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AN INTERIOR-POINT METHOD FOR SYMMETRIC OPTIMIZATION BASED ON A NEW WIDE NEIGHBORHOOD

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Abstract: In this paper, we present a theoretical framework of an interior-point method for solving linear optimization problems over symmetric cones. First, we define a new neighborhood of the central path and show that the defined neighborhood is wider than the neighborhoods that are available. Then, the convergence of the algorithm is investigated and, using an elegant analysis and Euclidean Jordan algebra as a tool is shown that the iteration complexity coincides with the best-known one obtained by any feasible interior-point method that uses the Nesterov-Todd direction. Finally, numerical results show that the proposed algorithm is efficient and promising.

Key words: *symmetric optimization, wide neighborhood, Jordan algebras, polynomial complexity*

Mathematics Subject Classification: *90C51*

1 Introduction

Symmetric optimization (SO) problem optimizes a linear function of real variables subject to the vector of real variables lying in the intersection of a prescribed affine subspace and a symmetric cone (homogeneous and self-dual). Interior-point methods (IPMs), initialed by Karmarkar [8] in 1984 for linear optimization (LO), are yet an active and fertile area of research for the field of optimization. Over the last few decades, more and more attention has been attracted to these methods because of their powers to solve more general classes of optimization problems, such as semidefinite optimization (SDO) and even SO. Faybusovich [5, 6] has been pioneered in the extension of IPMs to the SO case. Several variants of IPMs were successfully extended to the SO [6, 11, 22–25, 28] as well as to the convex quadratic optimization over symmetric cones (CQSO) [10,29] and the linear complementarity problems over symmetric cones (SCLCP) [14]. Liu et al. [17] proposed a second-order Mehrotra-type predictor-corrector IPM for SO and derived the polynomial convergence. Wang et al. [26] presented a full-Newton step feasible IPM for $P_*(\kappa)$-LCP and obtained the currently best known iteration bound for small-update methods. Wang et al. [27] proposed an interior point algorithm and improved the complexity bound of IPMs for SDO using the Nesterov and Todd (NT) direction.

In an important class of the path-following IPMs, the iterates are allowed to move in within a neighborhood of the central path. In 2005, an interesting result was given by Ai and Zhang [2]. Their algorithm decomposes the classical Newton direction as a sum of two separate directions, and relying on this, a new wide neighborhood of the central path is defined. The authors proved that their algorithm has the same theoretical complexity as small neighborhood algorithms. Later, Li and Terlaky [19] and Feng and Fang [7], respectively, generalized the Ai-Zhang approach and its predictor-corrector version to the SDO. Some variants of this algorithm have been extended to horizontal LCP (HLCP) [21], SDO and $P_*(\kappa)$-LCP [12,13,15]. Zhang and Zhang [30] designed an IPM with the second-order corrector step for SO in a given negative infinity neighborhood of the central path and established the convergence of the proposed algorithm for commutative class of search directions. The aforementioned methods work in the Ai-Zhang wide neighborhood. Darvay and Takács [3] proposed a wide neighborhood interior-point algorithm for LO which the used neighborhood differs from the one introduced by Ai and Zhang. Very recent, Kheirfam [9] extended the Darvay-Takács technique to the SDO based on the NT search direction (NT-direction) which is introduced in [20].

Motivated by the above-mentioned works, especially by [3, 9], we present a primal-dual path-following feasible IPM for SO based on a new wide neighborhood of the central path. We first define our new neighborhood and show that it is even wider than the wide neighborhoods that are available. Our method can be essentially regarded as a generalization of the algorithms proposed in [3, 9] to the SO case. Using the Euclidean Jordan algebra as a tool, we analyse the algorithm and prove that it has the same iteration complexity as any feasible IPM for SO that uses the NT-direction.

The paper is organized as follows. Relevant concepts and properties of Euclidean Jordan algebras, as well as some main lemmas that are needed in during analysis of Algorithm, are reviewed in Section 2. In Section 3, we first explain the search directions and define our new neighborhood, and then state a theoretical framework of our feasible IPM. Several technical lemmas and the polynomial-time convergence is established in Section 4. In Section 5, we provide some numerical results. Some concluding remarks are given in Section 6.

2 Euclidean Jordan Algebra

We assume that the reader has some familiarity with Euclidean Jordan algebras. We briefly recall some of the definitions and key results that are needed. For a comprehensive study, we refer the reader to the book by Faraut and Korányi [4].

Let $\mathcal{J}$ be a finite-dimensional vector space on real field $\mathbb{R}$, along with a bilinear map $\circ : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$. Then $(\mathcal{J}, \circ)$ (shortly denoted by $\mathcal{J}$) is said to be a Jordan algebra if $x \circ y = y \circ x$ and $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ where $x^2 = x \circ x$, for all $x, y \in \mathcal{J}$. A Jordan algebra $\mathcal{J}$ is Euclidean if there exists a symmetric positive definite quadratic form Q on $\mathcal{J}$ such that $Q(x \circ y, z) = Q(x, y \circ z)$. $e \in \mathcal{J}$ is an identity element if for any $x \in \mathcal{J}, e \circ x = x \circ e = x$. An element $c \in \mathcal{J}$ is idempotent if $c \neq 0$ and $c^2 = c$. Two idempotents c_1 and c_2 are orthogonal if $c_1 \circ c_2 = 0$. An idempotent element is said to be primitive if it cannot be written as the sum of two other idempotents. The set of primitive idempotents $\{c_1, \dots, c_k\}$ is a Jordan frame if $c_1 + \cdots + c_k = e$ and $c_i \circ c_j = 0, i \neq j$. Denote the corresponding cone of squares by $\mathcal{K} := \{x^2 : x \in \mathcal{J}\}$, which is a symmetric cone. $deg(x)$ denotes the the degree of $x \in \mathcal{J}$ and is defined as the smallest integer k such that the set $\{e, x, \dots, x^k\}$ is linearly dependent. The rank of $\mathcal{J}$, denoted by r, is $r = \max\{deg(x), x \in \mathcal{J}\}$. It is well-known that each element $x \in \mathcal{J}$ has a spectral decomposition $x = \lambda_1 c_1 + \cdots + \lambda_r c_r$, where $\lambda_i (i = 1, \dots, r)$ are called the eigenvalues of x and $\{c_1, \cdots, c_r\}$ forms a Jordan frame [4, Theorem III.1.2]. That is, the number of the idempotents in a Jordan algebra is exactly the rank of the Euclidean Jordan algebra. We denote by x^{-1} the inverse of x and is defined as $x \circ x^{-1} = x^{-1} \circ x = e$, and it can be decomposed as $x^{-1} = \lambda_1^{-1} c_1 + \cdots + \lambda_r^{-1} c_r$. Similarly, the spectral decomposition of $\sqrt{x}$ is as $\sqrt{x} = \sqrt{\lambda_1} c_1 + \cdots + \sqrt{\lambda_r} c_r$. Some other functions are defined in terms of the eigenvalues as $\mathrm{Tr}(x) := \sum_{i=1}^{r} \lambda_i$ and $\lambda_{\min} = \min_{1 \leq i \leq r} \lambda_i$. Leaning on the function $\mathrm{Tr}(\cdot)$, we define $\langle x, s\rangle := \mathrm{Tr}(x \circ s)$ and $\|x\|_F := \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i=1}^{r} \lambda_i^2}$. Since the inner product " $\circ$ " defined in $\mathcal{J}$ is a bilinear mapping, thus we can define a linear operator $L : \mathcal{J} \to \mathcal{J}$ such that $L_x s = x \circ s$ for any $s \in \mathcal{J}$. We say $x \in \mathcal{J}$ and $s \in \mathcal{J}$ operator commute if $L_x L_s = L_s L_x$. Moreover, it can be proven that x and s operator commute if and only if they share the same Jordan frame [23]. For any $x \in \mathcal{J}$, we define the quadratic representation of x as $Q_x := 2L_x^2 - L_{x^2}$, where $L_x^2 = L_x L_x$. In the sequel, we list some fundamental results which will be used in during analysis.

Lemma 2.1 ([23, Proposition 21]). *Let $x, s, p \in \mathrm{int}\mathcal{K}$, where $int\mathcal{K}$ denotes the interior of $\mathcal{K}$, and define $\tilde{x} := Q_p x$ and $\tilde{s} := Q_{p^{-1}} s$. Then $Q_{x^{\frac{1}{2}}} s$ and $Q_{\tilde{x}^{\frac{1}{2}}} \tilde{s}$ have the same eigenvalues.*

Lemma 2.2 ([16, Lemma 5.12]). *For $x, s \in \mathcal{J}$, then $\|(x + s)^+\|_F \le \|x^+\|_F + \|s^+\|_F$.*

Lemma 2.3 ([22, Lemma 2.9]). *Let $\mathcal{J}$ be any Euclidean Jordan algebra and $x, s \in \mathcal{J}$, then $\|x \circ s\|_F \le \|x\|_F \|s\|_F$.*

Lemma 2.4 ([23, Lemma 30]). *Let $x, s \in int\mathcal{K}$. If x and s operator commute, then $Q_{x^{\frac{1}{2}}} s = Q_{s^{\frac{1}{2}}} x = x \circ s$.*

3 SO Problem and the New Wide Neighborhood

Let $\mathcal{J}$ be a Euclidean Jordan algebra of rank r and the symmetric cone $\mathcal{K}$. We consider the primal-dual pair of SO problems in standard forms

$$\min\{\langle c, x\rangle : \ Ax = b, \ x \in \mathcal{K}\}, \qquad (P)$$

$$\max\{b^T y : \ A^T y + s = c, \ s \in \mathcal{K}\}, \qquad (D)$$

where c and the rows of A belong to $\mathcal{J}$, and $b \in \mathbb{R}^m$. Without any loss of generality, we assume that $\mathrm{rank}(A) = m$. We also denote

$$\mathcal{F}^0 := \{(x, y, s) : Ax = b, A^T y + s = c, x \in int\mathcal{K}, s \in int\mathcal{K}\}.$$

Throughout this paper, we suppose that the interior point condition (IPC) holds for both (P) and (D), i.e., $\mathcal{F}^0 \neq \emptyset$. It is shown in [6] that, under this assumption, a necessary and sufficient optimality condition for (P) and (D) is

$$\begin{aligned} Ax &= b, \quad x \in \mathcal{K}, \\ A^T y + s &= c, \quad s \in \mathcal{K}, \\ x \circ s &= 0, \end{aligned} \tag{3.1}$$

where the last equation is called the complementarity condition. A common aspect in primal-dual IPMs is to replace the complementarity condition $x \circ s = 0$ by the centrality equation $x \circ s = \mu e$, where $\mu > 0$. Hence, we obtain the system which defines the central path:

$$\begin{aligned} Ax &= b, \quad x \in \mathcal{K}, \\ A^T y + s &= c, \quad s \in \mathcal{K}, \\ x \circ s &= \mu e. \end{aligned} \tag{3.2}$$

It is well-known that system (3.2) has a unique solution for a fixed $\mu > 0$ if the condition $\mathcal{F}^0 \neq \emptyset$ is satisfied [6]. If μ tends to zero, then the central path converges to the primal-dual optimal solution. It is proved in [23, Lemma 28] that $x \circ s = \mu e$ if and only if $Q_p x \circ Q_{p^{-1}} s = \mu e$, for any $p \in \mathcal{J}$ invertible and $x, s \in int\mathcal{K}$. Letting $\tilde{x} = Q_p x, \tilde{s} = Q_{p^{-1}} s, \tilde{A} = AQ_{p^{-1}}, \tilde{c} = Q_{p^{-1}} c$, the system (3.2) can be rewritten as follows:

$$\begin{aligned} \tilde{A}\tilde{x} &= b, \\ \tilde{A}^T y + \tilde{s} &= \tilde{c}, \\ \tilde{x} \circ \tilde{s} &= \mu e. \end{aligned} \tag{3.3}$$

Let $\mathcal{C}(x, s)$ denote the set of all $p \in int\mathcal{K}$ such that the scaled elements operator commute, i.e.,

$$\mathcal{C}(x, s) := \{p \in int\mathcal{K} : Q_p x \text{ and } Q_{p^{-1}} s \text{ operator commute}\}.$$

This class includes the xs and sx search directions and the NT direction as special subclass. In this article, we study the case of NT direction, which p is satisfied with $p = [Q_{x^{\frac{1}{2}}}(Q_{x^{\frac{1}{2}}} s)^{-\frac{1}{2}}]^{-\frac{1}{2}} = [Q_{s^{-\frac{1}{2}}}(Q_{s^{\frac{1}{2}}} x)^{\frac{1}{2}}]^{-\frac{1}{2}}$. Note that in such a way, we have $\tilde{x} = \tilde{s}$ [23]. Let $\psi : [0, \infty) \to [0, \infty)$ be a continuously differentiable function such that $\psi'(t) > 0, \forall t \geq 0$. For $t \in \mathbb{R}^n$, we define $\psi(t) = (\psi(t_i))_{1 \leq i \leq n}$. In this way, the scaled system (3.3) can be equivalently written as

$$\begin{aligned} \tilde{A}\tilde{x} &= b, \\ \tilde{A}^T y + \tilde{s} &= \tilde{c}, \\ \psi\big(\tfrac{\tilde{x} \circ \tilde{s}}{\tau\mu}\big) &= \psi(e), \end{aligned} \tag{3.4}$$

where the target is a point on the central path corresponds to $\mu := \tau\mu$ and $\tau \in (0, 1)$ is the centering parameter.

Given a primal-dual feasible point (x, y, s), then applying Newton's approach to system (3.4) leads to the following linear system:

$$\begin{aligned} \tilde{A}\Delta\tilde{x} &= 0, \\ \tilde{A}^T \Delta y + \Delta\tilde{s} &= 0, \\ \tilde{s} \circ \Delta\tilde{x} + \tilde{x} \circ \Delta\tilde{s} &= \tau\mu\big(\psi'\big(\tfrac{\tilde{x} \circ \tilde{s}}{\tau\mu}\big)\big)^{-1} \circ \Big(\psi(e) - \psi\big(\tfrac{\tilde{x} \circ \tilde{s}}{\tau\mu}\big)\Big), \end{aligned} \tag{3.5}$$

where $\Delta\tilde{x} = Q_p \Delta x, \Delta\tilde{s} = Q_{p^{-1}} \Delta s$. Considering $\psi(t) = \sqrt{t}$, the Newton system (3.5) becomes

$$\begin{aligned} \tilde{A}\Delta\tilde{x} &= 0, \\ \tilde{A}^T \Delta y + \Delta\tilde{s} &= 0, \\ \tilde{s} \circ \Delta\tilde{x} + \tilde{x} \circ \Delta\tilde{s} &= 2\Big(\sqrt{\tau\mu}(\tilde{x} \circ \tilde{s})^{\frac{1}{2}} - \tilde{x} \circ \tilde{s}\Big). \end{aligned} \tag{3.6}$$

Based on Ai's original idea [1], we decompose the Newton system (3.6) into the following two systems:

$$\begin{aligned} \tilde{A}\Delta\tilde{x}_- &= 0, \\ \tilde{A}^T \Delta y_- + \Delta\tilde{s}_- &= 0, \\ \tilde{x} \circ \Delta\tilde{s}_- + \Delta\tilde{x}_- \circ \tilde{s} &= 2\big(\sqrt{\tau\mu}(\tilde{x} \circ \tilde{s})^{\frac{1}{2}} - \tilde{x} \circ \tilde{s}\big)^-, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \tilde{A}\Delta\tilde{x}_+ &= 0, \\ \tilde{A}^T \Delta y_+ + \Delta\tilde{s}_+ &= 0, \\ \tilde{x} \circ \Delta\tilde{s}_+ + \Delta\tilde{x}_+ \circ \tilde{s} &= 2\big(\sqrt{\tau\mu}(\tilde{x} \circ \tilde{s})^{\frac{1}{2}} - \tilde{x} \circ \tilde{s}\big)^+. \end{aligned} \tag{3.8}$$

Note that for $a \in \mathbb{R}$, $a^+ = \max\{a, 0\}$ and $a^- = \min\{a, 0\}$. Since $\sqrt{\tau\mu}(\tilde{x} \circ \tilde{s})^{\frac{1}{2}} - \tilde{x} \circ \tilde{s} = \left(\sqrt{\tau\mu}(\tilde{x} \circ \tilde{s})^{\frac{1}{2}} - \tilde{x} \circ \tilde{s}\right)^+ + \left(\sqrt{\tau\mu}(\tilde{x} \circ \tilde{s})^{\frac{1}{2}} - \tilde{x} \circ \tilde{s}\right)^-$, the classical Newton direction is as simply $(\Delta\tilde{x}, \Delta y, \Delta\tilde{s}) = (\Delta\tilde{x}_+, \Delta y_+, \Delta\tilde{s}_+) + (\Delta\tilde{x}_-, \Delta y_-, \Delta\tilde{s}_-)$. Let $\alpha := (\alpha_1, \alpha_2) \in \mathbb{R}^2$ with $0 < \alpha_1 \le 1$ and $0 < \alpha_2 \le 1$, be the step sizes taken along $(\Delta\tilde{x}_-, \Delta y_-, \Delta\tilde{s}_-)$ and $(\Delta\tilde{x}_+, \Delta y_+, \Delta\tilde{s}_+)$, respectively. Thus, we consider the new iterate as follows

$$(\tilde{x}(\alpha), \tilde{y}(\alpha), \tilde{s}(\alpha)) := (\tilde{x}, y, \tilde{s}) + \alpha_1(\Delta\tilde{x}_-, \Delta y_-, \Delta\tilde{s}_-) + \alpha_2(\Delta\tilde{x}_+, \Delta y_+, \Delta\tilde{s}_+). \tag{3.9}$$

In this way, by symmetry of Q_p we get

$$\begin{aligned}\tilde{\mu}(\alpha) &= \frac{1}{r}\langle\tilde{x}(\alpha), \tilde{s}(\alpha)\rangle = \frac{1}{r}\langle Q_p x(\alpha), Q_{p^{-1}} s(\alpha)\rangle \\ &= \frac{1}{r}\langle x(\alpha), Q_p Q_{p^{-1}} s(\alpha)\rangle = \frac{1}{r}\langle x(\alpha), s(\alpha)\rangle = \mu(\alpha),\end{aligned}$$

and in a manner similar, we will have $\tilde{\mu} = \mu$. Throughout the paper, we will use these two equations in the position that are required.

In a number of the usually primal-dual IPMs, the iterates are allowed to move in within a neighborhood of the central path. The so-called negative infinity neighborhood that is a wide neighborhood defined as

$$\mathcal{N}_\infty^-(1-\gamma) = \{(x, y, s) \in \mathcal{F}^0 : \lambda_{\min}(Q_{x^{\frac{1}{2}}} s) \ge \gamma\mu\},$$

where $0 < \gamma < 1$. Another popular wide neighborhood, based on Ai's idea [1], is defined in [18] as follows:

$$\mathcal{N}(\tau, \beta) = \{(x, y, s) \in \mathcal{F}^0 : \left\|(\tau\mu e - Q_{x^{\frac{1}{2}}} s)^+\right\|_F \le \beta\tau\mu\},$$

where neighborhood parameters $\tau, \beta \in (0, 1)$ are given constants.

An important ingredient of this paper is to define a new neighborhood for the central path as follows:

$$\mathcal{N}(\sqrt{\tau}, \sqrt{\beta}) := \{(x, y, s) \in \mathcal{F}^0 : \left\|\left(\sqrt{\tau\mu}e - \sqrt{Q_{x^{\frac{1}{2}}} s}\right)^+\right\|_F \le \sqrt{\beta\tau\mu}\},$$

where $0 < \tau < 1$ and $0 < \beta < 1$ are given constants. In view of Lemma 2.1, it follows that the neighborhood $\mathcal{N}(\sqrt{\tau}, \sqrt{\beta})$ is scaling invariant, i.e., (x, y, s) is in the neighborhood if and only if $(\tilde{x}, y, \tilde{s})$ is.

The next lemma shows that the neighborhood $\mathcal{N}(\sqrt{\tau}, \sqrt{\beta})$ is even a wider neighborhood than neighborhood $\mathcal{N}(\tau, \beta)$.

Lemma 3.1. *We have,* $\mathcal{N}(\tau, \beta) \subseteq \mathcal{N}(\sqrt{\tau}, \sqrt{\beta})$.

Proof. Let the spectral decomposition of $\tilde{x}\circ\tilde{s}$ be as $\tilde{x}\circ\tilde{s} = \lambda_1 c_1 + \cdots + \lambda_r c_r$, where $\{c_1, \ldots, c_r\}$ is a Jordan frame and the real numbers $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of $\tilde{x} \circ \tilde{s}$, which satisfy the following inequalities:

$$\tau\mu - \lambda_1 \ge \cdots \ge \tau\mu - \lambda_k \ge 0 > \tau\mu - \lambda_{k+1} \ge \cdots \ge \tau\mu - \lambda_r,$$

and this will be used throughout the paper. By Lemmas 2.1, 2.4 and $(x, y, s) \in \mathcal{N}(\tau, \beta)$, thus $(\tilde{x}, y, \tilde{s}) \in \mathcal{N}(\tau, \beta)$, it follows that

$$\sum_{i=1}^{k}(\tau\mu - \lambda_i)^2 = \left\|(\tau\mu e - Q_{x^{\frac{1}{2}}} s)^+\right\|_F^2 \le \beta^2\tau^2\mu^2. \tag{3.10}$$

Thus, for any $i = 1, \dots, k$, we have

$$(1-\beta)\tau\mu \le \lambda_i \le (1+\beta)\tau\mu,$$

and then

$$\sqrt{\lambda_i} + \sqrt{\tau\mu} \ge \sqrt{(1-\beta)\tau\mu} + \sqrt{\tau\mu}. \tag{3.11}$$

Using the fact that

$$\tau\mu - \lambda_i = (\sqrt{\lambda_i} + \sqrt{\tau\mu})(\sqrt{\tau\mu} - \sqrt{\lambda_i})$$

together with (3.10) and (3.11), we can write that

$$\begin{aligned}
\left\|\left(\sqrt{\tau\mu}e - \sqrt{Q_{x^{\frac{1}{2}}}s}\right)^+\right\|_F^2 &= \sum_{i=1}^k \left(\sqrt{\tau\mu} - \sqrt{\lambda_i}\right)^2 = \sum_{i=1}^k \frac{(\tau\mu - \lambda_i)^2}{\left(\sqrt{\tau\mu} + \sqrt{\lambda_i}\right)^2} \\
&\le \frac{1}{(1+\sqrt{1-\beta})^2\tau\mu} \sum_{i=1}^k (\tau\mu - \lambda_i)^2 \\
&= \frac{1}{(1+\sqrt{1-\beta})^2\tau\mu} \left\|\left(\tau\mu e - Q_{x^{\frac{1}{2}}}s\right)^+\right\|_F^2 \\
&\le \frac{\beta^2\tau^2\mu^2}{(1+\sqrt{1-\beta})^2\tau\mu} = \frac{\beta^2}{(1+\sqrt{1-\beta})^2}\tau\mu \\
&= (1-\sqrt{1-\beta})^2\tau\mu \le \beta^2\tau\mu \le \beta\tau\mu,
\end{aligned}$$

where the last inequality is due to $0 < \beta < 1$. If we take the square root of both sides of the above inequality, we obtain

$$\left\|\left(\sqrt{\tau\mu}e - \sqrt{Q_{x^{\frac{1}{2}}}s}\right)^+\right\|_F \le \sqrt{\beta\tau\mu}.$$

This means that $(x, y, s) \in \mathcal{N}(\sqrt{\tau}, \sqrt{\beta})$ and hence the inclusion holds. This completes the proof of the lemma. □

We now are in a position to state the theoretical framework of the interior-point algorithm.

Algorithm 1

Step 1 Choose an initial point $(x^0, y^0, s^0) \in \mathcal{N}(\sqrt{\tau}, \sqrt{\beta})$ with $\mu_0 > 0$.
Select an accuracy parameter $\varepsilon > 0$ and
neighborhood parameters $\tau \le \frac{1}{19}$ and $\beta \le \frac{1}{19}$. Let $k = 0$.

Step 2 If $\mu_k \le \varepsilon$, then stop, else go to Step 3.

Step 3 Take $p = [Q_{(x^k)^{\frac{1}{2}}}(Q_{(x^k)^{\frac{1}{2}}}s^k)^{-\frac{1}{2}}]^{-\frac{1}{2}}$. Set $\tilde{x}^k = Q_p x^k$ and $\tilde{s}^k = Q_{p^{-1}} s^k$.

Step 4 Compute $(\Delta\tilde{x}^k_-, \Delta y^k_-, \Delta\tilde{s}^k_-)$ and $(\Delta\tilde{x}^k_+, \Delta y^k_+, \Delta\tilde{s}^k_+)$ by solvingthe scaled
Newton systems (3.7) and (3.8), respectively.

Step 5 Choose step size vector $\alpha^k = (\alpha^k_1, \alpha^k_2)$ such that
$(\tilde{x}^{k+1}, y^{k+1}, \tilde{s}^{k+1}) := (\tilde{x}(\alpha^k), y(\alpha^k), \tilde{s}(\alpha^k)) \in \mathcal{N}(\sqrt{\tau}, \sqrt{\beta})$.

Step 6 Let $(x^{k+1}, y^{k+1}, s^{k+1}) = (Q_{p^{-1}}\tilde{x}^{k+1}, y^{k+1}, Q_p\tilde{s}^{k+1})$ and
$\mu_{k+1} = \frac{\langle x^{k+1}, s^{k+1}\rangle}{r}$. Set $k := k + 1$ and go to Step 2.

Figure 1. Primal – dual interior – point algorithm.

4 Analysis of the Algorithm

Let us define

$$v := Q_p x \ [= Q_{p^{-1}} s],\ d\tilde{x}_- := Q_p \Delta\tilde{x}_-,\ d\tilde{s}_- := Q_{p^{-1}} \Delta\tilde{s}_-,\ d\tilde{x}_+ := Q_p \Delta\tilde{x}_+,$$

and

$$d\tilde{s}_+ := Q_{p^{-1}} \Delta\tilde{s}_+,\ \Delta\tilde{x}(\alpha) = \alpha_1 \Delta\tilde{x}_- + \alpha_2 \Delta\tilde{x}_+,\ \Delta\tilde{s}(\alpha) = \alpha_1 \Delta\tilde{s}_- + \alpha_2 \Delta\tilde{s}_+.$$

Using these notations and invoking the third equations of systems (3.7), (3.8) and the equality (3.9), we obtain

$$\tilde{x}(\alpha) \circ \tilde{s}(\alpha) = h(\alpha) + \Delta\tilde{x}(\alpha) \circ \Delta\tilde{s}(\alpha), \tag{4.1}$$

where

$$h(\alpha) = v^2 + 2\alpha_1 v \circ (\sqrt{\tau\mu} e - v)^- + 2\alpha_2 v \circ (\sqrt{\tau\mu} e - v)^+.$$

Proposition 4.1. *We have* $\langle \Delta\tilde{x}(\alpha), \Delta\tilde{s}(\alpha) \rangle = 0$

Proof. It follows directly from the first two equations of systems (3.7), (3.8) and using the definitions of $\Delta\tilde{x}(\alpha)$ and $\Delta\tilde{s}(\alpha)$. □

Lemma 4.2. *Let* $(\tilde{x}, y, \tilde{s}) \in \mathcal{N}(\sqrt{\tau}, \sqrt{\beta})$ *and* $\tilde{\mu}(\alpha) = \frac{\langle \tilde{x}(\alpha), \tilde{s}(\alpha) \rangle}{r}$ *with* $\alpha = (\alpha_1, \alpha_2) \in (0,1]^2$. *Then, we have* $\tilde{\mu}(\alpha) \geq (1 - 2\alpha_1)\tilde{\mu}$.

Proof. Using (4.1) and Proposition 4.1, we have

$$\begin{aligned}
\tilde{\mu}(\alpha) &= \frac{1}{r}\langle \tilde{x}(\alpha), \tilde{s}(\alpha) \rangle = \frac{1}{r}\mathrm{Tr}(h(\alpha)) + \frac{1}{r}\langle \Delta\tilde{x}(\alpha), \Delta\tilde{s}(\alpha) \rangle \\
&= \tilde{\mu} + \frac{2\alpha_1}{r}\mathrm{Tr}\big(v \circ (\sqrt{\tau\mu} e - v)^-\big) + \frac{2\alpha_2}{r}\mathrm{Tr}\big(v \circ (\sqrt{\tau\mu} e - v)^+\big) \\
&= \tilde{\mu} + \frac{2\alpha_1}{r}\sum_{i=k+1}^{r} \sqrt{\lambda_i}(\sqrt{\tau\mu} - \sqrt{\lambda_i}) + \frac{2\alpha_2}{r}\sum_{i=1}^{k} \sqrt{\lambda_i}(\sqrt{\tau\mu} - \sqrt{\lambda_i}) \\
&\geq \tilde{\mu} + \frac{2\alpha_1}{r}\sum_{i=k+1}^{r} -\lambda_i \geq \tilde{\mu} + \frac{2\alpha_1}{r}\sum_{i=1}^{r} -\lambda_i = \tilde{\mu} - \frac{2\alpha_1}{r}\mathrm{Tr}(\tilde{x} \circ \tilde{s}) \\
&= (1 - 2\alpha_1)\tilde{\mu}.
\end{aligned}$$

The proof is completed. □

Lemma 4.3. *Let* $(\tilde{x}, y, \tilde{s}) \in \mathcal{N}(\sqrt{\tau}, \sqrt{\beta})$*, then*

(i) $\mathrm{Tr}((\tau\mu e - v^2)^-) \leq -(1 - \tau) r\mu$.

(ii) $\lambda_i(v \circ (\sqrt{\tau\mu} e + v)^{-1}) \geq \frac{1-\sqrt{\beta}}{2-\sqrt{\beta}}, \quad i = 1, \ldots, r.$

Proof. We have

$$\begin{aligned}
\mathrm{Tr}((\tau\mu e - v^2)^-) &= \mathrm{Tr}(\tau\mu e - v^2) - \mathrm{Tr}((\tau\mu e - v^2)^+) \\
&\leq \mathrm{Tr}(\tau\mu e - v^2) = \tau\mu r - \mathrm{Tr}(v^2) \\
&= \tau\mu r - r\tilde{\mu} = -(1 - \tau) r\mu.
\end{aligned}$$

This completes the proof of the first part of the lemma. For (ii), since $v^2 = \tilde{x}\circ\tilde{s} = \lambda_1 c_1 + \cdots + \lambda_r c_r$, thus we get $\sqrt{\tau\mu}e + v = (\sqrt{\tau\mu} + \sqrt{\lambda_1})c_1 + \cdots + (\sqrt{\tau\mu} + \sqrt{\lambda_r})c_r$. These imply that $\frac{\sqrt{\lambda_i}}{\sqrt{\tau\mu}+\sqrt{\lambda_i}}, i = 1, \ldots, r$ are the eigenvalues of $v\circ(\sqrt{\tau\mu}e+v)^{-1}$. On the other hand, from the fact that $(\tilde{x}, y, \tilde{s}) \in \mathcal{N}(\sqrt{\tau}, \sqrt{\beta})$ it follows that $\sqrt{\lambda_i} \geq (1-\sqrt{\beta})\sqrt{\tau\mu}, i = 1, \ldots, r$. Hence, we will have

$$\lambda_i(v\circ(\sqrt{\tau\mu}e + v)^{-1}) = \frac{\sqrt{\lambda_i}}{\sqrt{\tau\mu}+\sqrt{\lambda_i}} \geq \frac{1-\sqrt{\beta}}{2-\sqrt{\beta}},\ i = 1, \ldots, r$$

where the inequality concludes from the fact that the right-hand side of the equality is increasing for $\sqrt{\lambda_i} \geq (1-\sqrt{\beta})\sqrt{\tau\mu}$. This proves the second claim, and the proof is completed. □

The following lemma ensures that Algorithm 1 reduces the duality gap if the generated iterates by the algorithm belong to the neighborhood presented.

Lemma 4.4. *Let $(\tilde{x}, y, \tilde{s}) \in \mathcal{N}(\sqrt{\tau}, \sqrt{\beta})$ and $\tilde{\mu}(\alpha) = \frac{\langle\tilde{x}(\alpha),\tilde{s}(\alpha)\rangle}{r}$ with $\alpha = (\alpha_1, \alpha_2) \in (0, 1]^2$. Then, we have*

$$\tilde{\mu}(\alpha) \leq \Big(1 - \frac{2\alpha_1(1-\sqrt{\beta})(1-\tau)}{2-\sqrt{\beta}} + \frac{2\alpha_2\tau\sqrt{\beta}}{\sqrt{r}}\Big)\mu.$$

Proof. In a manner similar to the proof of Lemma 4.2 and using $\tilde{\mu} = \mu$, we obtain

$$\begin{aligned}
\tilde{\mu}(\alpha) &= \mu + \frac{2\alpha_1}{r}\mathrm{Tr}\big(v\circ(\sqrt{\tau\mu}e - v)^-\big) + \frac{2\alpha_2}{r}\mathrm{Tr}\big(v\circ(\sqrt{\tau\mu}e - v)^+\big)\\
&= \mu + \frac{2\alpha_1}{r}\mathrm{Tr}\big(\big(v\circ(\sqrt{\tau\mu}e + v)^{-1}\big)\circ(\tau\mu e - v^2)^-\big)\\
&\qquad\qquad + \frac{2\alpha_2}{r}\mathrm{Tr}\big(v\circ(\sqrt{\tau\mu}e - v)^+\big)\\
&\leq \mu + \frac{2\alpha_1}{r}\lambda_{\max}\big(v\circ(\sqrt{\tau\mu}e + v)^{-1}\big)\mathrm{Tr}\big((\tau\mu e - v^2)^-\big)\\
&\qquad\qquad + \frac{2\alpha_2}{r}\sum_{i=1}^{k}\sqrt{\lambda_i}\big(\sqrt{\tau\mu} - \sqrt{\lambda_i}\big)\\
&\leq \mu - \frac{2\alpha_1(1-\sqrt{\beta})(1-\tau)}{2-\sqrt{\beta}}\mu + \frac{2\alpha_2\sqrt{\tau\mu}}{r}\sum_{i=1}^{k}\big(\sqrt{\tau\mu} - \sqrt{\lambda_i}\big)\\
&= \mu - \frac{2\alpha_1(1-\sqrt{\beta})(1-\tau)\mu}{2-\sqrt{\beta}} + \frac{2\alpha_2\sqrt{\tau\mu}}{r}\mathrm{Tr}\big(e\circ(\sqrt{\tau\mu}e - \sqrt{\tilde{x}\circ\tilde{s}})^+\big)\\
&\leq \mu - \frac{2\alpha_1(1-\sqrt{\beta})(1-\tau)\mu}{2-\sqrt{\beta}} + \frac{2\alpha_2\sqrt{\tau\mu}}{\sqrt{r}}\big\|(\sqrt{\tau\mu}e - \sqrt{\tilde{x}\circ\tilde{s}})^+\big\|_F\\
&\leq \Big(1 - \frac{2\alpha_1(1-\sqrt{\beta})(1-\tau)}{2-\sqrt{\beta}} + \frac{2\alpha_2\tau\sqrt{\beta}}{\sqrt{r}}\Big)\mu,
\end{aligned}$$

where the second inequality is due to Lemma 4.3 and the fact that $\sqrt{\lambda_i} \leq \sqrt{\tau\mu}$ for $i = 1, \ldots, k$. The third inequality follows from the Cauchy-Schwartz inequality and the last inequality becomes from the fact that $(\tilde{x}, y, \tilde{s}) \in \mathcal{N}(\sqrt{\tau}, \sqrt{\beta})$. The proof is complete. □

Corollary 4.5. *Let $(\tilde{x}, y, \tilde{s}) \in \mathcal{N}(\sqrt{\tau}, \sqrt{\beta}), \tau \leq \frac{1}{19}$ and $\beta \leq \frac{1}{19}$. If $\alpha_1 = \alpha_2\sqrt{\frac{\beta\tau}{2r}}$ and $0 < \alpha_2 \leq 1$, then*

$$\tilde{\mu}(\alpha) \leq \Big(1 - \frac{1}{6}\alpha_1\Big)\mu.$$

Proof. Using Lemma 4.4, we have

$$\begin{aligned}\tilde{\mu}(\alpha) &\le \Big(1 - \frac{2\alpha_1(1-\sqrt{\beta})(1-\tau)}{2-\sqrt{\beta}} + \frac{2\alpha_2\tau\sqrt{\beta}}{\sqrt{r}}\Big)\mu\\ &= \Big(1 - \Big(\frac{2(1-\sqrt{\beta})(1-\tau)}{2-\sqrt{\beta}} - 2\sqrt{2\tau}\Big)\alpha_1\Big)\mu\\ &\le \Big(1 - \Big(\frac{36(\sqrt{19}-1)}{19(2\sqrt{19}-1)} - \frac{2\sqrt{2}}{\sqrt{19}}\Big)\alpha_1\Big)\mu\\ &\le \Big(1 - \frac{1}{6}\alpha_1\Big)\mu,\end{aligned}$$

where the second inequality follows from $\beta \le \frac{1}{19}$ and $\tau \le \frac{1}{19}$. The proof is completed. □

Lemma 4.6. *Let $x, s \in int\mathcal{K}$ and $(\tilde{x}, y, \tilde{s}) \in \mathcal{N}(\sqrt{\tau}, \sqrt{\beta})$. Then*

(i) $\|(L_{\tilde{x}}L_{\tilde{s}})^{-\frac{1}{2}}(\sqrt{\tau\mu}v - v^2)^+\|_F^2 \le \beta\tau\mu$.

(ii) $\|(L_{\tilde{x}}L_{\tilde{s}})^{-\frac{1}{2}}(\sqrt{\tau\mu}v - v^2)^-\|_F^2 \le r\mu$.

Proof. Let $\tilde{x} \circ \tilde{s} = \lambda_1 c_1 + \cdots + \lambda_r c_r$ be the spectral decomposition of $\tilde{x} \circ \tilde{s}$. In this way, we have $L_{\tilde{x}}L_{\tilde{s}}c_i = \tilde{x} \circ (\tilde{s} \circ c_i) = \lambda_i c_i, i = 1, \ldots, r$. From this, we deduce that

$$(L_{\tilde{x}}L_{\tilde{s}})^{-1}c_i = \frac{1}{\lambda_i}c_i,$$

and

$$\lambda_{\max}\big((L_{\tilde{x}}L_{\tilde{s}})^{-1}\big) = \frac{1}{\lambda_{\min}(\tilde{x} \circ \tilde{s})} = \frac{1}{\lambda_{\min}\big(Q_{\tilde{x}^{\frac{1}{2}}}\tilde{s}\big)} \le \frac{1}{(1-\sqrt{\beta})^2\tau\mu}, \tag{4.2}$$

where the second equality is due to Lemma 2.4 and the inequality follows from the fact that $(\tilde{x}, y, \tilde{s}) \in \mathcal{N}(\sqrt{\tau}, \sqrt{\beta})$. Therefore, we have

$$\begin{aligned}\|(L_{\tilde{x}}L_{\tilde{s}})^{-\frac{1}{2}}(\sqrt{\tau\mu}v - v^2)^+\|_F^2 &= \sum_{i=1}^{k}\frac{\big(\sqrt{\tau\mu\lambda_i} - \lambda_i\big)^2}{\lambda_i}\\ &= \sum_{i=1}^{k}(\sqrt{\tau\mu} - \sqrt{\lambda_i})^2 = \big\|\big(\sqrt{\tau\mu}e - \sqrt{Q_{\tilde{x}^{\frac{1}{2}}}\tilde{s}}\big)^+\big\|_F^2 \le \beta\tau\mu.\end{aligned}$$

This proves the assertion (i). In a manner similar to the case (i), we obtain

$$\|(L_{\tilde{x}}L_{\tilde{s}})^{-\frac{1}{2}}(\sqrt{\tau\mu}v - v^2)^-\|_F^2 = \sum_{i=k+1}^{r}\frac{\big(\sqrt{\tau\mu\lambda_i} - \lambda_i\big)^2}{\lambda_i} \le \sum_{i=1}^{r}\lambda_i = \mathrm{Tr}(\tilde{x} \circ \tilde{s}) = r\mu.$$

This completes the proof. □

Lemma 4.7 ([23, Lemmas 33 and 36])**.** *Let $u, v \in \mathcal{J}$ and G a positive definite matrix which is symmetric with respect to the scalar product $\langle \cdot, \cdot \rangle$. Then*

$$\|u\|_F\|v\|_F \le \frac{1}{2}\Big(\|G^{-\frac{1}{2}}u\|_F^2 + \|G^{\frac{1}{2}}v\|_F^2\Big).$$

The following lemma gives an upper bound on the Frobenius norm of $\Delta\tilde{x}(\alpha)\circ\Delta\tilde{s}(\alpha)$.

Lemma 4.8. *If* $G = L_{\tilde{s}}^{-1}L_{\tilde{x}}, \alpha_1 = \alpha_2\sqrt{\frac{\beta\tau}{2r}}$ *and* $(\tilde{x}, y, \tilde{s}) \in \mathcal{N}(\sqrt{\tau}, \sqrt{\beta})$*, then we have*

$$\left\|\Delta\tilde{x}(\alpha)\circ\Delta\tilde{s}(\alpha)\right\|_F \leq 3\alpha_2^2\beta\tau\mu.$$

Proof. Using the third equations of systems (3.7) and (3.8), we deduce that

$$\Delta\tilde{x}(\alpha)\circ\tilde{s} + \tilde{x}\circ\Delta\tilde{s}(\alpha) = 2\alpha_1(\sqrt{\tau\mu}v - v^2)^- + 2\alpha_2(\sqrt{\tau\mu}v - v^2)^+.$$

Multiplying the above equation by $(L_{\tilde{x}}L_{\tilde{s}})^{-\frac{1}{2}}$ and in view of $L_{\tilde{x}}\tilde{s} = \tilde{x}\circ\tilde{s}$, we obtain

$$\begin{aligned} G^{-\frac{1}{2}}\Delta\tilde{x}(\alpha) + G^{\frac{1}{2}}\Delta\tilde{s}(\alpha) = 2\alpha_1(L_{\tilde{x}}L_{\tilde{s}})^{-\frac{1}{2}}(\sqrt{\tau\mu}v - v^2)^- \\ +2\alpha_2(L_{\tilde{x}}L_{\tilde{s}})^{-\frac{1}{2}}(\sqrt{\tau\mu}v - v^2)^+. \end{aligned}$$

Taking norm-squared on both sides of the above equation and using Proposition 4.1, we can find that

$$\begin{aligned} \|G^{-\frac{1}{2}}\Delta\tilde{x}(\alpha)\|_F^2 + \|G^{\frac{1}{2}}\Delta\tilde{s}(\alpha)\|_F^2 &= \big\|2\alpha_1(L_{\tilde{x}}L_{\tilde{s}})^{-\frac{1}{2}}(\sqrt{\tau\mu}v - v^2)^- \\ &\qquad +2\alpha_2(L_{\tilde{x}}L_{\tilde{s}})^{-\frac{1}{2}}(\sqrt{\tau\mu}v - v^2)^+\big\|_F^2 \\ &= 4\alpha_1^2\big\|L_{\tilde{x}}L_{\tilde{s}})^{-\frac{1}{2}}(\sqrt{\tau\mu}v - v^2)^-\big\|_F^2 \\ &\qquad +4\alpha_2^2\big\|L_{\tilde{x}}L_{\tilde{s}})^{-\frac{1}{2}}(\sqrt{\tau\mu}v - v^2)^+\big\|_F^2 \\ &\leq 4\alpha_1^2 r\mu + 4\alpha_2^2\beta\tau\mu = 6\alpha_2^2\beta\tau\mu, \end{aligned}$$

where the inequality is due to Lemma 4.6. By invoking Lemmas 2.3, 4.7 and the above inequality, we conclude that

$$\begin{aligned} \left\|\Delta\tilde{x}(\alpha)\circ\Delta\tilde{s}(\alpha)\right\|_F &\leq \|\Delta\tilde{x}(\alpha)\|_F\|\Delta\tilde{s}(\alpha)\|_F \\ &\leq \frac{1}{2}\Big(\|G^{-\frac{1}{2}}\Delta\tilde{x}(\alpha)\|_F^2 + \|G^{\frac{1}{2}}\Delta\tilde{s}(\alpha)\|_F^2\Big) \leq 3\alpha_2^2\beta\tau\mu. \end{aligned}$$

This follows the desired result. ☐

Lemma 4.9. *If* $(\tilde{x}, y, \tilde{s}) \in \mathcal{N}(\sqrt{\tau}, \sqrt{\beta}), \alpha_1 = \alpha_2\sqrt{\frac{\beta\tau}{2r}}$ *and* $\alpha_2 = 1$*, then*

$$\left\|\big(\tau\tilde{\mu}(\alpha)e - h(\alpha)\big)^+\right\|_F \leq \beta\tau\tilde{\mu}(\alpha).$$

Proof. By Corollary 4.5, it is clear that $\tilde{\mu}(\alpha) \leq \mu$. For $i = 1, \ldots, k$, we have

$$\begin{aligned} \tau\tilde{\mu}(\alpha) - \lambda_i(h(\alpha)) &\leq \tau\tilde{\mu}(\alpha) - \frac{\tilde{\mu}(\alpha)}{\mu}\lambda_i(h(\alpha)) \\ &= \frac{\tilde{\mu}(\alpha)}{\mu}\big(\tau\mu - \lambda_i(h(\alpha))\big) \\ &= \frac{\tilde{\mu}(\alpha)}{\mu}\big(\tau\mu - \lambda_i(v^2 + 2v\circ(\sqrt{\tau\mu}e - v)\big) \\ &= \frac{\tilde{\mu}(\alpha)}{\mu}\big(\tau\mu + \lambda_i - 2\sqrt{\tau\mu}\sqrt{\lambda_i}\big) \\ &= \frac{\tilde{\mu}(\alpha)}{\mu}\big(\sqrt{\tau\mu} - \sqrt{\lambda_i}\big)^2 > 0. \end{aligned}$$

On the other hand, for $i = k+1, \dots, r$, we get $\tau\tilde{\mu}(\alpha) - \lambda_i(h(\alpha)) \le \tau\mu - \tau\mu = 0$. Therefore, we obtain

$$\begin{aligned}
\left\|\left(\tau\tilde{\mu}(\alpha)e - h(\alpha)\right)^+\right\|_F &\le \frac{\tilde{\mu}(\alpha)}{\mu}\left\|\left((\sqrt{\tau\mu}e - v)^+\right)^2\right\|_F \\
&\le \frac{\tilde{\mu}(\alpha)}{\mu}\left\|(\sqrt{\tau\mu}e - v)^+\right\|_F^2 \\
&= \frac{\tilde{\mu}(\alpha)}{\mu}\left\|(\sqrt{\tau\mu}e - \sqrt{\tilde{x} \circ \tilde{s}})^+\right\|_F^2 \\
&= \frac{\tilde{\mu}(\alpha)}{\mu}\left\|\left(\sqrt{\tau\mu}e - \sqrt{Q_{\tilde{x}^{\frac{1}{2}}}\tilde{s}}\right)^+\right\|_F^2 \\
&= \frac{\tilde{\mu}(\alpha)}{\mu}\left\|\left(\sqrt{\tau\mu}e - \sqrt{Q_{x^{\frac{1}{2}}}s}\right)^+\right\|_F^2 \\
&\le \beta\tau\tilde{\mu}(\alpha),
\end{aligned}$$

where the first equality is due to the definition of the vector v, the second equality follows from Lemma 2.4, the third equality becomes from the fact that the neighborhood $\mathcal{N}(\sqrt{\tau}, \sqrt{\beta})$ is scaling invariant and the last inequality is obtained from the assumption $(x, y, s) \in \mathcal{N}(\sqrt{\tau}, \sqrt{\beta})$. The proof is complete. □

Lemma 4.10. *Let* $(\tilde{x}, y, \tilde{s}) \in \mathcal{N}(\sqrt{\tau}, \sqrt{\beta}), \alpha_1 = \alpha_2\sqrt{\frac{\beta\tau}{2r}}$ *and* $\alpha_2 = 1$*, then*

$$\lambda_{\min}(\tilde{x}(\alpha) \circ \tilde{s}(\alpha)) \ge (1 - 4\beta)\tau\mu.$$

Proof. For $i = 1, \dots, k$, we have

$$\begin{aligned}
\lambda_{\min}(h(\alpha)) &= \lambda_{\min}\left(v^2 + 2\alpha_1 v \circ (\sqrt{\tau\mu}e - v)^- + 2\alpha_2 v \circ (\sqrt{\tau\mu}e - v)^+\right) \\
&\ge (1 - 2\alpha_2)\lambda_{\min} + 2\alpha_2\sqrt{\tau\mu\lambda_{\min}} \\
&\ge (1 - 2\alpha_2)(1 - \sqrt{\beta})^2\tau\mu + 2\alpha_2(1 - \sqrt{\beta})\tau\mu = (1 - \beta)\tau\mu.
\end{aligned}$$

Let $i = k+1, \dots, r$ and $\alpha_1 < \frac{1}{2}$, then we have

$$\begin{aligned}
\lambda_{\min}(h(\alpha)) &\ge (1 - 2\alpha_1)\lambda_{\min} + 2\alpha_1\sqrt{\tau\mu\lambda_{\min}} \\
&\ge (1 - 2\alpha_1)\tau\mu + 2\alpha_1\tau\mu = \tau\mu \\
&\ge (1 - \beta)\tau\mu.
\end{aligned}$$

Using (4.1), we obtain

$$\begin{aligned}
\lambda_{\min}(\tilde{x}(\alpha) \circ \tilde{s}(\alpha)) &\ge \lambda_{\min}(h(\alpha)) - \left\|\Delta\tilde{x}(\alpha) \circ \Delta\tilde{s}(\alpha)\right\|_F \\
&\ge (1 - \beta)\tau\mu - 3\beta\tau\mu = (1 - 4\beta)\tau\mu.
\end{aligned}$$

where the second inequality follows from the above inequalities and Lemma 4.8. The proof is complete. □

The following lemma gives a sufficient condition for which all the generated iterates by Algorithm lie in the neighborhood $\mathcal{N}(\sqrt{\tau}, \sqrt{\beta})$.

Lemma 4.11. *Let* $(\tilde{x}, y, \tilde{s}) \in \mathcal{N}(\sqrt{\tau}, \sqrt{\beta}), \beta \le \frac{1}{19}$ *and* $\tau \le \frac{1}{19}$*. If* $\alpha_1 = \alpha_2\sqrt{\frac{\beta\tau}{2r}}$ *and* $\alpha_2 = 1$*, then* $(\tilde{x}(\alpha), y(\alpha), \tilde{s}(\alpha)) \in \mathcal{N}(\sqrt{\tau}, \sqrt{\beta})$.

Proof. To prove the lemma, we need to show

$$\big\|\big(\sqrt{\tau\tilde{\mu}(\alpha)}e-\sqrt{Q_{\tilde{x}(\alpha)^{\frac{1}{2}}}\tilde{s}(\alpha)}\big)^{+}\big\|_F\leq\sqrt{\tau\beta\tilde{\mu}(\alpha)}$$

and $(\tilde{x}(\alpha),\tilde{s}(\alpha))\in int\mathcal{K}\times int\mathcal{K}$. To this end, using Lemmas 2.4, 4.10 and (4.2), we have

$$\begin{aligned}
\big\|\big(\sqrt{\tau\tilde{\mu}(\alpha)}e-\sqrt{Q_{\tilde{x}(\alpha)^{\frac{1}{2}}}\tilde{s}(\alpha)}\big)^{+}\big\|_F &= \big\|\big(\sqrt{\tau\tilde{\mu}(\alpha)}e-\sqrt{\tilde{x}(\alpha)\circ\tilde{s}(\alpha)}\big)^{+}\big\|_F\\
&\leq \frac{1}{\lambda_{\min}(\sqrt{\tau\tilde{\mu}(\alpha)}e+\sqrt{\tilde{x}(\alpha)\circ\tilde{s}(\alpha)})}\big\|\big(\tau\tilde{\mu}(\alpha)e-\tilde{x}(\alpha)\circ\tilde{s}(\alpha)\big)^{+}\big\|_F\\
&= \frac{1}{\sqrt{\tau\tilde{\mu}(\alpha)}+\sqrt{\lambda_{\min}(\tilde{x}(\alpha)\circ\tilde{s}(\alpha))}}\big\|\big(\tau\tilde{\mu}(\alpha)e-\tilde{x}(\alpha)\circ\tilde{s}(\alpha)\big)^{+}\big\|_F\\
&\leq \frac{1}{\sqrt{\tau\tilde{\mu}(\alpha)}}\big\|\big(\tau\tilde{\mu}(\alpha)e-\tilde{x}(\alpha)\circ\tilde{s}(\alpha)\big)^{+}\big\|_F\\
&= \frac{1}{\sqrt{\tau\tilde{\mu}(\alpha)}}\big\|\big(\tau\tilde{\mu}(\alpha)e-h(\alpha)-\Delta\tilde{x}(\alpha)\circ\Delta\tilde{s}(\alpha)\big)^{+}\big\|_F\\
&\leq \frac{1}{\sqrt{\tau\tilde{\mu}(\alpha)}}\Big(\big\|\big(\tau\tilde{\mu}(\alpha)e-h(\alpha)\big)^{+}\big\|_F+\big\|\big(-\Delta\tilde{x}(\alpha)\circ\Delta\tilde{s}(\alpha)\big)^{+}\big\|_F\Big)\\
&\leq \frac{1}{\sqrt{\tau\tilde{\mu}(\alpha)}}\Big(\big\|\big(\tau\tilde{\mu}(\alpha)e-h(\alpha)\big)^{+}\big\|_F+\big\|\Delta\tilde{x}(\alpha)\circ\Delta\tilde{s}(\alpha)\big\|_F\Big)\\
&\leq \frac{1}{\sqrt{\tau\tilde{\mu}(\alpha)}}\Big(\beta\tau\tilde{\mu}(\alpha)+3\alpha_2^2\beta\tau\mu\Big)\\
&\leq \frac{1}{\sqrt{\tau\tilde{\mu}(\alpha)}}\Big(\beta\tau\tilde{\mu}(\alpha)+\frac{3\alpha_2^2\beta\tau}{1-2\alpha_1}\tilde{\mu}(\alpha)\Big)\\
&= \Big(1+\frac{3\alpha_2^2}{1-2\alpha_1}\Big)\sqrt{\beta}\sqrt{\beta\tau\tilde{\mu}(\alpha)}\\
&\leq \Big(\frac{4-\sqrt{2\beta\tau}}{1-\sqrt{2\beta\tau}}\Big)\sqrt{\beta}\sqrt{\beta\tau\tilde{\mu}(\alpha)}\leq\sqrt{\beta\tau\tilde{\mu}(\alpha)},
\end{aligned}$$

where the third equality is due to (4.1), the third inequality follows from Lemma 2.2, the fifth inequality becomes from Lemmas 4.9 and 4.8, the sixth inequality concludes from Lemma 4.2, in the second last inequality we have used the assumptions $\alpha_1=\alpha_2\sqrt{\frac{\beta\tau}{2r}}$ and $\alpha_2=1$ and the last inequality is due to the fact that $f(t)=\frac{4-t}{1-t}$ is monotonically increasing, so, using $t=\sqrt{2\beta\tau}$, we get $g(\sqrt{2\beta\tau})\sqrt{\beta}\leq 0.973<1$. The proof is completed. □

Now we are in a position to present our complexity results.

Theorem 4.12. *Suppose that $\beta\leq\frac{1}{19},\tau\leq\frac{1}{19},\alpha_1=\alpha_2\sqrt{\frac{\beta\tau}{2r}}$ and $\alpha_2=1$ are fixed for all iterations. Then, Algorithm 1 terminates in $\mathcal{O}(\sqrt{r}\log\frac{\mu_0}{\varepsilon})$ iterations with an ε-optimal solution such that $\mu_k\leq\varepsilon$.*

Proof. Since the assumptions of Lemma 4.11 hold, we conclude that $(\tilde{x}(\alpha),y(\alpha),\tilde{s}(\alpha))\in\mathcal{N}(\sqrt{\tau},\sqrt{\beta})$. Furthermore, according to Corollary 4.5, we also have

$$\tilde{\mu}(\alpha)\leq\Big(1-\frac{1}{6}\sqrt{\frac{\beta\tau}{2r}}\Big)\mu.$$

Therefore,

$$\tilde{\mu}_k \leq \Big(1 - \frac{1}{6}\sqrt{\frac{\beta\tau}{2r}}\Big)^k \mu_0.$$

So, after k steps the duality gap, will be less that ε if

$$\Big(1 - \frac{1}{6}\sqrt{\frac{\beta\tau}{2r}}\Big)^k \mu_0 \leq \varepsilon.$$

Taking logarithms gives

$$k \log\Big(1 - \frac{1}{6}\sqrt{\frac{\beta\tau}{2r}}\Big) \leq -\log\frac{\mu_0}{\varepsilon}.$$

Since $\log(1+\xi) \leq \xi, \xi \geq -1$, using $\xi = -\frac{1}{6}\sqrt{\frac{\beta\tau}{2r}}$, we obtain that the above inequality holds if

$$k \geq 6\sqrt{\frac{2r}{\beta\tau}}\log\frac{\mu_0}{\varepsilon},$$

this proves the lemma. □

5 Numerical Results

In this section, we compare the proposed primal-dual algorithm with the Ai-Zhang's primal-dual algorithm [2]. These two algorithms will be denoted, respectively, by Algor. 1 and Algor. 2. We present some numerical results for the test problems given in Table 1 that are taken from the standard NETLIB test set for LO. All of our tests are run on an Intel Core i3 (3.40 GHz) under Windows XP and MATLAB 7.8.0 (R2009a). We select $\alpha_1 = \alpha_2\sqrt{\frac{\beta\tau}{2r}}$ and $\alpha_2 = 1$ for both Algorithms. Moreover, we choose the parameters according to the given default values in algorithms, that is, $\tau = \beta = \frac{1}{19}$ for Algor. 1 and $\tau = \frac{1}{4}, \beta = \frac{1}{2}$ for Algor. 2. Both algorithms stop if the relative duality gap satisfies

$$\frac{\langle x, s\rangle}{1 + \langle c, x\rangle} \leq 10^{-8}.$$

Table 1 lists the names of the test problems, the number of iterations (iter), the total CPU time (time) and the relative duality gap (regap) when Algorithms terminate. Based on the obtained results, we conclude that our wide neighborhood algorithm outperforms Algor. 2.

6 Concluding Remarks

In this paper, we have presented a primal-dual IPM for SO based on a new large neighborhood of the central path, which differs from those that are available. We have focused our attention on the analysis of the theoretical properties of the proposed algorithm and have proved that its complexity bound coincides with the best-known one obtained by any feasible interior-point method for SO. We highlighted the practical efficiency of the method by providing numerical results on the selected set of test problems from NETLIB. Due to the numerical results, we concluded that our algorithm is promising and efficient than Algorithm 2.

Table 1

name	row	column	Algor. 1			Algor. 2		
			time	iter	regap	time	iter	regap
adlittle	56	138	0.8105	21	2.414e-11	0.8222	21	2.3415e-9
afiro	27	51	0.0559	15	1.6355e-8	0.0886	19	3.7222e-9
bandm	305	472	33.6426	33	1.1582e-7	36.9278	36	5.3629e-9
blend	74	114	0.4042	17	2.8080e-8	0.5355	21	3.0453e-9
kb2	43	68	0.1194	13	6.1825e-8	0.1574	17	8.6648e-9
lotfi	153	366	4.2149	23	2.5303e-6	5.3554	30	6.9311e-9
share2b	96	162	0.9056	20	4.7176e-9	1.0762	22	4.6318e-9
share1b	117	253	5.1022	42	2.4320e-7	6.1116	51	6.0652e-9
grow7	140	301	1.3978	11	1.6875e-8	1.9361	16	9.3579e-9
sc50a	50	78	0.1474	16	6.8262e-9	0.2223	19	9.5413e-9
sc105	105	163	0.5216	15	6.2696e-7	0.6932	20	2.8704e-9
sc205	205	317	2.6861	18	3.5369e-7	3.2702	22	6.5849e-9
sc50b	50	78	0.1464	13	9.9829e-8	0.1418	18	3.0459e-9
beaconfd	173	295	3.5275	18	4.2074e-10	3.4556	19	3.7559e-9
brandy	220	303	7.9932	33	1.1478e-8	8.5047	35	8.8508e-9
e226	223	472	21.6580	36	2.5837e-8	22.6352	38	2.5032e-9
scagr7	129	185	2.2305	20	2.7814e-10	1.9981	19	3.3285e-9
scagr25	471	671	79.2994	27	4.8663e-11	72.7878	25	8.0152e-9
scfxm1	330	600	39.2637	40	3.5502e-8	41.8576	43	2.7347e-9
scfxm2	660	1200	498.4486	47	2.2702e-8	566.4426	52	3.3547e-9
scsd6	147	1350	274.3768	19	1.0537e-7	347.2908	24	3.1264e-9
sctap1	300	610	61.2735	34	4.2291e-9	78.5686	40	5.1767e-9
seba	515	1036	381.5771	43	2.2863e-8	396.3231	50	7.2605e-9
agg	488	615	33.3743	31	7.5871e-9	35.8101	31	4.3443e-9
agg2	516	758	52.7349	28	1.5198e-9	57.3524	29	4.5172e-9
agg3	516	756	59.6196	31	5.8488e-9	72.6941	33	5.9559e-9
boeing1	351	726	46.0068	24	6.8585e-6	70.0903	33	5.7699e-9
boeing1	166	305	5.2150	25	1.9039e-7	6.3456	32	3.9990e-9
capri	271	496	20.2361	33	5.9191e-8	21.8378	34	6.4709e-9

Table 1: Comparison with Ai-Zhang's algorithm given in [2]

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