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ON Q-TENSORS

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Abstract: One of the central problems in the theory of linear complementarity problems (LCPs) is to study the class of Q-matrices since it characterizes the solvability of LCP. Recently, Song and Qi [Ann. of Appl. Math. 33 (2017) 308-323] extended the concept of Q-matrix to the case of tensor, called Q-tensor, which characterizes the solvability of the corresponding tensor complementarity problem – a generalization of LCP. They investigated properties of Q-tensors, and proposed the following question: Whether or not a nonzero solution of the tensor complementarity problem contains at least two nonzero components if the involved tensor is a semi-positive Q-tensor. In this paper, we make further studies for Q-tensors. We extend two famous results related to Q-matrices to the tensor space, i.e., we show that within the class of strong P_0-tensors or nonnegative tensors, four classes of tensors, i.e., R_0-tensors, R-tensors, ER-tensors and Q-tensors, are all equivalent. We also construct several examples to show that three famous results related to Q-matrices cannot be extended to the tensor space; and one of which gives a negative answer to the question mentioned above raised by Song and Qi.

Key words: *Q-tensor, tensor complementarity problem, strong P_0-tensor, nonnegative tensor*

Mathematics Subject Classification: *15A18, 15A69, 65F15, 65F10, 90C33, 65K10*

1 Introduction

For any given $A \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, the linear complementarity problem [6], denoted by LCP(q, A), is to find a vector $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad Ax + q \geq 0, \quad x^T(Ax + q) = 0.$$

If LCP(q, A) has a solution for every vector $q \in \mathbb{R}^n$, we say that A is a *Q-matrix* [28]. Since the solvability of LCP(q, A) is very important, much research in the theory of LCP(q, A) has been devoted to finding constructive characterizations of subclasses of Q-matrices; and fruitful results have been obtained [1, 7, 13, 20, 21, 24, 30].

A real m-th order n-dimensional tensor can be denoted by $\mathscr{A} = (a_{i_1\cdots i_m})$ with $a_{i_1\cdots i_m} \in \mathbb{R}$ for all $i_j \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$. Obviously, $\mathscr{A}$ is a matrix when $m = 2$. As an extension of matrix, tensor has been widely studied (see an excellent survey by Kolda and Bader [26] and references therein). For any given real m-th order n-dimensional tensor $\mathscr{A}$ and vector $q \in \mathbb{R}^n$, the tensor complementarity problem [2, 4, 8, 14, 18, 27, 32–36], denoted by TCP$(q, \mathscr{A})$, is to find a vector $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad \mathscr{A}x^{m-1} + q \geq 0, \quad x^T(\mathscr{A}x^{m-1} + q) = 0,$$

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where $\mathscr{A}x^{m-1} \in \mathbb{R}^n$ with

$$\left(\mathscr{A}x^{m-1}\right)_i := \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m},\ \forall i \in \{1, 2, \ldots, n\}.$$

Recently, Song and Qi [32] extended the concept of Q-matrix to the case of tensor, i.e., a real m-th order n-dimensional tensor $\mathscr{A}$ is called a *Q-tensor* if $\mathrm{TCP}(q, \mathscr{A})$ has a solution for every vector $q \in \mathbb{R}^n$. Furthermore, they proved that several classes of tensors are the subclasses of Q-tensors.

In this paper, we consider several famous results related to Q-matrices and investigate whether they can be extended to the tensor space or not. In Section 2, after reviewing some basic concepts and known related results, we give a sufficient condition to judge whether a tensor is a Q-tensor or not.

Recall that a matrix $A \in \mathbb{R}^{n\times n}$ is called a *P_0-matrix* if all its principal minors are nonnegative; an *R_0-matrix* if $\mathrm{LCP}(0, A)$ has a unique solution; and an *R-matrix* if $\mathrm{LCP}(te, A)$ has a unique solution for each scalar $t \geq 0$, where $e \in \mathbb{R}^n$ denotes the vector of all ones. In 1979, Agangic and Cottle [1] proved that within the class of P_0-matrices, both the classes of R-matrices and R_0-matrices are equivalent to the class of Q-matrices. Since the classes of matrices mentioned above play important roles in the field of variational inequalities and complementarity problems [10, 15, 16, 19, 22, 25], it is important to extend Agangic-Cottle's result to the tensor space. It is known that P_0-matrix, R_0-matrix and R-matrix have been extended to the case of tensor by Song and Qi [32, 33], called P_0-tensor, R_0-tensor and R-tensor, respectively. Recently, Ding, Luo and Qi [8] gave another extension of P_0-matrix, which is called P_0'-tensor in this paper. A natural question is whether Agangic-Cottle's result can be extended to the tensor space or not. In Section 3, we answer this question. We clarify that within the class of P_0-tensors (or P_0'-tensors), the above equivalence cannot be extended to the tensor space. In order to extend Agangic-Cottle's result to the tensor space, we introduce a new class of tensors, called strong P_0-tensors, which is a generalization of P_0-matrices; and show that within the class of strong P_0-tensors, the classes of R_0-tensors, R-tensors and ER-tensors are all equivalent to the class of Q-tensors, where the class of ER-tensors was recently introduced by Wang, Huang and Bai [36]. In this section, we also discuss the relationships among P_0-tensors, P_0'-tensors and strong P_0-tensors.

Recall that a matrix $A \in \mathbb{R}^{n\times n}$ is said to be *nonnegative* if all its elements are nonnegative; and a real m-th order n-dimensional tensor $\mathscr{A}$ is said to be *nonnegative* if all its elements are nonnegative. In 1994, Danao [7] proved that within the class of nonnegative matrices, the class of Q-matrices coincides with the class of R_0-matrices. Since many tensors from practical problems are nonnegative, nonnegative tensors have been extensively studied in the last ten years [3, 5, 12, 17, 29, 37]. It is interesting whether Danao's result can be extended to the tensor space or not. In Section 4, we answer this question. We show that within the class of nonnegative tensors, the classes of R_0-tensors, R-tensors and ER-tensors are all equivalent to the class of Q-tensors. In addition, we also discuss the relationship between the class of nonnegative tensors and the class of strong P_0-tensors.

Recall that a matrix $A \in \mathbb{R}^{n\times n}$ is said to be *semi-monotone* if for every $x \in \mathbb{R}^n$ with $0 \neq x \geq 0$, there exists an index i such that $x_i > 0$ and $(Ax)_i \geq 0$; and *copositive* if $x^TAx \geq 0$ for all $x \in \mathbb{R}^n$ satisfying $x \geq 0$. In Section 5, we consider two results obtained by Pang [30] which are related to semi-monotone matrices and Q-matrices; and a result obtained by Jeter and Pye [20] which is related to copositive matrices and Q-matrices. We illustrate that these three results cannot be extended to the tensor space by using several examples. In addition, the final conclusions are given in Section 6.

In the rest of this paper, we assume that $m \geq 3$ and $n \geq 2$ are two integers unless otherwise specialized; and use $\mathbb{T}_{m,n}$ to denote the set of all real m-th order n-dimensional tensors. For any positive integer n, we denote $[n] := \{1, 2, \ldots, n\}$ and $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x \geq 0\}$. We use $\mathbf{Q}$ to denote the set of real Q-matrices; and $\mathcal{Q}$ to denote the set of real Q-tensors, i.e., $A \in \mathbf{Q}$ means that A is a real Q-matrix; and $\mathscr{A} \in \mathcal{Q}$ means that $\mathscr{A}$ is a real Q-tensor. If the sizes of the considered matrices or tensors need to be specified, we use $\mathbb{R}^{n\times n} \bigcap \mathbf{Q}$ to denote the set of all real Q-matrices of $n \times n$; and $\mathbb{T}_{m,n} \bigcap \mathcal{Q}$ to denote the set of all real m-th order n-dimensional Q-tensors, i.e., $A \in \mathbb{R}^{n\times n} \bigcap \mathbf{Q}$ means that A is a real Q-matrix of $n \times n$; and $\mathscr{A} \in \mathbb{T}_{m,n} \bigcap \mathcal{Q}$ means that $\mathscr{A}$ is a real m-th order n-dimensional Q-tensor. Similar notations will be used for other classes of matrices or tensors.

2 Preliminaries

In this section, we review definitions and properties of several structured tensors, which are useful for our subsequent discussions. We also give a sufficient condition to judge whether a tensor is a Q-tensor or not.

Definition 2.1. A tensor $\mathscr{A} \in \mathbb{T}_{m,n}$ is said to be

(i) **a P_0-tensor** if for each $x \in \mathbb{R}^n \setminus \{0\}$, there exists an index $i \in [n]$ such that

$$x_i \neq 0 \quad \text{and} \quad x_i(\mathscr{A}x^{m-1})_i \geq 0;$$

(ii) **a P_0'-tensor** if for each $x \in \mathbb{R}^n \setminus \{0\}$, there exists an index $i \in [n]$ such that

$$x_i \neq 0 \quad \text{and} \quad x_i^{m-1}(\mathscr{A}x^{m-1})_i \geq 0.$$

The concept of P_0-tensor was introduced by Song and Qi [33]; and the concept of P_0'-tensor was introduced by Ding, Luo and Qi [8] with the name of P_0-tensor. Since it is different from P_0-tensor defined by Song and Qi [33] (see the next section), in this paper, we call P_0-tensor introduced by Ding, Luo and Qi [8] as P_0'-tensor to avoid confusion. When $m = 2$, a P_0-tensor or a P_0'-tensor reduces to a P_0-matrix. We will use $\mathcal{P}_0$ ($\mathcal{P}_0'$) to denote the set of real P_0-tensors (P_0'-tensors); and $\mathbb{T}_{m,n} \bigcap \mathcal{P}_0$ ($\mathbb{T}_{m,n} \bigcap \mathcal{P}_0'$) to denote the set of all real m-th order n-dimensional P_0-tensors (P_0'-tensors).

Definition 2.2. (i) A tensor $\mathscr{A} \in \mathbb{T}_{m,n}$ is called an R-tensor, if there exists no $(x, t) \in (\mathbb{R}^n_+ \setminus \{0\}) \times \mathbb{R}_+$ such that

$$\begin{cases} (\mathscr{A}x^{m-1})_i + t = 0, & \text{if } x_i > 0, \\ (\mathscr{A}x^{m-1})_i + t \geq 0, & \text{if } x_i = 0. \end{cases} \tag{2.1}$$

(ii) A tensor $\mathscr{A} \in \mathbb{T}_{m,n}$ is called an R_0-tensor, if the system (2.1) has no solution when $t = 0$, i.e., there exists no $x \in \mathbb{R}^n_+ \setminus \{0\}$ such that

$$\begin{cases} (\mathscr{A}x^{m-1})_i = 0, & \text{if } x_i > 0, \\ (\mathscr{A}x^{m-1})_i \geq 0, & \text{if } x_i = 0. \end{cases}$$

(iii) A tensor $\mathscr{A} \in \mathbb{T}_{m,n}$ is called an ER-tensor, if there exists no $(x, t) \in (\mathbb{R}^n_+ \setminus \{0\}) \times \mathbb{R}_+$ such that

$$\begin{cases} (\mathscr{A}x^{m-1})_i + tx_i = 0, & \text{if } x_i > 0, \\ (\mathscr{A}x^{m-1})_i \geq 0, & \text{if } x_i = 0. \end{cases}$$

The concepts of R-tensor and R_0-tensor were introduced by Song and Qi in [32]; and the concept of ER-tensor was introduced by Wang, Huang and Bai in [36]. It is obvious that an R-tensor or ER-tensor is an R_0-tensor. When $m = 2$, an R-tensor (R_0-tensor) reduces to an R-matrix (R_0-matrix) [6, 23]; and an ER-tensor reduces to an ER-matrix [36]. We will use $\mathcal{R}$ ($\mathcal{R}_0, \mathcal{ER}$) to denote the set of real R-tensors (R_0-tensors, $\mathcal{ER}$-tensors); and $\mathbb{T}_{m,n} \bigcap \mathcal{R}$ ($\mathbb{T}_{m,n} \bigcap \mathcal{R}_0, \mathbb{T}_{m,n} \bigcap \mathcal{ER}$) to denote the set of all real m-th order n-dimensional R-tensors (R_0-tensors, ER-tensors).

Definition 2.3 ([32])**.** A tensor $\mathscr{A} \in \mathbb{T}_{m,n}$ is said to be **semi-positive** if for each $x \in \mathbb{R}^n_+ \setminus \{0\}$, there exists an index $i \in [n]$ such that $x_i > 0$ and $(\mathscr{A}x^{m-1})_i \geq 0$.

Clearly, every P_0-tensor (P_0'-tensor) is certainly semi-positive. It is shown that the class of semi-positive R_0-tensors is a subclass of Q-tensors [32]. When $m = 2$, a semi-positive tensor reduces to a semi-monotone matrix; and the set of all semi-monotone matrices is denoted by $\mathbf{L}_1$ (or $\mathbf{E}_0$) [9, 11].

Definition 2.4. A tensor $\mathscr{A} \in \mathbb{T}_{m,n}$ is said to be **copositive** if $\mathscr{A}x^m \geq 0$ for all $x \in \mathbb{R}^n_+$.

When $\mathscr{A} \in \mathbb{T}_{m,n}$ is symmetric, such a concept was first introduced by Qi [31]. When $m = 2$, every copositive tensor reduces to a copositive matrix [6].

Proposition 2.1. *Suppose that $\mathscr{A} \in \mathbb{T}_{m,n}$. Then the following results hold.*

(i) *If $\mathscr{A} \in \mathcal{R}$, then $\mathscr{A} \in \mathcal{Q}$.*

(ii) *If $\mathscr{A}$ is semi-positive, then*

$$\mathscr{A} \in \mathcal{R}_0 \iff \mathscr{A} \in \mathcal{ER} \iff \mathscr{A} \in \mathcal{R}.$$

(iii) *If $\mathscr{A} \in \mathcal{ER}$, then $\mathscr{A} \in \mathcal{Q}$.*

(iv) *If $\mathscr{A}$ is nonnegative, then $\mathscr{A} \in \mathcal{Q}$ if and only if $a_{i\cdots i} > 0$ for all $i \in [n]$.*

In Proposition 2.1, the results (i)-(iv) come from [32, Theorem 3.2], [36, Theorem 3.3], [36, Corollary 4.1], and [32, Theorem 3.5], respectively.

At the end of this section, we give a characterization of Q-tensor, which is an extension of [24, Proposition 2.1].

Theorem 2.2. *Let $\mathscr{A} = (a_{i_1\cdots i_m}) \in \mathbb{T}_{m,n}$ with $i_1, \ldots, i_m \in [n]$. Denote $\mathscr{A}_{1\cdots 1} := (a_{i_1\cdots i_m}) \in \mathbb{T}_{m,n-1}$ with $i_1, \ldots, i_m \in [n] \setminus \{1\}$ and $\mathscr{A}_{2\cdots 2} := (a_{i_1\cdots i_m}) \in \mathbb{T}_{m,n-1}$ with $i_1, \ldots, i_m \in [n] \setminus \{2\}$. Suppose that $a_{1i_2\cdots i_m} = a_{2i_2\cdots i_m}$ for all $i_2, \ldots, i_m \in [n]$, and both $\mathscr{A}_{1\cdots 1}$ and $\mathscr{A}_{2\cdots 2}$ are Q-tensors. Then $\mathscr{A}$ is a Q-tensor.*

Proof. For any $q = (q_1, \ldots, q_n)^T \in \mathbb{R}^n$, we consider the following two cases.

Case 1. $q_2 \leq q_1$. In this case, we denote

$$\mathcal{N} := \{(i_2, \ldots, i_m) : i_2, \ldots, i_m \in [n] \setminus \{1\}\} \quad \text{and} \quad q_{-1} := (q_2, \ldots, q_n)^T.$$

Then, for any $x = (0, \hat{x}^T)^T \in \mathbb{R} \times \mathbb{R}^{n-1}$ with $\hat{x} := (x_2, \ldots, x_n)^T \in \mathbb{R}^{n-1}$, it follows that $\mathscr{A}_{1\cdots 1}\hat{x}^{m-1} = ((\mathscr{A}_{1\cdots 1}\hat{x}^{m-1})_1, \ldots, (\mathscr{A}_{1\cdots 1}\hat{x}^{m-1})_{n-1})^T \in \mathbb{R}^{n-1}$ satisfying

$$(\mathscr{A}_{1\cdots 1}\hat{x}^{m-1})_i = \sum_{(i_2,\ldots,i_m)\in\mathcal{N}} a_{i+1 i_2\cdots i_m} x_{i_2} \cdots x_{i_m}$$

$$= \sum_{i_2,\ldots,i_m\in[n]} a_{i+1 i_2\cdots i_m} x_{i_2}\cdots x_{i_m} = (\mathscr{A}x^{m-1})_{i+1} \tag{2.2}$$

for all $i \in [n-1]$; and

$$\begin{aligned}(\mathscr{A}x^{m-1})_1 &= \sum_{i_2,\ldots,i_m\in[n]} a_{1 i_2\cdots i_m} x_{i_2}\cdots x_{i_m} \\ &= \sum_{i_2,\ldots,i_m\in[n]} a_{2 i_2\cdots i_m} x_{i_2}\cdots x_{i_m} = (\mathscr{A}x^{m-1})_2 \end{aligned}\tag{2.3}$$

since $a_{1i_2\cdots i_m} = a_{2i_2\cdots i_m}$ for all $i_2,\ldots,i_m \in [n]$.

Since $\mathscr{A}_{1\cdots 1} \in \mathbb{T}_{m,n-1}$ is a Q-tensor, it follows that $\mathrm{TCP}(q_{-1},\mathscr{A}_{1\cdots 1})$ has a solution, say $\hat{y} := (\bar{y}_2,\ldots,\bar{y}_n)^T$. Then, $\hat{y} \in \mathbb{R}^{n-1}_+$ and for any $i \in [n-1]$,

$$0 \le (\mathscr{A}_{1\cdots 1}\hat{y}^{m-1} + q_{-1})_i = (\mathscr{A}_{1\cdots 1}\hat{y}^{m-1})_i + q_{i+1}, \tag{2.4}$$

$$\begin{aligned} 0 &= \hat{y}_i(\mathscr{A}_{1\cdots 1}\hat{y}^{m-1} + q_{-1})_i = \bar{y}_{i+1}(\mathscr{A}_{1\cdots 1}\hat{y}^{m-1})_i + \bar{y}_{i+1}(q_{-1})_i \\ &= \bar{y}_{i+1}q_{i+1} + \bar{y}_{i+1}\sum_{(i_2,\ldots,i_m)\in\mathcal{N}} a_{i+1 i_2\cdots i_m}\bar{y}_{i_2}\cdots\bar{y}_{i_m}. \end{aligned}\tag{2.5}$$

Let $y := (0,\hat{y})^T$. Then, we have that $y \in \mathbb{R}^n_+$,

$$\begin{aligned} (\mathscr{A}y^{m-1}+q)_1 &= (\mathscr{A}y^{m-1})_1 + q_1 \\ &= (\mathscr{A}y^{m-1})_2 + q_1 \qquad \text{(by (2.3))} \\ &\ge (\mathscr{A}y^{m-1})_2 + q_2 \qquad \text{(by } q_1 \ge q_2\text{)} \\ &= (\mathscr{A}_{1\cdots 1}\hat{y}^{m-1})_1 + q_2 \qquad \text{(by (2.2))} \\ &\ge 0, \qquad \text{(by (2.4))} \\ (\mathscr{A}y^{m-1}+q)_{i+1} &= (\mathscr{A}_{1\cdots 1}\hat{y}^{m-1})_i + q_{i+1} \qquad \text{(by (2.2))} \\ &\ge 0,\ \forall i \in [n-1], \qquad \text{(by (2.4))} \end{aligned}$$

and

$$\begin{aligned} y_1(\mathscr{A}y^{m-1}+q)_1 &= 0, \qquad \text{(since } y_1 = 0\text{)} \\ y_{i+1}(\mathscr{A}y^{m-1}+q)_{i+1} &= \bar{y}_{i+1}q_{i+1} + \bar{y}_{i+1}(\mathscr{A}_{1\cdots 1}\hat{y}^{m-1})_i \qquad \text{(by (2.2))} \\ &= \bar{y}_{i+1}q_{i+1} + \bar{y}_{i+1}\sum_{(i_2,\ldots,i_m)\in\mathcal{N}} a_{i+1 i_2\cdots i_m}\bar{y}_{i_2}\cdots\bar{y}_{i_m} \\ &= 0,\ \forall i \in [n-1], \qquad \text{(by (2.5))} \end{aligned}$$

Thus, y is a solution to $\mathrm{TCP}(q,\mathscr{A})$.

Case 2. $q_1 \le q_2$. In this case, by using the condition that $\mathscr{A}_{2\cdots 2} \in \mathbb{T}_{m,n-1}$ is a Q-tensor, similar to the proof of Case 1, we can obtain that $\mathrm{TCP}(q,\mathscr{A})$ has a solution.

Combining Case 1 with Case 2, we complete the proof of this theorem. □

3 Equivalent Classes of Q-Tensors within the Class of Strong P_0-Tensors

Let $\mathbf{P}_0$ $(\mathbf{R}_0, \mathbf{R}, \mathbf{Q})$ denote the set of all real P_0-matrices (R_0-matrices, R-matrices, Q-matrices); and $\mathbb{R}^{n\times n} \bigcap \mathbf{P}_0$ denotes the set of all real P_0-matrices of $n \times n$. In 1979, Agangic and Cottle [1] obtained the following results.

Proposition 3.1. *If* $A \in \mathbb{R}^{n\times n} \bigcap \mathbf{P}_0$*, then*

$$A \in \mathbf{R}_0 \quad \Longleftrightarrow \quad A \in \mathbf{R} \quad \Longleftrightarrow \quad A \in \mathbf{Q}.$$

Such a proposition gives two equivalent classes of Q-matrices within the class of P_0-matrices. After such a pioneer work, the problem on equivalent class of Q-matrices has been extensively studied in the literature. See, for example, Pang [30], Jeter and Pye [21], Gowda [13], etc.

In this section, we try to extend the above results to the case of tensor.

Note that both P_0-tensor and P_0'-tensor are extensions of P_0-matrix. It is natural to consider whether the following result holds or not:

If $\mathscr{A} \in \mathbb{T}_{m,n} \bigcap \mathcal{P}_0$ (or $\mathscr{A} \in \mathbb{T}_{m,n} \bigcap \mathcal{P}_0'$), then

$$\mathscr{A} \in \mathcal{R}_0 \quad \Longleftrightarrow \quad \mathscr{A} \in \mathcal{R} \quad \Longleftrightarrow \quad \mathscr{A} \in \mathcal{Q}. \tag{3.1}$$

It is regret that this conjecture does not hold, which can be seen by the following examples.

Example 3.2. Let $\mathscr{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{T}_{4,2}$, where $a_{1122} = a_{2222} = 1, a_{2112} = -1$ and all other $a_{i_1i_2i_3i_4} = 0$. Then $\mathscr{A} \in \mathbb{T}_{4,2} \bigcap \mathcal{P}_0 \bigcap \mathcal{Q}$, but $\mathscr{A} \notin \mathbb{T}_{4,2} \bigcap \mathcal{P}_0 \bigcap \mathcal{R}_0$.

We show that the results in Example 3.2 hold. Obviously, for any $x \in \mathbb{R}^2$,

$$\mathscr{A} x^3 = \begin{pmatrix} x_1x_2^2 \\ x_2^3 - x_1^2x_2 \end{pmatrix},$$

and hence,

$$x_1(\mathscr{A} x^3)_1 = x_1^2x_2^2 \quad \text{and} \quad x_2(\mathscr{A} x^3)_2 = x_2^4 - x_1^2x_2^2. \tag{3.2}$$

It is easy to see from (3.2) that for any $x \in \mathbb{R}^2$ with $x \neq 0$, if $x_1 \neq 0$, then $x_1(\mathscr{A} x^3)_1 = x_1^2x_2^2 \geq 0$; and if $x_1 = 0$, then $x_2 \neq 0$ since $x \neq 0$, and $x_2(\mathscr{A} x^3)_2 = x_2^4 \geq 0$. Thus, $\mathscr{A} \in \mathbb{T}_{4,2} \bigcap \mathcal{P}_0$. It is also easy to see from (3.2) that $(1,0)^T \in \mathbb{R}^2$ is a solution to TCP$(0, \mathscr{A})$, which, together with the definition of R_0-tensor, implies that $\mathscr{A} \notin \mathbb{T}_{4,2} \bigcap \mathcal{R}_0$. In the following, we show that $\mathscr{A} \in \mathbb{T}_{4,2} \bigcap \mathcal{Q}$. Let a and b be two nonnegative real numbers, we consider the following four cases:

C1. Let $q = (a^3, b^3)^T$, then $z = (0,0)^T$ is a solution to TCP$(q, \mathscr{A})$.

C2. Let $q = (a^3, -b^3)^T$, then $z = (0,b)^T$ is a solution to TCP$(q, \mathscr{A})$.

C3. Let $q = (-a^3, -b^3)^T$ with $(a,b) \neq (0,0)$, we show that TCP$(q, \mathscr{A})$ has a solution. In this case, in order to ensure that $(\mathscr{A} x^3)_i + q_i \geq 0$ for $i \in \{1,2\}$, it must hold that $x_1 \neq 0$ and $x_2 \neq 0$. So we need to show that the system of equations

$$0 = \mathscr{A} x^3 + q = \begin{pmatrix} x_1x_2^2 - a^3 \\ x_2^3 - x_1^2x_2 - b^3 \end{pmatrix} \tag{3.3}$$

has a nonnegative solution. From the first equation in (3.3) it follows that $x_1 = \frac{a^3}{x_2^2}$; and hence, the second equation in (3.3) becomes

$$(x_2^3)^2 - b^3x_2^3 - a^6 = 0.$$

It is easy to see that the above equation has a solution $x_2^* := [(b^3 + \sqrt{b^6 + 4a^6})/2]^{1/3} > 0$. Furthermore, $(a^3/(x_2^*)^2, x_2^*)^T$ is a solution to TCP$(q, \mathscr{A})$.

C4. Let $q = (-a^3, b^3)^T$. Similar to the proof given in the case C3, we can obtain that TCP$(q, \mathscr{A})$ has a solution in this case.

Combining the above four cases, we obtain that $\mathscr{A} \in \mathbb{T}_{4,2} \bigcap \mathcal{Q}$.

Example 3.2 demonstrates that (3.1) cannot be obtained under the assumption that $\mathscr{A} \in \mathbb{T}_{m,n} \bigcap \mathcal{P}_0$.

Example 3.3. Let $\mathscr{A} = (a_{i_1 i_2 i_3}) \in \mathbb{T}_{3,2}$, where $a_{122} = a_{222} = 1, a_{212} = -1$ and all other $a_{i_1 i_2 i_3} = 0$. Then $\mathscr{A} \in \mathbb{T}_{3,2} \bigcap \mathcal{P}_0' \bigcap \mathcal{Q}$, but $\mathscr{A} \notin \mathbb{T}_{3,2} \bigcap \mathcal{P}_0' \bigcap \mathcal{R}_0$.

We show that the results in Example 3.3 hold.

First, it is obvious that TCP$(0, \mathscr{A})$ is to find $x \in \mathbb{R}^2$ such that

$$x \geq 0, \quad \mathscr{A} x^2 = \begin{pmatrix} x_2^2 \\ x_2^2 - x_1 x_2 \end{pmatrix} \geq 0, \quad x^T \mathscr{A} x^2 = 0.$$

It is easy to see that $(1,0)^T$ is a solution to TCP$(0, \mathscr{A})$; and hence, $\mathscr{A} \notin \mathbb{T}_{3,2} \bigcap \mathcal{R}_0$.

Second, we show that $\mathscr{A}$ is a P_0'-tensor. For any $x \in \mathbb{R}^2$ with $x \neq 0$,

if $x_1 \neq 0$, then $x_1^2 (\mathscr{A} x^2)_1 = x_1^2 x_2^2 \geq 0$; and

if $x_1 = 0$, then $x_2 \neq 0$ and $x_2^2 (\mathscr{A} x^2)_2 = x_2^4 > 0$.

Thus, $\mathscr{A}$ is a P_0'-tensor.

Third, we prove that $\mathscr{A} \in \mathbb{T}_{3,2} \bigcap \mathcal{Q}$. Let a and b be two nonnegative real numbers.

C1. Let $q = (a^2, b^2)^T$. Obviously, $(0,0)^T$ is a solution to TCP$(q, \mathscr{A})$.

C2. Let $q = (-a^2, b^2)^T$ with $a \neq 0$. Take $z := ((a^2 + b^2)/a, a)^T$, then

$$z \geq 0, \;\; \mathscr{A} z^2 + q = \begin{pmatrix} a^2 - a^2 \\ a^2 - a \times \frac{a^2 + b^2}{a} + b^2 \end{pmatrix} = 0, \;\; z^T (\mathscr{A} z^2 + q) = 0.$$

Thus, z solves TCP$(q, \mathscr{A})$ in this case.

C3. Let $q = (a^2, -b^2)^T$. Take $z := (0, b)^T$, then

$$z \geq 0, \;\; \mathscr{A} z^2 + q = \begin{pmatrix} b^2 + a^2 \\ 0 \end{pmatrix} \geq 0, \;\; z^T (\mathscr{A} z^2 + q) = 0.$$

Thus, z solves TCP$(q, \mathscr{A})$ in this case.

C4. Let $q = (-a^2, -b^2)^T$ with $a \leq b$. Take $z := (0, b)^T$, then

$$z \geq 0, \;\; \mathscr{A} z^2 + q = \begin{pmatrix} b^2 - a^2 \\ 0 \end{pmatrix} \geq 0, \;\; z^T (\mathscr{A} z^2 + q) = 0.$$

Thus, z solves TCP$(q, \mathscr{A})$ in this case.

C5. Let $q = (-a^2, -b^2)^T$ with $0 \neq a \geq b$. Take $z := ((a^2 - b^2)/a, a)^T$, then

$$z \geq 0, \;\; \mathscr{A} z^2 + q = \begin{pmatrix} a^2 - a^2 \\ a^2 - a \times \frac{a^2 - b^2}{a} - b^2 \end{pmatrix} = 0, \;\; z^T (\mathscr{A} z^2 + q) = 0.$$

Thus, z solves TCP$(q, \mathscr{A})$ in this case.

Therefore, it follows from C1-C5 that $\mathrm{TCP}(q, \mathscr{A})$ has a solution for each $q \in \mathbb{R}^2$. Thus, $\mathscr{A} \in \mathbb{T}_{3,2} \bigcap \mathcal{Q}$.

Example 3.3 demonstrates that (3.1) cannot be obtained under the assumption that $\mathscr{A} \in \mathbb{T}_{m,n} \bigcap \mathcal{P}_0'$.

Combining Example 3.2 with Example 3.3, we obtain that (3.1) cannot be obtained within the class of P_0-tensors or P_0'-tensors.

In order to extend the result of Proposition 3.1 to the case of tensor, we introduce a new class of tensors in the following. Recall that a function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called a P_0-function if, for all $x, y \in \mathbb{R}^n$ with $x \neq y$, there is an index $i \in [n]$ such that

$$x_i \neq y_i \quad \text{and} \quad (x_i - y_i)[f_i(x) - f_i(y)] \geq 0.$$

It is well known that an affine mapping $f(x) := Ax + q$ with $q \in \mathbb{R}^n$ is a P_0-function if and only if $A \in \mathbb{R}^{n\times n}$ is a P_0-matrix. Inspired by such a result, we introduce a class of tensors which is defined as follows.

Definition 3.1. Given $\mathscr{A} \in \mathbb{T}_{m,n}$. If the mapping $f(x) := \mathscr{A}x^{m-1} + q$ with $q \in \mathbb{R}^n$ is a P_0-function, we call $\mathscr{A}$ is a strong P_0-tensor, abbreviated as SP_0-tensor, and denote the set of all real m-th order n-dimensional SP_0-tensors by $\mathbb{T}_{m,n} \bigcap \mathcal{SP}_0$.

Obviously, when $m = 2$, an SP_0-tensor reduces to a P_0-matrix. Thus, SP_0-tensor is an extension of P_0-matrix from the matrix space to the tensor space. It is easy to see from Definition 2.1(i) and Definition 3.1 that $\mathbb{T}_{m,n} \bigcap \mathcal{SP}_0 \subseteq \mathbb{T}_{m,n} \bigcap \mathcal{P}_0$.

In the following, we extend the results of Proposition 3.1 to the tensor space.

Theorem 3.4. *If $\mathscr{A} \in \mathbb{T}_{m,n} \bigcap \mathcal{SP}_0$, we have*

$$\mathscr{A} \in \mathcal{R}_0 \iff \mathscr{A} \in \mathcal{R} \iff \mathscr{A} \in \mathcal{ER} \iff \mathscr{A} \in \mathcal{Q}. \tag{3.4}$$

Proof. Since an SP_0-tensor is a P_0-tensor and every P_0-tensor is semi-positive, it follows from Proposition 2.1(ii) that

$$\mathscr{A} \in \mathcal{R}_0 \iff \mathscr{A} \in \mathcal{R} \iff \mathscr{A} \in \mathcal{ER}. \tag{3.5}$$

Thus, in order to show that (3.4) holds, we only need to show that

$$\mathscr{A} \in \mathcal{R}_0 \iff \mathscr{A} \in \mathcal{Q}. \tag{3.6}$$

Suppose that $\mathscr{A} \in \mathcal{R}_0$, then $\mathscr{A} \in \mathcal{R}$ by (3.5). This, together with Proposition 2.1(i), implies that $\mathscr{A} \in \mathcal{Q}$. Thus, in order to show that (3.6) holds, we only need to show that $\mathscr{A} \in \mathcal{R}_0$ under the condition that $\mathscr{A} \in \mathcal{Q} \bigcap \mathcal{SP}_0$. Suppose that $\mathscr{A} \in \mathcal{Q}$ but $\mathscr{A} \notin \mathcal{R}_0$. Then there exists a vector $\bar{x} \in \mathbb{R}^n_+ \setminus \{0\}$ such that

$$\begin{cases} (\mathscr{A}\bar{x}^{m-1})_i &= 0, \quad \text{if } \bar{x}_i > 0, \\ (\mathscr{A}\bar{x}^{m-1})_i &\geq 0, \quad \text{if } \bar{x}_i = 0. \end{cases}$$

Denote $\mathcal{I} = \{i \in [n] : \bar{x}_i = 0\}$ and $\mathcal{J} = \{i \in [n] : \bar{x}_i > 0\}$. Take $q \in \mathbb{R}^n$ satisfying $q_i > 0$ for any $i \in \mathcal{I}$ and $q_i < 0$ for any $i \in \mathcal{J}$. Since $\mathscr{A} \in \mathcal{Q}$, we can assume that $\bar{y}$ is a solution of $\mathrm{TCP}(q, \mathscr{A})$. It is obvious that $\bar{x} \neq \bar{y}$. Let λ be a positive real number.

For any $i \in \{i \in [n] : \bar{x}_i \neq \bar{y}_i\}$, we consider the following two cases.

If $i \in \mathcal{I}$, then $\bar{x}_i = 0$ and $\bar{y}_i > 0$, and hence, it follows that $(\mathscr{A}\bar{x}^{m-1})_i \geq 0$ and $(\mathscr{A}\bar{y}^{m-1}+q)_i = 0$. The above equality implies that $(\mathscr{A}\bar{y}^{m-1})_i < 0$ since $q_i > 0$ for any $i \in \mathcal{I}$. This further yields that $(\mathscr{A}(\lambda\bar{y})^{m-1})_i = \lambda^{m-1}(\mathscr{A}\bar{y}^{m-1})_i < 0$ for any $i \in \mathcal{I}$ since $\lambda > 0$. So, for any $i \in \mathcal{I}$,

$$(\bar{x}_i - \lambda\bar{y}_i)[(\mathscr{A}\bar{x}^{m-1})_i - (\mathscr{A}(\lambda\bar{y})^{m-1})_i] < 0$$

holds for any $\lambda > 0$.

If $i \in \mathcal{J}$, then $(\mathscr{A}\bar{x}^{m-1})_i = 0$ and $(\mathscr{A}\bar{y}^{m-1} + q)_i \geq 0$. The above inequality implies that $(\mathscr{A}\bar{y}^{m-1})_i > 0$ since $q_i < 0$ for any $i \in \mathcal{J}$, which yields that $(\mathscr{A}(\lambda\bar{y})^{m-1})_i = \lambda^{m-1}(\mathscr{A}\bar{y}^{m-1})_i > 0$ for any $\lambda > 0$. Now, we can choose sufficiently small $\lambda > 0$ such that $(\bar{x} - \lambda\bar{y})_i > 0$. So, for any $i \in \mathcal{J}$,

$$(\bar{x}_i - \lambda\bar{y}_i)[(\mathscr{A}\bar{x}^{m-1})_i - (\mathscr{A}(\lambda\bar{y})^{m-1})_i] < 0$$

holds for any sufficiently small $\lambda > 0$.

Thus, we can choose sufficiently small $\lambda > 0$ such that for any $i \in \{i \in [n] : \bar{x}_i \neq \bar{y}_i\}$,

$$(\bar{x}_i - \lambda\bar{y}_i)[(\mathscr{A}\bar{x}^{m-1})_i - (\mathscr{A}(\lambda\bar{y})^{m-1})_i] < 0,$$

which contradicts the condition that $\mathscr{A} \in \mathcal{SP}_0$. Therefore, $\mathscr{A} \in \mathcal{R}_0$; and the desired results are obtained. □

In the following, we discuss the relationships among three classes of P_0-type tensors. First, we construct the following example.

Example 3.5. Let $\mathscr{A} = (a_{i_1\cdots i_m}) \in \mathbb{T}_{m,n}$, where $a_{122\cdots 2} = 1$ and all other $a_{i_1\cdots i_m} = 0$. Then $\mathscr{A} \in \mathbb{T}_{m,n} \bigcap \mathcal{SP}_0 \bigcap \mathcal{P}_0 \bigcap \mathcal{P}_0'$.

We show that the results in Example 3.5 hold. Obviously, for any $x \in \mathbb{R}^n$, we have

$$\mathscr{A}x^{m-1} = \begin{pmatrix} x_2^{m-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n.$$

On one hand, for any $x, y \in \mathbb{R}^n$ with $x \neq y$, if there exists an index $i_0 \in \{2, \ldots, n\}$ such that $x_{i_0} \neq y_{i_0}$, then $(x_{i_0} - y_{i_0})[(\mathscr{A}x^{m-1})_{i_0} - (\mathscr{A}y^{m-1})_{i_0}] = 0$. Otherwise, we have that $x_i = y_i$ for all $i \in \{2, \ldots, n\}$, and hence, $x_1 \neq y_1$ and $x_2 = y_2$. Furthermore, $(x_1 - y_1)[(\mathscr{A}x^{m-1})_1 - (\mathscr{A}y^{m-1})_1] = (x_1 - y_1)(x_2^{m-1} - y_2^{m-1}) = 0$. So $\mathscr{A} \in \mathbb{T}_{m,n} \bigcap \mathcal{SP}_0$.

On the other hand, for any $x \in \mathbb{R}^n$ with $x \neq 0$, if there exists an index $i_0 \in \{2, \ldots, n\}$ such that $x_{i_0} \neq 0$, then $x_{i_0}(\mathscr{A}x^{m-1})_{i_0} = 0$ and $x_{i_0}^{m-1}(\mathscr{A}x^{m-1})_{i_0} = 0$. Otherwise, we have that $x_i = 0$ for all $i \in \{2, \ldots, n\}$, and hence, $x_1 \neq 0$ and $x_2 = 0$. Furthermore, $x_1(\mathscr{A}x^{m-1})_1 = x_1x_2^{m-1} = 0$ and $x_1^{m-1}(\mathscr{A}x^{m-1})_1 = x_1^{m-1}x_2^{m-1} = 0$. So $\mathscr{A} \in \mathbb{T}_{m,n} \bigcap \mathcal{P}_0 \bigcap \mathcal{P}_0'$.

Proposition 3.6. *We have that* $\mathbb{T}_{m,n} \bigcap \mathcal{SP}_0 \bigcap \mathcal{P}_0 \bigcap \mathcal{P}_0' \neq \emptyset$.

Proof. The results of the proposition hold directly from Example 3.5. □

Second, we consider the relationship between the class of P_0-tensors and the class of SP_0-tensor. We construct the following example.

Example 3.7. Let $\mathscr{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{T}_{4,2}$, where $a_{1122} = a_{2122} = 1$ and all other $a_{i_1i_2i_3i_4} = 0$. Then $\mathscr{A} \in \mathbb{T}_{4,2} \bigcap \mathcal{P}_0$, but $\mathscr{A} \notin \mathbb{T}_{4,2} \bigcap \mathcal{SP}_0$.

We show that the results in Example 3.7 hold. Obviously, for any $x \in \mathbb{R}^2$ with $x \neq 0$, we have

$$\mathscr{A}x^3 = \begin{pmatrix} x_1x_2^2 \\ x_1x_2^2 \end{pmatrix}.$$

If $x_1 \neq 0$, then $x_1(\mathscr{A}x^3)_1 = x_1^2x_2^2 \geq 0$; and if $x_1 = 0$, then $x_2 \neq 0$, and $x_2(\mathscr{A}x^3)_2 = x_1x_2^3 = 0$. So we obtain that $\mathscr{A} \in \mathbb{T}_{4,2} \bigcap \mathcal{P}_0$.

In addition, for any given $q \in \mathbb{R}^2$, let $f(x) = \mathscr{A}x^3 + q$ for any $x \in \mathbb{R}^2$; and take $\bar{x} = (1,1)^T$ and $\bar{y} = (1,-2)^T$, then it is easy to see that

$$\bar{x}_1 = \bar{y}_1, \quad \text{and} \quad \bar{x}_2 \neq \bar{y}_2,\ (\bar{x}_2 - \bar{y}_2)(f(\bar{x}) - f(\bar{y}))_2 = -9 < 0.$$

These demonstrate that $\mathscr{A} \notin \mathbb{T}_{4,2} \bigcap \mathcal{SP}_0$.

Proposition 3.8. *If $\mathscr{A}$ is an SP_0-tensor, then it is a P_0-tensor. But the converse is not true.*

Proof. The first result holds from the definition of SP_0-tensor given in Definition 3.1 and the definition of P_0-tensor given in Definition 2.1(i); and the second result holds from Example 3.7. □

Third, we consider the relationship between the class of P_0-tensors and the class of P_0'-tensors. We construct the following two examples.

Example 3.9. Let $\mathscr{A} = (a_{i_1i_2i_3}) \in \mathbb{T}_{3,2}$, where $a_{121} = 1, a_{211} = -1$ and all other $a_{i_1i_2i_3} = 0$. Then $\mathscr{A} \in \mathbb{T}_{3,2} \bigcap \mathcal{P}_0$, but $\mathscr{A} \notin \mathbb{T}_{3,2} \bigcap \mathcal{P}_0'$.

We show that the results in Example 3.9 hold. Obviously, for any $x \in \mathbb{R}^2$, we have

$$\mathscr{A}x^2 = \begin{pmatrix} x_2x_1 \\ -x_1^2 \end{pmatrix}.$$

On one hand, from $x_1(\mathscr{A}x^2)_1 = x_2x_1^2$ and $x_2(\mathscr{A}x^2)_2 = -x_2x_1^2$, it is easy to see that for any $x \in \mathbb{R}^2$ with $x \neq 0$, there exists an index $i \in \{1,2\}$ such that $x_i \neq 0$ and $x_i(\mathscr{A}x^2)_i \geq 0$, i.e., $\mathscr{A} \in \mathbb{T}_{3,2} \bigcap \mathcal{P}_0$. On the other hand, for any $\alpha > 0$ and $\beta < 0$, by taking $(x_1, x_2) := (\alpha, \beta)$, we have

$$x_1^2(\mathscr{A}x^2)_1 = x_2x_1^3 = \beta\alpha^3 < 0 \quad \text{and} \quad x_2^2(\mathscr{A}x^2)_2 = -x_2^2x_1^2 = -\beta^2\alpha^2 < 0,$$

and hence, $\mathscr{A} \notin \mathbb{T}_{3,2} \bigcap \mathcal{P}_0'$.

Example 3.10. Let $\mathscr{A} = (a_{i_1i_2i_3}) \in \mathbb{T}_{3,2}$, where $a_{122} = 1, a_{211} = -1$ and all other $a_{i_1i_2i_3} = 0$. Then $\mathscr{A} \in \mathbb{T}_{3,2} \bigcap \mathcal{P}_0'$, but $\mathscr{A} \notin \mathbb{T}_{3,2} \bigcap \mathcal{P}_0$.

We show that the results in Example 3.10 hold. Obviously, for any $x \in \mathbb{R}^2$, we have

$$\mathscr{A}x^2 = \begin{pmatrix} x_2^2 \\ -x_1^2 \end{pmatrix}.$$

On one hand, from $x_1^2(\mathscr{A}x^2)_1 = x_1^2x_2^2$ and $x_2^2(\mathscr{A}x^2)_2 = -x_2^2x_1^2$, it is easy to see that for any $x \in \mathbb{R}^2$ with $x \neq 0$, there exists an index $i \in \{1,2\}$ such that $x_i \neq 0$ and $x_i^2(\mathscr{A}x^2)_i \geq 0$, i.e.,

$\mathscr{A} \in \mathbb{T}_{3,2} \bigcap \mathcal{P}_0'$. On the other hand, for any $\alpha < 0$ and $\beta > 0$, by taking $(x_1, x_2) := (\alpha, \beta)$, we have

$$x_1(\mathscr{A}x^2)_1 = x_1 x_2^2 = \alpha\beta^2 < 0 \quad \text{and} \quad x_2(\mathscr{A}x^2)_2 = -x_2 x_1^2 = -\beta\alpha^2 < 0,$$

and hence, $\mathscr{A} \notin \mathbb{T}_{3,2} \bigcap \mathcal{P}_0$.

Proposition 3.11. (i) *We have $\mathbb{T}_{m,n} \bigcap \mathcal{P}_0' = \mathbb{T}_{m,n} \bigcap \mathcal{P}_0$ when m is even.* (ii) *There is no inclusion relation between the class of odd order P_0-tensors and the class of odd order P_0'-tensors.*

Proof. The result (i) holds directly from the definitions of P_0-tensor and P_0'-tensor given in Definition 2.1; and the result (ii) holds directly from Examples 3.9 and 3.10. □

Fourth, we consider the relationship between the class of SP_0-tensor and the class of P_0'-tensor. From Proposition 3.8 and Proposition 3.11(i), we obtain immediately the following results.

Proposition 3.12. *When m is even, every m-th order SP_0-tensor is an m-th order P_0'-tensor, but the converse is not true.*

In the following, we consider the odd order SP_0-tensors and odd order P_0'-tensors.

Lemma 3.13. *Let $\mathscr{A} \in \mathbb{T}_{m,n} \bigcap \mathcal{SP}_0$ with m being odd. Then, for any $i \in [n]$, we have that either $(\mathscr{A}x^{m-1})_i \equiv 0$ or $(\mathscr{A}x^{m-1})_i$ is a function of variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, but independent of the variable x_i.*

Proof. For any $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, suppose that a is an arbitrary fixed real number and $i_0 \in [n]$, we take $y = (x_1, \ldots, x_{i_0-1}, a, x_{i_0+1}, \ldots, x_n)^T$, then for any

$$i \in \mathcal{I} := \{1, \ldots, i_0 - 1, i_0 + 1, \ldots, n\},$$

we have $x_i = y_i$. For any $x_{i_0} \in \mathbb{R} \setminus \{a\}$, we have $x_{i_0} \neq y_{i_0}$, which, together with $\mathscr{A} \in \mathcal{SP}_0$, implies that

$$(x_{i_0} - a)[(\mathscr{A}x^{m-1})_{i_0} - (\mathscr{A}y^{m-1})_{i_0}] \geq 0. \tag{3.7}$$

For $-x$ and $-y$, we have $\mathscr{A}(-x)^{m-1} = \mathscr{A}x^{m-1}$ since m is odd; and $-x_{i_0} \neq -y_{i_0}$ and $-x_i = -y_i$ for any $i \in \mathcal{I}$. These and $\mathscr{A} \in \mathcal{SP}_0$ imply that

$$(a - x_{i_0})[(\mathscr{A}x^{m-1})_{i_0} - (\mathscr{A}y^{m-1})_{i_0}] \geq 0. \tag{3.8}$$

Combining (3.7) with (3.8), we obtain that for any $x_{i_0} \in \mathbb{R} \setminus \{a\}$,

$$(\mathscr{A}x^{m-1})_{i_0} = (\mathscr{A}y^{m-1})_{i_0}.$$

By the arbitrariness of x_{i_0}, the above equality implies that either $(\mathscr{A}x^{m-1})_{i_0} \equiv 0$ or $(\mathscr{A}x^{m-1})_{i_0}$ is independent of the variable x_{i_0}. Furthermore, the desired results holds by the arbitrariness of i_0. □

Proposition 3.14. *Let m be odd. Then $\mathbb{T}_{m,2} \bigcap \mathcal{SP}_0 \subseteq \mathbb{T}_{m,2} \bigcap \mathcal{P}_0'$.*

Proof. Given $\mathscr{A} \in \mathbb{T}_{m,2} \bigcap \mathcal{SP}_0$. Then $\mathscr{A} \in \mathbb{T}_{m,2} \bigcap \mathcal{P}_0$.

First, we show that for any $x \in \mathbb{R}^2$, there exists an index $i \in \{1, 2\}$ such that $(\mathscr{A}x^{m-1})_i \equiv 0$. We assume that $(\mathscr{A}x^{m-1})_i \not\equiv 0$ for all $i \in \{1, 2\}$. Since $\mathscr{A} \in \mathbb{T}_{m,2} \bigcap \mathcal{SP}_0$, it follows from Lemma 3.13 that

$$\mathscr{A}x^{m-1} = \begin{pmatrix} \alpha x_2^{m-1} \\ \beta x_1^{m-1} \end{pmatrix},$$

where $\alpha, \beta \in \mathbb{R} \setminus \{0\}$. Without loss of generality, we assume that $\bar{x} = (\bar{x}_1, \bar{x}_2)^T \in \mathbb{R}^2_+$ such that $(\mathscr{A}x^{m-1})_i \neq 0$ for all $i \in \{1, 2\}$. Take

$$\hat{x} = \begin{cases} (-\bar{x}_1, -\bar{x}_2)^T & \text{if } (\mathscr{A}\bar{x}^{m-1})_1 > 0 \text{ and } (\mathscr{A}\bar{x}^{m-1})_2 > 0, \\ (-\bar{x}_1, \bar{x}_2)^T & \text{if } (\mathscr{A}\bar{x}^{m-1})_1 > 0 \text{ and } (\mathscr{A}\bar{x}^{m-1})_2 < 0, \\ (\bar{x}_1, -\bar{x}_2)^T & \text{if } (\mathscr{A}\bar{x}^{m-1})_1 < 0 \text{ and } (\mathscr{A}\bar{x}^{m-1})_2 > 0, \\ (\bar{x}_1, \bar{x}_2)^T & \text{if } (\mathscr{A}\bar{x}^{m-1})_1 < 0 \text{ and } (\mathscr{A}\bar{x}^{m-1})_2 < 0, \end{cases}$$

then $\mathscr{A}\hat{x}^{m-1} = \mathscr{A}\bar{x}^{m-1}$ since m is odd; and hence, $\hat{x}_1(\mathscr{A}\hat{x}^{m-1})_1 < 0$ and $\hat{x}_2(\mathscr{A}\hat{x}^{m-1})_2 < 0$, which is a contradiction with $\mathscr{A} \in \mathcal{P}_0$.

Second, we show that $\mathscr{A} \in \mathbb{T}_{m,2} \bigcap \mathcal{P}'_0$. Without loss of generality, we assume that $(\mathscr{A}x^{m-1})_2 \equiv 0$ for any $x \in \mathbb{R}^2$. For any $x \in \mathbb{R}^2$ with $x \neq 0$,

if $x_2 \neq 0$, then $x_2^{m-1}(\mathscr{A}x^{m-1})_2 = 0$; and

if $x_2 = 0$, then $x_1 \neq 0$ and $x_1^{m-1}(\mathscr{A}x^{m-1})_1 = x_1^{m-1}(\alpha x_2^{m-1}) = 0$,

so $\mathscr{A} \in \mathcal{P}'_0$. Therefore, the desired result holds. □

From Example 3.5 and Proposition 3.14, we have obtained some relationship between $\mathbb{T}_{m,n} \bigcap \mathcal{SP}_0$ and $\mathbb{T}_{m,n} \bigcap \mathcal{P}'_0$. But this does not give a full characterization for the relationship between $\mathbb{T}_{m,n} \bigcap \mathcal{SP}_0$ and $\mathbb{T}_{m,n} \bigcap \mathcal{P}'_0$. We conjecture that it is possible that the class of SP_0-tensors is a proper subset of the class of P'_0-tensors, which needs to be further studied in the future.

At the end of this section, we give a characterization of SP_0-tensor. Song and Qi [33] gave the concept of principal sub-tensors and proved that the principal sub-tensors of every P_0-tensor is a P_0-tensor. A tensor $\mathscr{C} \in \mathbb{T}_{m,r}$ is called a principal sub-tensor of tensor $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{T}_{m,n}$ $(1 \leq r \leq n)$ if there is a set $\mathcal{J}$ that composed of r elements in $[n]$ such that $\mathscr{C} = (a_{i_1 i_2 \cdots i_m})$ for all $i_1, \ldots, i_m \in \mathcal{J}$.

Theorem 3.15. *Let $\mathscr{A} \in \mathbb{T}_{m,n}$ be an SP_0-tensor, then every principal sub-tensor of $\mathscr{A}$ is an SP_0-tensor.*

Proof. Let $\mathscr{A}_r^{\mathcal{J}}$ be an arbitrary principal sub-tensor of $\mathscr{A}$. Suppose that $\mathscr{A}_r^{\mathcal{J}}$ is not an SP_0-tensor, then for each pair of distinct vectors $x = (x_{j_1}, \ldots, x_{j_r})^T \in \mathbb{R}^r$ and $y = (y_{j_1}, \ldots, y_{j_r})^T \in \mathbb{R}^r$, it follows that the set $\mathcal{K} := \{j \in \mathcal{J} : x_j \neq y_j\}$ is nonempty and

$$(x_j - y_j)(\mathscr{A}_r^{\mathcal{J}} x^{m-1} - \mathscr{A}_r^{\mathcal{J}} y^{m-1})_j < 0, \quad \forall j \in \mathcal{K}.$$

Let $x^* = (x_1^*, \ldots, x_n^*)^T \in \mathbb{R}^n$ and $y^* = (y_1^*, \ldots, y_n^*)^T \in \mathbb{R}^n$ be defined by

$$x_i^* = \begin{cases} x_i & \text{if } i \in \mathcal{J}, \\ 0 & \text{otherwise,} \end{cases} \qquad y_i^* = \begin{cases} y_i & \text{if } i \in \mathcal{J}, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall i \in [n],$$

then it is easy to show that

$$(x_i^* - y_i^*)[(\mathscr{A}(x^*)^{m-1})_i - (\mathscr{A}(y^*)^{m-1})_i] < 0$$

holds for any $i \in \{i \in [n] : x_i^* \neq y_i^*\}$. This contradicts the condition that $\mathscr{A} \in \mathcal{SP}_0$. Thus, the desired result holds. □

4 Equivalent Classes of Q-Tensors within the Class of Nonnegative Tensors

The following result was obtained by Danao (see Theorem 4.8 in [7]).

Proposition 4.1. *If $A \in \mathbb{R}^{n\times n}$ is nonnegative, then*

$$A \in \mathbf{R}_0 \iff A \in \mathbf{Q}.$$

A natural question is whether the result of Proposition 4.1 can be extended to the case of tensor or not. In this section, we give a positive answer to this question, which is given as follows.

Theorem 4.2. *If $\mathscr{A} \in \mathbb{T}_{m,n}$ is nonnegative, we have*

$$\mathscr{A} \in \mathcal{R}_0 \iff \mathscr{A} \in \mathcal{R} \iff \mathscr{A} \in \mathcal{Q} \iff \mathscr{A} \in \mathcal{ER}. \tag{4.1}$$

Proof. We consider the following five cases.

(a) We show that $\mathscr{A} \in \mathcal{R}_0 \Rightarrow \mathscr{A} \in \mathcal{R}$. Suppose that $\mathscr{A}$ is not an R-tensor. By the definition of R-tensor, there exists $(\hat{x}, \hat{t}) \in (\mathbb{R}^n_+ \setminus \{0\}) \times \mathbb{R}_+$ such that

$$\begin{cases} (\mathscr{A}\hat{x}^{m-1})_i + \hat{t} = 0, & \text{if } \hat{x}_i > 0, \\ (\mathscr{A}\hat{x}^{m-1})_i + \hat{t} \geq 0, & \text{if } \hat{x}_i = 0. \end{cases}$$

If $\hat{t} = 0$, then the above contradicts the condition that $\mathscr{A} \in \mathcal{R}_0$. So $\hat{t} > 0$. Thus, for any $i \in [n]$, we have

$$\begin{cases} (\mathscr{A}\hat{x}^{m-1})_i = -\hat{t} < 0, & \text{if } \hat{x}_i > 0, \\ (\mathscr{A}\hat{x}^{m-1})_i + \hat{t} \geq 0, & \text{if } \hat{x}_i = 0, \end{cases}$$

which yields

$$\mathscr{A}\hat{x}^m = \hat{x}^T \mathscr{A}\hat{x}^{m-1} = \sum_{i=1}^n \hat{x}_i (\mathscr{A}(\hat{x})^{m-1})_i < 0. \tag{4.2}$$

In addition, since $\hat{x} \in \mathbb{R}^n_+ \setminus \{0\}$ and $\mathscr{A}$ is a nonnegative tensor, i.e., $a_{i_1\cdots i_m} \geq 0$ for all $i_1, \ldots, i_m \in [n]$, we have

$$\mathscr{A}\hat{x}^m = \sum_{i_1,\ldots,i_m=1}^n a_{i_1\cdots i_m} \hat{x}_{i_1} \cdots \hat{x}_{i_m} \geq 0,$$

which contradicts (4.2). So $\mathscr{A} \in \mathcal{R}_0 \Rightarrow \mathscr{A} \in \mathcal{R}$.

(b) We show that $\mathscr{A} \in \mathcal{R} \Rightarrow \mathscr{A} \in \mathcal{Q}$. Such a result holds from Proposition 2.1(i).

(c) We show that $\mathscr{A} \in \mathcal{Q} \Rightarrow \mathscr{A} \in \mathcal{R}_0$. Suppose that $\mathscr{A} \notin \mathcal{R}_0$. Then, by the definition of R_0-tensor, there exists $\hat{x} \in \mathbb{R}^n_+$ with $\hat{x} \neq 0$ such that

$$\begin{cases} (\mathscr{A}\hat{x}^{m-1})_i = 0, & \text{if } \hat{x}_i > 0, \\ (\mathscr{A}\hat{x}^{m-1})_i \geq 0, & \text{if } \hat{x}_i = 0. \end{cases}$$

Thus, there exists $j \in [n]$ such that $\hat{x}_j > 0$, and then, $(\mathscr{A}\hat{x}^{m-1})_j = 0$. Since $\mathscr{A} \in \mathcal{Q}$, it follows from Proposition 2.1(iv) that $a_{j\cdots j} > 0$. So,

$$0 = (\mathscr{A}\hat{x}^{m-1})_j = \sum_{i_2,\ldots,i_m=1}^{n} a_{ji_2\cdots i_m}\hat{x}_{i_2}\cdots\hat{x}_{i_m} \geq a_{j\cdots j}x_j\cdots x_j > 0,$$

which derives a contradiction. Thus, $\mathscr{A} \in \mathcal{Q} \Rightarrow \mathscr{A} \in \mathcal{R}_0$.

(d) We show that $\mathscr{A} \in \mathcal{R}_0 \;\Rightarrow\; \mathscr{A} \in \mathcal{ER}$. Suppose that $\mathscr{A}$ is not an ER-tensor. By the definition of ER-tensor, there exists $(\hat{x}, \hat{t}) \in (\mathbb{R}^n_+ \setminus \{0\}) \times \mathbb{R}_+$ such that

$$\begin{cases} (\mathscr{A}\hat{x}^{m-1})_i + \hat{t}\hat{x}_i & = & 0, & \text{if } \hat{x}_i > 0, \\ (\mathscr{A}\hat{x}^{m-1})_i & \geq & 0, & \text{if } \hat{x}_i = 0. \end{cases}$$

If $\hat{t} = 0$, then the above contradicts the condition that $\mathscr{A} \in \mathcal{R}_0$. So $\hat{t} > 0$. Denote $\mathcal{I} := \{i \in [n] : \hat{x}_i > 0\}$, then

$$\mathscr{A}\hat{x}^m = \hat{x}^T\mathscr{A}\hat{x}^{m-1} = \sum_{i=1}^{n}\hat{x}_i(\mathscr{A}\hat{x}^{m-1})_i = \sum_{i\in\mathcal{I}}\hat{x}_i(-\hat{t}\hat{x}_i) < 0. \tag{4.3}$$

In addition, since $\hat{x} \in \mathbb{R}^n_+ \setminus \{0\}$ and $\mathscr{A}$ is a nonnegative tensor, we have

$$\mathscr{A}\hat{x}^m = \sum_{i_1,\ldots,i_m=1}^{n} a_{i_1\cdots i_m}\hat{x}_{i_1}\cdots\hat{x}_{i_m} \geq 0,$$

which contradicts (4.3). So $\mathscr{A} \in \mathcal{R}_0 \Rightarrow \mathscr{A} \in \mathcal{ER}$.

(e) We show that $\mathscr{A} \in \mathcal{ER} \;\Rightarrow\; \mathscr{A} \in \mathcal{Q}$. Such a result holds from Proposition 2.1(iii).

Combining cases (a)-(e), we obtain that (4.1) holds within the class of nonnegative tensors. □

In Theorems 3.4 and 4.2, we have obtained several equivalent classes of Q-tensors within the class of SP_0-tensors and the class of nonnegative tensors, respectively. What is the relationship between the class of SP_0-tensors and the class of nonnegative tensors? We see the following two examples.

Example 4.3. Let $\mathscr{A} = (a_{i_1i_2i_3}) \in \mathbb{T}_{3,2}$, where $a_{111} = 1, a_{222} = 1$ and all other $a_{i_1i_2i_3} = 0$. Then $\mathscr{A}$ is nonnegative, but $\mathscr{A} \notin \mathbb{T}_{3,2}\bigcap\mathcal{SP}_0$.

We show that the results in Example 4.3 hold. On one hand, it is obvious that $\mathscr{A}$ is nonnegative. On the other hand, we have

$$\mathscr{A}x^2 = \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix}, \quad \forall x \in \mathbb{R}^2.$$

Take $x = (-2, -3)^T$, $y = (1, 2)^T$, then

$$\max\left\{(x_i - y_i)(\mathscr{A}x^2 - \mathscr{A}y^2)_i : i \in \{1, 2\}\right\} = \max\{-9, -12\} = -9 < 0,$$

and hence, $\mathscr{A} \notin \mathbb{T}_{3,2}\bigcap\mathcal{SP}_0$.

Example 4.4. Let $\mathscr{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{T}_{4,2}$, where $a_{1122} = -1, a_{2222} = 1$ and all other $a_{i_1i_2i_3i_4} = 0$. Then $\mathscr{A}$ is not nonnegative, but $\mathscr{A} \in \mathbb{T}_{4,2}\bigcap\mathcal{SP}_0$.

We show that the results in Example 4.4 hold. On one hand, it is obvious that $\mathscr{A}$ is not nonnegative. On the other hand, we have

$$\mathscr{A}x^3 = \begin{pmatrix} -x_1x_2^2 \\ x_2^3 \end{pmatrix}, \quad \forall x \in \mathbb{R}^2.$$

For any $x, y \in \mathbb{R}^2$ with $x \neq y$, we have

$$\begin{aligned} \max\left\{(x_i - y_i)[(\mathscr{A}x^3)_i - (\mathscr{A}y^3)_i)] : i \in \{1,2\}\right\} &\geq (x_2 - y_2)[(\mathscr{A}x^3)_2 - (\mathscr{A}y^3)_2] \\ &= (x_2 - y_2)(x_2^3 - y_2^3) \\ &= (x_2 - y_2)^2(x_2^2 + x_2y_2 + y_2^2) \\ &\geq 0, \end{aligned}$$

and hence, $\mathscr{A} \in \mathbb{T}_{4,2} \bigcap \mathcal{SP}_0$.

From Examples 4.3 and 4.4, it follows that the class of nonnegative tensors is different from the class of SP_0-tensors. Combining Theorems 3.4 and 4.2, we have obtained that (4.1) holds within the class of nonnegative tensors or the class of SP_0-tensors.

5 Q-Tensors, Semi-Positive Tensors and Copositive Tensors

In this section, we show that three known results related to Q-matrices cannot be extended to the tensor space by using several examples.

First, we consider the following result which was obtained by Pang [30].

Proposition 5.1. *Let $A \in \mathbb{R}^{n\times n} \bigcap \mathbf{L}_1 \bigcap \mathbf{Q}$. If x^* is a nonzero solution to LCP$(0, A)$, then x^* contains at least two nonzero components.*

Since it is not clear whether such a result can be extended to the case of tensor or not, Song and Qi proposed the following question (see Question 3.1 in [32]):

> Whether or not a nonzero solution x of TCP$(0, \mathscr{A})$ contains at least two nonzero components if $\mathscr{A}$ is a semi-positive Q-tensor.

We now answer this question by using Example 3.3. Let $\mathscr{A}$ be given by Example 3.3. On one hand, it has been showed that $\mathscr{A} \in \mathcal{P}_0' \bigcap \mathcal{Q}$, which implies that $\mathscr{A}$ is a semi-positive Q-tensor since every P_0'-tensor is a semi-positive tensor. On the other hand, it is obvious that $(1, 0)^T$ is a solution to TCP$(0, \mathscr{A})$. Therefore, by Example 3.3 we obtain that the results of Proposition 5.1 cannot be extended to the case of tensor.

Second, we consider the following result which was obtained by Pang [30].

Proposition 5.2. *Let $A \in \mathbb{R}^{n\times n} \bigcap \mathbf{L}_1 \bigcap \mathbf{Q}$. Then the system*

$$Mx = 0, \quad x > 0$$

is inconsistent.

A natural question is whether or not the system

$$\mathscr{A}x^{m-1} = 0, \quad x > 0 \tag{5.1}$$

is inconsistent for any semi-positive $\mathscr{A} \in \mathbb{T}_{m,n} \bigcap \mathcal{Q}$. We now answer this question by constructing the following example.

Example 5.3. Let $m \geq 3$ be odd and $\mathscr{A} = (a_{i_1 \cdots i_m}) \in \mathbb{T}_{m,2}$, where

$$a_{\underbrace{1\cdots1}_{m-i}\underbrace{2\cdots2}_{i}} = a_{\underbrace{2\cdots2}_{i+1}\underbrace{1\cdots1}_{m-1-i}} = (-1)^i C_{m-1}^i, \quad \forall i \in \{1, \ldots, m-1\}$$

and all other $a_{i_1 \cdots i_m} = 0$. Then $\mathscr{A} \in \mathbb{T}_{m,2} \bigcap \mathcal{Q}$ is semi-positive, but $(1,1)^T$ is a solution to the system (5.1).

We show that the results in Example 5.3 hold. Obviously, for any $x \in \mathbb{R}^n$, it follows that

$$\mathscr{A} x^{m-1} = \begin{pmatrix} (x_1 - x_2)^{m-1} \\ (x_1 - x_2)^{m-1} \end{pmatrix}.$$

It is easy to see that $(1,1)^T$ is a solution to the system (5.1). Moreover, for any $x \in \mathbb{R}^2_+$ with $x \neq 0$,

if $x_1 > 0$, then $x_1(\mathscr{A} x^2)_1 = x_1(x_1 - x_2)^{m-1} \geq 0$; and

if $x_1 = 0$, then $x_2 > 0$ and $x_2(\mathscr{A} x^2)_2 = x_2(x_1 - x_2)^{m-1} > 0$.

Thus, $\mathscr{A}$ is a semi-positive tensor. In the following, we prove that $\mathscr{A} \in \mathbb{T}_{m,2} \bigcap \mathcal{Q}$. For any $a, b \in \mathbb{R}_+$, we consider the following five cases.

C1. Let $q = (a^{m-1}, b^{m-1})^T$. Obviously, $(0,0)^T$ is a solution to $\mathrm{TCP}(q, \mathscr{A})$.

C2. Let $q = (-a^{m-1}, b^{m-1})^T$. Take $z := (a, 0)^T$, then

$$z \geq 0, \quad \mathscr{A} z^{m-1} + q = \begin{pmatrix} 0 \\ a^{m-1} + b^{m-1} \end{pmatrix} \geq 0, \quad z^T(\mathscr{A} z^{m-1} + q) = 0.$$

Thus, z solves $\mathrm{TCP}(q, \mathscr{A})$ in this case.

C3. Let $q = (a^{m-1}, -b^{m-1})^T$. Take $z := (0, b)^T$, then

$$z \geq 0, \quad \mathscr{A} z^{m-1} + q = \begin{pmatrix} a^{m-1} + b^{m-1} \\ 0 \end{pmatrix} \geq 0, \quad z^T(\mathscr{A} z^{m-1} + q) = 0.$$

Thus, z solves $\mathrm{TCP}(q, \mathscr{A})$ in this case.

C4. Let $q = (-a^{m-1}, -b^{m-1})^T$ with $a \leq b$. Take $z := (0, b)^T$, then

$$z \geq 0, \quad \mathscr{A} z^{m-1} + q = \begin{pmatrix} b^{m-1} - a^{m-1} \\ 0 \end{pmatrix} \geq 0, \quad z^T(\mathscr{A} z^{m-1} + q) = 0.$$

Thus, z solves $\mathrm{TCP}(q, \mathscr{A})$ in this case.

C5. Let $q = (-a^{m-1}, -b^{m-1})^T$ with $a \geq b$. Take $z := (a, 0)^T$, then

$$z \geq 0, \quad \mathscr{A} z^{m-1} + q = \begin{pmatrix} 0 \\ a^{m-1} - b^{m-1} \end{pmatrix} \geq 0, \quad z^T(\mathscr{A} z^{m-1} + q) = 0.$$

Thus, z solves $\mathrm{TCP}(q, \mathscr{A})$ in this case.

Combining cases C1-C5, we obtain that TCP$(q, \mathscr{A})$ has a solution for each $q \in \mathbb{R}^2$. Thus, $\mathscr{A} \in \mathbb{T}_{m,2} \bigcap \mathcal{Q}$.

Example 5.3 shows that the result of Proposition 5.2 cannot be extended to the case of tensor.

Third, we consider the following result which was obtained by Jeter and Pye [20].

Proposition 5.4. *If $M \in \mathbb{R}^{n\times n}$ is copositive and $n \leq 3$, then*

$$M \in \mathbf{Q} \quad \Longleftrightarrow \quad M \in \mathbf{R}_0.$$

A natural question is whether the above result can be extended to the tensor space or not. In order to answer this question, we use Examples 3.2 and 3.3. It has been proved that if $\mathscr{A}$ is given by Example 3.2 or Example 3.3, then $\mathscr{A} \in \mathcal{Q}$ but $\mathscr{A} \notin \mathcal{R}_0$. In addition,

if $\mathscr{A}$ is given by Example 3.2, then $x^T \mathscr{A} x^3 = x_2^4 \geq 0$ for any $x \in \mathbb{R}^2_+$ with $x \neq 0$; and

if $\mathscr{A}$ is given by Example 3.3, then $x^T \mathscr{A} x^2 = x_2^3 \geq 0$ for any $x \in \mathbb{R}^2_+$ with $x \neq 0$,

which imply that if $\mathscr{A}$ is given by Example 3.2 or Example 3.3, then $\mathscr{A}$ is copositive. Thus, the result of Proposition 5.4 cannot be extended to the case of tensor. Moreover, by using Example 5.3 we can also obtain the above result.

6 Conclusions

In this paper, we studied Q-tensor and gave a sufficient condition to judge whether a tensor is a Q-tensor or not. In order to extend Agangic-Cottle's result to the tensor space, we introduced the concept of SP_0-tensor and showed that within the class of SP_0-tensors, four classes of tensors, i.e., R_0-tensors, R-tensors, ER-tensors and Q-tensors, are all equivalent. We clarified that the above equivalence does not hold if SP_0-tensors is replaced by P_0-tensors or P_0'-tensors by constructing two examples; and discussed the relationships among three classes of P_0-type tensors, i.e., P_0-tensors, P_0'-tensors and SP_0-tensors. In order to extend Danao's result to the tensor space, we showed that the above equivalence holds within the class of nonnegative tensors. We also discussed the relationship between the class of SP_0-tensors and the class of nonnegative tensors. Moreover, by using several examples we also showed that three famous results, related to Q-matrices, semi-positive matrices and copositive matrices, cannot be extended to the tensor space. Since Q-matrix plays an important role in the field of structured matrices and the theory of complementarity problems, fruitful results related to Q-matrices have been obtained in the literature. We believe more results related to Q-matrices can be clarified whether they can be extended to the case of tensor or not.

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