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BRUALDI-TYPE BOUNDS ON NINMUM EIGENVALUES FOR FAN PRODUCT OF $\mathcal{M}$- TENSORS*

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Abstract: This paper is devoted to bound estimations on the minimum eigenvalues of $\mathcal{M}$-tensors. To this end, we explore Brualdi-type inclusion sets in the sense of $\mathcal{M}$-tensors, and establish some Brualdi-type bounds on the minimum eigenvalues for the Fan product of $\mathcal{M}$-tensors. Numerical examples show the validity of the conclusions.

Key words: *$\mathcal{M}$-tensors, Fan product, minimum eigenvalues*

Mathematics Subject Classification: *15A18, 15A42, 15A69*

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1 Introduction

Generally, tensor is a higher-order extension of matrix, and hence many concepts and related properties for matrices such as determinant, eigenvalue and singular value theory can be extended to higher order tensors by exploring their multilinear algebra properties [14, 18, 19]. Matrices with special structures such as nonnegative matrices and M-matrices can also be extended to higher order tensors and these are becoming the focus of tensor in recent research [3-7] and [17-29]. In particular, $\mathcal{M}$-tensors play important roles in the stability study of nonlinear autonomous systems via Lyapunov's direct method in automatic control [10, 11, 17], spectral hypergraph theory [35] and the sparsest solutions to tensor complementarity problems [16]. Meanwhile, many practical problems, such as the weak minimum principle in partial differential equations, characteristic functions in probability theory and the study of association schemes in combinatorial theory [12], are closely related to Fan product of M-matrices and Hadamard product of nonnegative matrices. So, Horn *et al.* [12] proposed lower bounds on the minimum eigenvalue for Fan product of M-matrices and upper bounds on the spectral radius for the Hadamard product of nonnegative matrices. Later, the improved bounds on the Hadamard product involving nonnegative matrices and Fan product of M-matrices were established in [8, 13, 15, 33]. Recently, Fan product of M-matrices and Hadamard product of nonnegative matrices were extended to higher order $\mathcal{M}$-tensors and nonnegative tensors [21, 24, 25]. Based on Gersgorin-type eigenvalue inclusion set and Perron-Frobenius theorem, Sun *et al.* [21] investigated some inequalities for the Hadamard product of tensors and obtained some bounds on spectral radius of the Hadamard

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product of tensors, and used them to estimate the spectral radius of a directly weighted hypergraph. Wang *et al.* [24] established lower bounds on the minimum eigenvalues for the Fan product of two $\mathcal{M}$-tensors. It should be noted that Brualdi-type inclusion set (Brauer-type inclusion set) is sharper than Gersgorin-type inclusion set [1,2]. By virtue of the close relationship between the nonnegative tensors and $\mathcal{M}$-tensors [32,35], we want to establish sharp Brualdi-type lower bounds for minimum eigenvalues for the Fan product of $\mathcal{M}$-tensors by diagonal similarity transformation methods.

This paper is organized as follows. In Section 2, we introduce important notation and recall fundamental results on tensor analysis. In Section 3, we first explore Brualdi-type inclusion sets in the sense of $\mathcal{M}$-tensors and show that the Fan product of two $\mathcal{M}$-tensors is an $\mathcal{M}$-tensor. In view of the characterizations of $\mathcal{M}$-tensors, we establish Brualdi-type inequalities on the minimum eigenvalues for the Fan product when the directed graph of Fan product is weakly connected. Furthermore, Brauer-type inequalities are proposed on the minimum eigenvalues for the Fan product when the directed graph of Fan product may be not weakly connected. Numerical experiments are provided to exhibit the efficiency of the obtained results.

2 Notation and Preliminaries

We begin with some fundamental notions and properties related to eigenvalue of tensors [14,18], which are needed in the subsequent analysis.

Definition 2.1. Let $\mathcal{A}$ be an m-order n-dimensional tensor. Assume that $\mathcal{A}x^{m-1}$ is not identical to 0.

(i) We say that $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \backslash \{0\})$ is an eigenvalue-eigenvector of $\mathcal{A}$ if

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

where $(\mathcal{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} x_{i_2} \ldots x_{i_m}$, $x^{[m-1]} = [x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1}]^T$, and (λ, x) is called an H-eigenpair if they are both real.

(ii) We call $\sigma(\mathcal{A})$ as the set of all eigenvalues of $\mathcal{A}$. Assume $\sigma(\mathcal{A}) \neq \emptyset$. Then the spectral radius and the minimum eigenvalue of $\mathcal{A}$ are denoted by

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}, \tau(\mathcal{A}) = \min\{\lambda \in \mathbb{R} : \lambda \in \sigma(\mathcal{A})\}.$$

Based on the connectivity of a graph associated with a polynomial map, Friedland *et al.* [9] defined weakly irreducible polynomial maps as follows.

Given a tensor $\mathcal{A} = (a_{i_1\ldots i_m})$, we associate $\mathcal{A}$ with a digraph $\Gamma_{\mathcal{A}}$ as follows. The vertex set of $\Gamma_{\mathcal{A}}$ is $V(\mathcal{A}) = \{1, \ldots, n\}$ and the arc set of $\Gamma_{\mathcal{A}}$ is $E(\mathcal{A}) = \{(i, j) : a_{i_1\ldots i_m} \neq 0, j \in \{i_2, \ldots, i_m\} \neq \{i, \ldots, i\}\}$. A directed graph $\Gamma_{\mathcal{A}}$ is called weakly connected if for each vertex $v_i \in V$, there exists a circuit such that v_i belongs to the circuit. A directed graph $\Gamma_{\mathcal{A}}$ is called strongly connected if for each ordered pair of distinct vertices v_i and v_j, there is a path from v_i to v_j. Further, the tensor $\mathcal{A}$ is called weakly irreducible if the directed graph $\Gamma_{\mathcal{A}}$ is strongly connected.

The Perron-Frobenius theorem for weakly irreducible nonnegative tensors was established in [9].

Lemma 2.2. *Let $\mathcal{A}$ be a weakly irreducible nonnegative tensor of order m and dimension n. Then, $\mathcal{A}$ has a positive eigenpair (λ, x), and x is unique up to a multiplicative constant.*

The following specially structured tensors are extended from matrices [32].

Definition 2.3. $\mathcal{A}$ is called a $\mathcal{Z}$-tensor if it can be written as $\mathcal{A} = c\mathcal{I} - \mathcal{B}$, where $c > 0$, $\mathcal{I}$ is a unit tensor with entries

$$\delta_{i_1 i_2 \dots i_m} = \begin{cases} 1, & \text{if } i_1 = i_2 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

and $\mathcal{B}$ is a nonnegative tensor. Furthermore, if $c \geq \rho(\mathcal{B})$, then $\mathcal{A}$ is said to be an $\mathcal{M}$-tensor, and if $c > \rho(\mathcal{B})$, then tensor $\mathcal{A}$ is said to be a strong $\mathcal{M}$-tensor. $\mathcal{A}$ is a weakly irreducible M-tensor if $\mathcal{B}$ is weakly irreducible.

It is easy to see that all off-diagonal entries of a $\mathcal{Z}$-tensor are nonpositive [32, 33], and (strong) $\mathcal{M}$-tensor is closely linked with the diagonal dominance defined below.

Definition 2.4. For an m-order n-dimensional tensor $\mathcal{A}$, it is called diagonally dominant if

$$|a_{i\dots i}| \geq \sum_{\delta_{ii_2\dots i_m}=0} |a_{ii_2\dots i_m}|, \quad \forall i \in N.$$

$\mathcal{A}$ is called strictly diagonally dominant if the strict inequality holds.

Let $\mathcal{A}$ be an m-order n-dimensional tensor and $D = \operatorname{diag}(d_1, \dots, d_n)$ be a positive diagonal matrix. Set

$$\mathcal{A}_D = \mathcal{A} \cdot D^{-(m-1)} \overbrace{D \dots D}^{m-1}$$

with $(\mathcal{A}_D)_{i_1\dots i_m} = a_{i_1\dots i_m} d_{i_1}^{-[m-1]} d_{i_2} \dots d_{i_m}$. Then we have the following conclusion.

Lemma 2.5 ([35]). *Suppose $\mathcal{A}$ is a $\mathcal{Z}$-tensor and its all diagonal elements are nonnegative (positive). Then, $\mathcal{A}$ is an (strong) $\mathcal{M}$-tensor if and only if there exists a positive diagonal matrix D such that $\mathcal{B} = \mathcal{A} \cdot D^{-(m-1)} \overbrace{D \dots D}^{m-1}$ is (strictly) diagonally dominant.*

Lemma 2.6 ([30]). *Let $\mathcal{A}, \mathcal{B}$ be order m dimension n tensors. If there is a diagonal nonsingular matrix D such that $\mathcal{B} = \mathcal{A} \cdot D^{-(m-1)} \overbrace{D \dots D}^{m-1}$, then they have the same eigenvalues.*

For weakly connected tensors and general tensors, Bu *et al.* [1, 2] gave Brualdi-type eigenvalue inclusion sets and Brauer-type eigenvalue inclusion sets.

Lemma 2.7 (Theorem 3.1 of [1], Theorem 3.3 of [2]). *Let $\mathcal{A} = (a_{ii_2\dots i_m})$ be an m-order n-dimensional tensor such that the directed graph $\Gamma_{\mathcal{A}}$ is weakly connected. Then,*

$$\sigma(\mathcal{A}) \subseteq \bigcup_{\gamma \in C(\mathcal{A})} \{z \in C : \prod_{i\in\gamma} |z - a_{i\dots i}| \leq \prod_{i\in\gamma} r_i(\mathcal{A})\}.$$

If $r_i(\mathcal{A}) \neq 0$ and the directed graph $\Gamma_{\mathcal{A}}$ is not weakly connected, then

$$\sigma(\mathcal{A}) \subseteq \bigcup_{\substack{a_{i_1 i_2 \dots i_m} \neq 0 \\ (i_2,\dots,i_m)\neq(i_1,\dots,i_1)}} \{z \in C : \prod_{j=1}^{m} |z - a_{i_j\dots i_j}| \leq \prod_{j=1}^{m} r_{i_j}(\mathcal{A})\},$$

where $r_i(\mathcal{A}) = \sum_{\delta_{ii_2\dots i_m}=0} |a_{ii_2\dots i_m}|$ and $C(\mathcal{A})$ is the set of circuits for $\Gamma_{\mathcal{A}}$.

To end this section, we give the definition for the Fan product of tensors.

Definition 2.8. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{M}$-tensors. Fan product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $\mathcal{A}\star\mathcal{B}=\mathcal{D}=(d_{i_1i_2\ldots i_m})$, where

$$d_{i_1i_2\ldots i_m}=\begin{cases} a_{i\ldots i}b_{i\ldots i}, & \text{if } i_1=i_2=\ldots=i_m=i,\\ -a_{i_1i_2\ldots i_m}b_{i_1i_2\ldots i_m}, & \text{otherwise.}\end{cases}$$

3 Bounds on the Minimum Eigenvalue for the Fan Product of $\mathcal{M}$-Tensors

In this section, we shall give some inequalities on the minimum eigenvalue for the Fan product of $\mathcal{M}$-tensors. We begin our work by collecting some properties of $\mathcal{M}$-tensors, which is crucial for further considerations.

Lemma 3.1. *Let $\mathcal{P}=(p_{i_1i_2i_3\ldots i_m})$ be an $\mathcal{M}$-tensor of order m dimension n.*

(i) *If the directed graph $\Gamma_{\mathcal{P}}$ is weakly connected, then there exists a circuit $\gamma\in C(\mathcal{P})$ such that*

$$\prod_{i\in\gamma}(p_{i\ldots i}-\tau(\mathcal{P}))\le\prod_{i\in\gamma}\widetilde{r}_i(\mathcal{P});\tag{3.1}$$

(ii) *If $r_i(\mathcal{P})\neq 0$ and $\Gamma_{\mathcal{P}}$ is not weakly connected, then there exists $p_{i_1i_2\ldots i_m}\neq 0$ with $(i_2,\ldots,i_m)\neq(i_1,\ldots,i_1)$ such that*

$$\prod_{j=1}^{m}(p_{i_j\ldots i_j}-\tau(\mathcal{P}))\le\prod_{j=1}^{m}\widetilde{r}_{i_j}(\mathcal{P})$$

where $\widetilde{r}_i(\mathcal{P})=\sum\limits_{\delta_{ii_2\ldots i_m}=0}-p_{ii_2\ldots i_m}$ and $\widetilde{r}_{i_j}(\mathcal{P})=\sum\limits_{\delta_{i_1i_2\ldots i_m}=0}-p_{i_1i_2\ldots i_m}$.

Proof. Let $\tau(\mathcal{P})$ be the minimum eigenvalue of $\mathcal{P}$. Then, it follows from Lemma 2.7 that

$$\prod_{i\in\gamma}|p_{i\ldots i}-\tau(\mathcal{P})|\le\prod_{i\in\gamma}r_i(\mathcal{P}),\tag{3.2}$$

where $r_i(\mathcal{P})=\sum\limits_{\delta_{ii_2\ldots i_m}=0}|p_{ii_2\ldots i_m}|$. Since $\mathcal{P}$ is an $\mathcal{M}$-tensor, it follows from Lemma 4.1 in [21] that

$$\tau(\mathcal{P})\le\min_{i\in N}p_{i\ldots i}\text{ and }\widetilde{r}_i(\mathcal{P})=r_i(\mathcal{P}).$$

So, (3.2) is equivalent to

$$\prod_{i\in\gamma}(p_{i\ldots i}-\tau(\mathcal{P}))\le\prod_{i\in\gamma}\widetilde{r}_i(\mathcal{P}).$$

(ii) A similar argument to the proof of Part (i) leads the desired result. □

Lemma 3.2. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two $\mathcal{M}$-tensors of order m dimension n. Then, $\mathcal{P}\star\mathcal{Q}$ is an $\mathcal{M}$-tensor. Furthermore, if $\mathcal{P}$ and $\mathcal{Q}$ are strong $\mathcal{M}$-tensors, then $\mathcal{P}\star\mathcal{Q}$ is a strong $\mathcal{M}$-tensor.*

Proof. By the definition of $\mathcal{P} \star \mathcal{Q}$, it holds that

$$\mathcal{P} \star \mathcal{Q} = \begin{cases} p_{i\dots i} q_{i\dots i}, & \text{if } i_2 = i_3 \dots = i_m = i, \\ -p_{ii_2\dots i_m} q_{ii_2\dots i_m}, & \text{otherwise.} \end{cases}$$

Since $\mathcal{P}$ and $\mathcal{Q}$ are $\mathcal{M}$-tensors, by Lemma 2.5, there exist positive diagonal matrices C, D such that

$$\mathcal{A} = \mathcal{P} \cdot C^{-(m-1)} \overbrace{C \dots C}^{m-1}, \quad \mathcal{B} = \mathcal{Q} \cdot D^{-(m-1)} \overbrace{D \dots D}^{m-1}$$

with

$$|p_{i\dots i}| = |a_{i\dots i}| \geq \sum_{\delta_{ii_2\dots i_m}=0} |a_{i\dots i_m}| = \sum_{\delta_{ii_2\dots i_m}=0} |p_{i\dots i_m}| c_i^{-(m-1)} c_{i_2} \dots c_{i_m},$$

$$|q_{i\dots i}| = |b_{i\dots i}| \geq \sum_{\delta_{ii_2\dots i_m}=0} |b_{i\dots i_m}| = \sum_{\delta_{ii_2\dots i_m}=0} |q_{i\dots i_m}| d_i^{-(m-1)} d_{i_2} \dots d_{i_m}.$$

Certainly,

$$\begin{aligned} &|p_{i\dots i} q_{i\dots i}| = |a_{i\dots i} b_{i\dots i}| \\ \geq &\sum_{\delta_{ii_2\dots i_m}=0} (|p_{ii_2\dots i_m}| c_i^{-(m-1)} c_{i_2} \dots c_{i_m}) \sum_{\delta_{ii_2\dots i_m}=0} (|q_{ii_2\dots i_m}| d_i^{-[m-1]} d_{i_2} \dots d_{i_m}) \\ \geq &\sum_{\delta_{ii_2\dots i_m}=0} |p_{ii_2\dots i_m}| c_i^{-(m-1)} c_{i_2} \dots c_{i_m} |q_{ii_2\dots i_m}| d_i^{-(m-1)} d_{i_2} \dots d_{i_m} \\ = &\sum_{\delta_{ii_2\dots i_m}=0} |p_{ii_2\dots i_m} q_{ii_2\dots i_m}| (c_i d_i)^{-(m-1)} c_{i_2} d_{i_2} \dots c_{i_m} d_{i_m}. \end{aligned} \tag{3.3}$$

From (3.3), there exists a positive diagonal matrix $U = diag(c_1 d_1, c_2 d_2, \dots, c_n d_n)$ such that

$$|p_{i\dots i} q_{i\dots i}| \geq \sum_{\delta_{ii_2\dots i_m}=0} p_{ii_2\dots i_m} q_{ii_2\dots i_m} (u_i)^{-(m-1)} u_{i_2} \dots u_{i_m}.$$

It follows from Lemma 2.5 that $\mathcal{P} \star \mathcal{Q}$ is an $\mathcal{M}$-tensor. Similar to the argument for the first conclusion, we can obtain the second conclusion. □

In terms of $\tau(\mathcal{P})$ and $\tau(\mathcal{Q})$, Wang *et al.* [24] established lower bounds on the minimum eigenvalue for the Fan product of two $\mathcal{M}$-tensors. However, the minimum eigenvalues of $\mathcal{M}$-tensors are not easy to calculate [32]. Compared with the minimum eigenvalues of $\mathcal{M}$-tensors, the spectral radius of the nonnegative tensor can be calculated by many algorithms [3, 9, 20, 31, 34]. Therefore, we introduce a nonnegative tensor associated with the $\mathcal{M}$-tensor and use its spectral radius to estimate the lower bounds on the minimum eigenvalues for the Fan product.

Let $\mathcal{P} = (p_{i_1 i_2 \dots i_m})$ be a strong $\mathcal{M}$-tensor of order m dimension n. Then $p_{ii\dots i} > 0$ since $\mathcal{P}$ is a strong M-tensor. Define the related nonnegative tensors of P as

$$J_{\mathcal{P}} = \mathcal{D}^{-1}(\mathcal{D} - \mathcal{P}),$$

where $\mathcal{D}$ is a diagonal tensor whose diagonal entries are the same as that of tensor $\mathcal{P}$ and

$$(J_{\mathcal{P}})_{ii_2\dots i_m} = \begin{cases} 0, & \text{if } i = i_2 = \dots = i_m, \\ -\frac{p_{ii_2\dots i_m}}{p_{ii\dots i}}, & \text{otherwise.} \end{cases}$$

Obviously, $J_{\mathcal{P}}$ is a nonnegative tensor.

3.1 Bounds for the weakly connected digraph $\Gamma_{(\mathcal{P}\star\mathcal{Q})}$

Based on the characterizations of $\mathcal{M}$-tensors, we can obtain the following Brualdi-type inequalities.

Theorem 3.3. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two strong $\mathcal{M}$-tensors of order m and dimension n. If Fan product $\mathcal{P}\star\mathcal{Q}$ is a tensor such that the directed graph $\Gamma_{(\mathcal{P}\star\mathcal{Q})}$ is weakly connected, there exists a circuit $\gamma\in C(\mathcal{P}\star\mathcal{Q})$ such that*

$$\tau(\mathcal{P}\star\mathcal{Q})\leq\min_{i\in N}p_{i\ldots i}q_{i\ldots i}, \tag{3.4}$$

$$\prod_{i\in\gamma}(p_{i\ldots i}q_{i\ldots i}-\tau(\mathcal{P}\star\mathcal{Q}))\leq\prod_{i\in\gamma}(p_{i\ldots i}q_{i\ldots i}\rho(J_{\mathcal{P}})\rho(J_{\mathcal{Q}})). \tag{3.5}$$

Proof. From Lemma 4.1 of [21], (3.4) holds. For (3.5), we break the proof into two cases.

Case 1. $\mathcal{P}$ and $\mathcal{Q}$ are both weakly irreducible. Then, $J_{\mathcal{P}}$ and $J_{\mathcal{Q}}$ are weakly irreducible nonnegative tensors. From Lemma 2.2, there exist two positive vectors u,v such that

$$\rho(J_{\mathcal{P}})u_i^{[m-1]}=J_{\mathcal{P}}u^{m-1},\rho(J_{\mathcal{Q}})v_i^{[m-1]}=J_{\mathcal{Q}}v^{m-1},$$

equivalently,

$$\frac{\sum\limits_{\delta_{ii_2\ldots i_m}=0}|p_{ii_2\ldots i_m}|u_{i_2}\ldots u_{i_m}}{p_{i\ldots i}u_i^{[m-1]}}=\rho(J_{\mathcal{P}}),\quad\frac{\sum\limits_{\delta_{ii_2\ldots i_m}=0}|q_{ii_2\ldots i_m}|v_{i_2}\ldots v_{i_m}}{q_{i\ldots i}v_i^{[m-1]}}=\rho(J_{\mathcal{Q}}). \tag{3.6}$$

Set $D=\text{diag}(u_1v_1,\ldots,u_nv_n)$. It is obvious that D is a positive diagonal matrix. It follows from lemma 2.3 that $\sigma(\mathcal{P}\star\mathcal{Q})=\sigma(D^{1-m}(\mathcal{P}\star\mathcal{Q})D)$. By Lemma 3.1, (3.6), there exists a circuit $\gamma\in C(\mathcal{P}\star\mathcal{Q})$ such that

$$\begin{aligned}\prod_{i\in\gamma}(p_{i\ldots i}q_{i\ldots i}-\tau(\mathcal{P}\star\mathcal{Q}))&\leq\prod_{i\in\gamma}\sum_{\delta_{ii_2\ldots i_m}=0}\frac{-p_{ii_2\ldots i_m}q_{ii_2\ldots i_m}u_{i_2}\ldots u_{i_m}v_{i_2}\ldots v_{i_m}}{u_i^{[m-1]}v_i^{[m-1]}}.\\&\leq\prod_{i\in\gamma}\Big(\sum_{\delta_{ii_2\ldots i_m}=0}\frac{|p_{ii_2\ldots i_m}|u_{i_2}\ldots u_{i_m}}{u_i^{[m-1]}}\Big)\Big(\sum_{\delta_{ii_2\ldots i_m}=0}\frac{|q_{ii_2\ldots i_m}|v_{i_2}\ldots v_{i_m}}{v_i^{[m-1]}}\Big)\\&=\prod_{i\in\gamma}(p_{i\ldots i}q_{i\ldots i}\rho(J_{\mathcal{P}})\rho(J_{\mathcal{Q}})).\end{aligned}$$

Thus, (3.5) is established.

Case 2. Either $\mathcal{P}$ or $\mathcal{Q}$ is weakly reducible. Let $\mathcal{S}$ be order m dimension n tensor with

$$s_{ii_2\ldots i_m}=\begin{cases}1, & \text{if } i_2=i_3=\cdots=i_m\neq i,\\ 0, & \text{otherwise}.\end{cases}$$

Then both $\mathcal{P}-\epsilon\mathcal{S},\mathcal{Q}-\epsilon\mathcal{S},J_{\mathcal{P}}+\frac{\epsilon}{p_{i\ldots i}}\mathcal{S}$ and $J_{\mathcal{Q}}+\frac{\epsilon}{q_{i\ldots i}}\mathcal{S}$ are weakly irreducible tensors for any $\epsilon>0$. Now, we claim that $\mathcal{P}-\epsilon\mathcal{S}$ and $\mathcal{Q}-\epsilon\mathcal{S}$ are both strong $\mathcal{M}$-tensors when $\epsilon>0$ is sufficiently small.

Since $\mathcal{P}$ and $\mathcal{Q}$ are strong $\mathcal{M}$-tensors, by Lemma 2.5, there exist positive diagonal matrices C,D such that

$$\mathcal{A}=\mathcal{P}\cdot C^{-(m-1)}\overbrace{C\ldots C}^{m-1},\quad\mathcal{B}=\mathcal{Q}\cdot D^{-(m-1)}\overbrace{D\ldots D}^{m-1}$$

with

$$|p_{i\dots i}| = |a_{i\dots i}| > \sum_{\delta_{ii_2\dots i_m}=0} |a_{i\dots i_m}| = \sum_{\delta_{ii_2\dots i_m}=0} |p_{ii_2\dots i_m}| c_i^{-(m-1)} c_{i_2}\dots c_{i_m},$$
$$|q_{i\dots i}| = |b_{i\dots i}| > \sum_{\delta_{ii_2\dots i_m}=0} |b_{i\dots i_m}| = \sum_{\delta_{ii_2\dots i_m}=0} |q_{ii_2\dots i_m}| d_i^{-(m-1)} d_{i_2}\dots d_{i_m}.$$

Set

$$L = \max_{\substack{i,j\in N\\ i\neq j}} \{\frac{c_j^{[m-1]}}{c_i^{[m-1]}}, \frac{d_j^{[m-1]}}{d_i^{[m-1]}}\}$$

and

$$\epsilon_0 = \min_{\substack{i,j\in N\\ i\neq j}} \Big\{\frac{|p_{i\dots i}| - \sum_{\delta_{ii_2\dots i_m}=0} |p_{ii_2\dots i_m}| c_i^{-(m-1)} c_{i_2}\dots c_{i_m}}{(n-1)L}, \frac{|q_{i\dots i}| - \sum_{\delta_{ii_2\dots i_m}=0} |q_{ii_2\dots i_m}| d_i^{-(m-1)} d_{i_2}\dots d_{i_m}}{(n-1)L}\Big\}.$$

For any $0 < \epsilon < \epsilon_0$, it holds that $\mathcal{P} - \epsilon\mathcal{S}, \mathcal{Q} - \epsilon\mathcal{S}$ and $(\mathcal{P} - \epsilon\mathcal{S}) \star (\mathcal{Q} - \epsilon\mathcal{S})$ are strong $\mathcal{M}$-tensors by Lemma 3.2. Noting that $\mathcal{P}$ and $\mathcal{Q}$ are two strong $\mathcal{M}$-tensors, for the circuit $\gamma \in C(\mathcal{P} \star \mathcal{Q})$, we get $\gamma \in C((\mathcal{P} - \epsilon\mathcal{S}) \star (\mathcal{Q} - \epsilon\mathcal{S}))$. Substituting $\mathcal{P} - \epsilon\mathcal{S}$ and $\mathcal{Q} - \epsilon\mathcal{S}$ for $\mathcal{P}$ and $\mathcal{Q}$ and letting $\epsilon \to 0$ on (3.7), we can obtain the desired results by the continuity of $\rho(J_{\mathcal{P}} + \frac{\epsilon}{p_{i\dots i}}\mathcal{S})$ and $\rho(J_{\mathcal{Q}} + \frac{\epsilon}{q_{i\dots i}}\mathcal{S})$. □

By the information of the absolute maximum in the off-diagonal elements, we are at the position to establish the following conclusions.

Theorem 3.4. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two strong $\mathcal{M}$-tensors of order m and dimension n. If Fan product $\mathcal{P} \star \mathcal{Q}$ is a tensor such that the directed graph $\Gamma_{(\mathcal{P}\star\mathcal{Q})}$ is weakly connected, there exists a circuit $\gamma \in C(\mathcal{P} \star \mathcal{Q})$ such that*

$$\tau(\mathcal{P} \star \mathcal{Q}) \leq \min_{i\in N} p_{i\dots i} q_{i\dots i},$$

$$\prod_{i\in\gamma} (p_{i\dots i} q_{i\dots i} - \tau(\mathcal{P} \star \mathcal{Q})) \leq \prod_{i\in\gamma} (\alpha_i \beta_i p_{i\dots i} q_{i\dots i} \rho(J_{\mathcal{P}}) \rho(J_{\mathcal{Q}}))^{\frac{1}{2}}, \tag{3.7}$$

where $\alpha_i = \max\limits_{\delta_{ii_2\dots i_m}=0} -p_{ii_2\dots i_m}$ and $\beta_i = \max\limits_{\delta_{ii_2\dots i_m}=0} -q_{ii_2\dots i_m}$.

Proof. We break the proof into two cases.

Case 1. $\mathcal{P}$ and $\mathcal{Q}$ are both weakly irreducible. Then, $J_{\mathcal{P}}$ and $J_{\mathcal{Q}}$ are weakly irreducible nonnegative tensors. It follow from Lemma 2.2 that there exist two positive eigenvectors $u = (u_i^2), v = (v_i^2)$ such that

$$\rho(J_{\mathcal{P}}) u_i^{2[m-1]} = J_{\mathcal{P}} u^{2(m-1)}, \rho(J_{\mathcal{Q}}) v_i^{2[m-1]} = J_{\mathcal{Q}} v^{2(m-1)},$$

equivalently,

$$\frac{\sum_{\delta_{ii_2\dots i_m}=0} |p_{ii_2\dots i_m}| u_{i_2}^2 \dots u_{i_m}^2}{p_{i\dots i} u_i^{2[m-1]}} = \rho(J_{\mathcal{P}}), \frac{\sum_{\delta_{ii_2\dots i_m}=0} |q_{ii_2\dots i_m}| v_{i_2}^2 \dots v_{i_m}^2}{q_{i\dots i} v_i^{2[m-1]}} = \rho(J_{\mathcal{Q}}). \tag{3.8}$$

Without loss of generality, we assume that $u, v \in \mathbb{R}^n_{++}$. Set $D = \mathrm{diag}(u_1 v_1, \dots, u_n v_n)$. Then, $\sigma(\mathcal{P} \star \mathcal{Q}) = \sigma(D^{1-m}(\mathcal{P} \star \mathcal{Q})D)$. By Lemma 3.1, (3.9) and the definitions of α_i, β_i, there exists a circuit $\gamma \in C(\mathcal{P} \star \mathcal{Q})$ such that

$$\begin{aligned}\prod_{i\in\gamma}(p_{i\ldots i}q_{i\ldots i}-\tau(\mathcal{P}\star\mathcal{Q})) \leq\ & \prod_{i\in\gamma}\sum_{\delta_{ii_2\ldots i_m}=0}\frac{-p_{ii_2\ldots i_m}q_{ii_2\ldots i_m}u_{i_2}\ldots u_{i_m}v_{i_2}\ldots v_{i_m}}{u_i^{[m-1]}v_i^{[m-1]}}\\ \leq\ & \prod_{i\in\gamma}\Big(\sum_{\delta_{ii_2\ldots i_m}=0}\frac{p^2_{ii_2\ldots i_m}u^2_{i_2}\ldots u^2_{i_m}}{u_i^{2[m-1]}}\Big)^{\frac{1}{2}}\Big(\sum_{\delta_{ii_2\ldots i_m}=0}\frac{q^2_{ii_2\ldots i_m}v^2_{i_2}\ldots v^2_{i_m}}{v_i^{2[m-1]}}\Big)^{\frac{1}{2}}\\ \leq\ & \prod_{i\in\gamma}(\alpha_i\beta_i p_{i\ldots i}q_{i\ldots i}\rho(J_{\mathcal{P}})\rho(J_{\mathcal{Q}}))^{\frac{1}{2}},\end{aligned}$$

where the second inequality uses the Cauchy-Schwartz inequality. So, (3.8) holds.

Case 2. Either $\mathcal{P}$ or $\mathcal{Q}$ is weakly reducible. Similar to the proof of Theorem 3.3, we obtain the desired result. □

Theorem 3.5. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two strong $\mathcal{M}$-tensors of order m and dimension n. If Fan product $\mathcal{P}\star\mathcal{Q}$ is a tensor such that the directed graph $\Gamma_{(\mathcal{P}\star\mathcal{Q})}$ is weakly connected, there exists a circuit $\gamma\in C(\mathcal{P}\star\mathcal{Q})$ such that*

$$\tau(\mathcal{P}\star\mathcal{Q})\leq\min_{i\in N}p_{i\ldots i}q_{i\ldots i},$$

$$\prod_{i\in\gamma}(p_{i\ldots i}q_{i\ldots i}-\tau(\mathcal{P}\star\mathcal{Q}))\leq\prod_{i\in\gamma}\beta_i p_{i\ldots i}\rho(J_{\mathcal{P}}). \tag{3.9}$$

Proof. The proof is broken into two cases.

Case 1. $\mathcal{P}$ is weakly irreducible. Then, $J_{\mathcal{P}}$ is a weakly irreducible nonnegative tensor. By Lemma 2.2, there exists a positive eigenvector u such that (3.6) holds. Set $D=\mathrm{diag}(u_1,\ldots,u_n)$. Thus, $\sigma(\mathcal{P}\star\mathcal{Q})=\sigma(D^{1-m}(\mathcal{P}\star\mathcal{Q})D)$. By Lemma 3.1, (3.6) and the definitions of β_i, there exists a circuit $\gamma\in C(\mathcal{P}\star\mathcal{Q})$ such that

$$\begin{aligned}\prod_{i\in\gamma}(p_{i\ldots i}q_{i\ldots i}-\tau(\mathcal{P}\star\mathcal{Q}))&\leq\prod_{i\in\gamma}\sum_{\delta_{ii_2\ldots i_m}=0}\frac{-p_{ii_2\ldots i_m}q_{ii_2\ldots i_m}u_{i_2}\ldots u_{i_m}}{u_i^{[m-1]}}.\\ &\leq\prod_{i\in\gamma}\beta_i\sum_{\delta_{ii_2\ldots i_m}=0}\frac{|p_{ii_2\ldots i_m}|u_{i_2}\ldots u_{i_m}}{u_i^{[m-1]}}\\ &=\prod_{i\in\gamma}\beta_i p_{i\ldots i}\rho(J_{\mathcal{P}}),\end{aligned}\tag{3.10}$$

which implies (3.10).

Case 2. $\mathcal{P}$ is weakly reducible. Then $\mathcal{P}-\epsilon\mathcal{S}$ is a strong weakly irreducible $\mathcal{M}$-tensor and $J_{\mathcal{P}}+\frac{\epsilon}{p_{i\ldots i}}\mathcal{S}$ is a weakly irreducible nonnegative tensor when $\epsilon>0$ is sufficiently small. Noting that $\mathcal{P}$ is a strong $\mathcal{M}$-tensor, for the circuit $\gamma\in C(P\star\mathcal{Q})$, we get $\gamma\in C((\mathcal{P}-\epsilon\mathcal{S})\star\mathcal{Q})$. Substituting $\mathcal{P}-\epsilon\mathcal{S}$ for $\mathcal{P}$ and letting $\epsilon\to 0$ on (3.11), we can obtain the desired results by the continuity of $\rho(J_{\mathcal{P}}+\frac{\epsilon}{p_{i\ldots i}}\mathcal{S})$. □

Since Fan product is commutative, the inequality (3.10) remains correct if $\mathcal{P}$ and $\mathcal{Q}$ are switched. Moreover, the following result can be immediately obtained.

Theorem 3.6. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two strong $\mathcal{M}$-tensors of order m and dimension n. If Fan product $\mathcal{P}\star\mathcal{Q}$ is a tensor such that the directed graph $\Gamma_{(\mathcal{P}\star\mathcal{Q})}$ is weakly connected, there exists a circuit $\gamma\in C(\mathcal{P}\star\mathcal{Q})$ such that*

$$\tau(\mathcal{P}\star\mathcal{Q})\leq\min_{i\in N}p_{i\ldots i}q_{i\ldots i},$$

$$\prod_{i\in\gamma}(p_{i\ldots i}q_{i\ldots i}-\tau(\mathcal{P}\star\mathcal{Q}))\leq\prod_{i\in\gamma}\alpha_i q_{i\ldots i}\rho(J_{\mathcal{Q}}). \tag{3.11}$$

Remark 3.7. There exists two differences between the paper [24] and this paper. The first difference is that the paper [24] established lower bounds on the minimum eigenvalues for the Fan product by virtue of Gersgorin-type eigenvalue inclusion set and Perron-Frobenius theorem, whereas this paper proposed lower bounds on the minimum eigenvalues based on Brualdi-type eigenvalue inclusion set and diagonal similarity transformation method. The second difference lies in the fact that the paper [24] gave lower bounds in terms of $\tau(\mathcal{P})$ and $\tau(\mathcal{Q})$, whereas this paper provided lower bounds in the light of $\rho(J_{\mathcal{P}})$ and $\rho(J_{\mathcal{Q}})$, which are relatively easy to calculate.

The following example exhibits efficiency of Theorems 3.3-3.6.

Example 3.8. Let $\mathcal{P} = (p_{ijk}), \mathcal{Q} = (q_{ijk})$ be two tensors of order 3 dimension 3 with elements defined as follows:

$$\mathcal{P} = [P(1,:,:), P(2,:,:), P(3,:,:)], \mathcal{Q} = [Q(1,:,:), Q(2,:,:), Q(3,:,:)],$$

where

$$P(1,:,:) = \begin{pmatrix} 3 & 0 & -\frac{1}{3} \\ 0 & -1 & 0 \\ -\frac{1}{3} & 0 & -\frac{1}{2} \end{pmatrix}, P(2,:,:) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, P(3,:,:) = \begin{pmatrix} -\frac{1}{3} & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 5 \end{pmatrix},$$

$$Q(1,:,:) = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}, Q(2,:,:) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 4 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{3} \end{pmatrix}, Q(3,:,:) = \begin{pmatrix} 0 & 0 & -\frac{1}{3} \\ 0 & -\frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 2 \end{pmatrix}.$$

It is easy to see that $\mathcal{P}$ and $\mathcal{Q}$ are both strong $\mathcal{M}$-tensors. By computations, we get $\tau(\mathcal{P}) = 1.0560, \tau(\mathcal{Q}) = 0.7153, \rho(J_{\mathcal{P}}) = 0.6842, \rho(J_{\mathcal{Q}}) = 0.7328$. Obviously, the directed graph $\Gamma_{(\mathcal{P}\star\mathcal{Q})}$ is weakly connected and has three circuits: $3 \to 3, 3 \to 2 \to 3, 3 \to 1 \to 3$. By Theorem 3.6 of [24], we obtain

$$\tau(\mathcal{P}\star\mathcal{Q}) \geq \min_{1\leq i\leq 3}\{4.5585, 5.6146, 4.9931\} = 4.5585.$$

By Theorem 3.8 of [24], we get

$$\tau(\mathcal{P}\star\mathcal{Q}) \geq \min_{1\leq i\leq 3}\{6.8872, 9.4666, 8.8731\} = 6.8872.$$

By Theorem 3.11 of [24], we deduce

$$\tau(\mathcal{P}\star\mathcal{Q}) \geq \max\{\min_{1\leq i\leq 3}\{6.7153, 8.7153, 9.3577\}, \min_{1\leq i\leq 3}\{7.0560, 10.0560, 8.028\}\} = 7.0560.$$

Comparing Theorem 3.6 of [24], by Theorem 3.3, for the circuit $3 \to 3$, we obtain

$$\tau(\mathcal{P}\star\mathcal{Q}) \geq 4.9862;$$

for the circuit $3 \to 2 \to 3$, it holds that

$$\tau(\mathcal{P}\star\mathcal{Q}) \geq 5.4173;$$

for the circuit $3 \to 1 \to 3$, we deduce

$$\tau(\mathcal{P}\star\mathcal{Q}) \geq 4.7173.$$

Because we don't know the information of the circuit γ, and our aim is to estimate lower bound of the smallest eigenvalue, we choose the smallest estimation. Thus,

$$\tau(\mathcal{P}\star\mathcal{Q}) \geq \min_{\gamma\in C(\mathcal{P}\star\mathcal{Q})}\{4.9862, 5.4173, 4.7173\} = 4.7173.$$

Comparing Theorem 3.8 of [24], by Theorem 3.4, we get

$$\tau(\mathcal{P}\star\mathcal{Q}) \geq \min_{\gamma\in C(\mathcal{P}\star\mathcal{Q})}\{8.8804, 9.0645, 7.8788\} = 7.8788.$$

Comparing Theorem 3.11 of [24], by Theorems 3.5 and 3.6, we deduce

$$\begin{aligned}&\tau(\mathcal{P}\star\mathcal{Q}) \geq \max\{\min_{\gamma\in C(\mathcal{P}\star\mathcal{Q})}\{8.2895, 8.8761, 7.5607\},\\ &\min_{\gamma\in C(\mathcal{P}\star\mathcal{Q})}\{9.2672, 9.2257, 8.1358\}\} = 8.1358.\end{aligned}$$

Numerical results show that the lower bounds in Theorems 3.3-3.6 are tighter than those of Theorems 3.10, 3.12 and 3.11 in [24].

3.2 Bounds for the general directed graph $\Gamma_{(\mathcal{P}\star\mathcal{Q})}$

When the directed graph $\Gamma_{(\mathcal{P}\star\mathcal{Q})}$ may not be weakly connected, Brualdi-type inclusion sets can not be satisfied. To overcome the difficulties, based on Brauer-type eigenvalue inclusion sets [2], we establish Brauer-type inequalities for the Fan product of two $\mathcal{M}$-tensors.

Theorem 3.9. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two strong $\mathcal{M}$-tensors of order m and dimension n. If $r_i(\mathcal{P}\star\mathcal{P}) \neq 0$, there exists a entry $p_{i_1\cdots i_m}q_{i_1\cdots i_m} \neq 0$ with $(i_2,\ldots,i_m) \neq (i_1,\ldots,i_1)$ such that*

$$\tau(\mathcal{P}\star\mathcal{Q}) \leq \min_{i\in N} p_{i\ldots i}q_{i\ldots i},$$

$$\prod_{j=1}^{m}(p_{i_j\ldots i_j}q_{i_j\ldots i_j} - \tau(\mathcal{P}\star\mathcal{Q})) \leq \prod_{j=1}^{m}[p_{i_j\ldots i_j}q_{i_j\ldots i_j}\rho(J_{\mathcal{P}})\rho(J_{\mathcal{Q}})]. \tag{3.12}$$

Proof. If $\mathcal{P}$ and $\mathcal{Q}$ are both weakly irreducible, similar to the proof of Theorem 3.3, by Lemma 3.1, we obtain (3.13).

If either $\mathcal{P}$ or $\mathcal{Q}$ is weakly reducible, Then both $\mathcal{P}-\epsilon\mathcal{S}$, $\mathcal{Q}-\epsilon\mathcal{S}$, $J_{\mathcal{P}}+\frac{\epsilon}{p_{i\ldots i}}\mathcal{S}$ and $J_{\mathcal{Q}}+\frac{\epsilon}{q_{i\ldots i}}\mathcal{S}$ are weakly irreducible tensors for any $\epsilon > 0$. Similar to the proof of Theorem 3.3, we claim that $\mathcal{P}-\epsilon\mathcal{S}$ and $\mathcal{Q}-\epsilon\mathcal{S}$ are both strong $\mathcal{M}$-tensors when $\epsilon > 0$ is sufficiently small. Observing that $\mathcal{P}$ and $\mathcal{Q}$ are strong $\mathcal{M}$-tensors, for $-p_{i_1\cdots i_m}q_{i_1\cdots i_m} \neq 0$ in $\mathcal{P}\star\mathcal{Q}$, we get $-(p_{i_1\cdots i_m}-\epsilon)(q_{i_1\cdots i_m}-\epsilon) \neq 0$ in $(\mathcal{P}-\epsilon\mathcal{S})\star(\mathcal{Q}-\epsilon\mathcal{S})$. Substituting $\mathcal{P}-\epsilon\mathcal{S}$ and $\mathcal{Q}-\epsilon\mathcal{S}$ for $\mathcal{P}$ and $\mathcal{Q}$ and letting $\epsilon\to 0$, we can obtain the desired results by the continuity of $\rho(J_{\mathcal{P}}+\frac{\epsilon}{p_{i\ldots i}}\mathcal{S})$ and $\rho(J_{\mathcal{Q}}+\frac{\epsilon}{q_{i\ldots i}}\mathcal{S})$. So, (3.13) holds. □

Based on Lemma 3.1 and Theorems 3.4-3.6, we propose Brauer-type inequalities for the Fan product of two $\mathcal{M}$-tensors.

Theorem 3.10. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two strong $\mathcal{M}$-tensors of order m and dimension n. If $r_i(\mathcal{P}\star\mathcal{P}) \neq 0$, there exists a entry $p_{i_1\cdots i_m}q_{i_1\cdots i_m} \neq 0$ with $(i_2,\ldots,i_m) \neq (i_1,\ldots,i_1)$ such that*

$$\tau(\mathcal{P}\star\mathcal{Q}) \leq \min_{i\in N} p_{i\ldots i}q_{i\ldots i},$$

$$\prod_{j=1}^{m}(p_{i_j\ldots i_j}q_{i_j\ldots i_j} - \tau(\mathcal{P}\star\mathcal{Q})) \leq \prod_{j=1}^{m}(\alpha_{i_j}\beta_{i_j}p_{i_j\ldots i_j}q_{i_j\ldots i_j}\rho(J_{\mathcal{P}})\rho(J_{\mathcal{Q}}))^{\frac{1}{2}}.$$

Theorem 3.11. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two strong $\mathcal{M}$-tensors of order m and dimension n. If $r_i(\mathcal{P}\star\mathcal{P})\neq 0$, there exists a entry $p_{i_1\cdots i_m}q_{i_1\cdots i_m}\neq 0$ with $(i_2,\dots,i_m)\neq(i_1,\dots,i_1)$ such that*

$$\tau(\mathcal{P}\star\mathcal{Q})\leq\min_{i\in N}p_{i\dots i}q_{i\dots i},$$

$$\prod_{j=1}^{m}(p_{i_j\dots i_j}q_{i_j\dots i_j}-\tau(\mathcal{P}\star\mathcal{Q}))\leq\prod_{j=1}^{m}\beta_{i_j}p_{i_j\dots i_j}\rho(J_{\mathcal{P}}).$$

Theorem 3.12. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two strong $\mathcal{M}$-tensors of order m and dimension n. If $r_i(\mathcal{P}\star\mathcal{P})\neq 0$, there exists a entry $p_{i_1\cdots i_m}q_{i_1\cdots i_m}\neq 0$ with $(i_2,\dots,i_m)\neq(i_1,\dots,i_1)$ such that*

$$\tau(\mathcal{P}\star\mathcal{Q})\leq\min_{i\in N}p_{i\dots i}q_{i\dots i},$$

$$\prod_{j=1}^{m}(p_{i_j\dots i_j}q_{i_j\dots i_j}-\tau(\mathcal{P}\star\mathcal{Q}))\leq\prod_{j=1}^{m}\alpha_{i_j}q_{i_j\dots i_j}\rho(J_{\mathcal{Q}}).$$

4 Conclusion

In this paper, we generalized important inequalities on the minimum eigenvalue for the Fan product from matrices to tensors. Based on characterizations of $\mathcal{M}$-tensors, we established Brualdi-type (Brauer-type) bounds on the minimum eigenvalues for the Fan product of two $\mathcal{M}$-tensors. Numerical experiments are provided to exhibit the efficiency of the obtained results.

References

[1] C. Bu, Y. Wei, L. Sun and J. Zhou, Brualdi-type eigenvalue inclusion sets of tensors, *Linear Algebra Appl.* 480 (2015) 168-175.

[2] C. Bu, X. Jin, H. Li and C. Deng, Brauer-type eigenvalue inclusion sets and the spectral radius of tensors, *Linear Algebra Appl.* 512 (2017) 234–248.

[3] H. Chen and Y. Wang, On computing the minimal H-eigenvalue of sign-structured tensors, *Front. Math. China* 12 (2017) 1289–1202.

[4] H. Chen, Y. Chen, G. Li and L. Qi, A semidefinite program approach for computing the maximum eigenvalue of a class of structured tensors and its applications in hypergraphs and copositivity test, *Numer. Linear Algebra Appl.* 25 (2018) e2125.

[5] H. Chen, L. Qi, Y. Song, Column sufficient tensors and tensor complementarity problems, *Front. Math. China* 13 (2018) 255-276

[6] W. Ding, L. Qi and Y. Wei, $\mathcal{M}$-tensors and nonsingular $\mathcal{M}$-tensors, *Linear Algebra Appl.* 439 (2013) 3264–3278.

[7] W. Ding and Y. Wei, Solving multi-linear systems with $\mathcal{M}$-tensors, *J. Sci. Comput.* 68 (2016) 689–715.

[8] F. Fang, Bounds on eigenvalues of Hadamard product and the Fan product of matrices, *Linear Algebra Appl.* 425 (2007) 7–15.

[9] S. Friedland, S. Gaubert and L. Han, Perron-Frobenius theorem for nonnegative multilinear forms and extensions,*Linear Algebra Appl.* 438 (2013) 738–749.

[10] L. Gao, D. Wang and G. Wang, Further results on exponential stability for impulsive switched nonlinear time-delay systems with delayed impulse effects, *Appl. Math. Comput.* (2015) 186–200.

[11] L. Gao, D. Wang, Input-to-state stability and integral inputto-state stability for impulsive switched systems with time-delay under asynchronous switching, *Nonlinear Anal-Hybri* (2016) 55–71.

[12] R. Horn and C. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1985.

[13] Y. Li, F. Chen and D. Wang, New lower bounds on eigenvalue of the Hadamard product of an M-matrix and its inverse, *Linear Algebra Appl.* 430 (2009) 1423–1431.

[14] L. Lim, Singular values and eigenvalues of tensors: a variational approach. in: *Proceedings of the IEEE Internationaln Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, 2005, pp. 129–132.

[15] Q. Liu, G. Chen and L. Zhao, Some new bounds on the spectral radius of matrices, *Linear Algebra Appl.* 432 (2010) 936–948.

[16] Z. Luo, L. Qi, and N. Xiu, The sparsest solutions to $\mathcal{Z}$-tensor complementarity problems, *Optim. Lett.* 11 (2017) 471–482.

[17] Q. Ni, L. Qi and F. Wang, An eigenvalue method for testing the positive definiteness of a multivariate form, *IEEE Trans. Automat. Control* 53 (2008) 1096–1107.

[18] L. Qi, Eigenvalues of a real supersymmetric tensor, *J. Symb. Comput.* 40 (2005) 1302–1324.

[19] L. Qi and Z. Luo, *Tensor Analysis: Spectral Theory and Special Tensors*, SIAM, 2017.

[20] L. Qi, F. Wang and Y. Wang, Z-eigenvalue methods for a global polynomial optimization problem, *Math. Progaram.* 118 (2009) 301–316.

[21] L. Sun, B. Zheng, J. Zhou and H. Yang, Some inequalities for the Hadamard product of tensors, *Linear Multilinear Algebra* 66 (2018) 1199–1214.

[22] G. Wang, G. Zhou and L. Caccetta, Z-eigenvalue inclusion theorems for tensors, *Discrete Contin. Dyn. Syst.* Ser. B. 22 (2017) 187–198.

[23] G. Wang, G. Zhou and L. Caccetta, Sharp Brauer-type eigenvalue inclusion theorems for tensors, *Pacific J. Optim.* 14 (2018) 227–244.

[24] G. Wang, Y. Wang and L. Liu, Bound estimations on the eigenvalues for Fan Product of $\mathcal{M}$-tensors, *Taiwan. J. Math.* 23 (2019) 751–766.

[25] G. Wang, Y. Wang and Y. Zhang, Some inequalities for the Fan product of $\mathcal{M}$-tensors, *J. Inequal. Appl.* 2018 (2018) 257.

[26] G. Wang, Y. Wang and Y. Wang, Some Ostrowski-type bound estimations of spectral radius for weakly irreducible nonnegative tensors, Linear Multilinear Algebra, (2018) DOI: 10.1080/03081087.2018.1561823.

[27] X. Wang, H. Chen and Y. Wang, Solution structures of tensor complementarity problem, *Front. Math. China* 13 (2018) 935-9-45.

[28] Y. Wang, G. Zhou and L. Caccetta, Convergence analysis of a block improvement method for polynomial optimization over unit spheres, *Numer. Linear Algebra Appl.* 22 (2015) 1059–1076.

[29] Y. Wang, K. Zhang and H. Sun, Criteria for strong $\mathcal{H}$-tensors, *Front. Math. China* 11 (2016) 577–592.

[30] Y. Yang and Q. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors I, *SIAM. J. Matrix. Anal. Appl.* 31 (2010) 2517–2530.

[31] K. Zhang and Y. Wang, An $\mathcal{H}$-tensor based iterative scheme for identifying the positive definiteness of multivariate homogeneous forms, *J. Comput. Appl. Math.* 305 (2016) 1–10.

[32] L. Zhang, L. Qi and G. Zhou, $\mathcal{M}$-tensors and some applications, *SIAM. J. Matrix. Anal. Appl.* 35 (2014) 437–452.

[33] D. Zhou, G. Chen, G. Wu and X. Zhang, On some new bounds for eigenvalues of the Hadamard product and the Fan product of matrices, *Linear Algebra Appl.* 438 (2013) 1415–1426.

[34] G. Zhou, G. Wang, L. Qi and A. Alqahtani, A fast algorithm for the spectral radii of weakly reducible nonnegative tensors, *Numer. Linear Algebra Appl.* 25 (2018) e2134.

[35] J. Zhou, L. Sun, Y. Wei and C. Bu, Some characterizations of $\mathcal{M}$-tensors via digraphs, *Linear Algebra Appl.* 495 (2016) 190–198.

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