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P-STRICT FEASIBILITY OF EQUILIBRIUM PROBLEMS IN REFLEXIVE BANACH SPACES

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Abstract: In this paper, P-strict feasibility of a equilibrium problem as a novel notation is introduced. Moreover, we discuss that the P-strict feasibility is a sufficient condition for guaranteeing the nonemptiness and boundedness of the solution set for the equilibrium problem with a pseudomonotone bifunction in reflexive Banach spaces. Our results generalize and extend some known results.

Key words: *P-strict feasibility, equilibrium problems, pseudomonotone*

Mathematics Subject Classification: *47H05, 49J40*

1 Introduction

It is well known that the equilibrium problem provides a unifying model for lots of mathematical problems such as variational inequalities, complementarity problems, fixed point problems etc. Because of its wide applications, the equilibrium problem has been studied intensively in recent years. For details, we refer the readers to [1–3, 9–12] and the references therein.

The issue on the nonemptiness and boundedness of the solution sets is among the most interesting and important topics in the field of optimization problems. It is found that strict feasibility is a useful condition guaranteeing the nonemptiness and boundedness of the solution sets. In [4,6–8,13–16], the authors have considered that a monotone-type variational inequality or complementarity problem has a nonempty and bounded solution set if and only if it is strictly feasible by implementing various different approaches. Recently, Hu and Fang [9] extended the concept of strict feasibility to the generalized system and proved that a monotone generalized system is solvable whenever it is strictly feasible. Later, Hu and Fang [11] further studied that under the suitable conditions, the equilibrium problem has a nonempty and bounded solution set if and only if it is strictly feasible. Very recently, Luo [14] introduced quasi-strict feasibility for the generalized mixed variational inequality and studied

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the sufficiency of quasi-strict feasibility for ensuring its solution set to be nonempty and bounded.

Motivated and inspired by the above works, in this paper we first introduce a new concept, i. e., P-strict feasibility for the equilibrium problem and verify the P-strict feasibility is equivalent to its strict feasibility if the bifunction satisfies some conditions. Then, we investigate that the P-strict feasibility is a sufficient condition for the solution set of the equilibrium problem to be nonempty and bounded if the underlying bifunction is pseudomonotone in reflexive Banach spaces. Finally, we give several examples to support our main results. In comparison to Theorem 3.1 in [11], Theorem 3.4 of this paper relaxes the condition that the bifunction is proper and weakens the monotone-type assumption.

The rest of this paper is organized as follows. In Sect. 2, we recall some basic notations and present some preliminary results. In Sect. 3, we introduce the concept of P-strict feasibility for the equilibrium problem. We discuss the P-strict feasibility is a sufficient condition guaranteeing the nonemptiness and boundedness of the solution set for the pseudomonotone equilibrium problem in reflexive Banach spaces. Finally, we conclude this paper in Sect. 4.

2 Notations and Preliminaries

Throughout this paper, let us denote X to be a reflexive Banach space with the dual space X^*, $\|x\|$ to be the norm of $x \in X$, and $\langle \phi, x\rangle$ to be the dual pair between $\phi \in X^*$ and $x \in X$. Let $K \subset X$ be a nonempty, closed, and convex subset, $\varphi : K \times K \to \bar{\mathbb{R}}$ be a bifunction, where $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. The equilibrium problem is to find $x \in K$ such that

$$\varphi(x, y) \geq 0, \quad \forall y \in K. \tag{2.1}$$

In the following, we use $\mathrm{EP}(\varphi, K)$ and $\mathrm{S}(\varphi, K)$ to denote the problem (2.1) and its solution set, respectively.

The $\mathrm{EP}(\varphi, K)$ includes the variational inequalities as its special cases: If $\varphi(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in K$, where $T : K \to X^*$ is a single-valued mapping, then (2.1) reduces to the following classical variational inequality problem $\mathrm{VI}(T, K)$: to find $x \in K$ such that

$$\langle Tx, y - x\rangle \geq 0, \quad \forall y \in K.$$

It is well known that $\mathrm{EP}(\varphi, K)$ is closely related to the following dual equilibrium problem, denoted by $\mathrm{DEP}(\varphi, K)$, which consists of finding $x \in K$ such that

$$\varphi(y, x) \leq 0, \quad \forall y \in K. \tag{2.2}$$

The symbol "$\to$" and "$\rightharpoonup$" are used to denote the strong and weak convergence, respectively. Let

$$\mathrm{barr}(K) := \{x^* \in X^* : \sup_{x \in K} \langle x^*, x\rangle < \infty\} \tag{2.3}$$

denoting the barrier cone of K. The recession cone of K is the closed and convex cone defined by

$$K_\infty := \{d \in X : \exists t_n \downarrow 0, \exists x_n \in K, t_n x_n \rightharpoonup d\}. \tag{2.4}$$

It is known that, given $x_0 \in K$,

$$K_\infty = \{d \in X : x_0 + \lambda d \in K \text{ for all } \lambda > 0\}. \tag{2.5}$$

For a nonempty set D in X, $\mathrm{int}(D)$ denotes the interior of D.

Definition 2.1 ([11])**.** Let K be a nonempty, closed, and convex subset of X with $\mathrm{int}(\mathrm{barr}\, K) \neq \emptyset$. We say that $\mathrm{EP}(\varphi, K)$ is strictly feasible if

$$\mathcal{F}_K^+ := \{x \in K : \varphi(x, x+d) > 0, \quad \forall d \in K_\infty \setminus \{0\}\} \neq \emptyset.$$

Definition 2.2. A bifunction $\varphi : K \times K \to \bar{\mathbb{R}}$ is said to be

(i) monotone on K if
$$\varphi(x, y) + \varphi(y, x) \leq 0, \quad \forall x, y \in K;$$

(ii) pseudomonotone on K if
$$\varphi(x, y) \geq 0 \text{ implies } \varphi(y, x) \leq 0, \quad \forall x, y \in K$$
or equivalently
$$\varphi(x, y) > 0 \text{ implies } \varphi(y, x) < 0, \quad \forall x, y \in K.$$

Some preliminary results are quoted below.

Lemma 2.3 ([6])**.** *Let K be a nonempty, closed, and convex subset in X with* $\mathrm{int}(\mathrm{barr}\, K) \neq \emptyset$*, then there does not exist $\{x_n\} \subset K$ with each $\|x_n\| \to \infty$ such that $\frac{x_n}{\|x_n\|} \rightharpoonup 0$. If additionally K is a cone, then there does not exist $\{d_n\} \subset K$ with each $\|d_n\| = 1$ such that $d_n \rightharpoonup 0$.*

Lemma 2.4. *Let X be a reflexive Banach space with the dual space X^*, K be a nonempty, closed, and convex subset in X and $\varphi : K \times K \to \bar{\mathbb{R}}$ be a bifunction satisfying the following conditions:*

(i) *φ is pseudomonotone;*

(ii) *for every $x, y \in K$ and $t \in [0, 1]$,*
$$\lim_{t \to 0^+} \frac{\varphi(x + t(y - x), x)}{t} = -\varphi(x, y).$$

Then $x \in K$ is a solution of $\mathrm{EP}(\varphi, K)$ *if and only if it is a solution of* $\mathrm{DEP}(\varphi, K)$.

Proof. Let $x^* \in K$ solves $\mathrm{EP}(\varphi, K)$. Then
$$\varphi(x^*, y) \geq 0, \quad \forall y \in K.$$
Since φ is pseudomonotone,
$$\varphi(y, x^*) \leq 0, \quad \forall y \in K.$$
Thus, x^* also solves $\mathrm{DEP}(\varphi, K)$.

Conversely, let $x^* \in K$ such that
$$\varphi(y, x^*) \leq 0, \quad \forall y \in K.$$
For any $z \in K$, we take $z_t = x^* + t(z - x^*) \in K$ for $t \in (0, 1)$. It follows that
$$\varphi(z_t, x^*) \leq 0, \quad \forall z \in K.$$

By (ii), we have

$$\lim_{t\to 0^+}\frac{\varphi(z_t, x^*)}{t} = -\varphi(x^*, z), \quad \forall z \in K,$$

and so

$$\varphi(x^*, z) \geq 0, \quad \forall z \in K.$$

Hence, x^* also solves $\mathrm{EP}(\varphi, K)$. This completes the proof. □

Lemma 2.5 ([5])**.** *Let K be a nonempty and convex subset of a Hausdorff topological vector space E and $G : K \to 2^E$ be a set-valued mapping from K into E satisfying the following properties:*

(i) *G is a KKM mapping for every finite subset A of K, $\mathrm{co}(A) \subset \bigcup_{x\in A} G(x)$;*

(ii) *$G(x)$ is closed in E for every $x \in K$;*

(iii) *$G(x_0)$ is compact in E for some $x_0 \in K$.*

Then $\bigcap_{x\in K} G(x) \neq \emptyset$.

3 Main Results

In this section, we first introduce a new notation, i. e., P-strict feasibility for $\mathrm{EP}(\varphi, K)$ in reflexive Banach spaces. Then, we shall establish the equivalence between P-strict feasibility and its strict feasibility whenever the bifunction satisfies some conditions.

Definition 3.1. Let K be a nonempty, closed, and convex subset of X and $\varphi : K \times K \to \bar{\mathbb{R}}$ be a bifunction. The $\mathrm{EP}(\varphi, K)$ is called P-strictly feasible iff,

$$\mathcal{P}_K := \{x \in K : \varphi(x, y) > -\infty, \quad \forall y \in K\} \neq \emptyset.$$

Theorem 3.2. *Let X be a reflexive Banach space with the dual space X^*, K be a nonempty, closed, and convex subset in X with $\mathrm{int}(\mathrm{barr}\, K) \neq \emptyset$, and $\varphi : K \times K \to \bar{\mathbb{R}}$ be a bifunction satisfying the following conditions:*

(i) *$\varphi(x, x) = 0$ for all $x \in K$;*

(ii) *for every $x, y \in K$ and $t \in [0, 1]$,*

$$\lim_{t\to 0^+}\frac{\varphi(x + t(y - x), x)}{t} = -\varphi(x, y);$$

(iii) *for every $x \in K$, $\varphi(x, \cdot)$ is lower semicontinuous.*

Then the following statements are equivalent:

(i) *$\mathrm{EP}(\varphi, K)$ is strictly feasible;*

(ii) *$\mathrm{EP}(\varphi, K)$ is P-strictly feasible.*

Proof. The "only if" part: If the conclusion does not hold, then for all $x \in K$, there exists $d_x \in K_\infty \setminus \{0\}$ such that $y_x = x + d_x \in K$ satisfying

$$\varphi(x, x + d_x) = \varphi(x, y_x) = -\infty. \tag{3.1}$$

Since EP(φ, K) is strictly feasible, let $x_0 \in \mathcal{F}_K^+$. Thus, $\varphi(x_0, x_0 + d_x) > 0$, which is a contradiction with (3.1) as $x_0 \in K$.

The "if" part: Suppose that EP(φ, K) is P-strictly feasible. Now we claim that for all $\lambda > 0$,

$$\{x \in K : \varphi(x, x + \lambda d) > 0, \quad \forall d \in K_\infty \setminus \{0\}\} \neq \emptyset. \tag{3.2}$$

If the claim (3.2) does not hold, then there exists a sequence $\{d_n\} \subset K_\infty \setminus \{0\}$ such that

$$\varphi(x, x + \lambda d_n) \leq 0, \quad \forall x \in K.$$

Without loss of generality, we may assume that $\|d_n\| = 1$ for each n. Hence, $d_n \rightharpoonup d_0$ as $n \to \infty$. Since K_∞ is a closed and convex cone, it is weakly closed and so $d_0 \in K_\infty$. By Lemma 2.3, we have $d_0 \neq 0$.
Combining with $d_n \rightharpoonup d_0$ and the lower semicontinuity of $\varphi(x, \cdot)$, it follows that

$$\varphi(x, x + \lambda d_0) \leq 0. \tag{3.3}$$

Since EP(φ, K) is P-strictly feasible, let $x_0 \in \mathcal{P}_K$. Now we claim that for any $\lambda > 0$,

$$\varphi(x_0, x_0 + \lambda d_0) < 0. \tag{3.4}$$

In fact, if not, then $\varphi(x_0, x_0 + \lambda d_0) = 0$ as (3.3) holds. The assumption (i) implies that $d_0 = 0$ because $\lambda > 0$ is arbitrary. It leads to a contradiction, so the claim (3.4) is verified.

Since $x_0 \in K$, for any $y \in K$, we have $x_t = x_0 + t(y - x_0) \in K$ for all $t \in [0, 1]$. It follows from the assumption (ii) that

$$\lim_{t \to 0^+} \frac{\varphi(x_t, x_0)}{t} = -\varphi(x_0, y) < +\infty.$$

Thus, for small enough $t > 0$, $\frac{1}{t}\varphi(x_t, x_0) < +\infty$. It turns out that

$$\varphi(x_t, x_0) \leq 0. \tag{3.5}$$

By y is arbitrary, we can take $y = x_0 + d_0 \in K$ in (3.5). It yields that for any small enough $t > 0$,

$$\varphi(x_0 + t d_0, x_0) \leq 0. \tag{3.6}$$

The inequality (3.4) and the assumption (ii) implies that

$$\begin{aligned} & \lim_{t \to 0^+} \frac{\varphi(x_0 + t^2 d_0, x_0)}{t} \\ = \; & \lim_{t \to 0^+} \frac{\varphi(x_0 + t(x_0 + t d_0 - x_0), x_0)}{t} \\ = \; & -\varphi(x_0, x_0 + t d_0) \\ > \; & 0. \end{aligned} \tag{3.7}$$

Setting $k = t^2$. Since $0 \leq t \leq 1$, $0 \leq k \leq t \leq 1$. Hence, (3.7) shows that for sufficient small $k > 0$, we have $\varphi(x_0 + k d_0, x_0) > 0$. It leads to a contradiction with (3.6). So the claim (3.2) is proved. Hence, EP(φ, K) is strict feasibility. This completes the proof. □

In the following, we shall discuss that the P-strict feasibility is a sufficient condition for $\mathrm{EP}(\varphi, K)$ to have a nonempty and bounded solution set whenever $\varphi(\cdot,\cdot)$ enjoys the pseudomonotonicity assumption in reflexive Banach spaces. In order to obtain the result, we first give the following Theorem 3.3.

Theorem 3.3. *Let X be a reflexive Banach space with the dual space X^*, K be a nonempty, closed, and convex subset in X with $\mathrm{int}(\mathrm{barr}\, K) \neq \emptyset$ and $\varphi : K \times K \to \bar{\mathbb{R}}$ be a bifunction satisfying the following conditions:*

(i) *φ is pseudomonotone and $\varphi(x, x) = 0$ for all $x \in K$;*

(ii) *for every $x \in K$, $\varphi(x, \cdot)$ is quasiconvex and lower semicontinuous.*

If $\mathcal{F}_K^+ \neq \emptyset$, then the solution set of $\mathrm{DEP}(\varphi, K)$ is nonempty and bounded.

Proof. First, we claim that there exists a bounded set $C \subset K$, such that for every $x \in K \setminus C$, there exists some $y \in C$ satisfying

$$\varphi(y, x) > 0. \tag{3.8}$$

If the above claim (3.8) does not hold, then there exists a sequence $\{x_n\} \subset K$, such that for each n, $\|x_n\| \geq n$ and $\varphi(y, x_n) \leq 0$ for every $y \in K$ with $\|y\| \leq n$. Without loss of generality, we may assume that $d_n = \frac{x_n}{\|x_n\|} \rightharpoonup d_0$ as $n \to \infty$. Then $d_0 \in K_\infty$ by the definition of the recession cone. Since $\mathrm{int}(\mathrm{barr}\, K) \neq \emptyset$, $d_0 \neq 0$ by Lemma 2.3.

For all $\|y\| < n$, the assumption (i) and (ii) imply that

$$\begin{aligned} & \varphi(y, y + d_0) \\ \leq\ & \liminf_{n\to\infty} \varphi\left(y, y + \frac{1}{\|x_n\|}(x_n - y)\right) \\ \leq\ & 0 \end{aligned}$$

and so $\mathcal{F}_K^+ = \emptyset$, which is a contradiction. Hence, the claim (3.8) is proved.

Let $G : K \to 2^K$ be a set-valued mapping defined by

$$G(y) := \{x \in K : \varphi(y, x) \leq 0\}, \quad \forall y \in K.$$

For any $x_n \in G(y)$ with $x_n \to x_0$, one has

$$\varphi(y, x_n) \leq 0.$$

It follows from the lower semicontinuity of $\varphi(y, \cdot)$ that

$$\varphi(y, x_0) \leq \liminf_{n\to\infty} \varphi(y, x_n) \leq 0.$$

This shows that $x_0 \in G(y)$ and so $G(y)$ is closed. Next we need to prove that G is a KKM mapping from K to K. If not, then there exist $t_1, t_2, ..., t_n \in [0, 1]$, $y_1, y_2, ..., y_n \in K$ and $\widetilde{y} = t_1 y_1 + t_2 y_2 + ... + t_n y_n \in \mathrm{co}\{y_1, y_2, ..., y_n\}$ such that $\widetilde{y} \notin \cup G_{i\in\{1,2,...,n\}}(y_i)$. Then

$$\varphi(y_i, \widetilde{y}) > 0, \quad \forall i = 1, 2, \ldots, n.$$

By φ is pseudomonotone, we have

$$\varphi(\widetilde{y}, y_i) < 0, \quad \forall i = 1, 2, \ldots, n.$$

From $\varphi(\widetilde{y}, \cdot)$ is quasiconvex, it turns out that

$$0 = \varphi(\widetilde{y}, \widetilde{y}) < 0,$$

which is a contradiction. Thus, we know that G is a KKM mapping.

We may assume that C is a bounded, closed, and convex set (otherwise, consider the closed and convex hull of C instead of C). Let $\{y_1, \ldots, y_m\}$ be finite number of points in K and let $M := \mathrm{co}(C \cup \{y_1, \ldots, y_m\})$. Then the reflexivity of the space X yields that M is weakly compact convex. Consider the set-valued mapping G' defined by $G'(y) := G(y) \cap M$ for every $y \in M$. Then each $G'(y)$ is a weakly compact convex subset of M and G' is a KKM mapping. We claim that

$$\emptyset \neq \bigcap_{y \in M} G'(y) \subset C. \tag{3.9}$$

Indeed, by Lemma 2.5, the intersection in (3.9) is nonempty. Moreover, if there exists some $x_0 \in \bigcap_{y\in M} G'(y)$ but $x_0 \notin C$, then by the claim (3.8), we have $\varphi(y, x_0) > 0$ for some $y \in C$. Thus, $x_0 \notin G(y)$ and so $x_0 \notin G'(y)$, a contradiction to the choice of x_0.

Let $z \in \bigcap_{y\in M} G'(y)$. Then $z \in C$ by (3.9), hence $z \in \bigcap_{i=1}^{m}(G(y_i) \cap C)$. Thus the collection $\{G(y) \cap C : y \in K\}$ has finite intersection property. Since for each $y \in K$, $G(y) \cap C$ is weakly compact, it follows that $\bigcap_{y\in K}(G(y) \cap C)$ is nonempty which coincides with the solution set of $\mathrm{DEP}(\varphi, K)$. This completes the proof. □

In the following Theorem 3.4, we shall discuss the P-strict feasibility is a sufficient condition for ensuring the nonemptiness and boundedness of the solution set of the $\mathrm{EP}(\varphi, K)$ with a pseudomonotone bifunction in reflexive Banach spaces.

Theorem 3.4. *Let X be a reflexive Banach space with the dual space X^*, K be a nonempty, closed, and convex subset in X with $\mathrm{int}(\mathrm{barr}\, K) \neq \emptyset$ and $\varphi : K \times K \to \bar{\mathbb{R}}$ be a bifunction satisfying the following conditions:*

(i) *φ is pseudomonotone and $\varphi(x, x) = 0$ for all $x \in K$;*

(ii) *for every $x, y \in K$ and $t \in [0, 1]$,*

$$\lim_{t\to 0^+} \frac{\varphi(x + t(y - x), x)}{t} = -\varphi(x, y);$$

(iii) *for every $x \in K$, $\varphi(x, \cdot)$ is quasiconvex and lower semicontinuous.*

If $\mathrm{EP}(\varphi, K)$ is P-strictly feasible, then the solution set of $\mathrm{EP}(\varphi, K)$ is nonempty and bounded.

Proof By Theorem 3.2, Theorem 3.3 and Lemma 2.4, we obtain that $\mathrm{S}(h, K)$ is nonempty and bounded. This completes the proof. □

Remark 3.5. Theorem 3.4 presents the P-strict feasibility is a sufficient condition for $\mathrm{EP}(\varphi, K)$ to have a nonempty and bounded solution set in reflexive Banach spaces. In comparison to Theorem 3.1 of [11], Theorem 3.4 weakens the monotone-type assumption and relaxes the condition to the case that $\varphi : K \times K \to \bar{\mathbb{R}}$.

Corollary 3.6. *Let X be a reflexive Banach space with the dual space X^*, K be a nonempty, closed, and convex subset in X with $\mathrm{int}(\mathrm{barr}\, K) \neq \emptyset$ and $\varphi : K \times K \to \mathbb{R} \cup \{+\infty\}$ be a bifunction satisfying the following conditions:*

(i) *φ is pseudomonotone and $\varphi(x,x)=0$ for all $x\in K$;*

(ii) *for every $x,y\in K$ and $t\in[0,1]$,*

$$\lim_{t\to 0^+}\frac{\varphi(x+t(y-x),x)}{t}=-\varphi(x,y);$$

(iii) *for every $x\in K$, $\varphi(x,\cdot)$ is quasiconvex and lower semicontinuous.*

Then the solution set of $\mathrm{EP}(\varphi,K)$ *is nonempty and bounded.*

Proof. Since $\varphi(\cdot,\cdot)$ is proper, $\mathrm{EP}(\varphi,K)$ is P-strictly feasible. By Theorem 3.4, $\mathrm{S}(\varphi,K)$ is nonempty and bounded. This completes the proof. □

Remark 3.7. In fact, the assumption that "$\varphi: K\times K\to\mathbb{R}\cup\{+\infty\}$" in Corollary 3.6 is stronger than the condition that "$\mathcal{P}_K\neq\emptyset$". Naturally, we can know that even if there is no any strict feasibility condition, the nonemptiness and boundedness of $\mathrm{S}(\varphi,K)$ still can be guaranteed.

The following examples are to show that all the assumptions of Theorem 3.4 is essential.

Example 3.8. Let $X=\ell^2$ and so $X^*=\ell^2$. The norm and the inner product in ℓ^2 are defined by

$$\|x\|_2=(\sum_{n=1}^{\infty}|x_n|^2)^{\frac{1}{2}},\quad \forall x\in\ell^2,$$

and

$$\langle x,y\rangle=\sum_{n=1}^{\infty}x_ny_n,\quad \forall x,y\in\ell^2.$$

Let $K=\{x=(x_1,x_2,...,x_n,...)\in\ell^2:|x_n|\le n,\forall n\in\mathbb{N}\}$. Define $\varphi: K\times K\to\bar{\mathbb{R}}$ by

$$\varphi(x,y):=-\langle x,y-x\rangle\|x\|_2^2,\quad \forall x,y\in K.$$

It is easy to see that $\varphi(x,x)=0$ and the lower semicontinuity of $\varphi(x,\cdot)$ hold. Now we shall show that φ is not pseudomonotone on K. Take $x=(1,1,\dots,0,1,\dots)$ and $y=(0,0,\dots,1,0,\dots)$, we have

$$\begin{aligned}&\varphi(x,y)\\ =\;&-\langle(1,1,\dots,0,1,\dots),(-1,-1,\dots,1,-1,\dots)\rangle\|(1,1,\dots,0,1,\dots)\|_2^2\\ >\;&0,\end{aligned}$$

however,

$$\begin{aligned}&\varphi(y,x)\\ =\;&-\langle(0,0,\dots,1,0,\dots),(1,1,\dots,-1,1\dots)\rangle\|(0,0,\dots,1,0,\dots)\|_2^2\\ >\;&0.\end{aligned}$$

This implies that φ is not pseudomotone.

Then, we will prove the condition (ii) of Theorem 3.4 is satisfied. For all $t\in[0,1]$,

$$\lim_{t\to 0^+}\frac{\varphi(x+t(y-x),x)}{t}$$

$$
\begin{aligned}
&= \lim_{t\to 0^+} \frac{-\langle x+t(y-x), t(x-y)\rangle \|x+t(y-x)\|_2^2}{t}\\
&= \langle x, y-x\rangle \|x\|_2^2\\
&= -\varphi(x,y).
\end{aligned}
$$

For any $y_1, y_2 \in K$ and any $k \in [0,1]$, we have

$$
\begin{aligned}
&\varphi(x, ky_1+(1-k)y_2)\\
=\;& -\langle x, ky_1+(1-k)y_2-x\rangle \|x\|_2^2\\
=\;& -k\langle x, y_1-x\rangle \|x\|_2^2 - (1-k)\langle x, y_2-x\rangle \|x\|_2^2\\
=\;& k\varphi(x,y_1)+(1-k)\varphi(x,y_2),
\end{aligned}
$$

which implies that $\varphi(x,\cdot)$ is convex and so quasiconvex. It is easy to see that $\mathrm{EP}(\varphi, K)$ is P-strictly feasible. Indeed, for any $y \in K$, we have

$$\varphi(0,y) = 0 > -\infty,$$

which implies that $\mathcal{P}_K \neq \emptyset$. However, by the simple deduction, it yields that the solution set is the unbounded solution set K.

Remark 3.9. From Example 3.8, we can know that the pseudomonotonicity assumption in Theorem 3.4 is important.

Example 3.10. Let $X = \ell^p$ $(1 < p < +\infty)$ and so $X^* = \ell^q$ $(\frac{1}{p}+\frac{1}{q}=1)$. Let $K = \{x = (x_1, x_2, ..., x_n, ...) \in \ell^p : x_n \geq 0, \forall n \in \mathbb{N}\}$. Define $\varphi : K \times K \to \bar{\mathbb{R}}$ by

$$\varphi(x,y) := \|y-x\|_p, \quad \forall x, y \in K,$$

where $\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$. It is clear that $\varphi(x,x) = 0$ and the lower semicontinuity of $\varphi(x,\cdot)$ hold. Now we shall prove that φ is not pseudomonotone. Indeed,

$$\varphi(x,y) = \|y-x\|_p > 0,$$

however,

$$\varphi(y,x) = \|x-y\|_p > 0.$$

For all $t \in [0,1]$, we have

$$
\begin{aligned}
&\lim_{t\to 0^+} \frac{\varphi(x+t(y-x), x)}{t}\\
=\;& \lim_{t\to 0^+} \frac{t\|y-x\|_p}{t}\\
=\;& \|y-x\|_p\\
\neq\;& -\varphi(x,y),
\end{aligned}
$$

which shows that the assumption (ii) is also not satisfied.

For any $y_1, y_2 \in K$ and any $k \in [0,1]$, we have

$$
\begin{aligned}
&\varphi(x, ky_1+(1-k)y_2)\\
=\;& \|ky_1+(1-k)y_2-x\|_p
\end{aligned}
$$

$$\begin{aligned} &\leq k\|x-y_1\|_p+(1-k)\|x-y_2\|_p \\ &= k\varphi(x,y_1)+(1-k)\varphi(x,y_2), \end{aligned}$$

which implies that $\varphi(x,\cdot)$ is convex and so quasiconvex.

Next we will verify that $\mathrm{EP}(\varphi, K)$ is P-strictly feasible. Indeed, let $x \in K$, for any $y \in K$,

$$\varphi(x,y)=\|y-x\|_p \geq 0 > -\infty,$$

which implies that $\mathcal{P}_K \neq \emptyset$. Obviously, the solution set of $\mathrm{EP}(\varphi, K)$ is the unbounded set K.

Remark 3.11. Example 3.10 turns out that the pseudomonotonicity and the assumption (ii) of Theorem 3.4 is essential.

The following examples show that if φ is pseudomonotone, the P-strict feasibility is a sufficient condition for $\mathrm{S}(\varphi, K)$ to be nonempty and bounded.

Now let us consider another example in a Hilbert space ℓ^2.

Example 3.12. Let $X = \ell^2$ and so $X^* = \ell^2$. Let $K = \{x = (x_1, x_2, ..., x_n, ...) \in \ell^2 : |x_n| \leq n, \forall n \in \mathbb{N}\}$. Define $\varphi : K \times K \to \bar{\mathbb{R}}$ by

$$\varphi(x,y) := \langle x, y-x\rangle, \quad \forall x,y \in K.$$

It is easy to see that $\varphi(x,x) = 0$ and the lower semicontinuity of φ hold. Now we shall show that φ is pseudomonotone. Indeed,

$$\langle x, y-x\rangle \geq 0.$$

Then $\|y-x\|_2^2 = \langle y-x, y-x\rangle \geq 0$ implies that $\langle y, y-x\rangle \geq \langle x, y-x\rangle$ and so $\langle y, y-x\rangle \geq 0$. Thus, we have

$$\langle y, x-y\rangle \leq 0.$$

Then, we will prove the condition (ii) is satisfied. For all $t \in [0,1]$,

$$\begin{aligned} & \lim_{t\to 0^+} \frac{\varphi(x+t(y-x),x)}{t} \\ = & \lim_{t\to 0^+} \frac{\langle x+t(y-x), t(x-y)\rangle}{t} \\ = & \lim_{t\to 0^+} \langle x+t(y-x), x-y\rangle \\ = & -\varphi(x,y). \end{aligned}$$

For any $y_1, y_2 \in K$ and any $k \in [0,1]$, we have

$$\begin{aligned} & \varphi(x, ky_1+(1-k)y_2) \\ = & \langle x, ky_1+(1-k)y_2-x\rangle \\ = & \langle x, k(y_1-x)\rangle + \langle x, (1-k)(y_2-x)\rangle \\ = & k\varphi(x,y_1)+(1-k)\varphi(x,y_2), \end{aligned}$$

which implies that $\varphi(x,\cdot)$ is convex and so quasiconvex. Next we will verify that $\mathrm{EP}(\varphi, K)$ is quasi-strictly feasible. Indeed, for any $y \in K$, we have

$$\varphi(e_1,y) = \langle e_1, y-e_1\rangle = y_1 - 1 \geq -2 > -\infty,$$

which implies that $\mathcal{E}_K^+ \neq \emptyset$. Moreover, by a simple deduction, it yields that the solution set is the bounded single-point set $\{0\}$.

4 Conclusions

We first presented the P-strict feasibility for $\mathrm{EP}(\varphi, K)$ is equivalent to its strict feasibility if the bifunction satisfies some conditions. We studied that the P-strict feasibility is a sufficient condition for $\mathrm{S}(\varphi, K)$ to be nonempty and bounded if the bifunction φ enjoys pseudomonotonicity assumption in reflexive Banach spaces. We found that the P-strict feasibility is weaker than the assumption that $\varphi(\cdot,\cdot)$ is proper, so the stronger condition of Theorem 3.1 in [11] can be realxed. Further research works should be carried out to study the various strict feasibility for some other kinds of optimization problems.

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