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NONLINEAR DIFFERENTIAL VARIATIONAL INEQUALITIES WITH NONCONVEX SETS IN L^p SPACES*

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Abstract: In this paper, we use recent results on proximal analysis in Banach spaces [7,9] to prove the existence of solutions of a particular form of nonconvex differential variational inequalities (NDVI) in $L^p([0,1],\mathbb{R})$ spaces with $p \geq 2$. The proposed (NDVI) coincides with the well known nonconvex sweeping process when $p = 2$. Also, the convex sweeping process studied in Banach spaces in [6] is covered by our (NDVI) when the moving set is convex. Examples of nonconvex moving sets in $L^p([0,1],\mathbb{R})$ are stated. We also notice that the Lipschitz assumption on the moving set is weaker and easy to check relatively to the one used in [6].

Key words: *nonconvex differential variational inequality, generalized proximal normal cone, generalized uniformly V-prox-regularity, nonconvex sweeping process*

Mathematics Subject Classification: *49J52, 26E25, 49J40, 47J20*

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1 Introduction

In the present paper we consider the following nonlinear differential variational inequality: Find $u : I := [0,T](T > 0) \to L^p([0,1],\mathbb{R})$ such that $u(0) = u_0 \in C(0)$, $u(t) \in C(t), \forall t \in I$, and for any $x \in C(t)$

$$\left\langle f(t,u(t)) + \frac{d}{dt}\left(\|u(t)\|_{L^p}^{2-p}|u(t)|^{p-2}u(t)\right); x - u(t)\right\rangle_{L^{p'},L^p} \geq \frac{-\beta\dot{v}(t)}{2r}V(\left(\|u(t)\|_{L^p}^{2-p}|u(t)|^{p-2}u(t)\right); x),$$

where $p \geq 2$, $\frac{1}{p}+\frac{1}{p'} = 1$, $\beta > 0$, $f : I \times L^p([0,1],\mathbb{R}) \to L^{p'}([0,1],\mathbb{R})$, and $C : I \rightrightarrows L^p([0,1],\mathbb{R})$ is a bounded set-valued mapping with uniformly generalized V-prox-regular values w.r.t a positive real number $r > 0$, and for any $s, s' \in I$ and any $y \in X$ $|d_{C(s')}(y) - d_{C(s)}(y)| \leq |v(s') - v(s)|$, where $v : \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function. Here $V : L^{p'}([0,1],\mathbb{R}) \times L^p([0,1],\mathbb{R}) \to \mathbb{R}$ is defined by

$$V(u^*,u) = \|u^*\|_{L^{p'}}^2 - 2\langle u^*,u\rangle_{L^{p'},L^p} + \|u\|_{L^p}^2, \quad \forall u^* \in L^{p'}([0,1],\mathbb{R}), \forall u \in L^p([0,1],\mathbb{R}).$$

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- When $p = 2$, the (NDVI) becomes

$$\left\langle f(t, u(t)) + \frac{d}{dt}u(t); x - u(t) \right\rangle_{L^2,L^2} \geq \frac{-\beta \dot{v}(t)}{2r}\|u(t) - x\|^2,$$

  which is equivalent to the nonconvex sweeping process with perturbation:

$$-\frac{d}{dt}u(t) \in N^P(C(t); u(t)) + f(t, u(t)) \qquad (NSPP),$$

  where N^P is the proximal normal cone (see for instance [8]). This type of differential inclusion is widely studied in Hilbert spaces (see [8, 10] and the reference therein). Many works studied various extensions of (NSPP) in Hilbert spaces (see for instance Chapter 3 in [8] and the references therein).

- When $f \equiv 0$, $p = 2$, $r = +\infty$, and C has convex values, the (NDVI) becomes the convex sweeping process in the Hilbert space $L^2([0, 1], \mathbb{R})$ introduced and studied in [14].

- The case $p > 2$, $r = +\infty$, and C has convex values, the considered problem becomes

$$\int_0^1 \frac{d}{dt}\left(\|u(t)\|_{L^p}^{2-p}|u(t)(s)|^{p-2}u(t)(s)\right)[x(s) - u(t)(s)]\,ds \geq 0, \text{ a.e. on } I,$$

  and $\forall x \in C(t)$. This differential variational inequality in the Banach spaces $L^p([0, 1], \mathbb{R})$ ($p \geq 2$) has been introduced and studied in [6] in any uniformly convex and uniformly smooth spaces.

Our main objective in the present paper is to establish an existence result of (NDVI) for a large class of nonconvex sets in $L^p([0, 1], \mathbb{R})$ ($p \geq 2$). The contents of the paper will be as follows: In Section 2 we quote all needed concepts and results from [7, 9]. Section 3 is reserved to the proof of our main proofs. We end our paper with the conclusion section summarizing briefly our results and describing our perspectives on the subject.

2 Preliminaries

In all the paper, unless otherwise specified, the space X will denote $L^p([0, 1], \mathbb{R})$ with $p \geq 2$. Consider the functional $V : L^{p'}([0, 1], \mathbb{R}) \times L^p([0, 1], \mathbb{R}) \to \mathbb{R}$ such that

$$V(x^*, x) = \|x^*\|_{L^{p'}}^2 + \|x\|_{L^p}^2 - 2\langle x^*; x\rangle_{L^{p'},L^p}, \qquad (2.1)$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Using this functional V we define the generalised projection (see [1]).

Definition 2.1. Let S be a closed nonempty set in X and $x^* \in X^*$. If there exists a point $\bar{x} \in S$ satisfying $V(x^*, \bar{x}) = \inf_{x \in S} V(x^*, x)$, then $\bar{x}$ is called a generalized projection of x^* onto S. Then $\pi_S(x^*) := \{\bar{x} \in S : V(x^*, \bar{x}) = \inf_{x \in S} V(x^*, x)\}$.

We notice that, in the case $p = 2$ (i.e., the space X is Hilbert), the generalised projection π_S coincides with the well known metric projection Proj_S defined by $\mathrm{Proj}_S(x) = \{\bar{x} \in S : \|x - \bar{x}\|_{L^2} = \inf_{s \in S}\|s - \bar{x}\|_{L^2}\}$. Powerfully based on this concept of generalised projection π_S, the authors in [7] introduced and studied the concept of V-proximal normal cone $N^\pi(S; \bar{x})$ in smooth reflexive Banach spaces as follows:

$$N^\pi(S; \bar{x}) = \{x^* \in X^* : \exists \alpha > 0 \text{ such that } x \in \pi_S(J(x) + \alpha x^*)\}.$$

We recall from [2] that the duality mapping J on $L^p([0,1],\mathbb{R})$ has the following analytic representation:

$$J(x) = \|x\|_{L^p}^{2-p}|x|^{p-2}x, \quad \text{for all } x \in L^p([0,1],\mathbb{R}).$$

The class of nonconvex sets in X that will be used in our framework is introduced and studied in [9]. It is an extension of the well known concept of prox-regularity from Hilbert spaces to Banach spaces. Another extension of this concept from Hilbert spaces to Banach spaces can be found in [3–5] but it is not appropriate in our framework.

Definition 2.2 ([9])**.** Let E be a reflexive smooth Banach space, $S \subset E$, and let $\bar{x} \in S$. The set S will be called *uniformly generalized V-prox-regular* with respect to $r > 0$ if and only if for all $x \in S$ and for any nonzero $x^* \in N^\pi(S;x)$ the point x is a generalized projection of $Jx + r\frac{x^*}{\|x^*\|}$ on S, that is, $x \in \pi_S(Jx + r\frac{x^*}{\|x^*\|})$.

Example 2.3.

1. Any nonempty closed convex set is uniformly generalized V-prox-regular with respect to any $r > 0$;
2. The set $S := \mathbb{B} \cup (x_0 + \mathbb{B})$ (with $\|x_0\| > 4$) is a closed nonconvex set which is uniformly generalized V-prox-regular for some $r > 0$ (for its proof we refer to [9]).

We have to notice that a completely different approach, extending the concept of prox-regularity from Hilbert spaces to smooth Banach spaces, has been introduced and studied in [3–5]. Their extension is not the appropriate one in our setting because our main tool in the present framework is the generalised projection π_S with its nice properties for the class of nonconvex sets in Definition 2.2. However, they use in their extension the metric projection which is not, at all, appropriate in our approach.

The following proposition states an important property of uniformly generalized V-prox-regular sets which is needed in our proofs. Its proof is given in [9].

Proposition 2.4. *If S is a nonempty subset of a X, which is uniformly generalized V-prox-regular with respect to $r > 0$, then for any $x^* \in U_S^V(r) := \{x^* \in X^* : \inf_{s\in S} V(x^*, s) < r^2\}$ the generalized projection $\pi_S(x^*)$ exists.*

The results in the following lemma can be found in [2].

Lemma 2.5. *Let E be a p-uniformly smooth and q-uniformly convex Banach space. For any $R > 0$ there exist positive real numbers $\nu_R > 0$, $\beta_R > 0$, and $\omega_R > 0$ (depending only on R and the space E) such that*

1. $\|J(x) - J(y)\| \leq \nu_R \|x - y\|^{p-1}$, *for all* $\|x\| \leq R, \|y\| \leq R$,
2. $V(J(x);y) \leq \beta_R \|x - y\|^p$, *for all* $\|x\| \leq R, \|y\| \leq R$,
3. $\|x - y\|^q \leq \omega_R V(J(x);y)$, *for all* $\|x\| \leq R, \|y\| \leq R$.

The following proposition is needed in our proofs.

Proposition 2.6. *Let E be a reflexive smooth Banach space and let $S \subset X$ be a uniformly generalized V-prox-regular set with ratio $r > 0$. Assume that S is bounded (i.e., $S \subset l\mathbb{B}$). Then for any $x \in S$ and any nonzero $x^* \in N^\pi(S;x)$ we have:*

$$\langle \frac{x^*}{\|x^*\|}; y - x\rangle \leq \frac{2l + r}{r} d_S(y) + \frac{1}{2r} V(Jx;y), \quad \forall y \in E. \tag{2.2}$$

Proof. Let $r > 0$ as in Definition 2.2. Let any $x \in S$ and any nonzero $x^* \in N^\pi(S;x)$. By definition of uniform generalised V-prox-regularity, the point x is the generalised projection of $Jx + r\frac{x^*}{\|x^*\|}$ on S, that is,

$$V\left(Jx + r\frac{x^*}{\|x^*\|}; x\right) \leq V\left(Jx + r\frac{x^*}{\|x^*\|}; s\right), \quad \forall s \in S. \tag{2.3}$$

Since the functional $u \mapsto V(Jx + r\frac{x^*}{\|x^*\|}, u)$ is Lipschitz on S with constant $K := 4l + 2r$, then by Clarke penalisation in Proposition 6.3 on page 50 in [**?**]), we have

$$V\left(Jx + r\frac{x^*}{\|x^*\|}; x\right) \leq V\left(Jx + r\frac{x^*}{\|x^*\|}; y\right) + Kd_S(y), \quad \forall y \in E. \tag{2.4}$$

On the other hand we have

$$\begin{aligned}
&V\left(Jx + r\frac{x^*}{\|x^*\|}; x\right) - V\left(Jx + r\frac{x^*}{\|x^*\|}; y\right) \\
&= \|x\|^2 - \|y\|^2 - 2\left\langle Jx + r\frac{x^*}{\|x^*\|}; x - y\right\rangle \\
&= -\|x\|^2 - \|y\|^2 - 2r\left\langle \frac{x^*}{\|x^*\|}; x - y\right\rangle + 2\langle Jx; y\rangle \\
&= -V(Jx, y) - 2r\left\langle \frac{x^*}{\|x^*\|}; x - y\right\rangle.
\end{aligned}$$

Thus, the inequality (2.4) becomes

$$-V(Jx, y) - 2r\left\langle \frac{x^*}{\|x^*\|}; x - y\right\rangle \leq Kd_S(y), \quad \forall y \in E,$$

that is,

$$\left\langle \frac{x^*}{\|x^*\|}; y - x\right\rangle \leq \frac{2l + r}{r} d_S(y) + \frac{1}{2r} V(Jx, y), \quad \forall y \in E.$$

This completes the proof. □

3 Main Results

In this section, we will state and prove the main results of the paper. We start by proving the existence of approximate solutions of (NDVI) in the case $f \equiv 0$.

Theorem 3.1. *Let $T > 0$, $I := [0, T]$, and let $C : I \rightrightarrows X$ be a bounded set-valued mapping with uniformly generalized V-prox-regular values w.r.t some $r > 0$, and satisfying for any $t, t' \in I$ and any $u \in L^p([0,1], \mathbb{R})$*

$$|d_{C(t')}(u) - d_{C(t)}(u)| \leq |v(t') - v(t)|, \tag{3.1}$$

where $v : \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function. Then for any initial point $x_0 \in C(0)$, there exist $\beta > 0$, sequences of mappings $\theta_n : I \to I$, $u_n : I \to L^p([0,1], \mathbb{R})$, and $u_n^* : I \to L^{p'}([0,1], \mathbb{R})$, such that $\theta_n(t) \to t$ uniformly on I, $u_n(0) = x_0$, and for n sufficiently large we have

$$u_n(\theta_n(t)) \in C(\theta_n(t)), \quad \forall t \in I, \text{ and for a.e. on } I, \text{ and } \forall x \in C(\theta_n(t))$$

$$\left\langle \frac{d}{dt}u_n^*(t); x - u_n(\theta_n(t)) \right\rangle_{L^{p'},L^p} + \frac{\beta \dot{v}(t)}{2r} V\left(\|u_n(\theta_n(t))\|_{L^p}^{2-p} |u_n(\theta_n(t))|^{p-2} u_n(\theta_n(t)); x \right) \geq 0.$$

Proof. First, by the boundness assumption on C, we have for some $l > 0$ the inclusion

$$C(t) \subset l\mathbb{B}, \quad \forall t \in I.$$

Since the space $X = L^p([0,1], \mathbb{R})$ $(p \geq 2)$ is 2-uniformly smooth and p-uniformly convex, we use Lemma 2.5 to get a positive number $\beta_l > 0$ and β_l^* depending on the constant l and on the spaces X such that

$$V(J(x), y) \leq \beta_l \|x - y\|^2, \quad \forall x, y \in l\mathbb{B}. \tag{3.2}$$

Once again by Lemma 2.5 and the fact that $X^* = L^{p'}([0,1], \mathbb{R})$ $(1 < p' \leq 2)$ is p'-uniformly smooth and 2-uniformly convex, we can pick a positive number $\beta_l^* > 0$ depending on the constant l and on the spaces X^* such that

$$\|J(x) - J(y)\|^2 \leq \beta_l^* V(J(x), y) \quad \forall x, y \in l\mathbb{B}. \tag{3.3}$$

Without loss of generality we may assume that $T = 1$ and $\dot{v}(t) > 0$ for all $t \in I$. Consider $\forall n \in \mathbb{N}$ the following partition of I

$$I_{n,i+1} = (t_{n,i}, t_{n,i+1}], \quad t_{n,i} = \frac{i}{2^n}, \quad 0 \leq i \leq 2^n - 1, \quad I_{n,0} = \{0\}.$$

Put $\mu_n := \frac{1}{2^n}$, $\epsilon_{n,i} := \int_{t_{n,i}}^{t_{n,i+1}} \dot{v}(s)ds$, and $\epsilon_n := \max\{\epsilon_{n,i}; \quad 0 \leq i \leq 2^n - 1\}$. Clearly $\epsilon_n \downarrow 0$ and hence we can fix some $n_0 \geq 1$ sufficiently large so that

$$\epsilon_n < \frac{r}{\sqrt{\beta_l}}, \text{ for any } n \geq n_0.$$

Define now by induction the following iterative scheme: For any $n \geq n_0$ let

$$\begin{aligned} u_{n,0}^* &:= u_0^* = J(x_0); \\ u_{n,i+1} &\in \pi(C(t_{n,i+1}); u_{n,i}^*), \text{ for } 0 \leq i \leq 2^n - 1; \\ u_{n,i+1}^* &:= J(u_{n,i+1}), \end{aligned}$$

and

$$\begin{aligned} u_n(t) &:= J^*(u_n^*(t)) \\ u_n^*(t) &:= u_{n,i}^* + \frac{(v(t) - v(t_{n,i}))}{\epsilon_{n,i}}(u_{n,i+1}^* - u_{n,i}^*), \text{ for all } t \in I_{n,i} \end{aligned}$$

and $u_n^*(0) = u_{n,0}^*$. First, we start by showing the well defindeness of the previous iterative scheme. To do that we have to prove: $\{J(u_{n,i})\}_n \subset U_{C(t_{n,i+1})}^V(r), \quad \forall n \geq n_0$, that is,

$$\inf_{x \in C(t_{n,i+1})} V(J(u_{n,i}), x) < r^2 \quad \forall n \geq n_0.$$

Observe that, the sequence $\{u^*_{n,i}\}_n$ is bounded by l. We use the absolute continuity of C to write for any $n \geq n_0$

$$\begin{aligned} d_{C(t_{n,i+1})}(u_{n,i}) &= d_{C(t_{n,i+1})}(u_{n,i}) - d_{C(t_{n,i})}(u_{n,i}) \\ &\leq \int_{t_{n,i}}^{t_{n,i+1}} \dot{v}(s)ds \leq \epsilon_{n,i} \leq \epsilon_n. \end{aligned}$$

So,

$$d_{C(t_{n,i+1})}(u_{n,i}) \leq \epsilon_n.$$

Using now (3.2) and the definition of $\{u^*_{n,i}\}_n$, we obtain

$$\inf_{x\in C(t_{n,i+1})} V(J(u_{n,i}), x) \leq \beta_l \inf_{x\in C(t_{n,i+1})} \|u_{n,i} - x\|^2 \leq \beta_l d^2_{C(t_{n,i+1})}(u_{n,i})$$

and hence

$$\inf_{x\in C(t_{n,i+1})} V(J(u_{n,i}), x) \leq \beta_l d^2_{C(t_{n,i+1})}(u_{n,i}) \leq \beta_l \epsilon^2_n.$$

Now, using the choice of n_0 we obtain

$$\inf_{x\in C(t_{n,i+1})} V(J(u_{n,i}), x) \leq \beta_l \epsilon^2_n < r^2,$$

that is, $\{J(u_{n,i})\}_n \subset U^V_{C(t_{n,i+1})}(r), \quad \forall n \geq n_0$. Using now Proposition 2.4 the generalized projection of $u^*_{n,i} := J(u_{n,i})$ on the set $C(t_{n,i+1})$ exists for any $n \geq n_0$ and hence the iterative scheme is well defined.

Using the definition of the V-proximal normal cone, we can write for a.e. $t \in I$

$$u^*_{n,i+1} - u^*_{n,i} \quad \in \quad -N^\pi\left(C(t_{n,i+1}); u_{n,i+1}\right),$$

which gives

$$-\dot{u}^*_n(t) = -\dot{v}(t)\frac{u^*_{n,i+1} - u^*_{n,i}}{\epsilon_{n,i}} \quad \in \quad N^\pi\left(C(t_{n,i+1}); u_{n,i+1}\right). \tag{3.4}$$

Since the dual space $X^* = L^{p'}([0,1], \mathbb{R})$ $(1 < p' \leq 2)$ is p'-uniformly smooth and 2-uniformly convex, we use Part (3) in Lemma 2.5 and the definition of the sequence $\{u^*_{n,i}\}_n$ to write for some $\omega^*_l > 0$

$$\begin{aligned} \|u^*_{n,i+1} - u^*_{n,i}\|^2 &\leq \omega^*_l V(J(u_{n,i}); u_{n,i+1}) = \omega^*_l \inf_{x\in C(t_{n,i+1})} V(J(u_{n,i}), x) \\ &\leq \omega^*_l \inf_{x\in C(t_{n,i+1})} \beta_l \|u_{n,i} - x\|^2 = \omega^*_l \beta_l d^2_{C(t_{n,i+1})}(u_{n,i}) \leq \omega^*_l \beta_l \epsilon^2_{n,i}, \end{aligned}$$

which ensures that

$$\left\| \frac{u^*_{n,i+1} - u^*_{n,i}}{\epsilon_{n,i}} \right\| \quad \leq \quad (\omega^*_l \beta_l)^{\frac{1}{2}}.$$

This ensures that

$$\|\dot{u}^*_n(t)\| = \frac{\dot{v}(t)}{\epsilon_{n,i}} \|u^*_{n,i+1} - u^*_{n,i}\| \leq (\omega^*_l \beta_l)^{\frac{1}{2}} \dot{v}(t), \quad \text{a.e. on } I, \quad \forall n \geq n_0. \tag{3.5}$$

Define on $I_{n,i+1}$ the functions $\theta_n : I \to I$ by $\theta_n(0) = 0$, and

$$\theta_n(t) = t_{n,i+1}, \text{ for all } t \in I_{n,i+1}.$$

Then the inclusion (3.4) with (3.5) ensure

$$-\dot{u}_n^*(t) \in N^\pi(C(\theta_n(t)); u_n(\theta_n(t))) \cap \beta \dot{v}(t)\mathbb{B}, \quad \text{a.e. } t \in I, \quad \forall n \geq n_0, \tag{3.6}$$

where $\beta := (\omega_l^* \beta_l)^{\frac{1}{2}} > 0$. Also, we have by construction

$$u_n(\theta_n(t)) \in C(\theta_n(t)), \forall t \in I \text{ and all } n \geq n_0. \tag{3.7}$$

Using now Proposition 2.6 to conclude

$$\langle -\dot{u}_n^*(t); x - u_n(\theta_n(t))\rangle \leq \frac{(2l+r)(\beta\dot{v}(t))}{r} d_{C(\theta_n(t))}(x) + \frac{\beta\dot{v}(t)}{2r} V(J(u_n(\theta_n(t))), x),$$

for all $x \in X$ and hence for any $x \in C(\theta_n(t))$ and a.e. on I, we obtain

$$\langle \dot{u}_n^*(t); x - u_n(\theta_n(t))\rangle + \frac{\beta\dot{v}(t)}{2r} V(J(u_n(\theta_n(t))), x) \geq 0.$$

Thus, the proof of Theorem 3.1 is achieved. □

Theorem 3.2. *Let $C : I \rightrightarrows X$ be a bounded set-valued mapping with uniformly generalized V-prox-regular values w.r.t some $r > 0$, and satisfying (3.1). If, in addition, assume that the following compactness condition is satisfied: $\forall t \in I$ and any bounded set A in X with $\gamma(A) > 0$, $L > 0$ one has*

$$\gamma(J(C(t)) \cap L\mathbb{B}) < \gamma(A), \tag{3.8}$$

where γ is either the Kuratowski or the Hausdorff measure of noncompactness. Then for any initial condition $x_0 \in C(0)$, the sequences of approximate solutions u_n and u_n^* obtained in Theorem 3.1 admit uniformly convergent subsequences to some $u \in AC(I, X)$ and $u^* \in AC(I, X^*)$, respectively, such that $\dot{u}_n^*$ weakly converges to $\dot{u}^*$ in $L^1(I, X^*)$. Here $AC(I, X)$ means the space of absolutely continuous functions defined from I to X.

Proof. Assume that the sequences u_n and u_n^* are defined as in the proof of Theorem 3.1. Due to the uniform continuity of the mapping J^* on bounded sets it will be sufficient to prove the uniform continuity of a subsequence of u_n^*. To do that, we use the well known Arzela-Ascoli theorem to prove the compactness of $\{u_n^*\}_n$.

First, we quote from Theorem 3.1 the following upper bound estimate for the expression $\|u_{n,i+1}^* - u_{n,i}^*\|$:

$$\|u_{n,i+1}^* - u_{n,i}^*\| \leq \beta\epsilon_{n,i},$$

where $\beta > 0$ is given as in the proof of Theorem 3.1.

By construction, the mappings u_n^* and u_n are continuous on all I and u_n^* is differentiable on $I \setminus \{t_{n,i}; 0 \leq i \leq 2^n - 1\}$ with $\dot{u}_n^*(t) = \frac{\dot{v}(t)}{\epsilon_{n,i}} \left[u_{n,i+1}^* - u_{n,i}^*\right]$, for all $t \in I \setminus \{t_{n,i}; 0 \leq i \leq 2^n - 1\}$. Let us verify that u_n^* is absolutely continuous on I. Obviously for any $t, t' \in I_{n,i}$ we have

$$u_n^*(t') - u_n^*(t) = \frac{v(t') - v(t)}{\epsilon_{n,i}} \left[u_{n,i+1}^* - u_{n,i}^*\right].$$

Hence for any $t, t' \in I_{n,i}$ with $t \leq t'$, we obtain

$$\|u_n^*(t') - u_n^*(t)\| \leq \frac{\|u_{n,i+1}^* - u_{n,i}^*\|}{\epsilon_{n,i}} [v(t') - v(t)] \leq \beta [v(t') - v(t)]. \tag{3.9}$$

Consequently, by addition the inequality (3.9) still valid for all $t, t' \in I$ with $t \leq t'$. This ensures that u_n^* is absolutely continuous on I, for any $n \geq n_0$.

On the other side, the absolute continuity of C gives

$$\begin{aligned} d_{C(t)}(u_n(t)) &= \left[d_{C(t)}(u_n(t)) - d_{C(\theta_n(t))}(u_n(t))\right] \\ &\quad + \left[d_{C(\theta_n(t))}(u_n(t)) - d_{C(\theta_n(t))}(u_n(\theta_n(t))\right] \\ &\leq \left(\int_t^{\theta_n(t)} \dot{v}(s)ds\right) + \|u_n(\theta_n(t)) - u_n(t)\|. \end{aligned}$$

The absolute continuity of u_n^* on I in (3.9) gives

$$\|u_n^*(\theta_n(t)) - u_n^*(t)\| \leq \beta \left(\int_t^{\theta_n(t)} \dot{v}(s)ds\right) \leq \beta\epsilon_n.$$

On the other hand, the dual space $X^* = L^{p'}([0,1], \mathbb{R})$ $(1 < p' \leq 2)$ is p'-uniformly smooth and hence by Part (1) in Lemma 2.5, we have for some postive real number $\nu_l^* > 0$

$$\|u_n(\theta_n(t)) - u_n(t)\| = \|J^*(u_n^*(\theta_n(t))) - J^*(u_n^*(t))\| \leq \nu_l^* \|u_n^*(\theta_n(t)) - u_n^*(t)\|^{p'-1}$$

Thus,

$$\|u_n(\theta_n(t)) - u_n(t)\| \leq \nu_l^* \left(\beta\epsilon_n\right)^{p'-1}$$

and hence

$$d_{C(t)}(u_n(t)) \leq \epsilon_n + \nu_l^* \left(\beta\epsilon_n\right)^{p'-1}.$$

This may be written as follows:

$$u_n(t) \in C(t) + \eta_n\mathbb{B}, \text{ for all } n \geq n_0 \text{ and all } t \in I,$$

where $\eta_n := \epsilon_n + \nu_l^* \left(\beta\epsilon_n\right)^{p'-1}$. This means that $u_n(t) = c_n(t) + \eta_n b_n(t)$ for some $c_n(t) \in C(t)$ and some $b_n(t) \in \mathbb{B}$. Using once again Part (1) in Lemma 2.5 for the space $X = L^p([0,1], \mathbb{R})$ $(p \geq 2)$, which is 2-uniformly smooth, we find a positive number $\nu_l > 0$ and we write

$$\begin{aligned} d_{J(C(t))}(u_n^*(t)) &= d_{J(C(t))}(J(u_n(t))) = d_{J(C(t))}(J(c_n(t) + \eta_n b_n(t))) \\ &\leq d_{J(C(t))}(J(c_n(t) + \eta_n b_n(t))) - d_{J(C(t))}(J(c_n(t))) \\ &\leq \|J(c_n(t) + \eta_n b_n(t))) - J(c_n(t))\| \\ &\leq \nu_l\eta_n. \end{aligned}$$

Hence

$$u_n^*(t) \in J(C(t)) + \nu_l\eta_n\mathbb{B}, \quad \forall t \in I, \forall n \geq n_0. \tag{3.10}$$

Now, we are able to prove that (u_n^*) has a convergent subsequence. From what precedes, we have for any $n \geq n_0$ the mapping u_n^* is absolutely continuous on I with $\|\dot{u}_n^*(t)\| \leq \beta\dot{v}(t)$ a.e. on I. So, by Arzela-Ascoli theorem, we have to prove that the set $B^*(t) = \{u_n^*(t); n \geq n_0\}$ is relatively compact in X^*, for all $t \in I$. We suppose by contradiction that for some $t_0 \in I$, the set $B^*(t_0)$ is not relatively compact in X^*. So, $\gamma(B^*(t_0)) > 0$. Using (3.8) and the boundedness of $B^*(t_0)$ there exists some $\bar{\delta} \in (0, 1]$ so that

$$\gamma(B^*(t_0)) - \gamma(J(C(t_0)) \cap (l+1)\mathbb{B}) \geq 2\bar{\delta}. \tag{3.11}$$

Pick now $n_1 \geq n_0$ such that $\eta_n \leq \eta_{n_1} < \frac{\bar{\delta}}{2\nu_l}$, for all $n \geq n_1$. Then (3.10) can be reformulated as follows:

$$u_n^*(t) \in J(C(t)) + \nu_l \eta_n \mathbb{B} \subset J(C(t)) \cap (l+1)\mathbb{B} + \nu_l \eta_{n_1} \mathbb{B}, \forall n \geq n_1, \forall t \in I,$$

which ensures

$$\{u_n^*(t); n \geq n_1\} \subset J(C(t)) \cap (l+1)\mathbb{B} + \nu_l \eta_{n_1} \mathbb{B}, \text{ for all } t \in I.$$

Thus,

$$\begin{aligned}
\gamma(B^*(t_0)) &= \gamma(\{u_n^*(t_0) : n \geq n_0\}) = \gamma(\{u_n^*(t_0) : n \geq n_1\}) \\
&\leq \gamma(J(C(t_0)) \cap (l+1)\mathbb{B}) + \gamma(\nu_l \eta_{n_1} \mathbb{B}) \\
&\leq \gamma(B^*(t_0)) - 2\bar{\delta} + 2\nu_l \eta_{n_1} \\
&< \gamma(B^*(t_0)) - 2\bar{\delta} + \bar{\delta} \\
&< \gamma(B^*(t_0)) - \bar{\delta}.
\end{aligned}$$

This contradicts the choice of $\bar{\delta} > 0$. Consequently, the set $B^*(t)$ is relatively compact in X^* for any $t \in I$. Thus, we conclude, by Arzela-Ascoli theorem that (u_n^*) has a subsequence (still denoted u_n^*) converging uniformly to some u^* and the sequence $(\dot{u}_n^*)$ converges weakly in $L^1(I, X^*)$ to $\dot{u}^*$. Using the fact that $\lim_n \theta_n(t) = t$, we can write $\lim_n u_n^*(\theta_n(t)) = \lim_n u_n^*(t) = u^*(t)$ uniformly on I. Hence, the sequence $u_n = J^*(u_n^*)$ converges uniformly to $u = J^*(u^*)$ on I and so the proof of the theorem is complete. □

Theorem 3.3. *Under the same assumptions of Theorem 3.2, the $(NDVI)$ has at least one absolutely continuous solution, that is, there exists $u : I \to L^p([0,1], \mathbb{R})$ such that $u(0) = u_0 \in C(0)$, $u(t) \in C(t), \forall t \in I$, and for a.e. $t \in I$ and for any $x \in C(t)$*

$$\begin{aligned}
&\left\langle \frac{d}{dt}\left(\|u(t)\|_{L^p}^{2-p}|u(t)|^{p-2}u(t)\right); x - u(t)\right\rangle_{L^{p'},L^p} \\
&\quad + \frac{\beta \dot{v}(t)}{2r} V(\left(\|u(t)\|_{L^p}^{2-p}|u(t)|^{p-2}u(t)\right); x) \geq 0.
\end{aligned} \tag{3.12}$$

Proof. Let u_n, u_n^*, and θ_n be as in the proof in Theorem 3.1. We wish to prove that u is a solution of our problem (NDVI). First, we have to show that $u(t) \in C(t)$, for all $t \in I$. Using once again the absolute continuity of C to write for all $t \in I$

$$\begin{aligned}
d_{C(t)}(u_n(\theta_n(t)) &= d_{C(t)}(u_n(\theta_n(t)) - d_{C(\theta_n(t))}(u_n(\theta_n(t)) \\
&\leq \int_t^{\theta_n(t)} \dot{v}(s) ds \leq \epsilon_n,
\end{aligned} \tag{3.13}$$

and so

$$\begin{aligned}
d_{C(t)}(u(t)) &\leq d_{C(t)}(u_n(\theta_n(t))) + \|u_n(\theta_n(t)) - u(t)\| \\
&\leq \epsilon_n + \|u_n(\theta_n(t))) - u(t)\| \to 0,
\end{aligned} \tag{3.14}$$

as $n \to \infty$. This ensures together with the closedness of the set $C(t)$, that $u(t) \in C(t)$, for all $t \in I$. Returning back to (3.6) we get

$$-\dot{u}_n^*(t)) \quad \in \quad N^\pi(C(\theta_n(t)); u_n(\theta_n(t))) \cap \beta\dot{v}(t)\mathbb{B}, \text{ a.e. on } I.$$

So, Proposition 2.6 ensures for a.e. $t \in I$ and for any $x \in C(t)$

$$\left\langle -\frac{\dot{u}_n^*(t))}{\beta\dot{v}(t)}; x - u_n(\theta_n(t)) \right\rangle \leq \frac{2l+r}{r} d_{C(\theta_n(t))}(x) + \frac{1}{2r} V(J(u_n(\theta_n(t))); x). \tag{3.15}$$

Let any $t \in I$ for which $\dot{u}_n^*(t)$ and $\dot{u}^*(t)$ exist and let $x \in C(t)$. Then we have

$$d_{C(\theta_n(t))}(x) = d_{C(\theta_n(t))}(x) - d_{C(t)}(x) \leq \epsilon_n,$$

and so $x \in C(\theta_n(t)) + \epsilon_n\mathbb{B}$, that is, $x = y_n(t) + \epsilon_n b_n(t)$ with $y_n(t) \in C(\theta_n(t))$ and $b_n(t) \in \mathbb{B}$. Hence (3.15) yields

$$\begin{aligned}
\left\langle -\frac{\dot{u}^*(t)}{\beta\dot{v}(t)}; x - u(t) \right\rangle &= \left\langle \frac{\dot{u}_n^*(t) - \dot{u}^*(t)}{\beta\dot{v}(t)}; x - u(t) \right\rangle + \left\langle -\frac{\dot{u}_n^*(t)}{\beta\dot{v}(t)}; x - u(t) \right\rangle \\
&= \left\langle \frac{\dot{u}_n^*(t) - \dot{u}^*(t)}{\beta\dot{v}(t)}; x - u(t) \right\rangle + \left\langle -\frac{\dot{u}_n^*(t)}{\beta\dot{v}(t)}; x - u_n(\theta_n(t)) \right\rangle \\
&\quad + \langle -\frac{\dot{u}_n^*(t)}{\beta\dot{v}(t)}; u_n(\theta_n(t)) - u(t) \rangle \\
&\leq \left\langle \frac{\dot{u}_n^*(t) - \dot{u}^*(t)}{\beta\dot{v}(t)}; x - u(t) \right\rangle + \left\langle -\frac{\dot{u}_n^*(t)}{\beta\dot{v}(t)}; u_n(\theta_n(t)) - u(t) \right\rangle \\
&\quad + \frac{2l+r}{r} d_{C(\theta_n(t))}(x) + \frac{1}{2r} V(J(u_n(\theta_n(t)))); x).
\end{aligned}$$

Using the weak convergence in X^* of $\dot{u}_n^*(t)$ to $\dot{u}^*(t)$ and the uniform convergence in X of $u_n(\theta_n(t))$ to $u(t)$ proved in Theorem 3.2, and the uniform converge of $\theta_n(t) \to t$, we can pass to the limit in the last inequality as n goes to infinity

$$\left\langle -\frac{\dot{u}^*(t)}{\beta\dot{v}(t)}; x - u(t) \right\rangle \quad \leq \quad \frac{2l+r}{r} d_{C(t)}(x) + \frac{1}{r} V(J(u(t)); x) = \frac{1}{r} V(J(u(t)); x)$$

and hence

$$\left\langle -\frac{\dot{u}^*(t)}{\dot{v}(t)}; x - u(t) \right\rangle \quad \leq \quad \frac{\beta}{r} V(J(u(t)); x), \quad \forall x \in C(t).$$

This guarantees for any $x \in C(t)$

$$\left\langle -\frac{d}{dt} J(u(t)); x - u(t) \right\rangle \quad \leq \quad \frac{\beta\dot{v}(t)}{2r} V(J(u(t)); x). \tag{3.16}$$

Since $J(u(t))) = \|u(t)\|_{L^p}^{2-p} |u(t)|^{p-2} u(t)$, the last inequality (3.16) yields

$$\left\langle \frac{d}{dt} \left(\|u(t)\|_{L^p}^{2-p} |u(t)|^{p-2} u(t) \right); x - u(t) \right\rangle_{L^{p'}, L^p} + \frac{\beta\dot{v}(t)}{2r} V\left((\|u(t)\|_{L^p}^{2-p} |u(t)|^{p-2} u(t)); x \right) \geq 0$$

for a.e. $t \in I$ and for any $x \in C(t)$ and hence the proof is complete. □

Remark 3.4.

- The assumptions on the considered space $X = L^p([0,1],\mathbb{R})$ $(p \geq 2)$, which are needed in our proofs are also satisfied for the spaces l^p, and $W^{k,p}$ spaces with $p \geq 2$ which are 2-uniformly smooth and p-uniformly convex Banach space (see for instance [2]).
- An inspection of our proofs shows that the assumption (3.1) can be replaced by:

$$|d_{C(t')}(u) - d_{C(t)}(u)| \leq \lambda(u)|v(t') - v(t)|, \quad \forall t, t' \in I, \quad \forall u \in X, \tag{3.17}$$

  where $\lambda : X \to [0, \infty)$ is a bounded function on bounded sets.
- We notice that our existence result in Theorem 3.3 extends Theorem 3.1 in [6] and Theorem 2.1 in [7] from the convex case to the nonconvex case in q-uniformly convex and 2-uniformly smooth Banach space. In the nonconvex case (uniformly generalized V-prox-regular sets) our result extends Theorem 4.1 in [10] from Hilbert spaces setting to q-uniformly convex and 2-uniformly smooth Banach space.

We close Section 3 with an existence result for nonlinear differential variational inequality (NDVI) when $f \not\equiv 0$.

Theorem 3.5. *Suppose that the assumptions in Theorem 3.3 are satisfied. Let $f : I \times X \rightrightarrows X^*$ be a continuous mapping with values in X^* such that $f(t,x) \in \kappa$ for all $(t,x) \in I \times X$ for some convex compact set κ in X^*. Then for any initial point $x_0 \in C(0)$, there exists at least one absolutely continuous solution of the following nonconvex differential variational inequality with bounded perturbation $(NDVIP)$: there exists $u : I \to X$ such that $u(t) = x_0 + \int_0^t \dot{u}(s)ds$ and $u(t) \in C(t), \forall t \in I$ with*

$$\left\langle f(t,u(t)) + \frac{d}{dt}\left(\|u(t)\|_{L^p}^{2-p}|u(t)|^{p-2}u(t)\right); x - u(t)\right\rangle_{L^{p'},L^p}$$
$$\geq \frac{-\beta\dot{v}(t)}{2r} V(\left(\|u(t)\|_{L^p}^{2-p}|u(t)|^{p-2}u(t)\right); x), \quad \forall x \in C(t).$$

Proof. Following the same ideas and lines in the proofs of Theorems 3.1-3.3 and we consider the following iterative scheme which is adapted to the case when $f \not\equiv 0$:

$$\begin{array}{rcl} u_{n,0}^* & := & u_0^* = J(x_0), \quad z_{n,0}^* := f(t_{n,0}, J^*(u_{n,0}^*)); \\ z_{n,i}^* & := & f(t_{n,i}, J^*(u_{n,i}^*)); \\ u_{n,i+1} & \in & \pi(C(t_{n,i+1}); u_{n,i}^* + \epsilon_{n,i} z_{n,i}^*), \text{ for } 0 \leq i \leq 2^n - 1; \\ u_{n,i+1}^* & := & J(u_{n,i+1}); \end{array}$$

and for all $t \in I_{n,i}$ we set

$$\begin{array}{rcl} z_n^*(t) & := & z_{n,i}^*; \\ u_n(t) & := & J^*(u_n^*(t)); \\ u_n^*(t) & := & u_{n,i}^* + \dfrac{(v(t) - v(t_{n,i}))}{\epsilon_{n,i}}(u_{n,i+1}^* - u_{n,i}^*), \end{array}$$

and $u_n^*(0) := u_{n,0}^*$ and $z_n^*(0) := z_{n,0}^*$. This algorithm has been used in Theorem 4.3 in [7] for the convex case. The proof of the theorem is a combination of the proofs of Theorem 3.1 and Theorem 4.3 in [7]. We sketch the combinations as follows:

1. We follow the same lines and ideas in the proof of the previous theorem to prove the well definedness of the iterative scheme, the compactness of the sequences $\{u_n^*\}_n$ and $\{u_n\}_n$, and the passage to the limit to show that u^* is a solution of (NDVI).
2. We use the ideas used in [6] to prove the compactness of the sequence of mappings $\{z_n^*\}_n$ to some limit z^*.

We state here a simple example showing how can we use our previous results and that cannot be covered by any existing result. □

Remark 3.6.

- For $X = L^p([0,1],\mathbb{R})$ with $p \geq 2$, we fix any point $\bar{x} \in X$ with $\|\bar{x}\| > 4$ and let $T > 0$. We define $C : I \rightrightarrows X$ as: $C(t) = S \cap [\mathbb{B} \cup (\bar{x} + \mathbb{B})]$, $\forall t \in I$, where S is a given convex compact set in X. Then, obviously, the set-valued mapping C is Lipschitz continuous in the sense of (3.1) and for any $t \in I$ we have C satisfies the compactness condition in (3.8). By Example 2.3 the set-valued mapping C has uniformly generalized V-prox-regular values in X. Therefore, all the assumptions of our main results in Theorems 3.1-3.3 are fulfilled and hence by Theorem 3.3 there exists at least one absolutely (in this case it is Lipschitz) solution of (NDVI) associated with C. We have to point out that this existence of solutions of (NDVI) cannot be derived from any existing result proved in previous works.
- Notice that the set-valued mapping C in the above example is constant. Nevertheless, we can construct other examples depending on time t. Take for example C of the form $C(t) := S + g(t)$ and $C(t) := h(t).S$ with $g : I \to X$ is a bounded Lipschitz single-valued mapping and $h : I \to [0, \infty)$ is a bounded real-valued function, and S is the set used in the above Example.

4 Conclusions

In the present work, we extended some existing results of differential variational inequality from the convex case to the nonconvex case and from Hilbert space settings to Banach spaces settings. Our main results can be summarised as follows: In the framework of Banach spaces $L^p([0,1],\mathbb{R})$ $(p \geq 2)$, which are 2-uniformly smooth and q-uniformly convex and under the absolute continuity of C, and with the uniform V-prox-regularity of the values of C, we proved:

- Existence of absolutely continuous solutions of (NDVI) with $f \equiv 0$ in Theorem 3.3.
- Existence of absolutely continuous solutions of (NDVI) with $f \not\equiv 0$ in Theorem 3.5.
- The absolute continuity assumption (3.1) is very easy to verify relatively to the assumptions used in [6, 7]. In [7], instead of (3.1) the authors utilized the assumption in terms of the function d^V:

$$|(d^V_{C(t')})^{\frac{1}{2}}(u_1^*) - (d^V_{C(t)})^{\frac{1}{2}}(u_2^*)| \leq k_1|t' - t| + k_2\|u_1^* - u_2^*\|, \tag{4.1}$$

for any $u_1^*, u_2^* \in X^*$ and any $t, t' \in I$, where $d^V_{C(t)}(u^*) = \inf_{x \in C(t)} V(u^*, x)$ and $k_1, k_2 > 0$.

In [6], the author used a different assumption in terms of the distance function:

$$|d^{\frac{2}{p}}_{C(t')}(u) - d^{\frac{2}{p}}_{C(t)}(u)| \leq \lambda\|t' - t\|, \forall t, t' \in I \text{ and } \forall u \in X. \tag{4.2}$$

Clearly, all the conditions (3.1), (4.1), and (4.2) coincide in the Hilbert space $L^2([0,1],\mathbb{R})$. Nevertheless, in Banach spaces $L^p([0,1],\mathbb{R})$ ($p \geq 2$), the condition (4.1) is very hard to verify even for simple expressions of C. The difficulty comes from the definition of the function d_S^V and the fact that d_S^V does not have all the nice properties of the distance function d_S. To compare (3.1) and (4.2), we take for Example $X = L^3(0,1,\mathbb{R})$, $T = 1$, and $C(t) = \mathbb{B} + h(t)$, with $h : [0,1] \to X$ is a Lipschitz single-valued mapping. Obviously, the condition (3.1) is satisfied and it can be verified easily. The condition (4.2) is not satisfied since the expression $\|u - h(t)\|$ cannot be bounded from below by a positive number for any $u \in X$ and any $t, t' \in [0,1]$.

- The case of $L^p([0,1],\mathbb{R})$ with $p \in (1,2)$, which is 2-uniformly convex and p-uniformly smooth with any $p \neq 2$ is the subject of a future work.

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