Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 16, No. 2, pp. 213–224, 2020

NOVEL STABILITY CRITERIA FOR NONLINEAR STOCHASTIC HOPFIELD NEURAL NETWORKS WITH TIME-VARYING DELAYS*

Yi-Fu Feng† and Jing Wang

Abstract: This paper investigates the stability analysis of nonlinear stochastic Hopfield neural networks (NSHNNs) with time-varying delays which are described by a Takagi-Sugeno (T-S) fuzzy model. By using an improved homogeneous matrix polynomials technique, novel convergent stability criteria are proposed. More importantly, as the selected design parameter becomes larger, less conservative stability criteria will be obtained since more algebraic property of the fuzzy weighting functions in the unit simplex can be utilized to deal with stability analysis. Finally, an illustrative example is provided to show the efficiency of the proposed approach.

Key words: *hopfield neural networks, stochastic dynamical systems, Takagi-Sugeno(T-S) fuzzy model, homogeneous matrix polynomials*

Mathematics Subject Classification: *93C10,93C42,93D05*

---

1 Introduction

In recent years, the Hopfield neural networks (HNNs) [15] have been extensively investigated due to their important applications in various fields such as combinatorial optimization, signal estimation, image processing and pattern recognition, etc. It is worth noting that these applications are built upon the stability of the equilibrium of HNNs. Therefore, stability analysis of HNNs is a necessary step for further applications of HNNs. More importantly, it has been realized that the interactions between neurons are generally asynchronous, which inevitably results in time delays for either biological or artificial neural networks. Thus, there are lots of stability results for various neural networks with several kinds of time delays [1, 6, 9, 11, 14, 17, 24]. It should be mentioned that the time delays in neural networks are different from network-induced delays investigated in [18, 20, 22]. Note that fuzzy logic theory has shown to be an appealing and efficient approach to deal with the analysis and synthesis problems for complex nonlinear systems. In [16], the well-known Takagi-Sugeno fuzzy model was proposed to transform a class of nonlinear dynamic systems into a set of linear sub-models by defining a linear input-output relationship as its consequence of individual plant rule. More recently, the T-S fuzzy model has been applied to various kinds of nonlinear systems and some nice results have been obtained [2, 3, 5, 7, 10, 13].

*This work was supported in part by the Education Department of Jilin Province, China, under Science and Technology Research Project (Grant No. 20150214).
†Corresponding author.

Moreover, there are many stochastic perturbations that affect the stability of neural networks as performing most of practical physical computation. If stochastic perturbations are taken into account, the underlying neural networks can be usually destabilized, which implies that the stability analysis of nonlinear stochastic neural networks is of paramount significance [4,12,23,25]. Some global stability criteria for T-S fuzzy Hopfield neural networks with stochastic perturbations and time delays have been provided by applying Lyapunov functionals and model transformation with free weighting matrices. It is the fact that stability is the basic requirement of control systems [19,21]. When studying stability analysis of T-S fuzzy Hopfield neural networks, simple decomposition techniques were usually applied to covert parameterized matrix inequalities into a set of linear matrix inequalities, which will lead to much conservatism.

In this paper, novel stability criteria for nonlinear stochastic Hopfield neural networks with time delays are developed. Benefit from more algebraic properties of the fuzzy weighting functions in the unit simplex can be utilized in stability analysis, the fuzzy weighting functions that govern the behavior of the fuzzy system are seen as dynamic constraints. These dynamic constraints are incorporated into the stability criteria through the usage of the homogeneous matrix polynomials technique. As the value of the selected design parameter increases, the conservatism will be further reduced and less conservative LMI-based stability criteria will also be obtained in a convergent sense.

The remainder of this paper is organized as follows. In Section 2, we introduce some model formulation and preliminaries. In Section 3, the main results for asymptotical stability of NSHNNs are presented. In Section 4, the system performance analysis is provided to demonstrate the effectiveness of the newly derived results. Finally, in Section 5, some conclusions are drawn.

Notations. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively; I is the identity matrix with appropriate dimensions; T stands for the transpose of a matrix; for symmetric matrices X and Y, the notation $X > Y (X \geq Y)$ means that $X - Y$ is positive definite(respectively, positive semi-definite); $\mathbb{Z}_+$ denotes the set of nonnegative integers $\{0, 1, 2, ...\}$ and $M! = M(M-1)(M-2)\cdots(2)(1)$ for $M \in \mathbb{Z}_+$; $\Delta_N = \{\alpha \in \mathbb{R}^N; \sum_{i=1}^N \alpha_i = 1; \alpha_i \geq 0\}$. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2 Problem Formulation and Preliminaries

2.1 Nonlinear stochastic Hopfield neural networks with time-varying delays described by a T-S fuzzy model

The mathematical model of Hopfield neural networks with time-varying delay can be expressed as follows:

$$\dot{u}_i(t) = -c_i u_i(t) + \textstyle\sum_{j=1}^n a_{ij} g_j(u_j(t - \tau(t))) + J_i, \tag{2.1}$$

where $i = 1, 2, ..., n$, $u_i(t)$ is the state variable of the ith neuron at time t; $c_i > 0$ represents the passive decay rate; a_{ij} is the synaptic connection weight; $g_j(\cdot)$ is the activation function of the neuron; J_i denotes the external input; $\tau(t)$ represents the time-varying delay of neural networks satisfying $0 < \tau(t) \leq h$ and $\dot{\tau}(t) < \sigma$.

We introduce the following assumption about $g_j(u_j(t))$.

Assumption 2.1.

$$0 \leq \frac{g_i(x) - g_i(y)}{x - y} \leq L_i, i = 1, 2, ..., n,$$

for all $x, y \in \mathbb{R}, x \neq y$ and denotes $L = \{L_1, L_2, ..., L_n\}$.

It is reasonable to assume that the Hopfield neural network (1) has only one equilibrium point denoted by

$$u^* = (u_1^*, u_2^*, \ldots, u_n^*).$$

Shifting the equilibrium to the origin by transformation $x(t) = u(t) - u^*$, one can derive the following system:

$$\frac{dx(t)}{dt} = -Cx(t) + Af(x(t - \tau(t))), \tag{2.2}$$

where

$$x(t) = [x_1(t), x_2(t), \ \cdots, x_n(t)]^T \in \mathbb{R}^n, \ C = (c_1, c_2, \cdots, c_n),$$
$$A = (a_{ij})_{n\times n}, \ f(x) = [f_1(x_1), f_2(x_2), ..., f_n(x_n)] \in \mathbb{R}^n$$

with $f_i(x_i) = g_i(x_i + u_i^*) - g_i(u_i^*)(i = 1, 2, ..., n)$. Under the above assumption, it is easy to get $|f_i(x_i(t))| \leq L_i|x_i(t)|(i = 1, 2, ..., n)$.

As mentioned previously, stochastic perturbations in neural networks are always unavailable in practice. Therefore, the ith rule of the T-S fuzzy neural network with stochastic perturbations can be described as:

Plant Rule i:
IF $\theta_1(t)$ is η_1^i, and ..., and $\theta_p(t)$ is η_p^i, **THEN**

$$dx(t) = [-C_i x(t) + A_i f(x(t - \tau(t)))]dt + [M_i x(t) + N_i x(t - \tau(t))]d\varpi(t), \tag{2.3}$$

where $i = 1, 2, ..., r$, η_i^i $(i = 1, 2, ..., p)$ is the fuzzy set, $\theta(t) = [\theta_1(t), ..., \theta_p(t)]^T$ is the premise variable vector, r is the number of IF-THEN rules. $\varpi(t)$ is a one-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$. $\phi \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ is the initial value of (3). C_i, A_i, M_i and N_i are constant known real matrices.

By using product of inference, singleton fuzzifier, and center-average defuzzifier, the defuzzified output of the stochastic T-S fuzzy system (3) is represented as follows:

$$\begin{aligned} dx(t) \quad = \quad & \sum_{i=1}^{r} \mu_i(\theta(t)) \{[-C_i x(t) + A_i f(x(t - \tau(t)))]dt \\ & + [M_i x(t) + N_i x(t - \tau(t))]d\varpi(t)\}, \end{aligned} \tag{2.4}$$

where $\mu_i(\theta(t)) = \frac{\nu_i(\theta(t))}{\sum_{j=1}^{r} \nu_j(\theta(t))}$, $\nu_i(\theta(t)) = \prod_{j=1}^{p} \eta_j^k(\theta_j(t))$ in which $\eta_j^i(\theta_j(t))$ is the grade of membership of θ_j^i in η_j^i. According to the theory of fuzzy sets, we have:

$$\mu_i(\theta(t)) \geq 0, \quad \sum_{i=1}^{r} \mu_i(\theta(t)) = 1. \tag{2.5}$$

For stochastic systems, Itô formula plays an important role in the stability analysis. Consider a general stochastic system $dx(t) = f(x(t), t)dt + g(x(t), t)d\varpi(t)$ on $t \geq t_0$ with initial value $x(t_0) = x_0 \in \mathbb{R}^n$, where $f :\in \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^{n\times m}$ and $g :\in \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^{n\times m}$. Let $\mathcal{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$ denote the family of all nonnegative functions $V(x, t)$ on $\mathbb{R}^n \times \mathbb{R}^+$ which are continuously twice differentiable in x and once differentiable in t, while the stochastic operator $\mathcal{L}V(x, t)$ is defined from $\mathbb{R}^n \times \mathbb{R}^+$ to $\mathbb{R}$ by

$$\mathcal{L}V(x, t) = V_t(x, t) + V_x(x, t)f(x, t) + \frac{1}{2}\text{trace}[g^T(x, t)V_{xx}(x, t)g(x, t)],$$

where

$$V_t(x,t) = \frac{\partial V(x,t)}{\partial t},\ V_x(x,t) = \left(\frac{\partial V(x,t)}{\partial x_1}, \cdots, \frac{\partial V(x,t)}{\partial x_n}\right), V_{xx}(x,t) = \left(\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

We conclude this section with the following lemma.

Lemma 2.1 ([8]). *For any positive definite symmetric constant matrix $M \in \mathbb{R}^{n\times n}$, the scalars $r_1 < r_2$ and a vector function $w : [r_1, r_2] \to \mathbb{R}^n$ such that the concerned integrations are well defined, then the following inequality holds:*

$$\left(\int_{r_1}^{r_2} w(s)ds\right)^T M \int_{r_1}^{r_2} w(s)ds \le r_{12} \int_{r_1}^{r_2} w^T(s) M w(s) ds$$

where $r_{12} = r_2 - r_1$.

2.2 Homogeneous matrix polynomials

The following definitions about homogeneous matrix polynomials are introduced.

The set Δ_r is defined as $\Delta_r = \{\alpha \in \mathbb{R}^r; \sum_{i=1}^r \alpha_i = 1; \alpha \ge 0\}$. Define $\alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_r^{k_r}$, $\alpha \in \triangle_r, k_i \in \mathbb{Z}_+, i = 1, 2, ..., r$ as the monomials, $k = k_1k_2\cdots k_r$, and $P_k \in \mathbb{R}^{n\times n}, \forall k \in \mathcal{K}(g)$ are matrix-valued coefficients. Here, $\mathcal{K}(g)$ is the set of r-tuples obtained as all possible combinations of nonnegative integers $k_i, i = 1, 2, ..., r$, such that $k_1 + k_2 + \cdots + k_r = g$. Since the number of vertices is equal to r, the number of elements in $\mathcal{K}(g)$ is given by

$$J(g) = \frac{(r + g - 1)!}{g!(r - 1)!}. \tag{2.6}$$

To give an example, for homogeneous polynomials with the degree $g = 3$ and $r = 2$ variables, the possible values of the partial degrees are $\mathcal{K}(4) = 03, 12, 21, 30$, $J(4) = 4$, corresponding to the generic from $P_3(\alpha) = \alpha_2^3 P_{03} + \alpha_1\alpha_2^2 P_{12} + \alpha_1^2\alpha_2 P_{21} + \alpha_1^3 P_{30}$.

By definition, for r-tuples k and k', one can write $k \ge k'$ if $k_i \ge k_i', (i = 1, ..., r)$. The usual operations of summation, $k + k'$, and subtraction, $k - k'$ (whenever $k \ge k'$), are defined componentwise. Consider the following definitions for the r-tuple $\chi_i \in \mathcal{K}(1)$ and the coefficient $\pi(k)$:

$$\chi_i = 0\cdots 0 \underbrace{1}_{i-\text{th}} 0\cdots 0,\ \pi(k) = (k_1!)(k_2!)\cdots(k_r!),$$

$$\mu_i = \mu_i(\theta(t)),\ \mu = (\mu_1(\theta(t)), \cdots, \mu_r(\theta(t)))^T,\ \mu^k = \mu_1^{k_1}\mu_2^{k_2}\cdots\mu_r^{k_r}.$$

The introduced homogeneous matrix polynomials contribute to stability analysis for nonlinear stochastic Hopfield neural networks.

3 Main Results

For simplicity, we use x, μ_i instead of $x(t), \mu_i(\theta(t))$, respectively, in the sequel. For convenience, let

$$C(\mu) = \sum_{i=1}^r \mu_i C_i,\ A(\mu) = \sum_{i=1}^r \mu_i A_i,\ M(\mu) = \sum_{i=1}^r \mu_i M_i,$$

$$N(\mu) = \sum_{i=1}^r \mu_i N_i,\ x_\tau = x(t - \tau(t)),\ \tau = \tau(t),$$

then the system (2.4) can be rewritten as:

$$dx = [-C(\mu)x + A(\mu)f(x_\tau)]dt + [M(\mu)x + N(\mu)x_\tau]d\varpi(t). \tag{3.1}$$

Theorem 3.1. *For given positive scalars h and σ, the stochastic Hopfield neural network (2.4) is globally asymptotically stable in the mean square sense, if there exist two positive diagonal matrices X and Y, real matrices $P > 0, Q > 0, R > 0, S > 0, W_{ik}, V_{ij}(i = 1, 2, ..., 6; j = 1, 2, ..., r)$, , such that the following LMIs hold:*

$$\sum_{1\le i\le r, k-2\chi_i\ge 0} \frac{d!}{\pi(k-2\chi_i)}\Delta_{ii} + \sum_{1\le i<j\le r, k-\chi_i-\chi_j\ge 0} \frac{d!}{\pi(k-\chi_i-\chi_j)}\Delta_{ij} < 0,$$
$$\forall k \in \mathcal{K}(d+2), d \in \mathbb{Z}_+, \tag{3.2}$$

where $\Delta_{ii} = \begin{bmatrix} \Delta_{ii}^{11} & \Delta_{ii}^{12} & \Delta_{ii}^{13} & \Delta_{ii}^{14} & \Delta_{ii}^{15} & \Delta_{ii}^{16} & V_{1i} \\ * & \Delta_{ii}^{22} & \Delta_{ii}^{23} & \Delta_{ii}^{24} & \Delta_{ii}^{25} & \Delta_{ii}^{26} & V_{2i} \\ * & * & \Delta_{ii}^{33} & \Delta_{ii}^{34} & 0 & \Delta_{ii}^{36} & V_{3i} \\ * & * & * & \Delta_{ii}^{44} & \Delta_{ii}^{45} & \Delta_{ii}^{46} & V_{4i} \\ * & * & * & * & \Delta_{ii}^{55} & \Delta_{ii}^{56} & V_{5i} \\ * & * & * & * & * & \Delta_{ii}^{66} & V_{6i} \\ * & * & * & * & * & * & -S \end{bmatrix}$, *and*

$\Delta_{ii}^{11} = W_{1i}C_i + C_i^TW_{1i}^T + V_{1i} + V_{1i}^T + hM_i^TSM_i + M_i^TPM_i$,
$\Delta_{ii}^{12} = W_{2i} + C_i^TW_{2i}^T + V_{2i}^T - V_{1i} + hM_iSN_i + M_i^TPN_i$,
$\Delta_{ii}^{13} = XL + W_{3i} + C_i^TW_{3i}^T + V_{3i}^T$, $\Delta_{ii}^{14} = W_{4i} + C_i^TW_{4i}^T - W_{1i}A_i + V_{4i}^T$,
$\Delta_{ii}^{15} = P + W_{1i} + C_i^TW_{5i}^T + V_{5i}^T$, $\Delta_{ii}^{16} = C_i^TW_{6i}^T + V_{6i}^T - V_{1i}$,
$\Delta_{ii}^{22} = -V_{2i} - V_{2i}^T + hN_i^TSN_i + N_i^TPN_i$, $\Delta_{ii}^{23} = -V_{3i}^T$, $\Delta_{ii}^{24} = YL - W_{2i}A_i - V_{4i}^T$,
$\Delta_{ii}^{25} = -V_{5i}^T$, $\Delta_{ii}^{26} = -V_{6i}^T - V_{2i}$, $\Delta_{ii}^{33} = Q - 2X$, $\Delta_{ii}^{34} = -W_{3i}A_i$, $\Delta_{ii}^{36} = -V_{3i}$,
$\Delta_{ii}^{44} = -(1-\sigma)Q - 2Y - W_{4i}A_i - A_i^TW_{4l}^T$, $\Delta_{ii}^{45} = -A_i^TW_{5i}^T$, $\Delta_{ii}^{46} = -A_i^TW_{6i}^T - V_{4i}$,
$\Delta_{ii}^{55} = hR + W_{5i} + W_{5i}^T$, $\Delta_{ii}^{56} = W_{6i}^T - V_{5i}$, $\Delta_{ii}^{66} = -\frac{1-\sigma}{h}R - V_{6i} - V_{6i}^T$,

$\Delta_{ij} = \begin{bmatrix} \Delta_{ij}^{11} & \Delta_{ij}^{12} & \Delta_{ij}^{13} & \Delta_{ij}^{14} & \Delta_{ij}^{15} & \Delta_{ij}^{16} & V_{1i}+V_{1j} \\ * & \Delta_{ij}^{22} & \Delta_{ij}^{23} & \Delta_{ij}^{24} & \Delta_{ij}^{25} & \Delta_{ij}^{26} & V_{2i}+V_{2j} \\ * & * & \Delta_{ij}^{33} & \Delta_{ij}^{34} & 0 & \Delta_{ij}^{36} & V_{3i}+V_{3j} \\ * & * & * & \Delta_{ij}^{44} & \Delta_{ij}^{45} & \Delta_{ij}^{46} & V_{4i}+V_{4j} \\ * & * & * & * & \Delta_{ii}^{55} & \Delta_{ij}^{56} & V_{5i}+V_{5j} \\ * & * & * & * & * & \Delta_{ij}^{66} & V_{6i}+V_{6j} \\ * & * & * & * & * & * & -2S \end{bmatrix}$, *and*

$\Delta_{ij}^{11} = W_{1i}C_j + C_i^TW_{1j}^T + V_{1i} + V_{1i}^T + hM_i^TSM_j + M_i^TPM_j + W_{1j}C_i + C_j^TW_{1i}^T + V_{1j} + V_{1j}^T + hM_j^TSM_i + M_j^TPM_i$,
$\Delta_{ij}^{12} = W_{2i} + C_i^TW_{2j}^T + V_{2i}^T - V_{1i} + hM_iSN_j + M_i^TPN_j + W_{2j} + C_j^TW_{2i}^T + V_{2j}^T - V_{1j} + hM_jSN_i + M_j^TPN_i$,
$\Delta_{ij}^{13} = XL + W_{3i} + C_i^TW_{3j}^T + V_{3i}^T + XL + W_{3j} + C_j^TW_{3i}^T + V_{3j}^T$,
$\Delta_{ij}^{14} = W_{4i} + C_i^TW_{4j}^T - W_{1i}A_j + V_{4i}^T + W_{4j} + C_j^TW_{4i}^T - W_{1j}A_i + V_{4j}^T$,
$\Delta_{ij}^{15} = P_i + W_{1i} + C_i^TW_{5j}^T + V_{5i}^T + P_j + W_{1j} + C_j^TW_{5i}^T + V_{5j}^T$,
$\Delta_{ij}^{16} = C_i^TW_{6j}^T + V_{6i}^T - V_{1i} + C_j^TW_{6i}^T + V_{6j}^T - V_{1j}$,
$\Delta_{ij}^{22} = -V_{2i} - V_{2i}^T + hN_i^TSN_j + N_i^TPN_j - V_{2j} - V_{2j}^T + hN_j^TSN_i + N_j^TPN_i$,
$\Delta_{ij}^{23} = -V_{3i}^T - V_{3j}^T$, $\Delta_{ij}^{24} = YL - W_{2i}A_j - V_{4i}^T + YL - W_{2j}A_i - V_{4j}^T$,
$\Delta_{ij}^{25} = -V_{5i}^T - V_{5j}^T$, $\Delta_{ij}^{26} = -V_{6i}^T - V_{2i} - V_{6j}^T - V_{2j}$,
$\Delta_{ij}^{33} = 2Q - 4X$, $\Delta_{ij}^{34} = -W_{3i}A_j - W_{3j}A_i$, $\Delta_{ij}^{36} = -V_{3i} - V_{3j}$,
$\Delta_{ij}^{44} = -2(1-\sigma)Q - 4Y - W_{4i}A_j - A_i^TW_{4j}^T - W_{4j}A_i - A_j^TW_{4i}^T$,

$\Delta_{ij}^{45} = -A_i^T W_{5j}^T - A_j^T W_{5i}^T, \Delta_{ij}^{46} = -A_i^T W_{6j}^T - V_{4i} - A_j^T W_{6i}^T - V_{4j},$
$\Delta_{ij}^{55} = 2hR + W_{5i} + W_{5i}^T + W_{5j} + W_{5j}^T, \Delta_{ij}^{56} = W_{6i}^T - V_{5i} + W_{6j}^T - V_{5j},$
$\Delta_{ij}^{66} = -2\frac{1-\sigma}{h} R - V_{6i} - V_{6i}^T - V_{6j} - V_{6j}^T.$

Proof. Choose a parameter-dependent Lyapunov-Krasovskii functional for the system (2.4) as follows:

$$V(x,t) = V_1(x,t) + V_2(x,t) + V_3(x,t) + V_4(x,t), \tag{3.3}$$

where

$$\begin{aligned} V_1(x,t) &= x^T P x, \\ V_2(x,t) &= \int_{t-\tau}^{t} f^T(x(s)) Q f(x(s)) ds, \\ V_3(x,t) &= \int_{-\tau}^{0} \int_{t+\theta}^{t} \vartheta^T(s) R \vartheta(s) ds d\theta, \\ V_4(x,t) &= \int_{-\tau}^{0} \int_{t+\theta}^{t} y^T(s) S y(s) ds d\theta, \end{aligned}$$

and $\vartheta(t) = -C(\mu)x + A(\mu) f(x_\tau),\ y(t) = M(\mu)x + N(\mu)x_\tau$.

By the Itô formula, we can calculate $\mathcal{L}V(x,t)$ along (2.4), then we have

$$\begin{aligned} \mathcal{L}V(x,t) \quad = \quad & 2x^T P \vartheta(t) + y^T(t) P y(t) + f^T(t) Q f(t) \\ & -(1-\dot{\tau}) f^T(t-\tau) Q f(t-\tau) + \tau g^T(t) R \vartheta(t) \\ & -(1-\dot{\tau}) \int_{t-\tau}^{t} \vartheta^T(s) R g(s) ds + \tau y^T(t) S y(t) \\ & -(1-\dot{\tau}) \int_{t-\tau}^{t} y^T(s) S y(s) ds. \end{aligned} \tag{3.4}$$

By applying Lemma 1, we obtain

$$-\int_{t-\tau}^{t} \vartheta^T(s) R \vartheta(s) ds \leq -\frac{1}{\tau} \left(\int_{t-\tau}^{t} \vartheta(s) ds \right)^T R \left(\int_{t-\tau}^{t} \vartheta(s) ds \right). \tag{3.5}$$

Introducing two diagonal positive definite matrices X and Y and using Assumption 1, we can obtain the following inequalities:

$$2f^T(x(t)) X L x(t) - 2f^T(x(t)) X f(x(t)) \geq 0, \tag{3.6}$$
$$2f^T(x(t-\tau)) Y L x(t-\tau) - 2f^T(x(t-\tau)) Y f(x(t-\tau)) \geq 0. \tag{3.7}$$

Moreover, it is easy to see that the following two equalities hold

$$0 = 2\xi^T(t) W(\mu) \cdot (\vartheta(t) + Cx - A f(x_\tau)), \tag{3.8}$$
$$0 = 2\xi^T(t) V(\mu) \cdot \left(x - x_\tau - \int_{t-\tau}^{t} \vartheta(s) ds - \int_{t-\tau}^{t} y(s) d\varpi(s) \right), \tag{3.9}$$

where $\xi(t) = \left[x^T, x_\tau^T, f^T(x(t)), f^T(x(t-\tau)), \vartheta^T(s), \upsilon^T \right]^T$, and $\upsilon = \left(\int_{t-\tau}^{t} \vartheta(s) ds \right)$,

$$W(\mu) = \left[\begin{array}{cccccc} W_1(\mu)^T & W_2(\mu)^T & W_3(\mu)^T & W_4(\mu)^T & W_5(\mu)^T & W_6(\mu)^T \end{array} \right]^T,$$

$$V(\mu) = \left[\begin{array}{cccccc} V_1(\mu)^T & V_2(\mu)^T & V_3(\mu)^T & V_4(\mu)^T & V_5(\mu)^T & V_6(\mu)^T \end{array}\right]^T,$$

with $W_i(\mu) = \sum_{j=1}^{r} \mu_j W_{ij}, V_i(\mu)^T = \sum_{j=1}^{r} \mu_j V_{ij},\ (i = 1, 2, ..., 6)$.

On the other hand, one has

$$\begin{aligned} -2\xi^T(t)V\int_{t-\tau}^{t} y(s)d\varpi(s) &\leq \frac{1}{1-\sigma}\xi^T(t)VS^{-1}V^T\xi(t) \\ &+(1-\sigma)\left(\int_{t-\tau}^{t} y(s)d\varpi(s)\right)^T S\left(\int_{t-\tau}^{t} y(s)d\varpi(s)\right). \end{aligned} \tag{3.10}$$

Thus

$$\begin{aligned} \mathcal{L}V(x,t) &\leq \xi^T(t)\left(\Xi + VS^{-1}V^T\right)\xi(t) - (1-\sigma)\int_{t-\tau}^{t} y^T(s)Sy(s)ds \\ &+ (1-\sigma)\left(\int_{t-\tau}^{t} y(s)d\varpi(s)\right)^T S\left(\int_{t-\tau}^{t} y(s)d\varpi(s)\right), \end{aligned} \tag{3.11}$$

where $\Xi = \left[\begin{array}{cccccc} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} \\ * & \Xi_{22} & \Xi_{23} & \Xi_{24} & \Xi_{25} & \Xi_{26} \\ * & * & \Xi_{33} & \Xi_{34} & 0 & \Xi_{36} \\ * & * & * & \Xi_{44} & \Xi_{45} & \Xi_{46} \\ * & * & * & * & \Xi_{55} & \Xi_{56} \\ * & * & * & * & * & \Xi_{66} \end{array}\right]$, with

$\Xi_{11} = W_1(\mu)C(\mu) + C(\mu)^T W_1^T(\mu) + V_1(\mu) + V_1^T(\mu) + hM(\mu)^T SM(\mu) + M(\mu)^T PM(\mu)$,
$\Xi_{12} = W_2(\mu) + C(\mu)^T W_2^T(\mu) + V_2^T(\mu) - V_1(\mu) + hM(\mu)SN(\mu) + M(\mu)^T P\bar{N}$,
$\Xi_{13} = XL + W_3(\mu) + C(\mu)^T W_3^T(\mu) + V_3^T(\mu)$, $\Xi_{14} = W_4(\mu) + C(\mu)^T W_4^T(\mu) - W_1(\mu)\bar{A} + V_4^T(\mu)$,
$\Xi_{15} = P + W_1(\mu) + C(\mu)^T W_5^T(\mu) + V_5^T(\mu)$, $\Xi_{16} = C(\mu)^T W_6^T(t\mu) + V_6^T(\mu) - V_1(\mu)$,
$\Xi_{22} = -V_2(\mu) - V_2^T(\mu) + hN(\mu)^T SN(\mu) + N(\mu)^T PN(\mu)$, $\Xi_{23} = -V_3^T(\mu)$,
$\Xi_{24} = YL - W_2(\mu)A(\mu) - V_4^T(\mu)$, $\Xi_{25} = -V_5^T(\mu)$, $\Xi_{26} = -V_6^T(\mu) - V_2(\mu)$, $\Xi_{33} = Q - 2X$,
$\Xi_{34} = -W_3(\mu)A(\mu)$, $\Xi_{36} = -V_3(\mu)$, $\Xi_{44} = -(1-\sigma)Q - 2Y - W_4(\mu)A(\mu) - A(\mu)^T W_4^T(\mu)$,
$\Xi_{45} = -A(\mu)^T W_5^T(\mu)$, $\Xi_{46} = -A(\mu)^T W_6^T(\mu) - V_4(\mu)$, $\Xi_{55} = hR + W_5(\mu) + W_5^T(\mu)$,
$\Xi_{56} = W_6^T(\mu) - V_5(\mu)$, $\Xi_{66} = -\frac{1-\sigma}{h}R - V_6(\mu) - V_6^T(\mu)$.

Using the well-known Schur complement lemma, it is easy to see that $\Xi + VS^{-1}V^T < 0$ is equivalent to:

$$\Delta = \left[\begin{array}{cc} \Xi & V \\ * & -S \end{array}\right] < 0, \tag{3.12}$$

On the other hand, recalling the fact that $\mu_1 + \mu_2 + \cdots + \mu_r = 1$, the expression of Δ can be represented as follows:

$$\begin{aligned} \Delta &= (\mu_1 + \mu_2 + \cdots + \mu_r)^d \Delta \\ &= \sum_{k\in\mathcal{K}(d+2)} \mu^k \left(\sum_{1\leq i\leq r, k-2\chi_i\geq 0} \frac{d!}{\pi(k-2\chi_i)}\Delta_{ii} + \sum_{1\leq i<j\leq r, k-\chi_i-\chi_j\geq 0} \frac{d!}{\pi(k-\chi_i-\chi_j)}\Delta_{ij} \right), \end{aligned} \tag{3.13}$$

where Δ_{ii} and Δ_{ij} have been defined above with $d \in \mathbb{Z}_+$.

Using the property of Itô isometry, we have

$$\mathbb{E}\left\{\left(\int_{t-\tau}^{t} y(s)d\varpi(s)\right)^T S\left(\int_{t-\tau}^{t} y(s)d\varpi(s)\right)\right\} = \mathbb{E}\left\{\int_{t-\tau}^{t} y^T(s)Sy(s)ds\right\}. \tag{3.14}$$

Then if the condition in (3.2) is satisfied, the following inequality holds

$$\mathbb{E}[\mathcal{L}V(x,t)] \le -\gamma \mathbb{E}|x(t)|^2. \tag{3.15}$$

Thus, it follows that the stochastic Hopfield neural network (2.4) is globally asymptotically stable in the mean square sense. The proof is completed. □

Remark 3.2. In the above theorem, novel LMI-based stability criteria are proposed by using an improved homogeneous matrix polynomials technique. Indeed, as the selected design parameter d becomes larger, less conservative stability criteria will be obtained. Due to the algebraic property of the fuzzy weighting functions in the unit simplex can be utilized in the process of stability analysis, the fuzzy weighting functions that govern the behavior of the fuzzy system are seen as dynamic constraints. These dynamic constraints are incorporated into the stability criteria through the usage of the homogeneous matrix polynomials technique. As the value of d increases, the conservatism will be further reduced and less conservative LMI-based stability criteria will also be obtained in a convergent sense. Moreover, this fact will also be illustrated in the following system performance analysis.

4 System Performance Analysis

Example 1. Consider the stochastic fuzzy HNNs (2.4) with $r = 2$. The underlying T-S fuzzy model of nonlinear stochastic Hopfield neural network is of the following form:
Plant Rules:
Rule 1: IF $\theta_1(t)$ is M_{k1}, THEN
$dx(t) = (-C_1x(t) + A_1f(x(t-\tau(t))))dt + [M_1x(t) + N_1x(t-\tau(t))]d\varpi(t)$,
Rule 2: IF $\theta_2(t)$ is M_{k2}, THEN
$dx(t) = (-C_2x(t) + A_2f(x(t-\tau(t))))dt + [M_2x(t) + N_2x(t-\tau(t))]d\varpi(t)$,
with $f(x) = \tanh(x)$ and $\tau(t) = h_{\max}(\frac{1+\sin(0.1t)}{2})$. The membership functions for rules 1 and 2 are $M_{k1} = \frac{1}{e^{-2\theta_1(t)}}$, $M_{k2} = 1 - M_{k1}$, and $C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}$, $C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.88 & 0.30 \\ 0.26 & -0.25 \end{bmatrix}$, $M_1 = M_2 = \begin{bmatrix} 2.7 & 0 \\ 0 & 2.6 \end{bmatrix}$, $N_1 = N_2 = \begin{bmatrix} 1.8 & 0 \\ 0 & 2.5 \end{bmatrix}$.

Table 1: Maximum values of $h_{\max}$ via several methods proposed in Theorem 1

Methods	$d = 0$(usual method)	$d = 1$	$d = 2$	$d = 4$	$d = 6$
$h_{\max}$	2.3s	2.9s	3.6s	5.2s	6.0s

Due to $\tau(t) = h_{\max}(\frac{1+\sin(0.1t)}{2})$, we have $0 \le \tau(t) \le h_{\max}$ and $\dot{\tau}(t) \le \frac{h_{\max}}{2}$. Using Theorem 1 with $d = 0, 1, 2, 4, 6$, respectively, the allowable values of $h_{\max}$ are calculated and are shown in Table 1. From Table 1, it can be found that the result of Theorem 1 with $d = 6$ is less conservative than other ones. Moreover, the conservatism is gradually reduced as the value of d increases. This fact illustrates that convergent stability criteria are developed by applying the approach given in this paper.

By choosing initial conditions of the underlying system as $x(0) = (0.8, -0.4)^T$ and $\tau(t) = 3.0 + 3.0\sin(0.1t)$, it is easy to see that $0 \le \tau(t) \le 6.0$ and $\dot{\tau}(t) \le 0.3$, i.e., the above

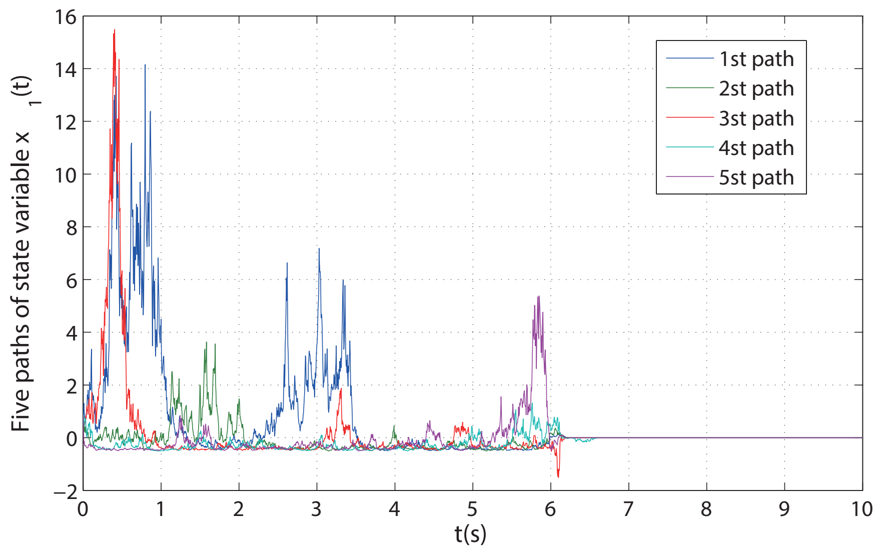

Figure 1: The trajectory of five paths of $x_1(t)$.

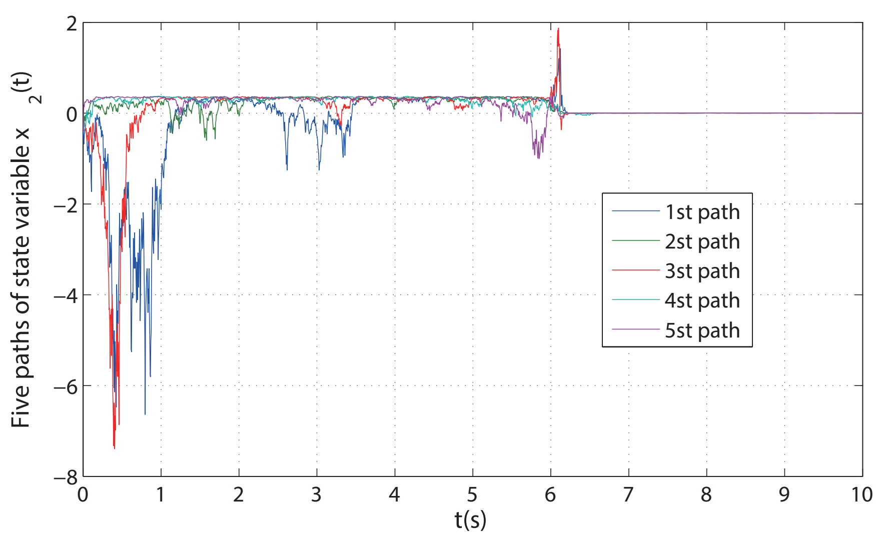

Figure 2: The trajectory of five paths of $x_2(t)$.

stochastic fuzzy Hopfield neural networks with time-varying delays can be ensured globally asymptotically stable in the mean square sense by using Theorem 1 with $d = 6$. On the other hand, Fig. 1 and Fig. 2 show the trajectories of five paths of $x_1(t)$ and $x_2(t)$, respectively. From Fig. 1 and Fig. 2, one can find that the above stochastic fuzzy Hopfield neural networks with time-varying delays is globally asymptotically stable in the mean square sense.

5 Conclusion

In this paper, novel stability criteria for nonlinear stochastic Hopfield neural networks with time-varying delays has been developed by using the Lyapunov stability theory and the stochastic analysis approach. With the purpose of reducing the conservatism, an improved homogeneous matrix polynomials technique is proposed and convergent stability criteria are developed. Finally, the system performance analysis has been given to demonstrate the effectiveness of the stability criteria proposed in this paper.

References

[1] M.S. Ali, Robust stability analysis of Takagi-Sugeno uncertain stochastic fuzzy recurrent neural networks with mixed time-varying delays, *Chinese Physics B.* 20 (2011) 080201.

[2] M.S. Ali, Novel delay-dependent stability analysis of Takagi Sugeno fuzzy uncertain neural networks with time varying delays, *Chinese Physics B.* 21 (2012) 070207.

[3] M.S. Ali, N. Gunasekaran and Q. Zhu, State estimation of T–S fuzzy delayed neural networks with Markovian jumping parameters using sampled-data control, *Fuzzy Sets and Systems.* 306 (2017) 87–104.

[4] G. Bao, Z. Zeng and Y. Shen, Region stability analysis and tracking control of memristive recurrent neural network, *Neural Networks.* 98 (2018) 51–58.

[5] R.-Y. Chen, A traceability chain algorithm for artificial neural networks using T–S fuzzy cognitive maps in blockchain, *Future Generation Computer Systems.* 80 (2018) 198–210.

[6] K. Guan, Global power-rate synchronization of chaotic neural networks with proportional delay via impulsive control, *Neurocomputing.* 283 (2018) 256–265.

[7] D.W. Gong, H.G. Zhang and Z.S. Wang, Novel stability and stabilization criteria for Takagi-Sugeno fuzzy time-delay systems, *Chinese Physics B.* 21 (2012) 030204.

[8] K. Gu, V. L. Kharitonov and J. Chen, *Stability of Time-delay Systems*, Birkhauser, 2003.

[9] C. Huang and J. Cao, Impact of leakage delay on bifurcation in high-order fractional BAM neural networks, *Neural Networks.* 98 (2018) 223–235.

[10] J. Jian and P. Wan, Global exponential convergence of fuzzy complex-valued neural networks with time-varying delays and impulsive effects, *Fuzzy Sets and Systems.* 338 (2018) 23–39.

[11] S.M. Lee, O.M. Kwon and H.H. Park, A new approach to stability analysis of neural networks with time-varying delay via novel Lyapunov–Krasovskii functional, *Chinese Physics B.* 19 (2010) 050507.

[12] H. Li, C. Li and T. Huang, Periodicity and stability for variable-time impulsive neural networks, *Neural Networks.* 94 (2017) 24–33.

[13] F. Sabahi, Introducing validity into self-organizing fuzzy neural network applied to impedance force control, *Fuzzy Sets and Systems.* 337 (2018) 113–127.

[14] K. Shi, X. Liu, Y. Tang, H. Zhu and S. Zhong, Some novel approaches on state estimation of delayed neural networks, *Information Sciences.* 372 (2016) 313–331.

[15] H.O. Silva and C.J.A. Bastos-Filho, Inter-domain routing for communication networks using hierarchical Hopfield neural networks, *Engineering Applications of Artificial Intelligence.* 70 (2018) 184–198.

[16] T. Takagi and M. Sugeno, Fuzzy identification of systems and its application to modeling and control, *IEEE Transactions on Systems, Man, Cybernetics.* 15 (1985) 116–132.

[17] P. Wan and J. Jian, Global convergence analysis of impulsive inertial neural networks with time-varying delays, *Neurocomputing.* 245 (2017) 68–76.

[18] Y.-L. Wang and Q.-L. Han, Network-based modelling and dynamic output feedback control for unmanned marine vehicles, *Automatica.* 91 (2018) 43–53.

[19] Y.-L. Wang, Q.-L. Han, M.-R. Fei and C. Peng, Network-based T-S fuzzy dynamic positioning controller design for unmanned marine vehicles, *IEEE Transactions on Cybernetics.* 48 (2018) 2750–2763.

.

[20] Y.-L. Wang, C.-C. Lim and P. Shi, Adaptively adjusted event-triggering mechanism on fault detection for networked control systems, *IEEE Transactions on Cybernetics.* 47 (2017) 2299–2311.

[21] Y.-L. Wang, P. Shi, C.-C. Lim and Y. Liu, Event-triggered fault detection filter design for a continuous-time networked control system, *IEEE Transactions on Cybernetics.* 46 (2016) 3414–3426

[22] Y.-L. Wang, T.-B. Wang and Q.-L. Han, Fault detection filter design for data reconstruction-based continuous-time networked control systems, *Information Sciences.* 328 (2016) 577–594.

[23] Y. Yang, Y. He, Y. Wang and M. Wu, Stability analysis of fractional-order neural networks: An LMI approach, *Neurocomputing.* 285 (2018) 82–93.

[24] H. Yao and J. Zhou, Global exponential stability of mixed discrete and distributively delayed cellular neural network, *Chinese Physics B.* 20 (2011) 110512.

[25] J. Zhu and J. Sun, Stability of quaternion-valued impulsive delay difference systems and its application to neural networks, *Neurocomputing.* 284 (2018) 63–69.

Manuscript received 11 April 2018
revised 26 October 2018
accepted for publication 20 November 2018

Yi-Fu Feng
College of Mathematics, Jilin Normal University
Siping Jilin 136000, China
E-mail address: yf19692004@163.com

Jing Wang
SINOPEC Petroleum Exploration and Development
Research Institute, Beijing 100083, China
E-mail address: 65111366@qq.com