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A TIGHTER M-EIGENVALUE LOCALIZATION SET FOR FOURTH-ORDER PARTIALLY SYMMETRIC TENSORS*

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Abstract: In this paper, an M-eigenvalue inclusion theorem for fourth-order partially symmetric tensors is proposed by choosing different component of M-eigenvector. We prove that the new upper bound is sharper than the existing upper bounds in the literature. Finally, numerical examples are illustrated to verify the theoretical results.

Key words: *M-eigenvalue, Z-eigenvalue, fourth-order partially symmetric tensors*

Mathematics Subject Classification: *15A06 74B20 47J25*

1 Introduction

For a positive integer $n \geq 2$, we denote the set $\{1, 2, \ldots, n\}$ by $[n]$, and denote the set of all complex (real) numbers by $\mathbb{C}(\mathbb{R})$. We call $\mathcal{C} = (c_{i_1 i_2 \ldots i_m})$ a real tensor of order m dimension n, denoted by $\mathbb{R}^{[m,n]}$, if

$$c_{i_1 i_2 \ldots i_m} \in \mathbb{R},\ i_j \in [n],\ j \in [m].$$

The tensor $\mathcal{C}$ is called nonnegative (positive) if $c_{i_1 i_2 \ldots i_m} \geq 0$ ($c_{i_1 i_2 \ldots i_m} > 0$).

The E-eigenvalue problem of tensors is firstly appeared in [10] and the tensor is symmetric.

Given a tensor $\mathcal{C} = (c_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}$, if there are $\lambda \in \mathbb{C}$ and $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n \backslash \{0\}$ such that

$$\mathcal{C}x^{m-1} = \lambda x \text{ and } x^T x = 1,$$

then (λ, x) is called an E-eigenpair of $\mathcal{C}$, where $\mathcal{C}x^{m-1}$ is an n-dimension vector whose i-th component is defined by

$$(\mathcal{C}x^{m-1})_i = \sum_{i_2,\ldots,i_m \in [n]} c_{i i_2 \ldots i_m} x_{i_2} \ldots x_{i_m}.$$

(λ, x) is called a Z-eigenpair if both of them are real.

The following definition about M-eigenvalue of tensors is firstly introduced in [11].

Given a tensor $\mathcal{C} = (c_{ijkl})$, if the coefficients have the following property

$$c_{ijkl} = c_{kjil} = c_{ilkj} = c_{klij},\ i, k \in [m],\ j, l \in [n],$$

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then we call tensor $\mathcal{C}$ is a fourth-order partially symmetric tensor.

For $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^m, y \in \mathbb{R}^n$, if

$$\begin{cases} \mathcal{C} \cdot yxy = \lambda x, \\ \mathcal{C}xyx\cdot = \lambda y, \\ x^T x = 1, \\ y^T y = 1 \end{cases} \tag{1.1}$$

holds, where $(\mathcal{C} \cdot yxy)_i = \sum_{k\in[m],\ j,l\in[n]} c_{ijkl} y_j x_k y_l$ and $(\mathcal{C}xyx\cdot)_l = \sum_{i,k\in[m],j\in[n]} c_{ijkl} x_i y_j x_k$, then the scalar λ is called an M-eigenvalue of the tensor $\mathcal{C}$, and x and y are called left and right M-eigenvectors of $\mathcal{C}$, respectively, which associated with the M-eigenvalue λ. The M-spectral radius $\rho(\mathcal{C})$ of $\mathcal{C}$ is defined as

$$\rho(\mathcal{C}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{C})\},$$

where $\sigma(\mathcal{C})$ is the M-spectrum of $\mathcal{C}$, which contains all M-eigenvalues of $\mathcal{C}$.

In [11], Qi et al. pointed out a Z-eigenvalue of a partially symmetric tensor is an M-eigenvalue. In recent years, Z-eigenvalue problem has received special attention, which has a wide range of practical application in statistical date analysis and engineering [8,16]. On the study of the bounds for Z-spectral radius of nonnegative tensors, Chang et al. [1] proposed the upper bounds. Song et al. [12] improved the upper bounds based on the relationship between the Gelfand formula and the spectral radius. For weakly symmetric and positive tensors, He et al. [5] presented the Ledermann-like upper bound for the largest Z-eigenvalue. For general tensors, Wang et al. [15] established Z-eigenvalue inclusion theorems, and the upper bounds for the largest Z-eigenvalue of a weakly symmetric nonnegative tensor was obtained.

At the same time, Qi et al. [11] also pointed out that an M-eigenvalue of a partially symmetric tensor is not necessarily a Z-eigenvalue. M-eigenvalue problem has a close connection to the strong ellipticity condition, which is essential in the theory of elasticity, since it guarantees the existence of solutions of basic boundary-value problems of elastostatics [6,7,13]. Han et al. [4] proposed the strong ellipticity condition to the rank-one positive definiteness of three second-order tensors, three fourth-order tensors, and a sixth-order tensor. Wang et al. [14] presented a practical method to compute the largest M-eigenvalue of a fourth-order partially symmetric tensor. Qi et al. [11] explored a necessary and sufficient condition of the strong ellipticity by introducing M-eigenvalues for ellipticity tensors and revealed that the strong ellipticity condition holds if and only if all the M-eigenvalues of the ellipticity tensor are positive. In [2], Che et al. gave the following theorems for the fourth-order partially symmetric tensors.

Theorem 1.1 ([2])**.** *Suppose the tensor $\mathcal{C}$ is a fourth-order partially symmetric tensor. Then*

$$\sigma(\mathcal{C}) \subseteq \Gamma(\mathcal{C}) = \bigcup_{i\in[m]} \Gamma_i(\mathcal{C}),$$

where $\Gamma_i(\mathcal{C}) = \{z \in \mathbb{C} : |z| \le R_i(\mathcal{C})\}$, and $R_i(\mathcal{C}) = \sum_{k\in[m],\ j,l\in[n]} |c_{ijkl}|$.

Theorem 1.2 ([2])**.** *Suppose the tensor $\mathcal{C}$ is a fourth-order partially symmetric tensor. Then*

$$\sigma(\mathcal{C}) \subseteq \mathcal{L}(\mathcal{C}) = \bigcup_{i\in[m]} \bigcap_{k\in[m],k\ne i} \mathcal{L}_{i,k}(\mathcal{C}),$$

where $\mathcal{L}_{i,k}(\mathcal{C}) = \{z \in \mathbb{C} : (|z| - (R_i(\mathcal{C}) - R_i^k(\mathcal{C})))|z| \le R_i^k(\mathcal{C}) R_k(\mathcal{C})\}$, and $R_i^k(\mathcal{C}) = \sum_{j,l\in[n]} |c_{ijkl}|$.

In this paper, inspired by the above references, we establish an inclusion theorem to identify the distribution of M-eigenvalues and give M-eigenvalue localization sets for fourth-order partially symmetric tensors by choosing different components of M-eigenvector. As an application, it is proven that the new upper bounds are sharper than the existing upper bounds in the literature. Finally, numerical examples are proposed to verify the theoretical results.

The remainder of this paper is organized as follows. In Section 2, we establish a new M-eigenvalue inclusion theorem and give relationships among these eigenvalue inclusion sets. In Section 3, we apply the inclusion theorem to estimate a sharper upper bound of the largest M-eigenvalue for nonnegative tensors.

2 A New M-Eigenvalue Inclusion Theorem

In this section, a new M-eigenvalue inclusion theorem is presented for fourth-order partially symmetric tensors. Furthermore, we establish relationships among different M-eigenvalue inclusion sets. Now, we introduce the following Lemma.

Lemma 2.1 ([9])**.** *Let $a, b, c \geq 0$ and $d > 0$. If $\frac{a}{b+c+d} \geq 1$, then*

$$\frac{a-(b+c)}{d} \geq \frac{a-b}{c+d} \geq \frac{a}{b+c+d}.$$

Enlightened by the ideas of H-eigenvalue inclusion theorem [10] and Z-eigenvalue inclusion theorems [15], we establish the following M-eigenvalue inclusion theorem.

Theorem 2.2. *Suppose the tensor $\mathcal{C}$ is a fourth-order partially symmetric tensor. Then*

$$\sigma(\mathcal{C}) \subseteq \mathcal{H}(\mathcal{C}) = \Big(\bigcup_{i\in[m]} \bigcap_{k\in[m],k\neq i} \widehat{\mathcal{H}}_{i,k}(\mathcal{C})\Big) \bigcup \Big(\bigcup_{i\in[m]} \bigcap_{k\in[m],k\neq i} (\overline{\mathcal{H}}_{i,k}(\mathcal{C}) \bigcap \Gamma_i(\mathcal{C}))\Big),$$

where

$$\widehat{\mathcal{H}}_{i,k}(\mathcal{C}) = \{z \in \mathbb{C} : |z| < R_i(\mathcal{C}) - R_i^k(\mathcal{C}),\ |z| < R_k^k(\mathcal{C})\},$$

$$\overline{\mathcal{H}}_{i,k}(\mathcal{C}) = \{z \in \mathbb{C} : (|z| - (R_i(\mathcal{C}) - R_i^k(\mathcal{C})))(|z| - R_k^k(\mathcal{C})) \leq R_i^k(\mathcal{C})(R_k(\mathcal{C}) - R_k^k(\mathcal{C}))\},$$

$$R_i^k(\mathcal{C}) = \sum_{j,l\in[n]} |c_{ijkl}|,$$

$\Gamma_i(\mathcal{C})$ and $R_i(\mathcal{C})$ are same as given in Theorem 1.1.

Proof. Let λ be an M-eigenvalue of the tensor $\mathcal{C}$ with corresponding left M-eigenvector $x \in \mathbb{R}^m$ and right M-eigenvector $y \in \mathbb{R}^n$. As x is a left M-eigenvector of the tensor $\mathcal{C}$ with $x^T x = 1$, we know that it has at least one nonzero component. Denote x_t by a component of x with the largest absolute value, that is

$$|x_t| = \max_{p\in[m]} |x_p| > 0.$$

In the following, we let $s \in [m]$ and $s \neq t$.

From (1.1), one has

$$\lambda x_t = (\mathcal{C} \cdot yxy)_t$$

$$\begin{aligned} &= \sum_{k\in[m],\ j,l\in[n]} c_{tjkl}y_jx_ky_l \\ &= \sum_{k\in[m],\ k\neq s,\ j,l\in[n]} c_{tjkl}y_jx_ky_l + \sum_{j,l\in[n]} c_{tjsl}y_jx_sy_l, \end{aligned}$$

which yields that

$$|\lambda||x_t| \leq \sum_{k\in[m],\ k\neq s,\ j,l\in[n]} |c_{tjkl}||x_t| + \sum_{j,l\in[n]} |c_{tjsl}||x_s|$$

Since

$$\sum_{k\in[m],\ k\neq s,\ j,l\in[n]} |c_{tjkl}| = R_t(\mathcal{C}) - R_t^s(\mathcal{C}), \quad \sum_{j,l\in[n]} |c_{tjsl}| = R_t^s(\mathcal{C}),$$

we have

$$(|\lambda| - (R_t(\mathcal{C}) - R_t^s(\mathcal{C})))|x_t| \leq R_t^s(\mathcal{C})|x_s|. \tag{2.1}$$

As follows, we break up the proof into two cases.

Case 1. If $|x_s| = 0$, then $|\lambda| \leq R_t(\mathcal{C}) - R_t^s(\mathcal{C})$ by (2.1).

(i) If $|\lambda| = R_t(\mathcal{C}) - R_t^s(\mathcal{C})$, it is obvious that $\lambda \in \overline{\mathcal{H}}_{t,s}(\mathcal{C}) \bigcap \Gamma_t(\mathcal{C})$.

(ii) If $|\lambda| < R_t(\mathcal{C}) - R_t^s(\mathcal{C})$ and $|\lambda| \geq R_s^s(\mathcal{C})$, together with (2.1), we have

$$(|\lambda| - (R_t(\mathcal{C}) - R_t^s(\mathcal{C})))(|\lambda| - R_s^s(\mathcal{C})) \leq 0 \leq R_t^s(\mathcal{C})(R_s(\mathcal{C}) - R_s^s(\mathcal{C})),$$

which means that $\lambda \in \overline{\mathcal{H}}_{t,s}(\mathcal{C})$. It is easy to see that $|\lambda| < R_t(\mathcal{C})$. Hence, $\lambda \in \overline{\mathcal{H}}_{t,s}(\mathcal{C}) \bigcap \Gamma_t(\mathcal{C})$.

If $|\lambda| < R_t(\mathcal{C}) - R_t^s(\mathcal{C})$ and $|\lambda| < R_s^s(\mathcal{C})$, we obtain $\lambda \in \widehat{\mathcal{H}}_{t,s}(\mathcal{C})$.

Case 2. If $|x_s| > 0$, one has

$$\begin{aligned} \lambda x_s =& (\mathcal{C} \cdot yxy)_s \\ &= \sum_{k\in[m],j,l\in[n]} c_{sjkl}y_jx_ky_l \\ &= \sum_{k\in[m],\ k\neq s,\ j,l\in[n]} c_{sjkl}y_jx_ky_l + \sum_{j,l\in[n]} c_{sjsl}y_jx_sy_l, \end{aligned}$$

which yields that

$$\lambda||x_s| \leq \sum_{k\in[m],\ k\neq s,\ j,l\in[n]} |c_{sjkl}||x_t| + \sum_{j,l\in[n]} |c_{sjsl}||x_s|.$$

Since

$$\sum_{k\in[m],\ k\neq s,\ j,l\in[n]} |c_{sjkl}| = R_s(\mathcal{C}) - R_s^s(\mathcal{C}), \quad \sum_{j,l\in[n]} |c_{sjsl}| = R_s^s(\mathcal{C}),$$

we have

$$(|\lambda| - R_s^s(\mathcal{C}))|x_s| \leq (R_s(\mathcal{C}) - R_s^s(\mathcal{C}))|x_t|. \tag{2.2}$$

(i) If $|\lambda| \geq R_t(\mathcal{C}) - R_t^s(\mathcal{C})$ or $|\lambda| \geq R_s^s(\mathcal{C})$, it follows from (2.1) and (2.2) that

$$(|\lambda| - (R_t(\mathcal{C}) - R_t^s(\mathcal{C})))(|\lambda| - R_s^s(\mathcal{C})) \leq R_t^s(\mathcal{C})(R_s(\mathcal{C}) - R_s^s(\mathcal{C})).$$

On the other hand, from (2.1), we can obtain $|\lambda| \leq R_t(\mathcal{C})$. Thus, we have $\lambda \in \overline{\mathcal{H}}_{t,s}(\mathcal{C}) \bigcap \Gamma_t(\mathcal{C})$.

(ii) If $|\lambda| < R_t(\mathcal{C}) - R_t^s(\mathcal{C})$ and $|\lambda| < R_s^s(\mathcal{C})$, we deduce $\lambda \in \widehat{\mathcal{H}}_{t,s}(\mathcal{C})$.

In summary, we have $\lambda \in \bigcap_{s\in[m],s\neq t}(\overline{\mathcal{H}}_{t,s}(\mathcal{C})\bigcap\Gamma_t(\mathcal{C}))$ or $\lambda \in \bigcap_{s\in[m],s\neq t}\widehat{\mathcal{H}}_{t,s}(\mathcal{C})$. As a result, $\sigma(\mathcal{C}) \subseteq \mathcal{H}(\mathcal{C})$ and the desired results follow. □

From Theorem 1.1, Theorem 1.2 and Theorem 2.2, we show the following relation of $\sigma(\mathcal{C})$, $\mathcal{L}(\mathcal{C})$, $\mathcal{H}(\mathcal{C})$ and $\Gamma(\mathcal{C})$.

Theorem 2.3. *Suppose the tensor $\mathcal{C}$ is a fourth-order partially symmetric tensor. Then*

$$\sigma(\mathcal{C}) \subseteq \mathcal{H}(\mathcal{C}) \subseteq \mathcal{L}(\mathcal{C}) \subseteq \Gamma(\mathcal{C}).$$

Proof. From Theorem 2.3 in [16], we obtain

$$\sigma(\mathcal{C}) \subseteq \mathcal{L}(\mathcal{C}) \subseteq \Gamma(\mathcal{C}).$$

Hence, we only need to prove $\mathcal{H}(\mathcal{C}) \subseteq \mathcal{L}(\mathcal{C})$. Indeed, let

$$\lambda \in \mathcal{H}(\mathcal{C}) = (\bigcup_{i\in[m]}\bigcap_{k\in[m],k\neq i}\widehat{\mathcal{H}}_{i,k}(\mathcal{C}))\bigcup(\bigcup_{i\in[m]}\bigcap_{k\in[m],k\neq i}(\overline{\mathcal{H}}_{i,k}(\mathcal{C})\bigcap\Gamma_i(\mathcal{C}))),$$

then

$$\lambda \in \bigcup_{i\in[m]}\bigcap_{k\in[m],k\neq i}\widehat{\mathcal{H}}_{i,k}(\mathcal{C})$$

or

$$\lambda \in \bigcup_{i\in[m]}\bigcap_{k\in[m],k\neq i}(\overline{\mathcal{H}}_{i,k}(\mathcal{C})\bigcap\Gamma_i(\mathcal{C})).$$

We now break up the proof into two cases.

Case 1. If $\lambda \in \bigcup_{i\in[m]}\bigcap_{k\in[m],k\neq i}\widehat{\mathcal{H}}_{i,k}(\mathcal{C})$, then there exists an index $i \in [m]$ such that, for any $k \in [m], k \neq i$,

$$|\lambda| < R_i(\mathcal{C}) - R_i^k(\mathcal{C})$$

and

$$|\lambda| < R_k^k(\mathcal{C}),$$

which yield that

$$(|\lambda| - (R_i(\mathcal{C}) - R_i^k(\mathcal{C})))|\lambda| \leq 0 \leq R_i^k(\mathcal{C})R_k(\mathcal{C}).$$

Therefore, $\lambda \in \bigcap_{k\in[m],k\neq i}\mathcal{L}_{i,k}(\mathcal{C}) \subseteq \mathcal{L}(\mathcal{C})$.

Case 2. If $\lambda \in \bigcup_{i\in[m]}\bigcap_{k\in[m],k\neq i}(\overline{\mathcal{H}}_{i,k}(\mathcal{C})\bigcap\Gamma_i(\mathcal{C}))$, then there exists an index $i \in [m]$ such that, for any $k \in [m], k \neq i$,

$$(|\lambda| - (R_i(\mathcal{C}) - R_i^k(\mathcal{C})))(|\lambda| - R_k^k(\mathcal{C})) \leq R_i^k(\mathcal{C})(R_k(\mathcal{C}) - R_k^k(\mathcal{C})) \quad (2.3)$$

and

$$|\lambda| \leq R_i(\mathcal{C}). \quad (2.4)$$

(i) If $R_i^k(\mathcal{C})(R_k(\mathcal{C}) - R_k^k(\mathcal{C})) = 0$, then

$$R_k^k(\mathcal{C}) \leq |\lambda| \leq R_i(\mathcal{C}) - R_i^k(\mathcal{C})$$

or

$$R_i(\mathcal{C}) - R_i^k(\mathcal{C}) \leq |\lambda| \leq R_k^k(\mathcal{C}).$$

(a) If $R_k^k(\mathcal{C}) \leq |\lambda| \leq R_i(\mathcal{C}) - R_i^k(\mathcal{C})$, one has

$$(|\lambda| - (R_i(\mathcal{C}) - R_i^k(\mathcal{C})))|\lambda| \leq 0 \leq R_i^k(\mathcal{C})R_k(\mathcal{C}).$$

Therefore, $\lambda \in \mathcal{L}_{i,k}(\mathcal{C})$.
(b) If $R_i(\mathcal{C}) - R_i^k(\mathcal{C}) \leq |\lambda| \leq R_k^k(\mathcal{C})$, one has $|\lambda| \leq R_k(\mathcal{C})$. It follows from (2.4) that

$$|\lambda| - (R_i(\mathcal{C}) - R_i^k(\mathcal{C})) \leq R_i^k(\mathcal{C}). \tag{2.5}$$

Furthermore,

$$(|\lambda| - (R_i(\mathcal{C}) - R_i^k(\mathcal{C})))|\lambda| \leq R_i^k(\mathcal{C})R_k(\mathcal{C}).$$

Therefore, $\lambda \in \mathcal{L}_{i,k}(\mathcal{C})$.
(ii) If $R_i^k(\mathcal{C})(R_k(\mathcal{C}) - R_k^k(\mathcal{C})) > 0$, then according to (2.3), we have

$$\frac{|\lambda| - (R_i(\mathcal{C}) - R_i^k(\mathcal{C}))}{R_i^k(\mathcal{C})} \frac{|\lambda| - R_k^k(\mathcal{C})}{R_k(\mathcal{C}) - R_k^k(\mathcal{C})} \leq 1. \tag{2.6}$$

From (2.5), we have

$$\frac{|\lambda| - (R_i(\mathcal{C}) - R_i^k(\mathcal{C}))}{R_i^k(\mathcal{C})} \leq 1. \tag{2.7}$$

(a) If $\frac{|\lambda| - R_k^k(\mathcal{C})}{R_k(\mathcal{C}) - R_k^k(\mathcal{C})} \leq 1$, then $|\lambda| \leq R_k(\mathcal{C})$. Furthermore, from (2.7), one has

$$(|\lambda| - (R_i(\mathcal{C}) - R_i^k(\mathcal{C})))|\lambda| \leq R_i^k(\mathcal{C})R_k(\mathcal{C}).$$

Thus, $\lambda \in \mathcal{L}_{i,k}(\mathcal{C})$.
(b) If $\frac{|\lambda| - R_k^k(\mathcal{C})}{R_k(\mathcal{C}) - R_k^k(\mathcal{C})} > 1$, then $|\lambda| > R_k(\mathcal{C})$. Moreover,

$$\frac{|\lambda|}{R_k(\mathcal{C})} = \frac{|\lambda|}{(R_k(\mathcal{C}) - R_k^k(\mathcal{C})) + R_k^k(\mathcal{C})} \leq \frac{|\lambda| - R_k^k(\mathcal{C})}{R_k(\mathcal{C}) - R_k^k(\mathcal{C})}. \tag{2.8}$$

If $|\lambda| > R_i(\mathcal{C}) - R_i^k(\mathcal{C})$, it follows from (2.6) and (2.8) that

$$\frac{|\lambda| - (R_i(\mathcal{C}) - R_i^k(\mathcal{C}))}{R_i^k(\mathcal{C})} \frac{|\lambda|}{R_k(\mathcal{C})} \leq \frac{|\lambda| - (R_i(\mathcal{C}) - R_i^k(\mathcal{C}))}{R_i^k(\mathcal{C})} \frac{|\lambda| - R_k^k(\mathcal{C})}{R_k(\mathcal{C}) - R_k^k(\mathcal{C})} \leq 1,$$

which means that

$$(|\lambda| - (R_i(\mathcal{C}) - R_i^k(\mathcal{C})))|\lambda| \leq R_i^k(\mathcal{C})R_k(\mathcal{C}).$$

Consequently, $\lambda \in \mathcal{L}_{i,k}(\mathcal{C})$.
If $|\lambda| \leq R_i(\mathcal{C}) - R_i^k(\mathcal{C})$, we have

$$(|\lambda| - (R_i(\mathcal{C}) - R_i^k(\mathcal{C})))|\lambda| \leq 0 \leq R_i^k(\mathcal{C})R_k(\mathcal{C}).$$

Thus, $\lambda \in \mathcal{L}_{i,k}(\mathcal{C})$.
In summary, we obtain $\lambda \in \bigcap_{k\in[m],k\neq i}\mathcal{L}_{i,k}(\mathcal{C}) \subseteq \mathcal{L}(\mathcal{C})$,
that is,

$$\sigma(\mathcal{C}) \subseteq \mathcal{H}(\mathcal{C}) \subseteq \mathcal{L}(\mathcal{C}) \subseteq \Gamma(\mathcal{C}).$$

□

Example 2.4. Consider the following fourth-order partially symmetric tensor

$$c_{ijkl} = \begin{cases} c_{1111} = -1, c_{1112} = 2, c_{1131} = 3, c_{1121} = -1, c_{1211} = 2, c_{1221} = 1, c_{1122} = 1, \\ c_{2111} = -1, c_{2211} = 1, c_{2112} = 1, c_{2131} = -2, c_{2222} = 2, \\ c_{3111} = 3, c_{3232} = -1, c_{3131} = -2, \\ c_{ijkl} = 0, \ otherwise. \end{cases}.$$

By computation, we obtain that the corresponding M-eigenvalue is -0.8805. From Theorem 1.1, we obtain

$$\Gamma(\mathcal{C}) = \bigcup_{i\in[m]} \Gamma_i(\mathcal{C}) = \{\lambda \in C : |\lambda| \le 11\}.$$

From Theorem 1.2, we obtain

$$\begin{aligned}\mathcal{L}(\mathcal{C}) &= \bigcup_{i\in[m]} \bigcap_{k\in[m],k\neq i} \mathcal{L}_{i,k}(\mathcal{C}) \\ &= \left\{\lambda \in C : |\lambda| \le 4 + \sqrt{34}\right\},\end{aligned}$$

where

$$\mathcal{L}_{1,2}(\mathcal{C}) = \left\{\lambda \in C : |\lambda| \le 4 + \sqrt{37}\right\},\ \mathcal{L}_{1,3}(\mathcal{C}) = \left\{\lambda \in C : |\lambda| \le 4 + \sqrt{34}\right\},$$

$$\mathcal{L}_{2,1}(\mathcal{C}) = \left\{\lambda \in C : |\lambda| \le \frac{7 + \sqrt{181}}{2}\right\},\ \mathcal{L}_{2,3}(\mathcal{C}) = \left\{\lambda \in C : |\lambda| \le \frac{5 + \sqrt{73}}{2}\right\},$$

$$\mathcal{L}_{3,1}(\mathcal{C}) = \left\{\lambda \in C : |\lambda| \le \frac{3 + \sqrt{141}}{2}\right\},\ \mathcal{L}_{3,2}(\mathcal{C}) = \left\{\lambda \in C : |\lambda| \le 2 + 3\sqrt{2}\right\}.$$

From Theorem 2.2, we obtain

$$\begin{aligned}\mathcal{H}(\mathcal{C}) &= \left(\bigcup_{i\in[m]} \bigcap_{k\in[m],k\neq i} \widehat{\mathcal{H}}_{i,k}(\mathcal{C})\right) \bigcup \left(\bigcup_{i\in[m]} \bigcap_{k\in[m],k\neq i} (\overline{\mathcal{H}}_{i,k}(\mathcal{C}) \bigcap \Gamma_i(\mathcal{C}))\right) \\ &= \left\{\lambda \in C : |\lambda| \le \frac{11 + \sqrt{61}}{2}\right\},\end{aligned}$$

where

$$\widehat{\mathcal{H}}_{1,2}(\mathcal{C}) = \{\lambda \in C : |\lambda| < 2\},\ \widehat{\mathcal{H}}_{1,3}(\mathcal{C}) = \{\lambda \in C : |\lambda| < 3\},$$

$$\widehat{\mathcal{H}}_{2,1}(\mathcal{C}) = \{\lambda \in C : |\lambda| < 4\},\ \widehat{\mathcal{H}}_{2,3}(\mathcal{C}) = \{\lambda \in C : |\lambda| < 3\},$$

$$\widehat{\mathcal{H}}_{3,1}(\mathcal{C}) = \{\lambda \in C : |\lambda| < 3\},\ \widehat{\mathcal{H}}_{3,2}(\mathcal{C}) = \{\lambda \in C : |\lambda| < 2\},$$

$$\overline{\mathcal{H}}_{1,2}(\mathcal{C}) = \left\{\lambda \in C : 5 - 2\sqrt{6} \le \lambda \le 5 + 2\sqrt{6}\right\},$$

$$\overline{\mathcal{H}}_{1,3}(\mathcal{C}) = \left\{\lambda \in C : \frac{11 - \sqrt{61}}{2} \le \lambda \le \frac{11 + \sqrt{61}}{2}\right\},$$

$$\overline{\mathcal{H}}_{2,1}(\mathcal{C}) = \left\{ \lambda \in C : \frac{9-\sqrt{73}}{2} \leq \lambda \leq \frac{9+\sqrt{73}}{2} \right\},$$

$$\overline{\mathcal{H}}_{2,3}(\mathcal{C}) = \left\{ \lambda \in C : 4-\sqrt{7} \leq \lambda \leq 4+\sqrt{7} \right\},$$

$$\overline{\mathcal{H}}_{3,1}(\mathcal{C}) = \{\lambda \in C : 0 \leq \lambda \leq 4+\sqrt{19}\},\ \overline{\mathcal{H}}_{3,2}(\mathcal{C}) = \{\lambda \in C : 2 \leq \lambda \leq 6\},$$

$$\Gamma_1(\mathcal{C}) = \{\lambda \in C : |\lambda| \leq 11\},\ \Gamma_2(\mathcal{C}) = \{\lambda \in C : |\lambda| \leq 7\},\ \Gamma_3(\mathcal{C}) = \{\lambda \in C : |\lambda| \leq 6\}.$$

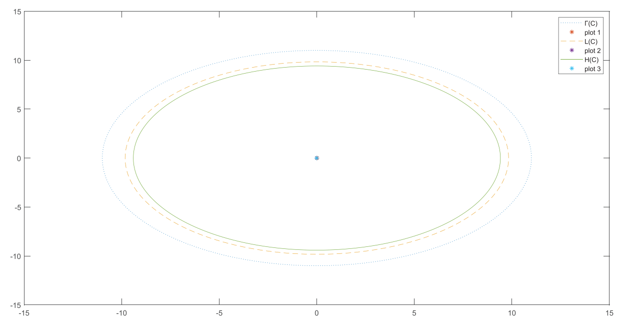

Figure 1: The comparisons of $\Gamma(\mathcal{C}),\ \mathcal{L}(\mathcal{C})$ *and* $\mathcal{H}(\mathcal{C})$

Remark 2.5. Example 2.4 and Figure 1 show the inclusion sets of $\Gamma(\mathcal{C})$, $\mathcal{L}(\mathcal{C})$ and $\mathcal{H}(\mathcal{C})$, where $\Gamma(\mathcal{C})$, $\mathcal{L}(\mathcal{C})$ and $\mathcal{H}(\mathcal{C})$ are represented by blue, yellow and green boundary, respectively, and the exact M-eigenvalues are plotted by $*$.

3 A Sharper Upper Bound on the Largest M-Eigenvalue of Nonnegative Fourth-Order Partially Symmetric Tensors

In this section, as an application of the sets in Theorem 2.2, we give a sharper upper bounds for M-spectral radius of nonnegative fourth-order partially symmetric tensors, which generalize the results of [1,12]. Now, we are in a position to recall some fundamental results of nonnegative tensors.

Lemma 3.1 ([1])**.** *Let $\mathcal{A}$ be an m-order and n-dimensional nonnegative tensor. Then*

$$\rho(\mathcal{A}) \leq \max_{i\in N} \sqrt{n} R_i(\mathcal{A}).$$

Lemma 3.2 ([12])**.** *Let $\mathcal{A}$ be an m-order and n-dimensional nonnegative tensor. Then*

$$\rho(\mathcal{A}) \leq \max_{i\in N} R_i(\mathcal{A}).$$

Lemma 3.3 ([3])**.** *The M-spectral radius of any nonnegative partially symmetric tensor is exactly its greatest M-eigenvalue. Furthermore, there is a pair of nonnegative M-eigenvectors corresponding to the M-spectral radius.*

Lemma 3.4 ([16]). *Suppose the tensor $\mathcal{C}$ is a nonnegative fourth-order partially symmetric tensor. Then*

$$\rho(\mathcal{C}) \le \max_{i\in[m]} \min_{k\in[m],\ k\neq i} \frac{1}{2}\left[R_i(\mathcal{C}) - R_i^k(\mathcal{C}) + \sqrt{(R_i(\mathcal{C}) - R_i^k(\mathcal{C}))^2 + 4R_i^k(\mathcal{C})R_k(\mathcal{C})}\right].$$

From Theorem 2.2, we can find a sharper bound for the largest M-eigenvalue of nonnegative fourth-order partially symmetric tensors.

Theorem 3.5. *Suppose the tensor $\mathcal{C}$ is a nonnegative fourth-order partially symmetric tensor. Then*

$$\rho(\mathcal{C}) \le \Omega_{max} = \max\{\rho_1(\mathcal{C}), \rho_2(\mathcal{C})\},$$

where

$$\rho_1(\mathcal{C}) = \max_{i\in[m]} \min_{k\in[m],\ k\neq i} \min\{R_i(\mathcal{C}) - R_i^k(\mathcal{C}), R_k^k(\mathcal{C})\},$$

$$\rho_2(\mathcal{C}) = \max_{i\in[m]} \min_{k\in[m],\ k\neq i} \min\{R_i(\mathcal{C}), \rho_3(\mathcal{C})\}$$

and

$$\rho_3(\mathcal{C}) = \frac{1}{2}\Big[R_i(\mathcal{C}) - R_i^k(\mathcal{C}) + R_k^k(\mathcal{C}) + \sqrt{(R_i(\mathcal{C}) - R_i^k(\mathcal{C}) + R_k^k(\mathcal{C}))^2 + 4(R_i^k(\mathcal{C})R_k(\mathcal{C}) - R_i(\mathcal{C})R_k^k(\mathcal{C}))}\Big].$$

Proof. Following Lemma 3.3, we obtain that $\rho(\mathcal{C})$ is the largest M-eigenvalue of $\mathcal{C}$. By Theorem 2.2, one has

$$\rho(\mathcal{C}) \in \bigcup_{i\in[m]} \bigcap_{k\in[m],k\neq i} \widehat{\mathcal{H}}_{i,k}(\mathcal{C})$$

or

$$\rho(\mathcal{C}) \in \bigcup_{i\in[m]} \bigcap_{k\in[m],k\neq i} (\overline{\mathcal{H}}_{i,k}(\mathcal{C}) \bigcap \Gamma_i(\mathcal{C})).$$

If $\rho(\mathcal{C}) \in \bigcup_{i\in[m]} \bigcap_{k\in[m],k\neq i} \widehat{\mathcal{H}}_{i,k}(\mathcal{C})$, then there exists an index $i \in [m]$ such that for $\forall\ k \in [m],\ k \neq i$,

$$\rho(\mathcal{C}) < R_i(\mathcal{C}) - R_i^k(\mathcal{C})$$

and

$$\rho(\mathcal{C}) < R_k^k(\mathcal{C}).$$

Then

$$\rho(\mathcal{C}) \le \min_{k\in[m],\ k\neq i} \min\{R_i(\mathcal{C}) - R_i^k(\mathcal{C}), R_k^k(\mathcal{C})\}.$$

Moreover, we obtain

$$\rho(\mathcal{C}) \le \max_{i\in[m]} \min_{k\in[m],\ k\neq i} \min\{R_i(\mathcal{C}) - R_i^k(\mathcal{C}), R_k^k(\mathcal{C})\}.$$

If $\rho(\mathcal{C}) \in \bigcup_{i\in[m]} \bigcap_{k\in[m],k\neq i} (\overline{\mathcal{H}}_{i,k}(\mathcal{C}) \bigcap \Gamma_i(\mathcal{C}))$, then there exists an index $i \in [m]$ such that for $\forall\ k \in [m],\ k \neq i$,

$$\rho(\mathcal{C}) \le R_i(\mathcal{C}) \tag{3.1}$$

and

$$(\rho(\mathcal{C}) - (R_i(\mathcal{C}) - R_i^k(\mathcal{C})))(\rho(\mathcal{C}) - R_k^k(\mathcal{C})) \le R_i^k(\mathcal{C})(R_k(\mathcal{C}) - R_k^k(\mathcal{C})). \tag{3.2}$$

From (3.2), it yields that

$$\rho(\mathcal{C}) \leq \rho_3(\mathcal{C}), \tag{3.3}$$

where

$$\rho_3(\mathcal{C}) = \frac{1}{2}\Big[R_i(\mathcal{C}) - R_i^k(\mathcal{C}) + R_k^k(\mathcal{C}) + \sqrt{(R_i(\mathcal{C}) - R_i^k(\mathcal{C}) + R_k^k(\mathcal{C}))^2 + 4(R_i^k(\mathcal{C})R_k(\mathcal{C}) - R_i(\mathcal{C})R_k^k(\mathcal{C}))}\Big].$$

From (3.1) and (3.3), we deduce

$$\rho(\mathcal{C}) \leq \min_{k\in[m],\ k\neq i} \min\{R_i(\mathcal{C}), \rho_3(\mathcal{C})\} \leq \max_{i\in[m]} \min_{k\in[m],\ k\neq i} \min\{R_i(\mathcal{C}), \rho_3(\mathcal{C})\},$$

and the desired result follows. □

In the following, we give two examples. According to the theorems in this paper, we can obtain the following results.

Example 3.6 ([1])**.** Consider 4 order 2 dimensional tensor $\mathcal{C} = (c_{ijkl})$ defined by

$$c_{ijkl} = \begin{cases} c_{1111} = \frac{1}{2}, c_{2222} = 3, \\ c_{ijkl} = \frac{1}{3}, otherwise. \end{cases}$$

It is easy to compute that $\rho(\mathcal{C}) = 3.1122$. And by Lemmas 3.1, 3.2, 3.4, we have $\rho(\mathcal{C}) \leq 7.5432$, $\rho(\mathcal{C}) \leq 5.3333$, $\rho(\mathcal{C}) \leq 4.7889$, respectively. Since Theorem 4.5, Theorem 4.6 and Theorem 4.7 of [15] are equivalent when $n = 2$, we have $\rho(\mathcal{C}) \leq 5.1822$ by these theorems. On the other hand, by Theorem 3.5, we have $\rho(\mathcal{C}) \leq 4.5776$.

Example 3.7 ([11])**.** Consider 4 order 2 dimensional tensor $\mathcal{C} = (c_{ijkl})$ defined by

$$c_{ijkl} = \begin{cases} c_{1111} = 1, c_{1112} = 2, c_{1121} = 2, c_{1212} = 3, \\ c_{1222} = 5, c_{1211} = 2, c_{1122} = 4, c_{1221} = 4, \\ c_{2111} = 2, c_{2112} = 4, c_{2121} = 3, c_{2122} = 5, \\ c_{2211} = 4, c_{2212} = 5, c_{2221} = 5, c_{2222} = 6. \end{cases}$$

According to the Theorem 7 of [11], it is easy to compute the corresponding M-eigenvalues, which are given as

$$0.0710,\ 15.2091,\ 0.3437,\ 0.1242,$$
$$-1.2765,\ -1.2765,\ 0.2765,\ 0.2765.$$

Then, $\rho(\mathcal{C}) = 15.2091$. On the other hand, by Lemmas 3.1, 3.2, 3.4 and Theorem 3.5, we have $\rho(\mathcal{C}) \leq 48.0833$, $\rho(\mathcal{C}) \leq 34$, $\rho(\mathcal{C}) \leq 30.3626$, $\rho(\mathcal{C}) \leq 29.4765$, respectively.

From the above examples, we can see the bound of Theorem 3.5 is sharper than the results in the corresponding references.

Conclusion

In this paper, we give an M-eigenvalue inclusion set for fourth-order partially symmetric tensors by choosing different components of an eigenvector. As an application of this result, we discuss a new upper bound for the M-spectral radius and prove that the new upper bound is sharper than the existing upper bounds in the references [1, 12, 15]. Finally, numerical examples are proposed to verify the theoretical results.

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