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ON GENERALIZED POLYNOMIAL VARIATIONAL INEQUALITY PART 1: EXISTENCE OF SOLUTIONS*

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Abstract: In this paper, we consider the generalized polynomial variational inequality (GPVI). By using the proposed regular condition, we establish a sufficient condition for the existence of solutions to the GPVI. The compactness and uniqueness of the solution set of the GPVI are investigated. We present an extension of Hartman-Stampacchia's theorem for the GPVI and use this result to propose sufficient conditions for the nonemptiness and compactness of the solution set of a class of the GPVI. Several illustrating examples are presented to compare obtained results with existing ones in [SIAM J. Control Optim. 33, (1995), 168–184] and others. Our main tools are the theory related to exceptional family of elements, structure of tensors, and recession cone.

Key words: *generalized polynomial variational inequality, generalized polynomial complementarity problem, existence (uniqueness) of solutions, exceptional family of elements*

Mathematics Subject Classification: *90C33, 15A69*

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1 Introduction

Variational inequalities (VI) theory, which was introduced by Stampacchia [20], is an important branch of the mathematical sciences. The VI provides us with a simple, natural general and unified framework to study a wide class of problems arising in pure and applied science. The VI has been studied extensively due to its successful applications in many fields including economic equilibrium, control theory, game theory, transportation science, and operations research.

Let $F, G : \mathbb{R}^n \to \mathbb{R}^n$ be two given polynomial maps and let K be a nonempty closed convex subset of $\mathbb{R}^n$. The so-called *generalized polynomial variational inequality* (GPVI) *defined by (F,G,K)*, denoted by GPVI(F, G, K), is to find a vector $x \in \mathbb{R}^n$ such that

$$G(x) \in K \text{ and } \langle F(x), y - G(x)\rangle \geq 0 \ \ \forall y \in K,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in real Euclidean space.

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In the case where G is the identity map, $G \equiv I\mathbb{R}^n$, GPVI$(F, I\mathbb{R}^n, K)$ reduces to *polynomial variational inequality* (PVI) which is denoted by PVI(F, K),

$$\text{Find } x \in K \text{ and } \langle F(x), y - x\rangle \geq 0 \;\; \forall y \in K,$$

(see, for instance, [2, 5, 6]). If F is a affine mapping, then PVI(F, K) reduces to the *generalized affine variational inequality* (GAVI). The GAVI and quadratic programming problems have been studied in detail in [5, 9, 10, 11, 12, 21, 22]. For the case where K is a cone, GPVI(F, G, K) is the following *generalized polynomial complementarity problem* (GPCP) (see [7, 19]), which denoted by GPCP(F, G, K),

$$\text{Find } x \in \mathbb{R}^n \text{ such that } G(x) \in K, \;\; F(x) \in K^* \text{ and } \langle F(x), G(x)\rangle = 0,$$

where K^* is the dual cone of K and GPVI$(F, id_{\mathbb{R}^n}, K)$ is a *polynomial complementarity problem* (see [1] and references therein).

Note that generalized polynomial variational inequality (GPVI) is a special case of the following *generalized variational inequality* (GVI): Find a vector $x \in \mathbb{R}^n$ such that

$$G(x) \in K \text{ and } \langle F(x), y - G(x)\rangle \geq 0 \;\; \forall y \in K,$$

where $F, G : \mathbb{R}^n \to \mathbb{R}^n$ are two given continuous maps and K is a nonempty closed convex subset of $\mathbb{R}^n$. This problem was firstly proposed by Noor [14]. The GVI has received considerable attention in recent three decades. Noor [15] also showed that the minimum of a differentiable hg-convex function on the hg-convex set K in $\mathbb{R}^n$ can be characterized by the GVI. Moreover, the problem GVI(F, G, K) is equivalent to a class of the *fixed point problems*: Finding $\bar{x}$ such that $\bar{x} = F(\bar{x})$, where $F(z) = z - G(z) + P_K(G(z) - F(z))$ and P_K is the projection of $\mathbb{R}^n$ onto K.

One of the central problems in the GVI theory is the existence of a solution. Research on the existence of a solution to the GVI has played a very important role in theory, algorithms, and practical applications of the problem. Since Noor [14] introduced the GVI problem, many authors have developed many numerical methods for the GVI problems. Under the assumption that functions F and G are locally Lipschitz continuous and G is injective, Pang and Yao [16] provided a sufficient condition for the existence of solution to GVI(F, G, K). Recently, Wang et al. [23] proposed a sufficient condition for the uniqueness of solutions of GPVI(F, G, K) by making use of properties of the involved polynomials. Some sufficient conditions for existence of GPCP(F, G, K) have been proposed in [7, 19]. However, some previous assumptions for the existence of GVI(F, G, K) in the above papers are rather strong when they are applied to GPVI(F, G, K); for instance, the map G must be assumed either injective [16] or surjective [15]. For the special cases where K is a compact set, establishing a Hartman-Stampacchia type theorem is very necessary to study algorithms.

In this paper, we propose a new regular condition for the GPVI. This concept is different from existing ones. By using the proposed regular condition, we establish sufficient conditions for the existence of the GPVI with G being an arbitrary polynomial map. The obtained results develop and complete some aspects of the corresponding ones in [2, 7, 16]. Our main tools are the theory related to exceptional family of elements, structure of tensors, and recession cone. We expect that the regular condition presented in the paper will be useful in the study of the GPVI.

The outline of the paper is as follows. Section 2 gives some preliminaries. In Subsection 3.1, we propose a new regular condition for the GPVI. By using this regular condition, we present sufficient conditions for the solution existence of the GPVI in Subsection 3.2. In Subsection 3.3, we investigate the existence for the special case where K is a compact set and applications.

2 Preliminaries

Throughout this paper, for any positive integer n, $\mathbb{R}^n$ denotes a real Euclidean space equipped with the scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. A tensor is a natural extension of a matrix (see [17]). For any given positive integers m and n with $m, n \geq 2$, we call $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$, where $a_{i_1 i_2 \dots i_m} \in \mathbb{R}$ for $i_j \in \{1, 2, \dots, n\}$ and $j \in \{1, 2, \dots, m\}$, an m-th order n-dimensional real square tensor; and denote the space of m-th order n-dimensional real square tensors by $\mathbb{R}^{[m,n]}$.

For any $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m,n]}$ and $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, $\mathcal{A}x^{m-1}$ is an n-dimensional vector whose ith component is given by

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, i_3, \dots, i_m = 1}^{n} a_{i i_2 i_3 \dots i_m} x_{i_2} x_{i_3} \dots x_{i_m}$$

for every $i \in \{1, 2, \dots, n\}$ and $\mathcal{A}x^m$ is a homogeneous polynomial of degree m, defined by

$$\mathcal{A}x^m := x^T (\mathcal{A}x^{m-1}) = \sum_{i_1, i_2, \dots, i_m = 1}^{n} a_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \dots x_{i_m}.$$

Denote by $\mathcal{P}^{[r,n]}$ the set of polynomial maps $H : \mathbb{R}^n \to \mathbb{R}^n$ such that there exists $\mathcal{C}^{(i)} \in \mathbb{R}^{[i,n]}$ for every $i \in \{2, \dots, r\}$ and $c \in \mathbb{R}^n$ satisfying

$$H(x) := \sum_{i=2}^{r} \mathcal{C}^{(i)} x^{i-1} + c.$$

For each $\mathcal{A}^{(r)} \in \mathbb{R}^{[r,n]}$, denote by

$$\Lambda(\mathcal{A}^{(r)}) := \{\lambda \in \mathbb{R} : \exists x \in \mathbb{R}^n \text{ such that } \|x\| = 1, \mathcal{A}^{(r)} x^{(r-1)} = \lambda x\}$$

the set of Z-eigenvalues of $\mathcal{A}^{(r)}$ (see [17, p. 5]).

Let

$$F(x) := \sum_{r=2}^{m} \mathcal{A}^{(r)} x^{r-1} + a \tag{2.1}$$

and

$$G(x) := \sum_{p=2}^{l} \mathcal{B}^{(p)} x^{p-1} + b, \tag{2.2}$$

where $\mathcal{A}^{(r)} \in \mathbb{R}^{[r,n]}$, $\mathcal{B}^{(p)} \in \mathbb{R}^{[p,n]}$ for every $r \in \{2, \dots, m\}$, $p \in \{2, \dots, l\}$ and $a, b \in \mathbb{R}^n$. Denote

$$F^{\infty}(x) := \mathcal{A}^{(m)} x^{m-1}$$

and

$$G^{\infty}(x) := \mathcal{B}^{(l)} x^{l-1}.$$

Let

$$\mathcal{A} := (\mathcal{A}^{(m)}, \ldots, \mathcal{A}^{(2)}, a) \in \mathbb{T}^{[m,n]} := \mathbb{R}^{[m,n]} \times \ldots \times \mathbb{R}^{[2,n]} \times \mathbb{R}^n;$$

$$\mathcal{B} := (\mathcal{B}^{(l)}, \ldots, \mathcal{B}^{(2)}, b) \in \mathbb{T}^{[l,n]} := \mathbb{R}^{[l,n]} \times \ldots \times \mathbb{R}^{[2,n]} \times \mathbb{R}^n.$$

The set of solutions of generalized variational inequality (GPVI) defined by *(F,G,K)* is denoted by $\mathrm{SOL}(F,G,K)$. We also denote by $\mathrm{SOL}(\mathcal{A}^{(m)}, \mathcal{B}^{(l)}, K^{\infty})$ the solution set $\mathrm{SOL}(F^{\infty}, G^{\infty}, K^{\infty})$ of generalized variational inequality (GPVI) defined by $(F^{\infty}, G^{\infty}, K^{\infty})$.

Tensor $\mathcal{A}^{(r)}$ is called positive definite on a set C if $\mathcal{A}^{(r)} x^r > 0$ for every $x \in C \setminus \{0\}$. We say that $\mathcal{A}^{(r)}$ is positive semi-definite on a set C if $\mathcal{A}^{(r)} x^r \geq 0$ for every $x \in C$.

Let $C \subset \mathbb{R}^n$ be a cone, denote

$$C^* := \{y \in \mathbb{R}^n : h^T y \geq 0 \ \ \forall h \in C\}.$$

Let $S \subset \mathbb{R}^n$ be a nonempty closed convex set. The recession cone of S is defined [18, p. 61] by

$$S^{\infty} := \{v \in \mathbb{R}^n : x + tv \in S \ \forall x \in S \ \forall t \geq 0\}.$$

It follows from the above definition that $S + S^{\infty} \subset S$. Clearly, $S \subset S + S^{\infty}$ since $0 \in S^{\infty}$. Thus, $S = S + S^{\infty}$. According to [18, Theorem 8.3],

$$S^{\infty} := \{v \in \mathbb{R}^n : \exists x \in S \text{ such that } x + tv \in S \ \forall t \geq 0\}.$$

Let F, G be two continuous functions. A set of points $\{x^k\} \subset \mathbb{R}^n$ is called an exceptional family of elements (see [23, 25]) for the pair (F,G) with respect to $\bar{x} \in \mathbb{R}^n$ if $\|x^k\| \to \infty$ as $k \to \infty$; and for each x^k, there exists a scalar $\alpha^k > 0$ such that $z^k := \alpha^k (x^k - \bar{x}) + G(x^k) \in K$ and

$$-\alpha^k (x^k - \bar{x}) - F(x^k) \in \mathcal{N}_K(z^k),$$

where $\mathcal{N}_K(z^k)$ is the normal cone of K at z^k.

The following is useful in our proofs.

Proposition 2.1 (see [23, 25])**.** *For two continuous mappings $F, G : \mathbb{R}^n \to \mathbb{R}^n$ and a nonempty, closed and convex set $K \subset \mathbb{R}^n$, there exists either a solution of* $\mathrm{GVI}(F,G,K)$ *or an exceptional family of elements with respect to any given $\bar{x} \in \mathbb{R}^n$ for the pair (F,G).*

We call F is copositive with respect to G on K if

$$\langle F(x), G(x) \rangle \geq 0$$

for every $x \in \mathbb{R}^n$ satisfying $G(x) \in K$.

We say that F is strictly monotone with respect to G on K if

$$\langle F(x) - F(y), G(x) - G(y) \rangle > 0$$

for every $x, y \in \mathbb{R}^n$ satisfying $x \neq y$ and $G(x), G(y) \in K$.

A well-known result on the existence and uniqueness of solutions for the GVI is proposed by Pang and Yao [16, Proposition 3.9] as follows.

Proposition 2.2. *Let K be a nonempty, closed and convex subset of $\mathbb{R}^n$, and let F and G be two continuous functions from $\mathbb{R}^n$ into itself with G being injective. Suppose that:*

(i) *there exists a vector $u \in G^{-1}(K)$ and positive scalars α and l such that*

$$\|G(x) - G(u) \leq \|x - u\|$$

holds for all $x \in G^{-1}(K)$ with $\|x\| \geq \alpha$;

(ii) *F is strongly monotone with respect to G on K , i.e., there is a scalar $c > 0$ such that*

$$\langle F(x) - F(y), G(x) - G(y)\rangle \geq c\|x - y\|^2$$

holds for all $G(x), G(y) \in K$ with $x \neq y$.

Then, GVI(F, G, K) *has a unique solution.*

3 Main existence results

In this section, we present sufficient conditions for existence of GPVI(F, G, K). A new regular condition is proposed in Subsection 3.1. The existence results are presented in Subsections 3.2 and 3.3.

3.1 A regular condition

For each positive integer pair (p, q), denote

$$\delta_{p,q} = \begin{cases} 1 & \text{if } p \geq q, \\ 0 & \text{if } p < q. \end{cases}$$

The following new concept plays a key role in proving the main results.

Definition 3.1. One says that GPVI(F, G, K) is *regular* if

$$\mathrm{SOL}(F^\infty + \delta_{l,m}\rho\, I,\ G^\infty + \delta_{m,l}\rho\, I,\ K^\infty) = \{0\}\ \ \forall \rho \geq 0, \tag{3.1}$$

where $I := id_{\mathbb{R}^n}$.

By the definition of the GPVI, we obtain that GPVI(F, G, K) is regular if and only if there exists no $(x, \rho) \in (K^\infty \setminus \{0\}) \times \mathbb{R}_+$ such that

$$\begin{aligned} &\mathcal{B}^{(l)}x^{l-1} + \delta_{m,l}\rho x \in K^\infty, \\ &\qquad \mathcal{A}^{(m)}x^{m-1} + \delta_{l,m}\rho x \in (K^\infty)^*,\ \left\langle \mathcal{A}^{(m)}x^{m-1} + \delta_{l,m}\rho x, \mathcal{B}^{(l)}x^{l-1} + \delta_{m,l}\rho x\right\rangle = 0. \end{aligned} \tag{3.2}$$

Remark 3.2. To characterize the existence of solutions for the problem GPCP(F, G, C), with C being a cone in $\mathbb{R}^n$, Ling et al. [7] presented the following concepts: The pair $(\mathcal{A}^{(m)}, \mathcal{B}^{(l)})$ is called *ER^C-tensor pair* if there exists no $(x, v, t) \in (C \setminus \{0\}) \times \mathbb{R}_+ \times \mathbb{R}_+$ such that

$$\mathcal{A}^{(m)}x^{m-1} + vx \in C^*,\ \mathcal{B}^{(l)}x^{l-1} + tx \in C,\ \left\langle \mathcal{A}^{(m)}x^{m-1} + vx, \mathcal{B}^{(l)}x^{l-1} + tx\right\rangle = 0. \tag{3.3}$$

Then, (3.3) is equivalent to

$$\mathrm{SOL}(F^\infty + vI, G^\infty + tI, C) = \{0\} \ \ \forall (v,t) \in \mathbb{R}_+ \times \mathbb{R}_+. \tag{3.4}$$

Clearly, when $C = K^\infty$, the regular condition (3.2) is weaker than (3.3). The following example illustrates the above relation.

Example 3.3. Consider the problem GPVI(F, G, K) with $n = 2$, $m = l = 2$, and $K := \{(z, 0) \in \mathbb{R}^2 : z \geq 0\}$. For each $x \in \mathbb{R}$, let $G(x) = (x_2 - 2, -2x_1 - 2x_2 + 7)$ and $F(x) = (x_2 - 1, -3x_1 + 5x_2 + 1)$. We obtain that $K^\infty = K$ and $(K^\infty)^* = \{(u_1, u_2) : u_1 \geq 0\}$. For each pair $(v, t) \in \mathbb{R}^2_+$, we have

$$G^\infty(x) + tx = (x_2 + tx_1, -2x_1 - 2x_2 + tx_2),$$

$$F^\infty(x) + vx = (x_2 + vx_1, -3x_1 + 5x_2 + vx_2).$$

Choose $(\bar v, \bar t) = (1, 0)$ and let any $\bar x = (\bar x_1, \bar x_2) \in \mathbb{R}^2$. Then, $\bar x \in \mathrm{SOL}(F^\infty + \bar v I, G^\infty + \bar t I, K^\infty)$ if and only if

$$\bar x_2 \geq 0, \ -2\bar x_1 - 2\bar x_2 = 0, \ \bar x_2 + \bar x_1 \geq 0, \ \bar x_2(\bar x_2 + \bar x_1) = 0.$$

We obtain that $\bar x_2 \geq 0$ and $\bar x_1 + \bar x_2 = 0$. Hence, $\mathrm{SOL}(F^\infty + \bar v I, G^\infty + \bar t I, K^\infty) \neq \{0\}$. This follows that (3.3) is not satisfied.

For any $\rho \geq 0$, let $\bar y = (\bar y_1, \bar y_2) \in \mathbb{R}^2$. Then, $\bar y \in \mathrm{SOL}(F^\infty + \rho I, G^\infty + \rho I, K^\infty)$ if and only if

$$\bar y_2 + \rho \bar y_1 \geq 0, \ -2\bar y_1 - 2\bar y_2 + \rho \bar y_2 = 0, \ (\bar y_2 + \rho \bar y_1)^2 = 0.$$

It implies that $\bar y = 0$. Hence, $\mathrm{SOL}(F^\infty + \rho I, G^\infty + \rho I, K^\infty) = \{0\}$ for every $\rho \geq 0$ and the condition (3.2) is satisfied.

The following concepts are used in the main theorem. The pair $(\mathcal{A}^{(m)}, \mathcal{B}^{(l)})$ is said to be a R_0^C*-tensor pair* (see [7, 24]) if there exists no $x \in C \setminus \{0\}$ such that

$$\mathcal{B}^{(l)} x^{l-1} \in C, \ \mathcal{A}^{(m)} x^{m-1} \in C^*, \ \left\langle \mathcal{A}^{(m)} x^{m-1}, \mathcal{B}^{(l)} x^{l-1} \right\rangle = 0. \tag{3.5}$$

Then, (3.5) is equivalent to

$$\mathrm{SOL}(F^\infty, G^\infty, C) = \{0\}. \tag{3.6}$$

If $\mathrm{SOL}(\mathcal{A}^{(m)}, I, \mathbb{R}^n) = \{0\}$, then $\mathcal{A}^{(m)}$ is called R_0*-tensor*, which has been used to study the existence and stability for tensor variational inequality and tensor complementary problem (see [1, 2]).

3.2 Existence under the regular condition

The main result is presented in the following theorem.

Theorem 3.4. *Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set and* GPVI(F, G, K) *be regular. If one of the following conditions is satisfied:*

(i) *$m = l$;*

(ii) *$m > l$ and $\mathcal{A}^{(m)}$ is positive definite;*

(iii) $m < l$ *and* $\mathcal{B}^{(l)}$ *is positive definite,*

then $\mathrm{SOL}(F, G, K)$ *is a nonempty and compact set.*

Proof. Suppose, on the contrary, that $\mathrm{GPVI}(F, G, K)$ has no solution. Then, it follows from Proposition 2.1 that there exists an exceptional family of elements for the pair (F, G) with respect to $0 \in \mathbb{R}^n$, i.e., there exist $\{x^k\} \subset \mathbb{R}^n$ satisfying $\|x^k\| \to \infty$ as $k \to \infty$ and $\sigma^k > 0$ such that $z^k := \sigma^k x^k + G(x^k) \in K$ and

$$-\sigma^k x^k - F(x^k) \in \mathcal{N}_K(\sigma^k x^k + G(x^k)).$$

By the definition of the normal cone, we obtain that

$$\langle \sigma^k x^k + F(x^k), y - \sigma^k x^k - G(x^k)\rangle \geq 0 \ \ \forall y \in K, \tag{3.7}$$

that is,

$$\langle \sigma^k x^k, y - F(x^k) - G(x^k)\rangle + \langle F(x^k), y - G(x^k)\rangle - (\sigma^k)^2\|x^k\|^2 \geq 0 \tag{3.8}$$

for every $y \in K$. By the fact that $\|x^k\| \to \infty$ as $k \to \infty$, we may assume that $\|x^k\| > 0$ for all $k \to \infty$.

Case 1: $m \geq l$. Dividing both sides of the inequality (3.8) by $\|x^k\|^{m+l-2}$, we obtain that

$$\begin{aligned}&\frac{\sigma^k}{\|x^k\|^{l-2}}\left\langle \frac{x^k}{\|x^k\|}, \frac{y - F(x^k) - G(x^k)}{\|x^k\|^{m-1}}\right\rangle \\ &+\left\langle \frac{F(x^k)}{\|x^k\|^{m-1}}, \frac{y - G(x^k)}{\|x^k\|^{l-1}}\right\rangle - \frac{(\sigma^k)^2}{\|x^k\|^{m+l-4}} \geq 0.\end{aligned} \tag{3.9}$$

Without loss of generality, we may assume that $\frac{x^k}{\|x^k\|} \to \bar{h}$ for some $\bar{h} \in \mathbb{R}^n$ with $\|\bar{h}\| = 1$. Denote:

$$\rho^k := \frac{\sigma^k}{\|x^k\|^{l-2}},$$

$$u^k := \left\langle \frac{x^k}{\|x^k\|}, \frac{y - F(x^k) - G(x^k)}{\|x^k\|^{m-1}}\right\rangle,$$

and

$$v^k := \left\langle \frac{F(x^k)}{\|x^k\|^{m-1}}, \frac{y - G(x^k)}{\|x^k\|^{l-1}}\right\rangle.$$

From (3.9), we have

$$\rho^k u^k + v^k - \frac{(\rho^k)^2}{\|x^k\|^{m-l}} \geq 0. \tag{3.10}$$

We now show that $\{\rho^k\}$ is bounded. Indeed, suppose, on the contrary, that $\rho^k \to +\infty$ as $k \to +\infty$. Consider the following two cases:

Case 1.1: $m = l$. Then, we obtain that

$$\begin{aligned}\lim_{k\to\infty} u^k &= \lim_{k\to\infty}\left\langle \frac{x^k}{\|x^k\|}, \frac{y - \sum_{r=2}^m \mathcal{A}^{(r)}(x^k)^{r-1} - a - \sum_{p=2}^l \mathcal{B}^{(p)}(x^k)^{p-1} - b}{\|x^k\|^{m-1}}\right\rangle \\ &= -\big\langle \bar{h}, \big(\mathcal{A}^{(m)} + \mathcal{B}^{(l)}\big)\bar{h}^{m-1}\big\rangle \\ &= -\big(\mathcal{A}^{(m)} + \mathcal{B}^{(l)}\big)\bar{h}^m\end{aligned}$$

and

$$\lim_{k\to\infty} v^k = \lim_{k\to\infty} \Big\langle \frac{\sum_{r=2}^{m} \mathcal{A}^{(r)}(x^k)^{r-1} + a}{\|x^k\|^{m-1}}, \frac{y - \sum_{p=2}^{l} \mathcal{B}^{(p)}(x^k)^{p-1} - b}{\|x^k\|^{l-1}} \Big\rangle$$
$$= -\big\langle \mathcal{A}^{(m)}\bar{h}^{m-1}, \mathcal{B}^{(l)}\bar{h}^{m-1}\big\rangle.$$

Dividing both sides of the equality (3.10) by $(\rho^k)^2$ and letting $k \to \infty$ yields $-1 \geq 0$, a contradiction.

Case 1.2: $m > l$. From (3.10) it follows that

$$\rho^k u^k + v^k \geq 0. \tag{3.11}$$

Then,

$$\lim_{k\to\infty} u^k = \lim_{k\to\infty} \Big\langle \frac{x^k}{\|x^k\|}, \frac{y - \sum_{r=2}^{m} \mathcal{A}^{(r)}(x^k)^{r-1} - a - \sum_{p=2}^{l} \mathcal{B}^{(p)}(x^k)^{p-1} - b}{\|x^k\|^{m-1}} \Big\rangle$$
$$= -\big\langle \bar{h}, \mathcal{A}^{(m)}\bar{h}^{m-1}\big\rangle$$
$$= -\mathcal{A}^{(m)}\bar{h}^{m}$$

and

$$\lim_{k\to\infty} v^k = -\big\langle \mathcal{A}^{(m)}\bar{h}^{m-1}, \mathcal{B}^{(l)}\bar{h}^{l-1}\big\rangle.$$

By the assumption that $\mathcal{A}^{(m)}$ is positive definite, we have $\mathcal{A}^{(m)}\bar{h}^m > 0$. Then, we obtain

$$\rho^k u^k + v^k \to -\infty$$

as $k \to +\infty$, contrary to the inequality (3.11).

Therefore, $\{\rho^k\}$ is bounded. Without loss of generality, we may assume that $\rho^k \to \bar{\rho}$ for some $\bar{\rho} \in \mathbb{R}_+$. Applying [18, Theorem 8.2] to $z^k = \rho^k \|x^k\|^{l-2} x^k + G(x^k) \in K$ and $\frac{1}{\|x^k\|^{l-1}} \to 0$, we have

$$\frac{1}{\|x^k\|^{l-1}} z^k = \rho^k \frac{x^k}{\|x^k\|} + \frac{G(x^k)}{\|x^k\|^{l-1}} \to \bar{\rho}\bar{h} + \mathcal{B}^{(l)}\bar{h}^{l-1} \in K^\infty.$$

Fix $w \in K$. For every $h \in K^\infty$, we have $z := w + h\|x^k\|^{l-1} \in K$. From (3.7) it follows that

$$\langle \rho^k \|x^k\|^{l-2} x^k + F(x^k), z - \rho^k \|x^k\|^{l-2} x^k - G(x^k)\rangle \geq 0,$$

that is,

$$\langle \rho^k \|x^k\|^{l-2} x^k + F(x^k), w + h\|x^k\|^{l-1} - \rho^k \|x^k\|^{l-2} x^k - G(x^k)\rangle \geq 0.$$

Dividing both sides of last inequality by $\|x^k\|^{m+l-2}$ and letting $k \to +\infty$ yields:

$$\big\langle \mathcal{A}^{(m)}\bar{h}^{m-1}, h - \mathcal{B}^{(l)}\bar{h}^{l-1} - \bar{\rho}\bar{h}\big\rangle \geq 0 \ \text{ if } m > l$$

and

$$\big\langle \mathcal{A}^{(m)}\bar{h}^{m-1} + \bar{\rho}\bar{h}, h - \mathcal{B}^{(l)}\bar{h}^{l-1} - \bar{\rho}\bar{h}\big\rangle \geq 0 \ \text{ if } m = l.$$

These lead to $0 \neq \bar{h} \in \mathrm{SOL}(F^\infty + \delta_{l,m}\bar{\rho}\, I,\ G^\infty + \delta_{m,l}\bar{\rho}\, I,\ K^\infty)$, contrary to the assumption that $\mathrm{GPVI}(F, G, K)$ is regular.

Case 2: $m < l$. Dividing both sides of the inequality (3.8) by $\|x^k\|^{m+l-2}$, we obtain that

$$\frac{\sigma^k}{\|x^k\|^{m-2}}\left\langle \frac{x^k}{\|x^k\|}, \frac{y - F(x^k) - G(x^k)}{\|x^k\|^{l-1}}\right\rangle + \left\langle \frac{F(x^k)}{\|x^k\|^{m-1}}, \frac{y - G(x^k)}{\|x^k\|^{l-1}}\right\rangle - \frac{(\sigma^k)^2}{\|x^k\|^{m+l-4}} \geq 0.$$

It implies that

$$\frac{\sigma^k}{\|x^k\|^{m-2}}\left\langle \frac{x^k}{\|x^k\|}, \frac{y - F(x^k) - G(x^k)}{\|x^k\|^{l-1}}\right\rangle + \left\langle \frac{F(x^k)}{\|x^k\|^{m-1}}, \frac{y - G(x^k)}{\|x^k\|^{l-1}}\right\rangle \geq 0. \tag{3.12}$$

Denote:

$$r^k := \frac{\sigma^k}{\|x^k\|^{m-2}},$$

$$w^k := \left\langle \frac{x^k}{\|x^k\|}, \frac{y - F(x^k) - G(x^k)}{\|x^k\|^{l-1}}\right\rangle,$$

and

$$v^k := \left\langle \frac{F(x^k)}{\|x^k\|^{m-1}}, \frac{y - G(x^k)}{\|x^k\|^{l-1}}\right\rangle.$$

From (3.12) it follws that

$$r^k w^k + v^k \geq 0. \tag{3.13}$$

We have

$$\begin{aligned}\lim_{k\to\infty} w^k &= \lim_{k\to\infty}\left\langle \frac{x^k}{\|x^k\|}, \frac{y - \sum_{r=2}^m \mathcal{A}^{(r)}(x^k)^{r-1} - a - \sum_{p=2}^l \mathcal{B}^{(p)}(x^k)^{p-1} - b}{\|x^k\|^{l-1}}\right\rangle \\ &= -\langle \bar{h}, \mathcal{B}^{(l)}\bar{h}^{l-1}\rangle \\ &= -\mathcal{B}^{(l)}\bar{h}^l\end{aligned}$$

and $\lim_{k\to\infty} v^k = -\langle \mathcal{A}^{(m)}\bar{h}^{m-1}, \mathcal{B}^{(l)}\bar{h}^{l-1}\rangle$. From the assumption that $\mathcal{B}^{(l)}$ is positive definite it follows that $\mathcal{B}^{(l)}\bar{h}^l > 0$. If $\{r^k\}$ is unbounded, then $r^k w^k + v^k \to -\infty$ as $k \to +\infty$, contrary to the inequality (3.13). Hence, $\{r^k\}$ is bounded. Without loss of generality, we may assume that $r^k \to \bar{r}$ for some $\bar{r} \in \mathbb{R}_+$. For every $h \in K^\infty$, for some $w \in K$, we have $\tilde{z} := w + h\|x^k\|^{l-1} \in K$. Substitute $z = \tilde{z}$ into the inequality (3.7) yields

$$\langle r^k\|x^k\|^{m-2}x^k + F(x^k), w + h\|x^k\|^{l-1} - r^k\|x^k\|^{m-2}x^k - G(x^k)\rangle \geq 0.$$

Dividing both sides of last inequality by $\|x^k\|^{m+l-2}$ and letting $k \to +\infty$ gives

$$\langle \mathcal{A}^{(m)}\bar{h}^{m-1} + \bar{r}\bar{h}, h - \mathcal{B}^{(l)}\bar{h}^{l-1}\rangle \geq 0.$$

That is, $0 \neq \bar{h} \in \mathrm{SOL}(F^\infty + \delta_{l,m}\bar{r}\, I,\ G^\infty + \delta_{m,l}\bar{r}\, I,\ K^\infty)$, contrary to the assumption that $\mathrm{GPVI}(F, G, K)$ is regular.

By the above cases, we obtain that $\mathrm{GPVI}(F, G, K)$ has a solution.

Next, we show the boundedness of the solution set of $\mathrm{SOL}(F, G, K)$. Suppose, on the contrary, that $\mathrm{SOL}(F, G, K)$ is unbounded. Then, there exists a sequence $\{y^k\} \subset \mathrm{SOL}(F, G, K)$ such that $\|y^k\| \to +\infty$ as $k \to +\infty$. Without loss of generality we may assume that

$\|y^k\| \to \infty$ as $k \to \infty$, $\|y^k\| \neq 0$ for all k, and $\|y^k\|^{-1}y^k \to \bar{v}$ with $\|\bar{v}\| = 1$. For each k, we have $G(y^k) \in K$ and

$$\langle F(y^k), y - G(y^k)\rangle \geq 0 \ \ \forall y \in K. \tag{3.14}$$

Applying [18, Theorem 8.2] to $G(y^k) \in K$ and $\frac{1}{\|y^k\|^{l-1}} \to 0$, we have

$$\frac{1}{\|y^k\|^{l-1}} G(y^k) = \frac{1}{\|y^k\|^{l-1}} \left(\sum_{p=2}^{l} \mathcal{B}^{(p)}(y^k)^{p-1} + b \right) \to \mathcal{B}^{(l)}\bar{v}^{l-1} \in K^\infty$$

as $k \to \infty$. Fix $w \in K$. For every $v \in K^\infty$, we have

$$z := w + v\|y^k\|^{l-1} \in K.$$

From (3.14) it follows that

$$\left\langle \sum_{r=2}^{m} \mathcal{A}^{(r)}(y^k)^{r-1} + a, z - \sum_{p=2}^{l} \mathcal{B}^{(p)}(y^k)^{p-1} - b \right\rangle \geq 0,$$

that is,

$$\left\langle \sum_{r=2}^{m} \mathcal{A}^{(r)}(y^k)^{r-1} + a, w + v\|y^k\|^{l-1} - \sum_{p=2}^{l} \mathcal{B}^{(p)}(y^k)^{p-1} - b \right\rangle \geq 0.$$

Dividing both sides of the last equality by $\|y^k\|^{m+l-2}$ and letting $k \to \infty$ yields

$$\left\langle \mathcal{A}^{(m)}\bar{v}^{m-1}, v - \mathcal{B}^{(l)}\bar{v}^{l-1} \right\rangle \geq 0.$$

Then, $0 \neq \bar{v} \in \mathrm{SOL}(\mathcal{A}^{(m)}, \mathcal{B}^{(l)}, K^\infty)$, contrary to the assumption that $\mathrm{GPVI}(F, G, K)$ is regular. Therefore, $\mathrm{SOL}(F, G, K)$ is bounded.

Let any a sequence $\{z^k\} \subset \mathrm{SOL}(F, G, K)$ such that $z^k \to \bar{z}$ as $k \to +\infty$ for some $\bar{z} \in \mathbb{R}^n$. Then, for some $y \in K$, we have

$$G(z^k) \in K \text{ and } \langle F(z^k), y - G(z^k)\rangle \geq 0 \ \ \forall k. \tag{3.15}$$

Since F and G are continuous and K is closed, passing the expressions in (3.15) to limits as $k \to \infty$, we obtain that

$$G(\bar{z}) \in K \text{ and } \langle F(\bar{z}), y - G(\bar{z})\rangle \geq 0.$$

This follows that $\mathrm{SOL}(F, G, K)$ is closed. Therefore, $\mathrm{SOL}(F, G, K)$ is a compact set. The proof is complete. □

The following example illustrates an application of Theorem 3.4.

Example 3.5. We consider the problem $\mathrm{GPVI}(F, G, K)$ with $n = 2$, $m = l = 4$,

$$K := \{(x_1, x_2) \in \mathbb{R}^2 : x_2^2 - x_1 \leq 0\},$$

$$F(x) = (2x_1^3 - 5x_1x_2 - 4x_2 + 1, -3x_2^3 + x_1x_2 - x_1 - 2),$$

and

$$G(x) = (x_1^3 + x_2^2 - x_1, x_2^3 + x_1x_2 + 1) \ \ \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

Then, K is nonempty since $(0,0) \in K$. From the fact that the function γ defined by $\gamma(x_1, x_2) := x_2^2 - x_1$ is a convex quadratic function it follows that K is a closed and convex set. According to [3, Lemma 1.1], we have $K^\infty = \{(x_1, 0) : x_1 \in \mathbb{R}_+\}$. Then, for each $\rho \geq 0$, we obtain that

$$F^\infty(x) + \rho x = (2x_1^3 + \rho x_1, -3x_2^3 + \rho x_2),$$

and

$$G^\infty(x) + \rho x = (x_1^3 + \rho x_1, x_2^3 + \rho x_2).$$

Suppose that $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \mathrm{SOL}(F^\infty + \rho I, G^\infty + \rho\, I, K^\infty)$, that is,

$$G^\infty(\bar{x}) + \rho\bar{x} \in K^\infty, \tag{3.16}$$

$$F^\infty(\bar{x}) + \rho\bar{x} \in (K^\infty)^*, \tag{3.17}$$

$$\langle F^\infty(\bar{x}) + \rho\bar{x}, G^\infty(\bar{x}) + \rho\bar{x}\rangle = 0. \tag{3.18}$$

From (3.16), we have $\bar{x}_1 \geq 0$ and $\bar{x}_2 = 0$. Then, for every $v = (v_1, 0) \in K^\infty$ with $v_1 \geq 0$, one has

$$\langle F^\infty(\bar{x}) + \rho\bar{x}, v\rangle = v_1(2\bar{x}_1^3 + \rho\bar{x}_1) \geq 0.$$

Hence, (3.17) is satisfied. By (3.18), we have

$$\bar{x}_1^2(2\bar{x}_1^2 + \rho)(\bar{x}_1^2 + \rho) = 0.$$

This follows $\bar{x}_1 = 0$. Thus $\bar{x} = (0,0)$ and the regular condition is satisfied. Therefore, the presented problem has a solution by using Theorem 3.4.

In the following corollary, we characterize the uniqueness of solutions of GPVI(F, G, K) under the assumptions, which is different from [23, Theorem 2].

Corollary 3.6. *Suppose that the assumptions in Theorem 3.4 are satisfied. Then,* GPVI(F, G, K) *has a unique solution provided that F is strictly monotone with respect to G on K.*

Proof. The emptiness of $\mathrm{SOL}(F, G, K) \neq \emptyset$ follows from by Theorem 3.4. We now prove that GPVI(F, G, K) has a unique solution. Indeed, suppose, on the contrary, that GPVI(F, G, K) has two different solutions $\bar{x}$ and $\hat{x}$. Then,

$$\langle F(\bar{x}), G(\hat{x}) - G(\bar{x})\rangle \geq 0 \text{ and } \langle F(\hat{x}), G(\bar{x}) - G(\hat{x})\rangle \geq 0.$$

It follows

$$\langle F(\hat{x}) - F(\bar{x}), G(\hat{x}) - G(\bar{x})\rangle \leq 0.$$

This contradicts the assumption that F is strictly monotone with respect to G on K. Therefore, the problem GPVI(F, G, K) has a unique solution. □

Remark 3.7. The assumptions that G is injective and F is strongly monotone with respect to G on K in Proposition 2.2 are omitted from Corollary 3.6. The following example illustrates an application of Corollary 3.6 and it also shows that [16, Proposition 3.9] cannot apply for this problem.

Example 3.8. Consider GPVI(F,G,K) with $n=2$, $m=l=4$, $K:=\{(x_1,0)\in\mathbb{R}^2 : x_1\in\mathbb{R}_+\}$, for all $x=(x_1,x_2)\in\mathbb{R}^2$, $F(x)=(x_1^3,-2x_2^3-x_1x_2+2x_2+1)$, and $G(x)=(x_1^3,x_2^3-3x_2-18)$. Then, $K^\infty=K$ and $(K^\infty)^*=K^*=\{(x_1,x_2):x_1\geq 0\}$.

Since K is a cone, this problem reduces to the problem GPCP(F,G,K) as follows: finding $x\in\mathbb{R}^2$ such that

$$G(x)\in K,\ F(x)\in K^*,\ \langle F(x),G(x)\rangle=0.$$

We can check that $\bar{x}=(\bar{x}_1,\bar{x}_2)=(0,3)$ is a solution of the above problem.

For every $x=(x_1,x_2),y=(y_1,y_2)\in\mathbb{R}^n$ satisfying $x\neq y$ and $G(x),G(y)\in K$. Since the equation $x_2^3-3x_2-18=(x-3)(x_2^2+3x_2+6)=0$ has a unique solution $x=3$. Hence, $x_2=y_2=3$ and $x_1\neq y_1$. Suppose that GPVI(F,G,K) have two different solutions $\bar{x}$ and $\hat{x}$. Then,

$$\hat{x}_1\neq\bar{x}_1,\ \hat{y}_1=\bar{y}_1=3,\ \langle F(\bar{x}),G(\hat{x})-G(\bar{x})\rangle\geq 0,\ \text{and}\ \langle F(\hat{x}),G(\bar{x})-G(\hat{x})\rangle\geq 0.$$

It follows

$$\langle F(\hat{x})-F(\bar{x}),G(\hat{x})-G(\bar{x})\rangle\leq 0.$$

This contradicts the fact that $\langle F(\bar{x}),G(\hat{x})-G(\bar{x})\rangle=(\hat{x}_1^3-\bar{x}_1^3)^2>0$. Therefore, F is strictly monotone with respect to G on K and this problem has a unique solution.

We have

$$F^\infty(x)+\rho x=(x_1^3+\rho x_1,-2x_2^3+\rho x_2)$$

and

$$G^\infty(x)+\rho x=(x_1^3+\rho x_1,x_2^3+\rho x_2).$$

Let any $\bar{z}=(\bar{z}_1,\bar{z}_2)\in\text{SOL}(F^\infty+\rho I,G^\infty+\rho\, I,K^\infty)$, that is,

$$G^\infty(\bar{z})+\rho\bar{z}\in K^\infty, \tag{3.19}$$

$$F^\infty(\bar{z})+\rho\bar{z}\in (K^\infty)^*, \tag{3.20}$$

$$\langle F^\infty(\bar{z})+\rho\bar{z},G^\infty(\bar{z})+\rho\bar{z}\rangle=0. \tag{3.21}$$

By (3.19), we have $\bar{z}_2=0$ and $\bar{z}_1\geq 0$. For every $v=(v_1,0)\in K^\infty$ with $v_1\geq 0$, we obtain

$$\langle F^\infty(\bar{z})+\rho\bar{z},v\rangle=v_1(\bar{z}_1^3+\rho\bar{z}_1)\geq 0.$$

and (3.20) is satisfied. From (3.21), we have

$$(\bar{z}_1^3+\rho\bar{z}_1)^2=0.$$

It implies $\bar{z}_1=0$. Hence, $\bar{z}=(0,0)$ and the presented problem is regular. Applying Corollary 3.6, this problem has a unique solution.

However, from the fact that $G(0,0)=G(0,\sqrt{3})=G(0,-\sqrt{3})=(0,-18)$ it follows that G is not injective; hence, [16, Proposition 3.9] cannot apply for this problem.

The following corollary characterizes the existence for GPCP(F,G,K).

Corollary 3.9. *Let* GPCP(F,G,K) *be regular. Suppose that the assumptions in Theorem 3.4 are satisfied. Then,* GPCP(F,G,K) *has a solution.*

Proof. This corollary follows immediately from Theorem 3.4. □

Remark 3.10. The regular condition used in Corollary 3.9 is weaker than one used in [7, Theorem 3.1] (see Remark 3.2).

3.3 Existence for the case where K is bounded and applications

For the case where K is compact, the existence for the problem GPVI(F, G, K) is proposed as follows.

Theorem 3.11. *If K is a nonempty compact convex set and if $\Lambda(\mathcal{B}^{(l)}) \subset \mathbb{R}_+ \setminus \{0\}$ then* GPVI(F, G, K) *has a solution.*

Proof. Suppose, on the contrary, that GPVI(F, G, K) has no solution. By using similar arguments as in the proof of Theorem 3.4, there exist $\{x^k\} \subset \mathbb{R}^n$ satisfying $\|x^k\| \to \infty$ as $k \to \infty$ and $\sigma^k > 0$ such that $z^k := \sigma^k x^k + G(x^k) \in K$. Without loss of generality, we may assume that $\frac{x^k}{\|x^k\|} \to \bar{h}$ for some $\bar{h} \in \mathbb{R}^n$. Denote

$$\rho^k := \frac{\sigma^k}{\|x^k\|^{l-2}}.$$

We consider the following two cases:

Case 1: $\{\rho^k\}$ is unbounded, that is, $\rho^k \to +\infty$ as $k \to +\infty$. Then, applying [18, Theorem 8.2] to $z^k = \sigma^k x^k + G(x^k) = \rho^k \|x^k\|^{l-2} x^k + G(x^k) \in K$ and $\frac{1}{\rho^k \|x^k\|^{l-1}} \to 0$, we have

$$\frac{1}{\rho^k \|x^k\|^{l-1}} z^k = \frac{x^k}{\|x^k\|} + \frac{1}{\rho^k}.\frac{G(x^k)}{\|x^k\|^{l-1}} \to \bar{h} \in K^\infty = \{0\}.$$

This contradicts the fact that $\|\bar{h}\| = 1$.

Case 2: $\{\rho^k\}$ is bounded. Without loss of generality, assume that $\rho^k \to \bar{\rho}$ for some $\bar{\rho} \in \mathbb{R}^n$ and $\bar{\rho} \geq 0$. Applying [18, Theorem 8.2] to $z^k = \rho^k \|x^k\|^{l-2} x^k + G(x^k) \in K$ and $\frac{1}{\|x^k\|^{l-1}} \to 0$, we have

$$\frac{1}{\|x^k\|^{l-1}} z^k = \rho^k \frac{x^k}{\|x^k\|} + \frac{G(x^k)}{\|x^k\|^{l-1}} \to \bar{\rho}\bar{h} + \mathcal{B}^{(l)} \bar{h}^{l-1} \in K^\infty = \{0\}.$$

This follows that

$$\mathcal{B}^{(l)} \bar{h}^{l-1} = -\bar{\rho}\bar{h},$$

that is, $-\bar{\rho} \leq 0$ is a eigenvalue of $\mathcal{B}^{(l)}$. This contradicts the assumption (i). Therefore, GPVI(F, G, K) has a solution. □

Remark 3.12. Applying Theorem 3.11 for $G = I$ with $\Lambda(G) = \{1\}$, we get well-known Hartman-Stampacchia's theorem for PVI (see [4]).

By using the result obtained in Theorem 3.11, we get the following important results.

Theorem 3.13. *Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set, $0 \in K$, and $\Lambda(\mathcal{B}^{(l)}) \subset \mathbb{R}_+ \setminus \{0\}$. Assume that F^∞ is copositive with respect to G^∞ on K^∞. The following statements are valid:*

(i) *If $(\mathcal{A}^{(m)}, \mathcal{B}^{(l)})$ is a $\mathrm{R}_0^{K^\infty}$-tensor pair then* SOL$(F + \tilde{F}, G + \tilde{G}, K)$ *is a nonempty compact set for every $(\tilde{F}, \tilde{G}) \in \mathcal{P}^{[m-1,n]} \times \mathcal{P}^{[l-1,n]}$;*

(ii) SOL$(F + c, G, K)$ *is a nonempty and compact set for every $c \in int\{G^\infty($SOL$(\mathcal{A}^{(m)}, \mathcal{B}^{(l)}, K^\infty))\}^*$.*

Proof. Suppose that $K \subset \mathbb{R}^n$ is a nonempty closed convex set, $\Lambda(\mathcal{B}^{(l)}) \subset \mathbb{R}_+ \setminus \{0\}$, $0 \in K$, and F^∞ is copositive with respect to G^∞ on K^∞.

(*i*) Suppose that $(\mathcal{A}^{(m)}, \mathcal{B}^{(l)})$ is a $\mathrm{R}_0^{K^\infty}$-tensor pair. For each $i = 1, 2, \ldots$, denote

$$K_i = \{z \in \mathbb{R}^n : z \in K, \|z\| \leq i\}.$$

Then, we may assume that K_i is nonempty compact set. For some $(\tilde{F}, \tilde{G}) \in \mathcal{P}^{[m-1,n]} \times \mathcal{P}^{[l-1,n]}$, by using Theorem 3.11, we have

$$\mathrm{SOL}(F + \tilde{F}, G + \tilde{G}, K_i) \neq \emptyset$$

for every i. Let any $x^i \in \mathrm{SOL}(F + \tilde{F}, G + \tilde{G}, K_i)$. We prove that $\{x^i\}$ is bounded. Indeed, suppose, on the contrary, that $\{x^i\}$ is unbounded. Then, we may assume that $x^i > 0$ for every i and $x^i/\|x^i\| \to \bar{v}$ for some $\bar{v} \in \mathbb{R}^n$. By the fact that $x^i \in \mathrm{SOL}(F + \tilde{F}, G + \tilde{G}, K_i)$, we have $G(x^i) + \tilde{G}(x^i) \in K_i$ and

$$\langle F(x^i) + \tilde{F}(x^i), z - G(x^i) - \tilde{G}(x^i)\rangle \geq 0 \tag{3.22}$$

for every $z \in K_i$. Since $G(x^i) + \tilde{G}(x^i) \in K$ and $\frac{1}{\|x^i\|^{l-1}} \to 0$, applying [18, Theorem 8.2], we have

$$\frac{1}{\|x^i\|^{l-1}} \left(\sum_{p=2}^{l} \mathcal{B}^{(p)}(x^i)^{p-1} + b + \tilde{G}(x^i) \right) \to \mathcal{B}^{(l)}\bar{v}^{l-1} \in K^\infty$$

as $i \to \infty$. Multiplying both sides of the inequality (3.22) by $\|x^i\|^{-(m+l-2)}$ and taking $i \to \infty$ yields

$$\left\langle \mathcal{A}^{(m)}\bar{v}^{m-1}, \mathcal{B}^{(l)}\bar{v}^{l-1} \right\rangle \leq 0. \tag{3.23}$$

Since F^∞ is copositive with respect to G^∞ on K^∞, we have

$$\left\langle \mathcal{A}^{(m)}\bar{v}^{m-1}, \mathcal{B}^{(l)}\bar{v}^{l-1} \right\rangle \geq 0.$$

By this and (3.23), we have

$$\left\langle \mathcal{A}^{(m)}\bar{v}^{m-1}, \mathcal{B}^{(l)}\bar{v}^{l-1} \right\rangle = 0. \tag{3.24}$$

For any $v \in K^\infty \setminus \{0\}$, one has $y^i := 0 + \frac{\|G(x^i) + \tilde{G}(x^i)\|}{\|v\|} v \in K$ and

$$\|y^i\| = \|G(x^i) + \tilde{G}(x^i)\| \leq i.$$

Hence, $y^i \in K_i$. By (3.22) one has

$$\begin{aligned}&\langle F(x^i) + \tilde{F}(x^i), y^i - G(x^i) - \tilde{G}(x^i)\rangle \\ =&\left\langle F(x^i) + \tilde{F}(x^i), \frac{\|G(x^i) + \tilde{G}(x^i)\|}{\|v\|} v - G(x^i) - \tilde{G}(x^i) \right\rangle \geq 0.\end{aligned}$$

Multiplying both sides of the last inequality by $\|x^i\|^{-(m+l-2)}$ and letting $i \to \infty$ yields

$$\left\langle \mathcal{A}^{(m)}\bar{v}^{m-1}, \frac{\|\mathcal{B}^{(l)}\bar{v}^{l-1}\|}{\|v\|} v \right\rangle \geq \left\langle \mathcal{A}^{(m)}\bar{v}^{m-1}, \mathcal{B}^{(l)}\bar{v}^{l-1} \right\rangle = 0.$$

Hence, $\langle \mathcal{A}^{(m)}\bar{v}^{m-1}, v\rangle \geq 0$, that is, $\mathcal{A}^{(m)}\bar{v}^{m-1} \in (K^\infty)^*$. By this and (3.24), we deduce that

$$\mathcal{B}^{(l)}\bar{v}^{l-1} \in K^\infty,\ \ \mathcal{A}^{(m)}\bar{v}^{m-1} \in (K^\infty)^* \ \text{ and } \langle \mathcal{A}^{(m)}\bar{v}^{m-1}, \mathcal{B}^{(l)}\bar{v}^{l-1}\rangle = 0,$$

that is,

$$0 \neq \bar{v} \in \mathrm{SOL}(\mathcal{A}^{(m)}, \mathcal{B}^{(l)}, K^\infty), \tag{3.25}$$

contrary to the assumption that $(\mathcal{A}^{(m)}, \mathcal{B}^{(l)})$ is a R_0^∞-tensor pair. Therefore, $\{x^i\}$ is bounded. We may assume, without loss of generality, that $x^i \to \bar{x}$ for some $\bar{x} \in \mathbb{R}^n$. Since $G(x^i) + \tilde{G}(x^i) \in K$ and since K is closed, we have $G(\bar{x}) + \tilde{G}(\bar{x}) \in K$. Passing (3.22) to limits as $i \to \infty$, we obtain

$$\langle F(\bar{x}) + \tilde{F}(\bar{x}), z - G(\bar{x}) - \tilde{G}(\bar{x})\rangle \geq 0. \tag{3.26}$$

Hence, $\bar{x} \in \mathrm{SOL}(F + \tilde{F}, G + \tilde{G}, K)$.

Suppose that $\mathrm{SOL}(F + \tilde{F}, G + \tilde{G}, K)$ is unbounded. Then, there exists $\{y^i\} \subset \mathrm{SOL}(F + \tilde{F}, G + \tilde{G}, K)$ such that $\|z^i\| \to \infty$ and $z^i/\|z^i\| \to \bar{y}$ for some $\bar{z} \in \mathbb{R}^n$. Repeating the above arguments, we obtain that $0 \neq \bar{z} \in \mathrm{SOL}(\mathcal{A}^{(m)}, \mathcal{B}^{(l)}, K^\infty)$, contrary to the assumption that $(\mathcal{A}^{(m)}, \mathcal{B}^{(l)})$ is a $\mathrm{R}_0^{K^\infty}$-tensor pair. Therefore, $\mathrm{SOL}(F+\tilde{F}, G+\tilde{G}, K)$ is a nonempty compact set.

(ii) For each $c \in \mathrm{int}\{G^\infty(\mathrm{SOL}(\mathcal{A}^{(m)}, \mathcal{B}^{(l)}, K^\infty))\}^*$. By the similar arguments in part (i) with $\tilde{F} \equiv c$ and $\tilde{G} \equiv 0$, we obtain (3.22)–(3.24). Repeating the arguments in part (i), one gets $\bar{v} \in \mathrm{SOL}(\mathcal{A}^{(m)}, \mathcal{B}^{(l)}, K^\infty)$. From (3.22), taking $z = 0$, we have

$$\langle F(x^i) + c, -G(x^i)\rangle \geq 0,$$

that is,

$$\langle c, G(x^i)\rangle \leq -\langle F(x^i), G(x^i)\rangle \leq 0.$$

Dividing both sides of the last inequality by $\|x^i\|^{l-1}$ and letting $i \to \infty$ yields

$$\langle c, \mathcal{B}^{(l)}\bar{v}^{l-1}\rangle \leq 0.$$

This contradicts the assumption that $c \in int\{G^\infty(\mathrm{SOL}(\mathcal{A}^{(m)}, \mathcal{B}^{(l)}, K^\infty))\}^*$. Therefore, $\mathrm{SOL}(F + c, g, K)$ is nonempty. The boundedness of $\mathrm{SOL}(F + c, g, K)$ follows from a similar analysis as in part (i). □

For the special case where $G = I$, from Theorem 3.13, we obtain the follows corollary.

Corollary 3.14. *Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set, $0 \in K$, F^∞ is copositive on K^∞. The following statements are valid:*

(i) *If $\mathcal{A}^{(m)}$ is a R_0-tensor on K^∞ then the solution set of* $\mathrm{PVI}(F + \tilde{F}, K)$ *is nonempty and compact for every $\tilde{F} \in \mathcal{P}^{[m-1,n]}$;*

(ii) *The solution set of $PVI(F + c, K)$ is nonempty and compact for every $c \in \mathrm{int}\{\mathrm{SOL}(\mathcal{A}^{(m)}, I, K^\infty)\}^*$.*

Remark 3.15. The obtained results in Theorem 3.13 and Corollary 3.14 are useful for studying the stability of parametric GPVIs and PVIs. This interesting topic will be concerned in our next studies. Part (ii) in Corollary 3.14 is a recent result obtained by Hieu [2, Theorem 4.1]. A better result, which can be easily obtained from Theorem 3.13, has been proposed by Ma et al. [8].

Conclusion

In the paper, to investigate the solution existence of the GPVI, we have proposed a new regular condition (Definition 3.1). We have presented sufficient conditions for the solution existence of the GPVI (Theorem 3.4). By this main theorem, we have obtained the existence results for the boundedness of the solution set and the special case where K is a compact set (Theorems 3.11 and 3.13).

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