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ROBUST OPTIMIZATION-BASED PD FEEDBACK CONTROLLER DESIGN FOR QUASI-LINEAR SYSTEMS: A PARAMETRIC METHOD*

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Abstract: This paper presents a parameterized control method aimed at the quasi-linear system, which gives general parameterization of designing a controller of proportional plus derivative (PD) feedback gain. The proposed approach is based on the solutions of generalized Sylvester equation and converts the closed-loop system into a linear constant one with an expected eigenstructure which is related to an arbitrarily constant matrix F containing the expected eigenvalues. Also, it provides all the degrees of freedom to improve the control accuracy and system performance. To further illustrate the superiority of this method, we give a robust optimized solution according to one of the indicators of robust optimization. An actual example demonstrates the effectiveness of the method.

Key words: *parametric method, quasi-linear system, PD feedback, robust control*

Mathematics Subject Classification: *93B51, 93D09*

1 Introduction

In recent years, the linear system has shown its limitation in describing problems because most of the actual systems are nonlinear. However, nonlinear systems are unstable and difficult to calculate. To compensate, the concept of the quasi-linear system has been proposed, which is a particular form of the nonlinear system. It can be easy to describe and calculate problems. With the development of science and technology, the research of the quasi-linear system is more and more extensive. In many fields, the quasi-linear system model is widely used and described problems effectively, such as industry process [11], mathematical field [25], power systems [2] and other applications [9, 15, 31]. Chen et al. proposed a solution by Lyapunov function and S-procedure approach to improving the stability of a time-varying system which is formulated as a quasi-linear system [8]. Castillo et al. aimed at first-order quasi-linear hyperbolic systems and designed the boundary observer by employing Lyapunov techniques [4]. Gan et al. built a quasi-linear model and optimized the parameters by combined genetic optimization and gradient-based optimization algorithm [3]. Based on the global practical trajectory tracking for a class of nonlinear systems, Li and Liu established a new tracking scheme [21]. Gan et al. built a quasi-linear model and optimized

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the parameters which used genetic optimization as well as gradient-based optimization algorithm [10]. Rumyantsev et al. presented a control law of linear feedback regulator to optimize the information constrained. Besides, this approach can apply to the real problem successfully [29]. Kabamba et al. designed a step tracking controller and simplified the technical process for the quasi-linear control theory [14].

However, some flaws can be seen in the proposed methods. Firstly, the counting process is a little complicated and difficult to compute. Secondly, the approaches mentioned above are not flexible and have a little degrees of freedom. Thirdly, the closed-loop system obtained by this method is usually nonlinear, and it can also be stabilized by the Lyapunov function. However, in this study, the proposed approach can stabilize the quasi-linear system effectively with proportional-plus-derivative state feedback.

It is well known that PD feedback is a widely used approach in controller designing. In particularly, proportional feedback can control part of the system in advance before the deviation is formed. However, excessive ones will weaken the anti-interference performance of the system. So designing an appropriate PD feedback controller is necessary. About the proportional-plus-derivative feedback, there has been a lot of research in the past few years. Ren et al. presented a proportional plus derivative feedback based on the problem of descriptor systems with uncertainties and designed a PD feedback controller to deal with it [26]. Ren and Zhang designed a PD controller to improve the stability of the closed-loop system [27]. Aktas et al. presented a PD feedback controller to improve performance better than nominal fixed gain controllers [1]. Also, there are other scholars have presented the PD feedback to kinds of systems and achieve the desired purpose [22, 23, 28].

This paper comes up with a parametric control approach for a quasi-linear system by PD feedback controller to stabilize a system. Compared with other methods for quasi-linear systems, this paper proposed an approach with higher freedom degrees of arbitrary parameters. The PD feedback controller with a parameterized form will be less computational complexity. It is more practical and advanced. The theoretical basis of the proposed method is the solution of a group of generalized Sylvester matrix equation [13, 24]. There are several advantages to this approach. First of all, the parametric approach can simplify the computational calculation process. Secondly, according to the parametric approach, its dynamic performance is determined by eigenvalues and eigenvectors. Because that a closed-loop system can be transformed into a linear constant one [5]. Thirdly, it can provide more degrees of freedom when control is achieved [6].

The present work is set out as follows. In Section 2 some necessary mathematical preliminaries and problem statement are given. The main results are provided in Section 3. Finally, a practical example is worked out in Section 4 to illustrate the effectiveness of the proposed approach.

2 Problem Description and Preliminary Results

2.1 Problem statement

According to the actual system mentioned above, we propose a class of quasi-linear system in this paper as following form

$$\dot{x} = A(\alpha, x)x + B(\alpha, x)u, \tag{2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$ are the state vector and control vector; α is a time-variant parameter vector which staisfies the following condition

$$\alpha(t) = \begin{bmatrix} \alpha_1(t) & \alpha_2(t) & \dots & \alpha_l(t) \end{bmatrix}^{\mathrm{T}} \in \Omega \subset \mathbb{R}^l, t \geq 0, \tag{2.2}$$

where Ω is a compact set. The values of the system parameter $\alpha = \alpha(t) \in \mathbb{R}^l$ are within some compact set Ω, $\alpha(t) \in \Omega \subset \mathbb{R}^l$. $A(\alpha, x) \in \mathbb{R}^{n\times n}$ and $B(\alpha, x) \in \mathbb{R}^{r\times n}$ are system coefficient matrices, we assume the following assumptions are satified.

Assumption 2.1. $B(\alpha, x)$ is consistently related to x and $\alpha(t) \in \Omega$.

Assumption 2.2. rank $\begin{bmatrix} sI - A(\alpha, x) & B(\alpha, x) \end{bmatrix} = n, \forall s \in \mathbb{C}$.

In order to achieve the purpose of the design to system (2.1), we propose a PD controller in the following form

$$u = K(\alpha, x)x - L(\alpha, x)\dot{x}, \tag{2.3}$$

where $K(\alpha, x)$ is the proportional feedback gain matrix, and $L(\alpha, x) \in \mathbb{R}^{r\times n}$ is the derivative feedback gain matrix

With the controller (2.3) applied to system (2.1), the closed-loop system is obtained as following form

$$E_c(\alpha, x)\dot{x} = A_c(\alpha, x)x, \tag{2.4}$$

with

$$E_c(\alpha, x) = I + B(\alpha, x)L(\alpha, x), \tag{2.5}$$

$$A_c(\alpha, x) = A(\alpha, x) + B(\alpha, x)K(\alpha, x). \tag{2.6}$$

Problem 2.1. Given the system (2.1) satisfying Assumptions 2.1-2.2, and an arbitrarily chosen matrix $F \in \mathbb{R}^{n\times n}$, find a right prime decomposition matrix $V(\alpha, x) \in \mathbb{R}^{n\times n}$, and a gain matrix $K(\alpha, x) \in \mathbb{R}^{r\times n}$ to satisfy the following equation

$$A_c(\alpha, x)V(\alpha, x) = E_c(\alpha, x)V(\alpha, x)F. \tag{2.7}$$

Remark 2.2. Let $\Gamma = \{\lambda_i, i = 1, \ldots, n', 1 \leq n' \leq n\}$ be the set of eigenvalues of matrix pair $(E_c(\alpha, x), A_c(\alpha, x))$, which is symmetric about the real axis, and the algebraic and geometric multiplicity of λ_i are denoted by a_i and b_i respectively. Then in the Jordan form of matrix pair $(E_c(\alpha, x), A_c(\alpha, x))$, corresponding with λ_i, there b_i sub-Jordan blocks of orders denoted by $p_{ij}, j = 1, 2, \ldots, b_i$, and the following relations hold

$$p_{i1} + p_{i2} + \cdots + p_{iq_i} = a_i,\ a_1 + a_2 + \cdots + a_{n'} = n.$$

Denote the right eigenvectors with respect to λ_i by $v_{ij}^k \in \mathbb{C}^n,\ k = 1, 2, \ldots, p_{ij},\ j = 1, 2, \ldots, b_i$. Then, by the definition we have

$$(A_c(\alpha, x) - \lambda_i E_c(\alpha, x))v_{ij}^k = E_c(\alpha, x)v_{ij}^{k-1}, v_{ij}^0 = 0. \tag{2.8}$$

$\boxed{2.2}$ Preliminary results

Consider a right coprime factorization (RCF) as follows

$$[sI - A(\alpha, x)]N(\alpha, x, s) = B(\alpha, x)D(\alpha, x, s), \tag{2.9}$$

among them, $N(\alpha, x, s) \in \mathbb{R}^{n\times r}[s]$ and $D(\alpha, x, s) \in \mathbb{R}^{r\times r}[s]$ are polynomial matrices. Denote $N(\alpha, x, s) = \left[n_{ij}(s)\right]_{n\times r}$ and $D(\alpha, x, s) = \left[d_{ij}(s)\right]_{n\times r}$, and

$$\begin{cases} \omega_1 & = \max \{\deg\ (n_{ij}(s)), i = 1, 2, \ldots, n, j = 1, 2, \ldots, r\}, \\ \omega_2 & = \max \{\deg\ (d_{ij}(s)), i = 1, 2, \ldots, r, j = 1, 2, \ldots, r\}, \\ \omega & = \max \{\omega_1, \omega_2\}, \end{cases} \tag{2.10}$$

where deg denotes the degree of the polynomial matrix. Then, $N(\alpha, x, s)$ and $D(\alpha, x, s)$ can be written as

$$\begin{cases} N(\alpha, x, s) = \sum_{i=0}^{\omega} N_i(\alpha, x)s^i, \\ D(\alpha, x, s) = \sum_{i=0}^{\omega} D_i(\alpha, x)s^i. \end{cases} . \tag{2.11}$$

$\boxed{3}$ Solution to Problem 2.1

$\boxed{3.1}$ Case of F arbitrary

We now are ready to deal with the Problem 2.1. First, we propose the following theorem.

Theorem 3.1. *Let $N(\alpha, x, s)$ and $D(\alpha, x, s)$ be polynomial matrices which are satisfying RCF* (2.9)*, then,*

1. *Problem 2.1 has a solution if and only if there is a matrix $Z_c \in \mathbb{R}^{r\times n}$ satisfying*

$$\det\ V(\alpha, x) \neq 0, \tag{3.1}$$

where

$$V(\alpha, x) = \sum_{i=0}^{\omega} N_i Z_c F^i. \tag{3.2}$$

2. *For maintaining the regularity of closed-loop system, we choose a matrix $L(\alpha, x)$ satisfying*

$$\det\ E_c(\alpha, x) \neq 0. \tag{3.3}$$

3. *When the conditions mentioned above are satisfied, $K(\alpha, x)$ can be obtained as*

$$K(\alpha, x) = (W(\alpha, x) + L(\alpha, x)V(\alpha, x)F)V^{-1}(\alpha, x), \tag{3.4}$$

where

$$W(\alpha, x) = \sum_{i=0}^{\omega} D_i Z_c F, \tag{3.5}$$

and $Z_c \in \mathbb{R}^{r\times n}$ is a group of arbitrary parameter matrix that represent the degrees of freedom in the solutions.

Proof. Taking $V(\alpha, x)$,$K(\alpha, x)$ and $L(\alpha, x)$ into Equation (2.7), we obtain

$$A(\alpha, x)V(\alpha, x) + B(\alpha, x)W_c(\alpha, x) = V(\alpha, x)F, \tag{3.6}$$

with

$$W_c(\alpha, x) = K(\alpha, x)V(\alpha, x) - L(\alpha, x)V(\alpha, x)F. \tag{3.7}$$

Using Equation (2.11), we have

$$\begin{aligned}(sI - A)N(\alpha, s) &= \sum_{i=0}^{\omega} N_i s^{i+1} - \sum_{i=0}^{\omega} AN_i s^i \\ &= N_\omega s^{\omega+1} + \sum_{i=1}^{\omega} N_{i+1} s^i - \sum_{i=1}^{\omega} AN_i s^i - AN_0 \\ &= N_\omega s^{\omega+1} + \sum_{i=1}^{\omega} (N_{i-1} - AN_i) s^i - AN_0\end{aligned}$$

and

$$BD(\alpha, s) = \sum_{i=0}^{\omega} B_0 D_i s^i = \sum_{i=1}^{\omega} BD_i s^i + BD_0.$$

We possess relations following

$$AN_0 = -BD_0, N_\omega = 0, \tag{3.8}$$

and

$$BD_i = N_{i-1} - AN_i, i = 1, 2, \ldots, \omega. \tag{3.9}$$

So, we can obtain

$$\begin{aligned}VF - AV &= \sum_{i=0}^{\omega} N_i Z_c F^{i+1} - A \sum_{i=1}^{\omega} AN_i Z_c F^i \\ &= N_\omega Z_c F^{\omega+1} + \sum_{i=1}^{\omega} N_{i-1} Z_c F^i - AN_0 Z \\ &= N_\omega Z_c F^{\omega+1} + \sum_{i=1}^{\omega} (N_{i-1} - AN_i) Z_c F^i - AN_0 Z_c \\ &= 0 + \sum_{i=1}^{\omega} BD_i Z_c F^i + BD_0 Z_c \\ &= B \sum_{i=0}^{\omega} D_i Z_c F^i \\ &= BW.\end{aligned}$$

According to the above deduction, we can obtain the expressions of $V(\alpha, x)$ and $W(\alpha, x)$, and we choose an arbitrary diagonal F and a derivative feedback gain matrix $L(\alpha, x)$ based on constraint condition. So, the generally completely parameterized expression of PD feedback gain matrices are given in Equation (3.4).

With the above description, the proof of Theorem 3.1 is completed. □

$\boxed{3.2}$ Case of F diagonal

In practical systems, F is usually chosen to be a diagonal one as follows

$$F = \text{diag } \{\lambda_1, \lambda_2, \dots, \lambda_n\}, \tag{3.10}$$

where $\lambda_i \in \mathbb{C}^-, i = 1, 2, \dots, n$ are a set of self-conjugate complex poles. In this situation, a class of general solutions of generalized Sylvester matrix Equation (3.5) can be written as

$$\begin{cases} V(\alpha, x) = \begin{bmatrix} v_1(\alpha, x) & v_2(\alpha, x) & \dots & v_n(\alpha, x) \end{bmatrix} \\ v_i(\alpha, x) = N(\alpha, s, \lambda_i) z_i^c, i = 1, 2, \dots, n \end{cases} \tag{3.11}$$

and

$$\begin{cases} W_c(\alpha, x) = \begin{bmatrix} \omega_1^c(\alpha, x) & \omega_2^c(\alpha, x) & \dots & \omega_n^c(\alpha, x) \end{bmatrix} \\ \omega_i^c(\alpha, x) = D(\alpha, s, \lambda_i) z_i^c, i = 1, 2, \dots, n \end{cases}. \tag{3.12}$$

Therefore, in this case, we possess the following corollary to deal with Problem 2.1.

Corollary 3.2. *Let $N(s)$ and $D(s)$ be a pair of polynomial matrices satisfying RCF (2.9), then,*

1. *Problem 2.1 possesses a solution if and only if there exists a set of parameter vector $z_i^c \in \mathbb{R}^r$, $i = 1, 2, \dots, n$, such that*
$$\det V(\alpha, s, \lambda_i, z_i^c, i = 1, 2, \dots, n) \neq 0. \tag{3.13}$$
2. *In order to satisfy the regularity of closed-loop system, we choose a matrix $K_1(\alpha, x)$ to satisfy*
$$\det E_c(\alpha, x) \neq 0. \tag{3.14}$$
3. *When it is meeting the conditions, we can calculate the proportional feedback gain $K(\alpha, x)$ in Equation (3.3).*

Proof. According to Theorem 3.1, when the matrix F is diagonal as the one shown in Equation (3.6), the matrices $V(\alpha, x)$ and $W_c(\alpha, x)$ can be given as Equations (3.7) and (3.8), the Corollary 3.2 can be proven. □

$\boxed{3.3}$ Robust criteria

In order to improve the robust stability of the calculated results, we optimize it to obtain more accurate results. The robust optimization satisifies the following criteria

$$\phi = \| \eta \|_2, \tag{3.15}$$

with

$$\eta = (\eta_1, \eta_2, \dots, \eta_n), \tag{3.16}$$

$$\eta_i = \frac{\| v_i \|_2 \| p_i \|_2}{\| p_i^{\mathrm{T}} v_i \|_2}. \tag{3.17}$$

The vectors v_i and p_i are column vectors of $V \in \mathbb{R}^{n \times n}$ and $P \in \mathbb{R}^{n \times n}$, which are desired closed-loop right and left eigenvectors in the allowable subspaces. And they satisfy the constraint as follows

$$P^{\mathrm{T}} V(\alpha, x) = I. \tag{3.18}$$

3.4 General procedure

We can propose a parametrized form of solution of PD feedback controller based on Theorems 3.1 and 3.2.

Step 1 Based on Theorem 3.1 or Corollary 3.2, we design a parameterized form of PD feedback controller to a type of quasi-linear system. Choose an appropriate diagonal matrix F which has desired eigenstructure, and usually a diagonal form. That is, all eigenvalues of F are on the left-half s-plane [12].

$$\lambda_i(F) \in \mathbb{C}^-, i = 1, 2, \ldots, n. \tag{3.19}$$

Step 2 According to (2.11) we can obtain the following solution

$$\begin{cases} N(s) = \operatorname{adj}(sI - A(\alpha, x))B(\alpha, x), \\ D(s) = \det(sI - A(\alpha, x))I_r. \end{cases} \tag{3.20}$$

Step 3 Based on the norm illustrated above, we use Function $fimnsearch$ to finish simulation and find a group of $Z_c' \in \mathbb{R}^{r\times n}$ which can minimize the value of ϕ in Equation (3.15). Considering the effect of non-optimized system dynamics and external disturbance in the closed-loop system, a disturbance is presented as following form

$$\dot{X} = \bar{A}_c X + \bar{G}w(t). \tag{3.21}$$

Step 4 We choose arbitrary matrices Z_c and L which can make E_c is constant. According to $V(\alpha, x)$ and $W(\alpha, x)$ are shown in Equations (3.2) and (3.5). Further we can calculate the proportional feedback gain K as Equation (3.4).

4 Example — A Two-Link Robot System

4.1 System description

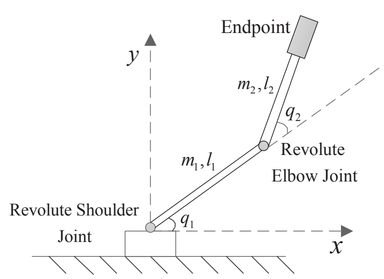

Figure 1: two-link robot with rigid links of machine model

Consider a two-link robot with rigid links as shown in Figure 1 [7], and the mathematical model is given as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + \xi(q, \dot{q}) = \tau, \tag{4.1}$$

where q and τ are respectively the coordinate vector and the control vector, both of dimension n. The control vector τ is in fact composed of the torques produced by the motors at the joint. $M(q)$ is the $n \times n$ symmetric and positive definite robot inertia matrix, $C(q, \dot{q})$ comprises the Coriolis and centrifugal effects, and $\xi(q, \dot{q})$ is some uniformly bounded piece-wisely continuous vector function.

The detailed derivation of the dynamics model can be found in [30] and [32]. As a result, $\xi(q,\dot{q})=0$. Then, let

$$x=\begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \tag{4.2}$$

we can obtain the following system

$$E\dot{x}=Ax+B\tau, \tag{4.3}$$

with

$$E=\begin{bmatrix} I & 0 \\ 0 & M(q) \end{bmatrix}, A=\begin{bmatrix} 0 & I \\ 0 & C(q,\dot{q}) \end{bmatrix}, B=\begin{bmatrix} 0 \\ I \end{bmatrix},$$

with

$$M(q)=\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, C(q,\dot{q})=-m_2 l_1 l_{c2} \sin\, q_2 \begin{bmatrix} \dot{q}_2 & \dot{q}_1+\dot{q}_2 \\ \dot{q}_1 & 0 \end{bmatrix}, \tag{4.4}$$

where

$$\begin{cases} M_{11} &= m_1 l_{c1}^2 + \frac{m_1 l_1^2 + m_2 l_2^2}{12} + m_2(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} \cos\, q_2) \\ M_{12} &= M_{21} = m_2(l_{c2}^2 + l_1 l_{c2} \cos\, q_2) + \frac{m_2 l_2^2}{12} \\ M_{22} &= m_2 l_{c2}^2 + \frac{m_2 l_2^2}{12} \end{cases}. \tag{4.5}$$

Consider Equation (3.21), to further illustrate the robust stability of optimization, we add a disturbance as following

$$\bar{G}=\begin{bmatrix} 0 & 0 & 2 & 0 \end{bmatrix}^{\mathrm{T}}, \tag{4.6}$$

$$w(t)=\begin{cases} 1, t\in[8,10] \\ 0,\ \text{else} \end{cases}. \tag{4.7}$$

4.2 Non-optimized solutions

Let $\tau=Kx-L\dot{x}$, we can obtain the following system

$$\dot{x}=Ax+B(Kx-L\dot{x}), \tag{4.8}$$

According to RCF (2.9), there are some solutions of mentioned system as

$$N(s)=\begin{bmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & s \end{bmatrix}^{\mathrm{T}}, \tag{4.9}$$

$$D(s)=\begin{bmatrix} 4s^2\cos\, q_2+\frac{20}{3}s^2+2s\dot{q}_2\sin\, q_2 & 2s^2\cos\, q_2+\frac{4}{3}s^2+2s\sin\, q_2(\dot{q}_1+\dot{q}_2) \\ 2s^2\cos\, q_2+\frac{4}{3}s^2+2s\dot{q}_1\sin\, q_2 & \frac{4}{3}s^2 \end{bmatrix}. \tag{4.10}$$

Choose

$$F=\mathrm{diag}\{-1,-2,-3,-4\}. \tag{4.11}$$

and

$$Z_c=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{bmatrix}, \tag{4.12}$$

according to Equation (3.2), $V(\alpha, x)$ can be calculated as

$$V(\alpha, x) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ -1 & -2 & -3 & -4 \\ 0 & -4 & -3 & -4 \end{bmatrix}, \tag{4.13}$$

further, based on Equation (3.7), we possess

$$W(\alpha, x) = \left[\begin{matrix} 4\cos q_2 + \frac{20}{3} - 2\dot{q}_2 \sin q_2 & 32\cos q_2 + \frac{112}{3} - 4\dot{q}_2 \sin q_2 - 8\sin q_2(\dot{q}_1 + \dot{q}_2) \\ 2\cos q_2 + \frac{20}{3} - 2\dot{q}_1 \sin q_2 & 8\cos q_2 + 16 - 4\dot{q}_1 \sin q_2 \end{matrix} \right.$$
$$\left. \begin{matrix} 54\cos q_2 + 72 - 6\dot{q}_2 \sin q_2 - 6\sin q_2(\dot{q}_1 + \dot{q}_2) & 96\cos q_2 + 128 - 8\dot{q}_2 \sin q_2 - 8\sin q_2(\dot{q}_1 + \dot{q}_2) \\ 18\cos q_2 + 24 - 6\dot{q}_1 \sin q_2 & 32\cos q_2 + \frac{128}{3} - 8\dot{q}_1 \sin q_2 \end{matrix} \right]. \tag{4.14}$$

According to Equation (3.3), let

$$L(\alpha, x) = \begin{bmatrix} 0 & 0 & -4\cos q_2 - \frac{17}{3} & -2\cos q_2 - \frac{4}{3} \\ 0 & 0 & -2\cos q_2 - \frac{4}{3} & -\frac{1}{3} \end{bmatrix}, \tag{4.15}$$

then, based on Equation (3.4), we can obtain the matrix $K(\alpha, x)$ as

$$K(\alpha, x) = \begin{bmatrix} -2 & -14 & 2\dot{q}_2 \sin q_2 + 1 & 2\sin q_2(\dot{q}_1 + \dot{q}_2) - 8 \\ 4 & -16 & 2\dot{q}_1 \sin q_2 + 4 & -11 \end{bmatrix}. \tag{4.16}$$

Under the controllers designed above, the closed-loop system can be obtained as

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & -14 & 1 & -8 \\ 4 & -16 & 4 & -11 \end{bmatrix} x. \tag{4.17}$$

4.3 Robust Solutions

This subsection is concerned with robust control design on the basic parametric eigenstructure assignment. The robustness indices as cost functions expressed in the frequency domain. The optimization is achieved via the gradient calculation of the robustness indices.

The initial selections of parameters Z_c, $N(s)$, $D(s)$, and F are chosen as the same as Section 4.2. According to Equation (3.15), we obtain that the initial value of ϕ is 31.3741. Many scholars have proposed various methods to solve the robust optimization, such as $fminsearch$ Function and distributionally robust optimization [17–19]. In the paper, we utilize Particle Swarm Optimization (PSO) to minimize the index ϕ in Equation (3.15). Based on the proposed approach, we choose the learning factors $c_1 = c_2 = 2$, the maximum number of iterations $M = 300$, the dimension of search space $D = 1$, the number of initialization group individuals $N = 30$, the inertia weight minimum $wmin = -0.5$, and the inertia weight max $wmax = 0.5$. Finally, through the optimization, the minimum value ϕ' is obtained as 7.2457. Further, we can cacaulate

$$Z_c' = \begin{bmatrix} -0.0063 & 0.7769 & 1.7635 & 0.4562 \\ 0.0014 & 3.3994 & -0.4030 & 1.9962 \end{bmatrix}. \tag{4.18}$$

Therefore, according to Equations (3.2), (3.4) and (3.18), we can obtain

$$W'(\alpha, x) = \left[\begin{array}{c} \frac{63\dot{q}_2 \sin q_2}{5000} - \frac{7 \sin q_2(\dot{q}_1 + \dot{q}_2)}{2500} - \frac{14 \cos q_2}{625} - \frac{301}{7500} \\ \frac{63\dot{q}_1 \sin q_2}{5000} - \frac{63\dot{q}_2 \cos q_2}{5000} - \frac{49}{7500} \end{array} \right.$$
$$\begin{array}{c} \frac{24766 \cos q_2}{625} - \frac{16997 \sin q_2(\dot{q}_1 + \dot{q}_2)}{1250} - \frac{7769\dot{q}_2 \sin q_2}{2500} + \frac{72839}{1875} \\ \frac{7769 \cos q_2}{1250} - \frac{7769\dot{q}_1 \sin q_2}{2500} + \frac{13921}{625} \end{array}$$
$$\begin{array}{c} \frac{7029 \cos q_2}{125} + \frac{1209 \sin q_2(\dot{q}_1 + \dot{q}_2)}{500} - \frac{10581 \cos q_2 \sin q_2}{1000} + \frac{50487}{500} \\ \frac{31743 \cos q_2}{1000} - \frac{10581\dot{q}_1 \sin q_2}{1000} + \frac{8163}{500} \end{array}$$
$$\left. \begin{array}{c} \frac{58172 \cos q_2}{625} + \frac{9981 \sin q_2(\dot{q}_1 + \dot{q}_2)}{625} - \frac{2281\dot{q}_2 \sin q_2}{625} + \frac{171088}{1875} \\ \frac{9124 \cos q_2}{625} - \frac{2281\dot{q}_1 \sin q_2}{625} + \frac{98096}{1875} \end{array} \right],$$

$$V'(\alpha, x) = \begin{bmatrix} -0.0063 & 0.7769 & 1.7635 & 0.4562 \\ 0.0014 & 3.3994 & -0.4030 & 1.9962 \\ 0.0063 & -1.5538 & -5.2905 & -1.8248 \\ -0.0014 & -6.7988 & 1.2090 & -7.9848 \end{bmatrix},$$

$$T(\alpha, x) = \begin{bmatrix} -226.5877 & 0.1240 & -0.2695 & -0.1067 \\ 51.7840 & 0.5600 & 0.0616 & -0.4766 \\ -75.5292 & 0.0307 & -0.2695 & -0.0537 \\ 17.2612 & 0.1401 & 0.0616 & 0.2382 \end{bmatrix},$$

$$K'(\alpha, x) = \begin{bmatrix} -3.2423 & -1.0873 & -4.0973 & -0.4348 \\ -1.0603 & -7.7577 & -0.4285 & -5.9027 \end{bmatrix}.$$

The derivative feedback matrix $L(\alpha, x)$ is the same as Equation (4.15). Based on Equation (2.6), the closed-loop system is obtained as

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3.2423 & -1.0873 & -4.0973 & -0.4348 \\ -1.0603 & -7.7577 & -0.4258 & -5.9027 \end{bmatrix} x. \tag{4.19}$$

Remark 4.1. PID controller has been attracting considerable research attention, e.g. [16] and [20]. PID controller is chosen as following form

$$u(t) = k_P e(t) + k_I \int_0^t e(s)\mathrm{d}s + k_D \dot{e}(t), \tag{4.20}$$

where $e(t) = q(t) - q_d$ is the position error, $q_d \in \mathbb{R}$ denotes the desired constant, k_P, k_I, k_D are the proportional, integral and derivative gains of the controller. The main purpose is to determine the control gain $k = [k_P\ k_I\ k_D]^{\mathrm{T}}$ under optimization. Applying the PID controller (4.20), the original system (4.1) can be rewritten as

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = \dfrac{1}{M(q)} u(t) - \dfrac{C(q, \dot{q})}{M(q)} x_2(t), \\ \dot{x}_3(t) = x_1(t) - q_d. \end{cases} \tag{4.21}$$

where $x_1(t) = q(t)$, $x_2(t) = \dot{q}(t)$, $u(t) = \tau(t)$. Then, we aim to find a control gain k to minimize cost function

$$J(k) = \int_0^2 [x_1(t) - q_d]^2 \mathrm{d}t. \tag{4.22}$$

The principle of the proposed approach and PID method is to transform the controller design problem into the parameters selection problem. We can see that there is an integral gain k_I in PID control which can eliminate the steady state error. However, comparing with PID controller design, the proposed method is much easier to calculate.

4.4 Simulation Analysis

In this subsection, the results of numerical simulations are presented to compare the performance of non-optimized and optimized solutions. The comparison of the time responses of states $x_1 - x_4$ between the non-optimized and optimized solutions with the disturbance are shown in Figures 2-5. From these figures we clearly see that the closed-loop system is stable and robust, furthermore, the optimized solutions leads to a better transient performance than the non-optimized.

The control of signals of the closed-loop system are shown in Figures 6 and 7. We can clearly see that the optimized solution leads to better transient performances at the cost of less control energy and magnitude of the control signals.

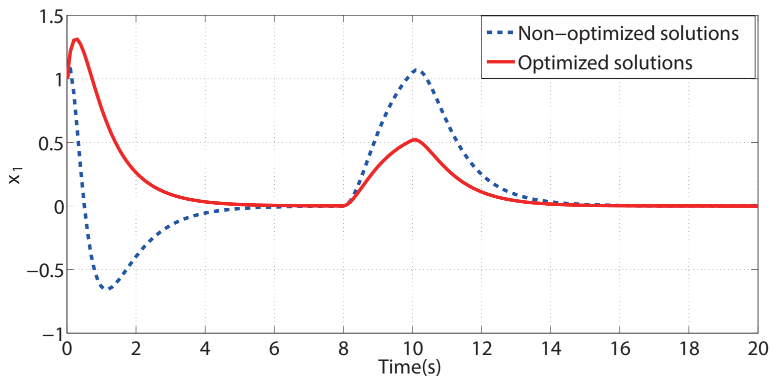

Figure 2: Comparison of the state variable x_1 between the non-optimized and optimized solutions

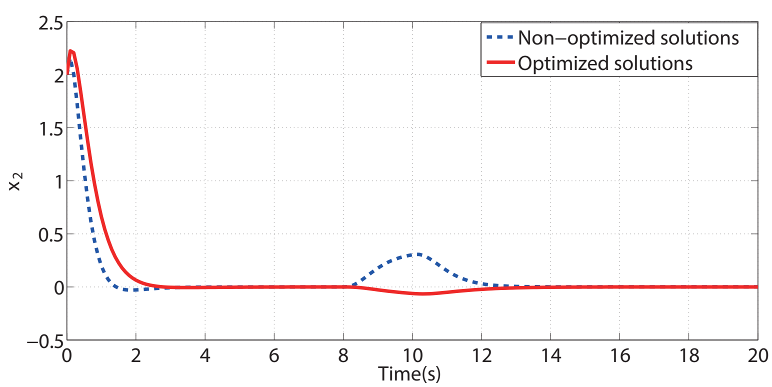

Figure 3: Comparison of the state variable u_2 between the non-optimized and optimized solutions

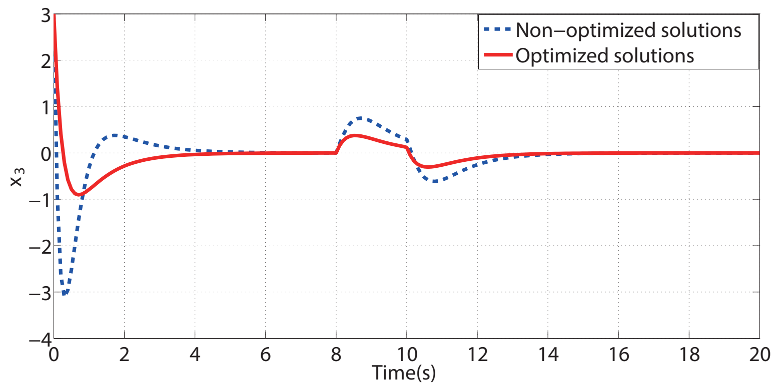

Figure 4: Comparison of the state variable x_3 between the non-optimized and optimized solutions

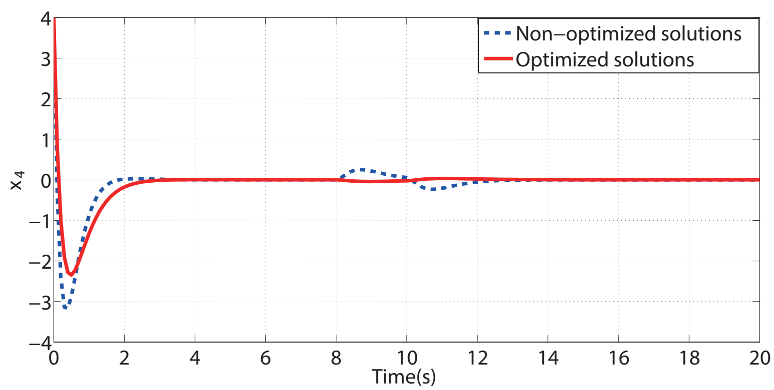

Figure 5: Comparison of the state variable x_4 between the non-optimized and optimized solutions

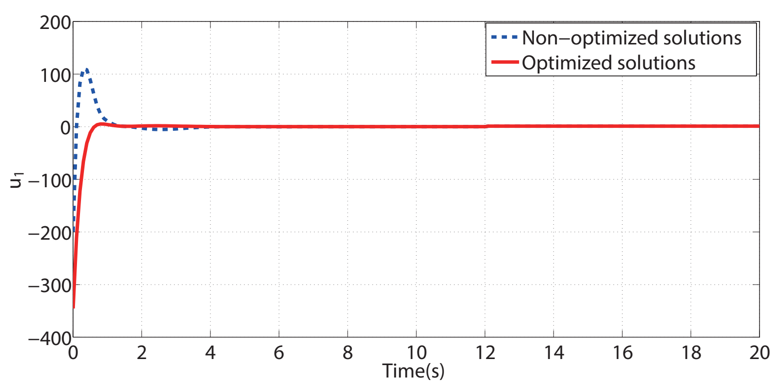

Figure 6: Comparison of the input variable u_1 between the non-optimized and optimized solutions

5 Conclusion

The paper designs a proportional plus derivative feedback controller for quasi-linear systems by parameterized approach and gives a general parametric form of the right eigenvector

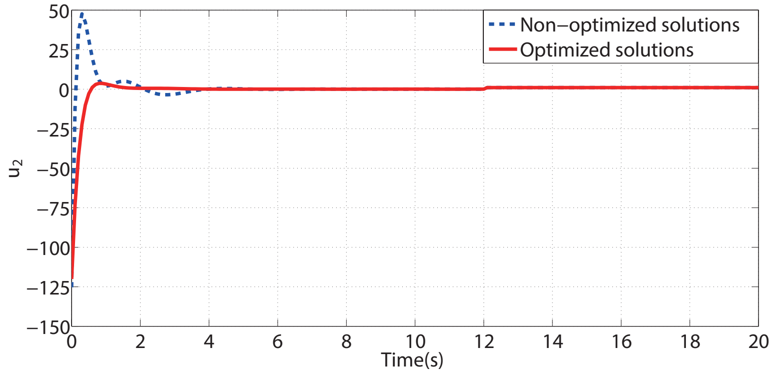

Figure 7: Comparison of the input variable x_2 between the non-optimized and optimized solutions

matrix. It is a significant result of the proposed approach that the closed-loop system can be converted into a linear constant one with desired eigenstructure. The proposed approach can simplify the design process for the complicated quasi-linear systems. An example of a two-link robot system is developed to show the proposed approach can effectively solve the positions and velocities control problem. The next major work is to find the pulse elimination conditions to improve the performance of the system.

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