K$, $\zeta_j \in \partial g_j(\overline{y})$, $j \in J(\overline{y})$, contradicting the first constraint of (IVWD). This means that $\overline{x}$ is a LU-Pareto solution of the problem (IVP) and completes the proof of this theorem. □

Theorem 4.6 (Restricted converse duality)**.** *Let* $(\overline{y}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu})$ *be feasible for the interval-valued vector Wolfe dual problem (IVWD). Further, assume that the objective functions* f_k^L, f_k^U, $k \in K$, *are strictly convex on* $D \cup Y$ *and* g_j, $j \in J(\overline{y})$, *are convex on* $D \cup Y$*. If there exists* $\overline{x} \in D$ *such that* $f(\overline{x}) = \vartheta(\overline{y}, \overline{\mu})$, *then* $\overline{x}$ *is a weak LU-Pareto solution of the interval-valued vector optimization problem (IVP).*

Proof. Proof of this theorem is similar to the proof of Theorem 4.5 and, therefore, it has been omitted in the paper. □

5 Conclusion

In the paper, we have considered a convex nonsmooth multiobjective programming problem with multiple interval-valued objective function. For such interval-valued multiobjective programming problem, its vector-valued Lagrange function and a saddle point have been defined. Then, the saddle point criteria have been established for the aforesaid interval-valued multiobjective programming problem under the assumption that the functions constituting it are convex. Further, for the considered nonsmooth multiobjective programming problem with multiple interval-valued objective function, its vector Wolfe dual problem with interval-valued objective function has been defined. Several duality results between both interval-valued considered nonsmooth multiobjective programming problems have been proved also under convexity hypotheses. To the best of our knowledge, there are no saddle point criteria and Wolfe duality results in the literature for such vector optimization problems, that is, for nonsmooth multiobjective programming problems with multiple interval-valued objective functions. Therefore, the results presented in this paper are new in the area of interval-valued multiobjective programming.

References

[1] I. Ahmad, D. Singh and B.A. Dar, Optimality conditions in multiobjective programming problems with interval valued objective functions, *Control Cybernet.* 44 (2015) 19–45.

[2] T. Antczak, Optimality conditions and duality results for a class of nonsmooth multiobjective programming problems with the interval-valued objective function, *Acta Math. Sci.* 37B (2017) 1133–1150.

[3] A.K. Bhurjee and G. Panda, Efficient solution of interval optimization problem, *Math. Meth. Oper. Res.* 76 (2012) 273–288.

[4] G. R. Bitran, Linear multiple objective problems with interval coefficients, *Management Sci.* 26 (1980) 694–706.

[5] S. Chanas and D. Kuchta, Multiobjective programming in optimization of interval objective functions - a generalized approach, *European J. Oper. Res.* 94 (1996) 594–598.

[6] E. Hosseinzade and H. Hassanpour, The Karush-Kuhn-Tucker optimality conditions in interval-valued multiobjective programming problems, *J. App. Math. Inform.* 29 (2011) 1157–1165.

[7] H. Ishihuchi and M. Tanaka, Multiobjective programming in optimization of the interval objective function, *European J. Oper. Res.* 48 (1990) 219–225.

[8] M. Jana and G. Panda, Solution of nonlinear interval vector optimization problem, *Oper. Res. Int. J.* 14 (2014) 71–85.

[9] A. Jayswal, I. Stancu-Minasian and I. Ahmad, On sufficiency and duality for a class of interval-valued programming problems, *App. Math. Comput.* 218 (2011) 4119–4127.

[10] A. Jayswal, I.Ahmad and J. Banerjee, Nonsmooth interval-valued optimization and saddle-point optimality criteria, *Bull. Malays. Math. Sci. Soc.* 39 (2016) 1391–1411.

[11] C. Jiang, X. Han and G.P. Liu, A nonlinear interval number programming method for uncertain optimization problems, *European J. Oper. Res.* 188 (2008) 1–13.

[12] C. Oliveira and C.H. Antunes, Multiple objective linear programming models with interval coefficients - an illustrated overview, *European J. Oper. Res.* 181 (2007) 1434–1463.

[13] R. Osuna-Gómez, Y. Chalco-Cano, B. Hernández-Jiménez and G. Ruiz-Garzón, Optimality conditions for generalized differentiable interval-valued functions, *Inform. Sci.* 321 (2015) 136–146.

[14] R.T. Rockafellar, *Convex Analysis*, Princeton University Press (1970).

[15] J. Rohn, Duality in interval linear programming, in: *Proceedings of an International Symposiumon Interval Mathematics*, Academic Press, New York, NY, USA, 1980, pp. 521–529.

[16] Y. Sun and L. Wang, Optimality conditions and duality in nondifferentiable interval-valued programming, *J. Ind. Manag. Optim.* 9 (2013) 131–142.

[17] Y. Sun, X. Xu and L. Wang, Duality and saddle-point type optimality for interval-valued programming, *Optim. Lett.* 8 (2014) 1077–1091.

[18] H-C. Wu, The Karush-Kuhn-Tucker optimality conditions in an optimization problem with interval-valued objective function, *European J. Oper. Res.* 176 (2007) 46–59.

[19] H-C. Wu, Wolfe duality for interval-valued optimization, *J. Optim. Theory Appl.* 138 (2008) 497–509.

[20] H-C. Wu, The Karush-Kuhn-Tucker optimality conditions in multiobjective programming problems with interval-valued objective functions, *European J. Oper. Res.* 196 (2009) 49–60.

[21] H-C. Wu, Duality theory for optimization problems with interval-valued objective functions, *J. Optim. Theory Appl.* 144 (2010) 615–628.

[22] J.K. Zhang, S.Y. Liu, L.F. Li and Q.X. Feng, The KKT optimality conditions in a class of generalized convex optimization problems with an interval-valued objective function. *Optim. Lett.* 8 (2014) 607–631.

[23] H-C. Zhou and Y-J Wang, Optimality condition and mixed duality for interval-valued optimization, in: *Fuzzy Information and Engineering, Advances in Intelligent and Soft Computing*, vol. 62, Proceedings of the Third International Conference on Fuzzy Information and Engineering (ICFIE 2009), Springer, 2009, pp. 1315–1323.

Manuscript received 28 December 2015
revised 9 February 2018
accepted for publication 22 July 2019

Tadeusz Antczak
Faculty of Mathematics and Computer Science
University of Łódź, Banacha 22, 90-238 Łódź, Poland
E-mail address: tadeusz.antczak@wmii.uni.lodz.pl