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EXISTENCE RESULTS FOR GENERALIZED VARIATIONAL-LIKE INEQUALITIES*

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Abstract: In this paper, we investigate the existence of solutions for generalized variational-like inequalities (for short GVLI). In reflexive Banach spaces, when the constraint set is bounded, we deduce an existence theorem for GVLI. In finite dimensional spaces, employing this result, when the constraint set is unbounded, we obtain an existence theorem for GVLI, where the mapping and the constraint set are simultaneously perturbed, which generalizes some recent results in this topic. Moreover, we show that there is a solution of GVLI in K_r under the same coercivity condition.

Key words: *variational-like inequality, perturbation, coercivity condition, existence*

Mathematics Subject Classification: *90C30, 90C33, 90C31*

1 Introduction

It is well known that the variational inequality theory, introduced in the early 1960s by Stampacchia, is a powerful tool for modelling a large variety of problems arising in elasticity, structural analysis, economics, decision process, optimization, physics, engineering science and so on (see, for example, [7, 9, 15, 23, 24, 27] and the references therein). A useful and important generalization of variational inequalities is the variational-like inequalities, which have been studied extensively by many authors in the literature; for instance, we refer the reader to [2, 4, 6, 8, 13, 14, 16, 18–20, 25, 26, 30–32].

Let $\mathbb{B}$ be a real Banach space with dual space $\mathbb{B}^*$. The value of the functional $f \in \mathbb{B}^*$ at the point $x \in \mathbb{B}$ is denoted by $\langle f, x\rangle$. Assume that K is a nonempty, closed and convex subset in $\mathbb{B}$. Let $T : K \rightrightarrows \mathbb{B}^*$ be a set-valued mapping and $\eta : K \times K \to \mathbb{B}$ be a nonlinear mapping. The generalized variational-like inequality associated with the mappings T and η, and the constraint set K, denoted by GVLI(T, η, K), consists of finding $x \in K$ such that

$$\exists x^* \in T(x) : \langle x^*, \eta(y, x)\rangle \geq 0, \quad \forall y \in K. \tag{1.1}$$

We denote the set of solutions of GVLI(T, η, K) by SOL(T, η, K). It is closely related to the weak generalized variational-like inequality, denoted by GVLI$_w(T, \eta, K)$, which consists of finding $x \in K$ such that

$$\forall y \in K, \exists x^* \in T(x) : \quad \langle x^*, \eta(y, x)\rangle \geq 0. \tag{1.2}$$

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The set of solutions of $\mathrm{GVLI}_w(T,\eta,K)$ is denoted by $\mathrm{SOL}_w(T,\eta,K)$. If T is single-valued, then $\mathrm{GVLI}_w(T,\eta,K)$ coincides with $\mathrm{GVLI}(T,\eta,K)$, which was introduced by Parida et al [26] in Euclidean spaces $\mathbb{R}^n$.

For convenience, $\mathrm{GVLI}(T,\eta,K)$ (resp. $\mathrm{GVLI}_w(T,\eta,K)$) in the case when $\eta(y,x)=y-x$ is denoted by $\mathrm{GVI}(T,K)$ (resp. $\mathrm{GVI}_w(T,K)$), which has been extensively studied, see for example [12, 22, 28].

Existence of solutions is one of basic problems in the theory of variational inequalities. Parida et al. [26] obtained some existence results of solutions for variational-like inequalities involving a single-valued mapping in Euclidean space $\mathbb{R}^n$. In most of cases it is difficult to model exactly a natural or social phenomenon because of existence of various of perturbation factors. Therefore, it is interesting to study the perturbation behavior of variational inequalities [5]. Recently, Li and He [22] introduced a perturbation way for $\mathrm{GVI}(T,K)$ in Euclidean spaces, that is, the mapping T is perturbed by a nonlinear continuous mapping, which is not necessarily monotone. Under suitable conditions, the authors obtained an existence result of solutions for $\mathrm{GVI}(T,K)$.

In this paper, following the idea of Li and He [22], a new perturbation way for $\mathrm{GVLI}(T,\eta,K)$ is introduced and studied. In reflexive Banach spaces, when the constraint set K is weakly compact, we deduce an existence theorem for $\mathrm{GVLI}(T,\eta,K)$. In finite dimensional spces, using this result, when the constraint set K is unbounded, we obtain an existence theorem for $\mathrm{GVLI}(T,\eta,K)$, where the mapping T and the constraint set K are simultaneously perturbed. Moreover, we show that there is a solution of $\mathrm{GVLI}(T,\eta,K)$ in K_r.

2 Preliminaries

Let $\mathbb{B}$ be a real Banach space with dual space $\mathbb{B}^*$, K be a nonempty, closed and convex subset in $\mathbb{B}$. The symbols $\to$ and $\rightharpoonup$ denote the strong convergence and weak convergence, respectively. For $r>0$, denote $K_r := \{x\in K : \|x\|\le r\}$. Let $\mathbf{B}(\theta,r) := \{x\in\mathbb{B} : \|x\|<r\}$ and $\bar{\mathbf{B}}(\theta,r) := \{x\in\mathbb{B} : \|x\|\le r\}$. Assume that X and Y are two Hausdorff topological spaces and $T : X \rightrightarrows Y$ is a set-valued mapping with nonempty values.

Definition 2.1. A set-valued mapping T is said to be upper continuous if, for any $x_0\in X$ and for each open set U containing $T(x_0)$, there is a neighborhood V of x_0 such that $T(x)\subseteq U$ for all $x\in V$.

Lemma 2.2 ([21, Lemma 2.1]). *If T is compact-valued, then T is upper semicontinuous if and only if for every net $\{x_i\}\subset X$ such that $x_i\to x_0\in X$ and for every $z_i\in T(x_i)$, there exist $z_0\in T(x_0)$ and a subnet $\{z_{i_j}\}$ of $\{z_i\}$ such that $z_{i_j}\to z_0$. If X and Y are metric spaces, instead of nets one consider sequences.*

In the sequel, we recall the upper semicontinuity of the marginal function associated with a set-valued mapping on topological spaces due to Aliprants and Border (see Lemma 17.30 of [1]). The same result on metric spaces can be found in the monograph of Aubin and Frankowska (see Definition 1.4.15 and item (ii) of Theorem 1.4.16 of [3]).

Definition 2.3. Let X and Y be two topological spaces. Given a set-valued mapping $T : X \rightrightarrows Y$ and a function $f : \mathrm{Graph}(T)\to\mathbb{R}$, the function $g : X\to\mathbb{R}\cup\{+\infty\}$ defined by

$$g(x) := \sup_{y\in T(x)} f(x,y)$$

is called the marginal function associated with T and f.

Lemma 2.4. *Let X and Y be two topological spaces. Let $T : X \rightrightarrows Y$ be a set-valued mapping with nonempty compact values, and $f : Graph(T) \to \mathbb{R}$ be a function. If f and T are both upper semicontinuous, then the marginal function associated with T and f is also upper semicontinuous.*

Lemma 2.5 ([21, Lemma 2.2]). *If $P \subset Q \subset X$, where Q is weakly compact and P is weakly sequently closed, then P is weakly compact.*

Definition 2.6. Let X be a Hausdorff topological real linear space and $M \subset X$. A set-valued mapping $G : M \rightrightarrows X$ is called to be a KKM mapping if, for every finite number of elements $x_1, x_2, \dots, x_n \in M$ one has $\text{co}\{x_1, x_2, \dots, x_n\} \subset \cup_{i=1}^{n} G(x_i)$.

The following two lemmas are due to Ky Fan, which will be used in the next section.

Lemma 2.7 ((Fan-KKM lemma) [11, Lemma 1]). *Let X be a Hausdorff topological real linear space, let $M \subset X$ be a nonempty set, and let $G : M \rightrightarrows X$ be a KKM mapping. If $G(x)$ is closed for every $x \in M$, and there exists $x_0 \in M$ such that $G(x_0)$ is compact, then $\cap_{x \in M} G(x) \neq \emptyset$.*

Lemma 2.8 ([10, Theorem 2]). *Let X be a compact space, Y a set, and $h : X \times Y \to \mathbb{R}$ a function that is concave on Y, convex on X, and for each $y \in Y$, the function $x \mapsto h(x, y)$ is lower semicontinuous on X. Then*

$$\sup_{y \in Y} \min_{x \in X} h(x, y) = \min_{x \in X} \sup_{y \in Y} h(x, y).$$

3 Existence Theorems for GVLI(T, η, K) in Banach Spaces

In this section, when the constraint set K is weakly compact, we deduce an existence theorem for GVLI(T, η, K), which is used in proving the main theorem in the next section. In the sequel, we may use the following assumptions:

- (H_K) Let K be a nonempty, bounded, closed and convex subset of a reflexive Banach space $\mathbb{B}$.
- (H_T) Let $T : K \rightrightarrows \mathbb{B}^*$ be a weak to norm upper semicontinuous mapping with nonempty, compact and convex values.
- (H_η) Let $\eta : K \times K \to \mathbb{B}$ be such that
  - (i) $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in K$;
  - (ii) for all $x, y \in K$ and all $z^* \in \mathbb{B}^*$, the mapping $y \mapsto \langle z^*, \eta(y, x)\rangle$ is convex;
  - (iii) for all $y \in K$, the mapping $x \mapsto \eta(y, x)$ is continuous from the weak topology to weak topology.

Remark 3.1. These assumptions were extensively used in the literature. For the reader's convenience, we list some references as follows:

- (i) Condition (H_T) was used in Theorem 3.4 of László [21];

(ii) Item (i) of condition (H_η) is the same as item (2) of Assumption 2.1 of Huang and Deng [18] (see also condition (H_η^2) of Costea et al. [8], the condition of Theorem 2.2 of Fang and Huang [13], the condition of Theorem 3.2 of Bai et al. [4]). From item (i) of (H_η), it is easy to see that $\eta(x,x)=0$ for each $x\in K$. Item (ii) of condition (H_η) is similar to item (3) of Assumption 2.1 of Huang and Deng [18] and the corresponding item in references mentioned above. Item (iii) of (H_η) is the same as item (iv) of Assumption 1.1 of Zeng et al. [31]. A similar version of condition (H_η) was been used in Corollary 1 of Ceng and Yao [6].

The proof of the following theorem follows the idea of Theorems 3.1 and 3.4 of László [21].

Theorem 3.2. *Assume that conditions (H_K), (H_T) and (H_η) hold. Then GVLI(T,η,K) admits a solution.*

Proof. Define a set-valued mapping $G: K \rightrightarrows K$ as follows:

$$G(y) := \{x\in K : \exists x^* \in T(x) \text{ such that } \langle x^*, \eta(y,x)\rangle \geq 0\}.$$

We first show that G is a KKM mapping. In fact, if not, then there is $\{y_1, y_2, \ldots, y_n\}\subset K$ such that $\mathrm{co}\{y_1,y_2,\ldots,y_n\} \not\subset \cup_{i=1}^n G(y_i)$, i.e., there exists $z=\sum_{i=1}^n \lambda_i y_i \in \mathrm{co}\{y_1,y_2,\ldots,y_n\}$ with $\lambda_i\geq 0$ for each i and $\sum_{i=1}^n\lambda_i = 1$ such that $z\notin \cup_{i=1}^n G(y_i)$. Thus, for every $i=1,2,\ldots,n$, we have $z\notin G(y_i)$ and so

$$\langle z^*, \eta(y_i,z)\rangle < 0, \quad \forall z^*\in T(z). \tag{3.1}$$

For any fixed $z^*\in T(z)$, by summing the multiplication of (3.1) by a scalar λ_i from $i=1$ to n, we get that

$$0 > \sum_{i=1}^n \lambda_i \langle z^*, \eta(y_i,z)\rangle \geq \left\langle z^*, \eta\left(\sum_{i=1}^n \lambda_i y_i, z\right)\right\rangle = \langle z^*, \eta(z,z)\rangle = 0,$$

where the first inequality follows from (3.1), the second inequality follows from item (ii) of condition (H_η), and the last equality follows from $\eta(z,z)=0$ (see item (ii) of Remark 3.1). This observation is a contradiction.

We next show that $G(y)$ is weakly compact for all $y\in K$. It is easy to see that $y\in G(y)$ for each $y\in K$, and so $G(y)\neq\emptyset$. For $y\in K$, let us consider a sequence $\{x_n\}\subset G(y)$ satisfying $x_n \rightharpoonup x\in K$. Thus, we know that there is $x_n^*\in T(x_n)$ such that

$$\langle x_n^*, \eta(y,x_n)\rangle \geq 0. \tag{3.2}$$

Since T is weak to norm upper semicontinuous on K, by Lemma 2.2, we deduce that there are $x^*\in T(x)$ and a subsequence $\{x_{n_k}^*\}$ of $\{x_n^*\}$ such that $x_{n_k}^*\to x^*$. By item (i) of condition (H_η), (3.2) is equivalent to

$$\langle x_n^*, \eta(x_n,y)\rangle \leq 0. \tag{3.3}$$

Taking the limit in (3.3) for the subsequence $\{x_{n_k}\}$, we have

$$\langle x^*, \eta(x,y)\rangle \leq \liminf_{k\to\infty} \langle x_n^*, \eta(x_n,y) \leq 0,$$

which can be written as

$$\langle x^*, \eta(y,x)\rangle \geq 0$$

by again item (i) of condition (H_η). Hence, $x \in G(y)$, which implies that $G(y)$ is weakly sequentially closed for all $y \in K$. Since $G(y) \subset K$ and K is weakly compact, by Lemma 2.5, we know that $G(y)$ is weakly compact for all $y \in K$, which implies that it is weakly closed as well. Consequently, it follows from Lemma 2.7 that $\cap_{y\in K} G(y) \neq \emptyset$. That is to say, there is $x \in K$ such that for each $y \in K$, there is $x^* \in T(x)$ satisfying

$$\langle x^*, \eta(y, x)\rangle \geq 0.$$

This observation shows that $x \in \mathrm{SOL}_w(T, \eta, K) \neq \emptyset$.

In order to obtain the conclusion, we only need to show that $\mathrm{SOL}_w(T, \eta, K) \subset \mathrm{SOL}(T, \eta, K)$. Let $x \in K$ be an element of $\mathrm{SOL}_w(T, \eta, K)$. If $x \notin \mathrm{SOL}(T, \eta, K)$, then for each $x^* \in T(x)$, there exists $y \in K$ such that $\langle x^*, \eta(y, x)\rangle < 0$ and so $\min_{y\in K}\langle x^*, \eta(y, x)\rangle < 0$. Since $T(x)$ is compact, one has

$$\max_{x^*\in T(x)} \min_{y\in K} \langle x^*, \eta(y, x)\rangle < 0. \tag{3.4}$$

Define a bi-function $h : K \times T(x) \to \mathbb{R}$ by $h(y, x^*) = \langle x^*, \eta(y, x)\rangle$. Then, it is easy to see that, for every $y \in K$, $h(y, \cdot) : T(x) \to \mathbb{R}$ is concave. Using items (ii) and (iii) of condition (H_η) and applying Lemma 2.8 for the bi-function h in the weak×norm topology, we have

$$\max_{x^*\in T(x)} \min_{y\in K} h(y, x^*) = \min_{y\in K} \max_{x^*\in T(x)} h(y, x^*). \tag{3.5}$$

This, together with (3.4), implies that

$$\min_{y\in K} \max_{x^*\in T(x)} h(y, x^*) < 0. \tag{3.6}$$

On the other hand, it follows from $x \in \mathrm{SOL}_w(T, \eta, K)$ that, for every $y \in K$,

$$\max_{x^*\in T(x)} h(y, x^*) = \max_{x^*\in T(x)} \langle x^*, \eta(y, x)\rangle \geq 0,$$

which leads to

$$\min_{y\in K} \max_{x^*\in T(x)} h(y, x^*) \geq 0. \tag{3.7}$$

This observation contradicts (3.6). Therefore, we have $\mathrm{SOL}_w(T, \eta, K) \subset \mathrm{SOL}(T, \eta, K)$. This, together with $\mathrm{SOL}_w(T, \eta, K) \neq \emptyset$, implies that $\mathrm{SOL}(T, \eta, K) \neq \emptyset$. This completes the proof. □

Remark 3.3. From the proof of Theorem 3.2, we have that $\mathrm{SOL}_w(T, \eta, K) \subset \mathrm{SOL}(T, \eta, K)$, provided that the conditions of Theorem 3.2 hold. Moreover, the inclusion relation $\mathrm{SOL}(T, \eta, K) \subset \mathrm{SOL}_w(T, \eta, K)$ is always true. Therefore, $\mathrm{SOL}_w(T, \eta, K)$ coincides with $\mathrm{SOL}(T, \eta, K)$ under the assumptions of Theorem 3.2.

4 Existence Results for GVLI(T, η, K) with Perturbation in Finite Dimensional Spaces

In this section, let $\mathbb{B}$ be a finite dimensional space. For convenience, let $\mathbb{B} = \mathbb{R}^n$. When the constraint set K is unbounded, we will investigate the existence of solutions for GVLI(T, η, K), in which the mapping T and the constraint set K are simultaneously perturbed. To this end, we need to introduce the following coercivity conditions, which will be used in Theorem 4.2.

(A1) There exists $r > 0$ such that, for every $x \in K \setminus K_r$, there is $y \in K$ with $\|y\| < \|x\|$ satisfying

$$\sup_{x^* \in T(x)} \langle x^*, \eta(y, x) \rangle \leq 0;$$

(A2) There exists $r > 0$ such that for every $x \in K \setminus K_r$, there is $y \in K$ with $\|y\| < \|x\|$ satisfying

$$\sup_{x^* \in T(x)} \langle x^*, \eta(y, x) \rangle < 0.$$

Remark 4.1. If T is single-valued, then coercivity condition (A2) is equivalent to condition (iii) of Theorem 3.4 of Parida et al. [26]. If $\eta(y, x) = y - x$, then conditions (A1) and (A2) becomes, respectively, to the following conditions:

(A1') There exists $r > 0$ such that for every $x \in K \setminus K_r$, there is $y \in K$ with $\|y\| < \|x\|$ satisfying

$$\sup_{x^* \in T(x)} \langle x^*, y - x \rangle \leq 0;$$

(A2') There exists $r > 0$ such that for every $x \in K \setminus K_r$, there is $y \in K$ with $\|y\| < \|x\|$ satisfying

$$\sup_{x^* \in T(x)} \langle x^*, y - x \rangle < 0,$$

which have been used extensively when GVI(T, K) is considered (see, for example, [17,22]).

It is easy to see that condition (A2) implies (A1), while the converse is not true. The following example shows that condition (A1) is strictly weaker than (A2) even in the case when $\eta(y, x) \neq y - x$ and T is single-valued.

Example 4.1. Let us consider the Euclidean space $\mathbb{R}^2$ with $x = (x_1, x_2)^T$. Let

$$K := \{x \in \mathbb{R}^2 : x_1 - 2 \leq x_2 \leq x_1, x_1 \geq 0, x_2 \geq 0\}.$$

Define

$$T(x) := (-x_1(-x_1 + x_2 + 2), -x_1 + x_2 + 2)^T$$

and

$$\eta(y, x) := (y_1^2 - x_1^2, y_2^2 - x_2^2)^T.$$

Then we can check that condition (A1) holds, while condition (A2) does not hold.

Indeed, take $r = 1$. For any $x \in K$ with $\|x\| > r = 1$, let $t \in (0, 1)$ be a constant small sufficiently and set

$$y = \begin{cases} (x_1, (1-t)x_2)^T, & \text{if } x_2 > x_1 - 2, \\ ((1-t)x_1, x_2)^T, & \text{if } x_2 = x_1 - 2. \end{cases} \tag{4.1}$$

Then we have $y \in K$ with $\|y\| < \|x\|$, and so

$$\eta(y, x) = \begin{cases} (0, -t(2-t)x_2^2)^T, & \text{if } x_2 > x_1 - 2, \\ (-t(2-t)x_1^2, 0)^T, & \text{if } x_2 = x_1 - 2. \end{cases} \tag{4.2}$$

Consequently,

$$\langle T(x), \eta(y,x)\rangle = \begin{cases} -t(2-t)x_2^2(-x_1+x_2+2), & \text{if } x_2 > x_1 - 2, \\ 0, & \text{if } x_2 = x_1 - 2. \end{cases} \tag{4.3}$$

It is easy to check that $\langle T(x), \eta(y,x)\rangle \leq 0$. This observation shows that condition (A1) holds.

For every $r > 0$, take $\bar{x} = (x_1, x_2)^T = (r+2, r)^T$. Then, clearly, $\bar{x} \in K \setminus K_r$. For each $y \in K$ with $\|y\| < \|\bar{x}\|$, we have

$$\langle T(\bar{x}), \eta(y, \bar{x})\rangle = (-x_1 + x_2 + 2)[-x_1(y_1 + x_1)(y_1 - x_1) + (y_2 + x_2)(y_2 - x_2)] = 0.$$

This observation shows that condition (A2) is not true.

Let $K \subset \mathbb{R}^n$ be a nonempty, closed and convex set, and $D \subset \mathbb{R}^n$ be a bounded, closed and convex set containing origin θ. Denote $K(\alpha) = K + \alpha D$ for any $\alpha \geq 0$. For $\varepsilon > 0$ and $m > 0$, define a continuous mapping $p : K(\alpha) \to \mathbb{R}^n$ such that

$$\|p(x)\| \leq \varepsilon, \quad \forall x \in K(\alpha)_m.$$

We denote the set of all such functions by $A(\varepsilon; K(\alpha)_m)$.

Theorem 4.2. *Let $K \subset \mathbb{R}^n$ be a nonempty, closed and convex set, and for some $\mu > 0$, $T : K(\mu) \rightrightarrows \mathbb{R}^n$ be an upper semicontinuous mapping with nonempty compact convex values. Suppose that η satisfies condition (H_η), in which K is replaced by $K(\mu)$. If coercivity condition (A2) holds, then for every $m > r$, there exists $\varepsilon \in (0, \mu)$ such that*

$$SOL(T + p, \eta, K(\alpha)) \cap \bar{\mathbf{B}}(\theta, m) \neq \emptyset, \quad \forall \alpha \in [0, \varepsilon), \forall p \in A(\varepsilon; K(\alpha)_m). \tag{4.4}$$

Proof. Suppose by contradiction that there exists $m > r$ such that, for every $\varepsilon > 0$, there exist $\alpha_\varepsilon \in [0, \varepsilon)$ and $p_\varepsilon \in A(\varepsilon; K(\alpha_\varepsilon)_m)$ satisfying

$$\mathrm{SOL}(T + p_\varepsilon, \eta, K(\alpha_\varepsilon)) \cap \bar{\mathbf{B}}(\theta, m) = \emptyset. \tag{4.5}$$

By the definition of $K(\alpha_\varepsilon)_m$, we know that $K(\alpha_\varepsilon)_m = \{x \in K(\alpha_\varepsilon) : \|x\| \leq m\}$ and so it is a bounded, closed and convex set. It follows from Theorem 3.2 that $\mathrm{GVLI}(T + p_\varepsilon, \eta, K(\alpha_\varepsilon)_m)$ admits a solution. Take $x_\varepsilon \in \mathrm{SOL}(T + p_\varepsilon, \eta, K(\alpha_\varepsilon)_m)$. Then, there exists $x_\varepsilon^* \in T(x_\varepsilon)$ such that

$$\langle x_\varepsilon^* + p_\varepsilon(x_\varepsilon), \eta(y, x_\varepsilon)\rangle \geq 0, \quad \forall y \in K(\alpha_\varepsilon)_m. \tag{4.6}$$

Since $x_\varepsilon \in K(\alpha_\varepsilon)_m$, we have $\|x_\varepsilon\| \leq m$. In what follows, let us consider two possible cases:

(*Case 1*) Suppose that there is some $\varepsilon > 0$ such that $\|x_\varepsilon\| < m$. We claim that

$$x_\varepsilon \in \mathrm{SOL}(T + p_\varepsilon, \eta, K(\alpha_\varepsilon)).$$

Indeed, for each $y \in K(\alpha_\varepsilon)$, there exists $t \in (0, 1)$ such that $z_t = x_\varepsilon + t(y - x_\varepsilon) \in K(\alpha_\varepsilon)_m$. It follows from (4.6) that

$$\begin{aligned} 0 &\leq \langle x_\varepsilon^* + p_\varepsilon(x_\varepsilon), \eta(z_t, x_\varepsilon)\rangle \\ &\leq t\langle x_\varepsilon^* + p_\varepsilon(x_\varepsilon), \eta(y, x_\varepsilon)\rangle + (1-t)\langle x_\varepsilon^* + p_\varepsilon(x_\varepsilon), \eta(x_\varepsilon, x_\varepsilon)\rangle \\ &= t\langle x_\varepsilon^* + p_\varepsilon(x_\varepsilon), \eta(y, x_\varepsilon)\rangle, \end{aligned} \tag{4.7}$$

where the first inequality follows from the fact that $z_t \in K(\alpha_\varepsilon)_m$, the second inequality follows from the item (ii) of condition (H_η) for replacing K by $K(\mu)$, and the equality follows from $\eta(x_\varepsilon, x_\varepsilon) = 0$ (see item (ii) of Remark 3.1). By the fact that $t \in (0,1)$ and the arbitrariness of $y \in K(\alpha_\varepsilon)$, it follows from (4.7) that $x_\varepsilon \in \mathrm{SOL}(T + p_\varepsilon, \eta, K(\alpha_\varepsilon))$. Thus,

$$x_\varepsilon \in \mathrm{SOL}(T + p_\varepsilon, \eta, K(\alpha_\varepsilon)) \cap \bar{\mathbf{B}}(\theta, m) \neq \emptyset.$$

(*Case 2*) Suppose that $\|x_\varepsilon\| = m$ for any $\varepsilon > 0$. Without loss of generality, assume that $\lim_{\varepsilon \to 0^+} x_\varepsilon = d$, where $\|d\| = m > r$. Since for any $\varepsilon > 0$, $x_\varepsilon \in K(\alpha_\varepsilon) = K + \alpha_\varepsilon D$, there exists $x'_\varepsilon \in K$ such that $\lim_{\varepsilon \to 0^+} \|x_\varepsilon - x'_\varepsilon\| = 0$. Thus, we get that $\lim_{\varepsilon \to 0^+} x'_\varepsilon = d$ and $d \in K$ by the closedness of the set K. It follows from $d \in K \setminus K_r$ and coercivity condition (A2) that there exists $y_0 \in K$ with $\|y_0\| < \|d\| = m$ satisfying

$$\sup_{d^* \in T(d)} \langle d^*, \eta(y_0, d) \rangle < 0. \tag{4.8}$$

By item (iii) of condition (H_η) and $x_\varepsilon \to d$, we have $\eta(y_0, x_\varepsilon) \to \eta(y_0, d)$. Since $x_\varepsilon \in K(\alpha_\varepsilon) \subset K(\varepsilon)$ and $\|x_\varepsilon\| = m$, we have $x_\varepsilon \in K(\varepsilon)_m$. This, together with $\sup_{x \in K(\varepsilon)_m} \|p_\varepsilon(x)\| \leq \varepsilon$, implies that

$$\lim_{\varepsilon \to 0^+} \langle p_\varepsilon(x_\varepsilon), \eta(y_0, x_\varepsilon) \rangle = 0. \tag{4.9}$$

Let $f : \mathrm{Graph}(T) \to \mathbb{R}$ be defined by

$$f(x, x^*) := \langle x^*, \eta(y_0, x) \rangle, \qquad \text{where } x^* \in T(x).$$

Then applying Lemma 2.4, we deduce that

$$g(x) := \sup_{x^* \in T(x)} f(x, x^*) = \sup_{x^* \in T(x)} \langle x^*, \eta(y_0, x) \rangle$$

is upper semicontinuous. Consequently, we have

$$\begin{aligned}
& \limsup_{\varepsilon \to 0^+} \left[\sup_{x'^*_\varepsilon \in T(x_\varepsilon)} \langle x'^*_\varepsilon, \eta(y_0, x_\varepsilon) \rangle + \langle p_\varepsilon(x_\varepsilon), \eta(y_0, x_\varepsilon) \rangle \right] \\
\leq \; & \limsup_{\varepsilon \to 0^+} \sup_{x'^*_\varepsilon \in T(x_\varepsilon)} \langle x'^*_\varepsilon, \eta(y_0, x_\varepsilon) \rangle + \lim_{\varepsilon \to 0^+} \langle p_\varepsilon(x_\varepsilon), \eta(y_0, x_\varepsilon) \rangle \\
\leq \; & \sup_{d^* \in T(d)} \langle d^*, \eta(y_0, d) \rangle + 0 \\
< \; & 0,
\end{aligned} \tag{4.10}$$

where the first inequality is obvious, the second inequality follows from the upper semicontinuity of g and (4.9), and the last inequality follows from (4.8). Thus, there is a constant $\delta > 0$ such that

$$\sup_{x'^*_\varepsilon \in T(x_\varepsilon)} \langle x'^*_\varepsilon, \eta(y_0, x_\varepsilon) \rangle + \langle p_\varepsilon(x_\varepsilon), \eta(y_0, x_\varepsilon) \rangle < 0, \quad \forall \varepsilon \in (0, \delta). \tag{4.11}$$

Since $\|y_0\| < m$, for any given $y \in K(\alpha_\varepsilon)$, there is $t \in (0,1)$ such that $z'_t = y_0 + t(y - y_0) \in K(\alpha_\varepsilon)_m$. For any $\varepsilon \in (0, \delta)$, by the fact that $x_\varepsilon \in \mathrm{SOL}(T + p_\varepsilon, \eta, K(\alpha_\varepsilon)_m)$, there exists $x^*_\varepsilon \in T(x_\varepsilon)$ such that (4.6) holds. Thus,

$$\begin{aligned}
0 \;\leq\; & \langle x^*_\varepsilon + p_\varepsilon(x_\varepsilon), \eta(z'_t, x_\varepsilon) \rangle \\
\leq\; & t \langle x^*_\varepsilon + p_\varepsilon(x_\varepsilon), \eta(y, x_\varepsilon) \rangle + (1-t) \langle x^*_\varepsilon + p_\varepsilon(x_\varepsilon), \eta(y_0, x_\varepsilon) \rangle
\end{aligned}$$

$$\leq \quad t\langle x_\varepsilon^* + p_\varepsilon(x_\varepsilon), \eta(y, x_\varepsilon)\rangle, \tag{4.12}$$

where the first inequality follows from the fact that $z_t' \in K(\alpha_\varepsilon)_m$, the second inequality follows from item (ii) of condition (H_η) for replacing K by $K(\mu)$, and the last inequality follows from (4.11). Since $t \in (0, 1)$ and $y \in K(\alpha_\varepsilon)$ is arbitrary, we have

$$\langle x_\varepsilon^* + p_\varepsilon(x_\varepsilon), \eta(y, x_\varepsilon)\rangle \geq 0, \quad \forall y \in K(\alpha_\varepsilon).$$

Thus, we conclude that

$$\langle x_\varepsilon^* + p_\varepsilon(x_\varepsilon), \eta(y, x_\varepsilon)\rangle \geq 0, \quad \forall y \in K.$$

This observation shows that $x_\varepsilon \in \mathrm{SOL}(T + p_\varepsilon, \eta, K(\alpha_\varepsilon))$ and so

$$x_\varepsilon \in \mathrm{SOL}(T + p_\varepsilon, \eta, K(\alpha_\varepsilon)) \cap \bar{\mathbf{B}}(\theta, m) \neq \emptyset.$$

In either case, we get a contradiction. This completes the proof. □

Remark 4.3. If $\eta(y, x) = y - x$, then Theorem 4.2 reduces to Theorem 1 of Wang and He [29]. Compared with Theorem 3.1 of Li and He [22], we not only generalize Theorem 3.1 of Li and He [22] from the generalized variational inequality to the generalized variational-like inequality, but also simultaneously consider the perturbation behavior of the mapping T and the constraint set K in Theorem 4.2.

Taking $p(x) = \theta$ for all $x \in K$ and $\alpha = 0$ in (4.4), we obtain the following existence result for GVLI(T, η, K).

Corollary 4.4. *Let $K \subset \mathbb{R}^n$ be a nonempty, closed and convex set, and $T : K \rightrightarrows \mathbb{R}^n$ be an upper semicontinuous mapping with nonempty compact convex values. If condition (H_η) and coercivity condition (A2) hold, then for every $m > r$, we have*

$$SOL(T, \eta, K) \cap \bar{\mathbf{B}}(\theta, m) \neq \emptyset.$$

We would like to mention that Corollary 4.4 can be further improved (see Theorem 4.6 below). As a preliminary, we give the following proposition.

Proposition 4.5. *Let $K \subset \mathbb{R}^n$ be a nonempty, closed and convex set, and $T : K \rightrightarrows \mathbb{R}^n$ be an upper semicontinuous mapping with nonempty compact convex values. If condition (H_η) and coercivity condition (A2) hold, then the set of solutions for GVLI(T, η, K), i.e., SOL(T, η, K), is nonempty and closed.*

Proof. Nonemptyness of $\mathrm{SOL}(T, \eta, K)$ follows from Corollary 4.4. In order to show that $\mathrm{SOL}(T, \eta, K)$ is closed, let us consider a convergent sequence $\{x_n\}$ in $\mathrm{SOL}(T, \eta, K)$, i.e., $x_n \to x_0$. It follows from $x_n \in \mathrm{SOL}(T, \eta, K)$ that there is $x_n^* \in T(x_n)$ such that, for any $y \in K$,

$$\langle x_n^*, \eta(y, x_n)\rangle \geq 0. \tag{4.13}$$

Without loss of generality, applying Lemma 2.2 and condition (H_T), we conclude that there is $x_0^* \in T(x_0)$ such that $x_n^* \to x_0^*$. It follows from item (iii) of condition (H_η) that for given $y \in K$, $\eta(y, x_n) \to \eta(y, x_0)$. Letting $n \to \infty$ in (4.13), we have $\langle x_0^*, \eta(y, x_0)\rangle \geq 0$. By the arbitrariness of $y \in K$, we know that $x_0 \in \mathrm{SOL}(T, \eta, K)$. This completes the proof. □

Theorem 4.6. *Let $K \subset \mathbb{R}^n$ be a nonempty, closed and convex set, and $T : K \rightrightarrows \mathbb{R}^n$ be an upper semicontinuous mapping with nonempty compact convex values. If condition (H_η) and coercivity condition (A2) hold, then there is a solution of GVLI(T, η, K) in K_r.*

Proof. Suppose on the contrary that $x \notin K_r$ for any $x \in \mathrm{SOL}(T, \eta, K)$. Then, for any $n \in \mathbb{N}$, it follows from Corollary 4.4 that there is $x_n \in \mathrm{SOL}(T, \eta, K)$ satisfying $r < \|x_n\| \leq r + \frac{1}{n}$. Letting n tend to infinity in this inequality, we get

$$\lim_{n \to \infty} \|x_n\| = r. \tag{4.14}$$

Since $\{x_n\}$ is a bounded sequence (because of $\|x_n\| \leq r + 1$ for any $n \in \mathbb{N}$), there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$. By the lower semicontinuity of $\| \cdot \|$, we have $\liminf_{k \to \infty} \|x_{n_k}\| \geq \|x_0\|$. Combining (4.14), we know that $\|x_0\| \leq r$. By the closedness of $\mathrm{SOL}(T, \eta, K)$, we conclude that $x_0 \in \mathrm{SOL}(T, \eta, K)$. This completes the proof. □

If $\eta(y, x) = y - x$, then we get the following corollary, which is an existence result of solutions for GVI(T, K).

Corollary 4.7. *Let $K \subset \mathbb{R}^n$ be a nonempty, closed and convex set, and $T : K \rightrightarrows \mathbb{R}^n$ be an upper semicontinuous mapping with nonempty compact convex values. If coercivity condition (A2') holds, then there is a solution of GVI(T, K) in K_r.*

5 Concluding Remarks

In the present paper, we mainly concern the existence results of GVLI(T, η, K). When the constraint set is bounded, we obtain an existence theorem for GVLI(T, η, K) in reflexive Banach spaces. When the constraint set is not necessarily bounded, we investigate the existence of solutions for GVLI(T, η, K) where the mapping T and the constraint set K are simultaneously perturbed in finite dimensional spaces. We would like to point out that the following issues are interesting in this topic.

Problem 1: Whether Theorem 4.2 remains true or not in reflexive (not necessarily finite dimensional) Banach spces?

Problem 2: As mentioned in the previous section, coercivity condition (A1) is strictly weaker than (A2), we want to know whether coercivity condition (A2) in Theorem 4.2 can be replaced by the weaker condition (A1) or not.

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