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TWO PROJECTION METHODS FOR SOLVING THE SPLIT COMMON NULL POINT PROBLEM IN TWO BANACH SPACES

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Abstract: In this article, we study the split common null point problem in two Banach spaces. Using metric resolvents and generalized resolvents of maximal monotone operators in Banach spaces, we prove strong convergence theorems under two projection methods for finding a solution of the split common null point problem in two Banach spaces. Using these results, we get new results which are connected with the split feasibility problem in two Banach spaces.

Key words: *split common null point problem, metric projection, generalized projection, metric resolvent, generalized resolvent, hybrid method*

Mathematics Subject Classification: *47H05, 47H09*

1 Introduction

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ux - Uy\rangle \geq \alpha \|Ux - Uy\|^2, \quad \forall x, y \in C.$$

Such a mapping U is called α-inverse strongly monotone. Let H_1 and H_2 be Hilbert spaces. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a bounded linear operator. Then the split feasibility problem [7] is to find $z \in H_1$ such that $z \in D \cap T^{-1}Q$. Byrne, Censor, Gibali and Reich [6] also considered the following problem: Given maximal monotone mappings $G : H_1 \to 2^{H_1}$, and $B : H_2 \to 2^{H_2}$, respectively, and a bounded linear operator $T : H_1 \to H_2$, the split common null point problem [6] is to find a point $z \in H_1$ such that

$$z \in G^{-1}0 \cap T^{-1}(B^{-1}0),$$

where $G^{-1}0$ and $B^{-1}0$ are null point sets of G and B, respectively. Defining $U = T^*(I - P_Q)T$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone

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operator [3], where T^* is the adjoint operator of T and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap T^{-1}Q$ is nonempty, then $z \in D \cap T^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda T^*(I - P_Q)T)z, \tag{1.1}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Furthermore, if $G^{-1}0 \cap T^{-1}(B^{-1}0)$ is nonempty, then for $\gamma > 0$, $z \in G^{-1}0 \cap T^{-1}(B^{-1}0)$ is equivalent to

$$z = J_\lambda(I - \gamma T^*(I - Q_\mu)T)z, \tag{1.2}$$

where J_λ and Q_μ are the resolvents of G for $\lambda > 0$ and B for $\mu > 0$, respectively. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility problem, the split common null point problem and the split common fixed point problem; see, for instance, [3, 6, 32, 33]. However, it is difficult to have such results outside Hilbert spaces. Takahashi [25, 26] and Hojo and Takahashi [9] extended the results of (1.1) and (1.2) in Hilbert spaces to Banach spaces; see Section 3.

In this article, we deal with the split common null point problem in two Banach spaces. We first prove strong convergence theorems under the hybrid method by Nakajo and Takahashi [14] for metric resolvents and generalized resolvents of maximal monotone operators with metric projections and generalized projections in two Banach spaces. Furthermore, using the shringking projection method by Takahashi, Takeuchi and Kubota [31] we prove strong convergence theorems for two resolvents of maximal monotone operators with two projections in two Banach spaces. Using these results, we get new results which are connected with the split feasibility problem in two Banach spaces.

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, i.e., $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$; see [8, 16].

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. The norm of E is said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. The norm of E is said to be uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [22, 23].

Lemma 2.1 ([22]). *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy \rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x - y, Jx - Jy \rangle = 0$, then $x = y$.*

Let E be a smooth Banach space and let J be the duality mapping on E. Define a function $\phi_E : E \times E \to \mathbb{R}$ by

$$\phi_E(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.2}$$

In the case when E is clear, ϕ_E is simply denoted by ϕ. Observe that, in a Hilbert space H, $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Furthermore, we know that for each $x, y, z, w \in E$,

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2; \tag{2.3}$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle; \tag{2.4}$$

$$2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w). \tag{2.5}$$

If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \quad \text{if and only if} \quad x = y. \tag{2.6}$$

The following lemma was proved by Kamimura and Takahashi [10].

Lemma 2.2 ([10]). *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the metric projection of E onto C. We know the following result.

Lemma 2.3 ([22]). *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = P_C x$;

(2) $\langle z - y, J(x - z) \rangle \geq 0, \quad \forall y \in C$.

For any $x \in E$, we also know that there exists a unique element $z \in C$ such that

$$\phi(z, x) = \min_{y \in C} \phi(y, x).$$

The mapping $\Pi_C : E \to C$ defined by $z = \Pi_C x$ is called the generalized projection of E onto C. The following results are well-known. For example, see [1, 2, 10].

Lemma 2.4 ([1, 2, 10]). *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = \Pi_C x$;

(2) $\langle z - y, Jx - Jz \rangle \geq 0, \quad \forall y \in C$.

Lemma 2.5 ([1,2,10]). *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x)$$

for all $y \in C$.

Let E be a Banach space and let B be a mapping of E into 2^{E^*}. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in E : Bx \neq \emptyset\}$. A multi-valued mapping B on E is said to be monotone if $\langle x - y, u^* - v^* \rangle \geq 0$ for all $x, y \in \mathrm{dom}(B)$, $u^* \in Bx$, and $v^* \in By$. A monotone operator B on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [5, 18]; see also Theorem 3.5.4 in [23].

Theorem 2.6 ([5, 18]). *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let B be a monotone operator of E into 2^{E^*}. Then B is maximal if and only if for any $r > 0$,*

$$R(J + rB) = E^*,$$

where $R(J + rB)$ is the range of $J + rB$.

Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rBx_r.$$

This equation has a unique solution x_r; see [23]. We define J_r by $x_r = J_r x$. Such a J_r is denoted by

$$J_r = (I + rJ^{-1}B)^{-1}$$

and is called the metric resolvent of B. For $r > 0$, the Yosida approximation $A_r : E \to E^*$ is defined by

$$A_r x = \frac{J(x - J_r x)}{r}, \quad \forall x \in E.$$

Lemma 2.7 ([4, 23]). *Let E be a uniformly convex and smooth Banach space and let $B \subset E \times E^*$ be a maximal monotone operator. Let $r > 0$ and let J_r and A_r be the metric resolvent and the Yosida approximation of B, respectively. Then, the following hold:*

(1) $\langle J_r x - u, J(x - J_r x) \rangle \geq 0, \quad \forall x \in E,\ u \in B^{-1}0$;

(2) $(J_r x, A_r x) \in B, \quad \forall x \in E$;

(3) $F(J_r) = B^{-1}0$.

For all $x \in E$ and $r > 0$, we also consider the following equation

$$Jx \in Jx_r + rBx_r.$$

This equation has a unique solution x_r; see [11]. We define Q_r by $x_r = Q_r x$. Such a Q_r is called the generalized resolvent of B. For $r > 0$, the Yosida approximation $B_r : E \to E^*$ is defined by

$$B_r x = \frac{Jx - JQ_r x}{r}, \quad \forall x \in E.$$

The set of null points of B is defined by $B^{-1}0 = \{z \in E : 0 \in Bz\}$. We know that $B^{-1}0$ is closed and convex; see [23]. In case a Banach space is a Hilbert space, we have that $J_r = Q_r$ for all $r > 0$. Such a J_r is simply called the resolvent of B.

Lemma 2.8 ([11]). *Let E be a uniformly convex and smooth Banach space and let $B \subset E \times E^*$ be a maximal monotone operator. Let $r > 0$ and let Q_r and B_r be the generalized resolvent and the Yosida approximation of B, respectively. Then, the following hold:*

(1) $\phi(u, Q_r x) + \phi(Q_r x, x) \le \phi(u, x), \quad \forall x \in E,\ u \in B^{-1}0$;

(2) $(Q_r x, B_r x) \in B, \quad \forall x \in E$;

(3) $F(Q_r) = B^{-1}0$.

For a sequence $\{C_n\}$ of nonempty, closed and convex subsets of a Banach space E, define $\text{s-Li}_n C_n$ and $\text{w-Ls}_n C_n$ as follows: $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in \text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies

$$C_0 = \underset{n}{\text{s-Li}}\, C_n = \underset{n}{\text{w-Ls}}\, C_n, \tag{2.7}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [12] and we write $C_0 = \text{M-lim}_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [12]. The following lemma was proved by Tsukada [34].

Lemma 2.9 ([34]). *Let E be a uniformly convex Banach space. Let $\{C_n\}$ be a sequence of nonempty, closed and convex subsets of E. If $C_0 =$M-$\lim_{n\to\infty} C_n$ exists and nonempty, then for each $x \in E$, $\{P_{C_n} x\}$ converges strongly to $P_{C_0} x$, where P_{C_n} and P_{C_0} are the mertic projections of E onto C_n and C_0, respectively.*

3 Four Results under the Hybrid Method

In this section, using the hybrid method by Nakajo and Takahashi [14] we obtain strong convergence theorems for finding a solution of the split common null point problem in two Banach spaces. See also [15,19] for the hybrid method. The following lemma was proved by Takahashi [26].

Lemma 3.1 ([26]). *Let E and F be strictly convex, reflexive and smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let A and B be maximal monotone operators of E into 2^{E^*} and F into 2^{F^*} such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$, respectively. Let J_λ^A and J_μ^B be the metric resolvents of A for $\lambda > 0$ and B for $\mu > 0$,* *respectively. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $\lambda, \mu, r > 0$ and $z \in E$. Then the following are equivalent:*

(i) $z = J_\lambda^A\big(z - rJ_E^{-1}T^*J_F(Tz - J_\mu^B Tz)\big)$;

(ii) $z \in A^{-1}0 \cap T^{-1}(B^{-1}0)$.

Hojo and Takahashi [9].also proved the following lemma..

Lemma 3.2 ([9]). *Let E and F be uniformly convex and smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let A and B be maximal monotone operators of E into 2^{E^*} and F into 2^{F^*} such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$, respectively. Let Q_λ^A and Q_μ^B be the generalized resolvents of A for $\lambda > 0$ and B for $\mu > 0$, respectively. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $\lambda, \mu, r > 0$ and $z \in E$. Then the following are equivalent:*

(i) $z = Q_\lambda^A J_E^{-1}\big(J_E z - rT^*(J_F Tz - J_F Q_\mu^B Tz)\big)$;

(ii) $z \in A^{-1}0 \cap T^{-1}(B^{-1}0)$.

Using the idea of Lemma 3.1, we can solve the split common null point problem for two metric resolvents of maximal monotone operators with metric projections in two Banach spaces. The following theorem was proved by Takahashi [27].

Theorem 3.3 ([27]). *Let E and F be uniformly convex and smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let A and B be maximal monotone operators of E into 2^{E^*} and F into 2^{F^*} such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$, respectively. Let J_λ^A and J_μ^B be the metric resolvents of A for $\lambda > 0$ and B for $\mu > 0$, respectively. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = x_n - \mu_n J_E^{-1}T^*J_F(Tx_n - J_{\mu_n}^B Tx_n), \\ y_n = J_{\lambda_n}^A z_n, \\ C_n = \{z \in E : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0\}, \\ D_n = \{z \in E : \langle y_n - z, J_E(z_n - y_n)\rangle \geq 0\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\lambda_n\}, \{\mu_n\} \subset (0, \infty)$ satisfy that for some $a, b, c \in \mathbb{R}$,

$$0 < a \leq \mu_n \leq b < \frac{1}{\|T\|^2} \text{ and } 0 < c \leq \lambda_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, where $z_0 = P_{A^{-1}0 \cap T^{-1}(B^{-1}0)} x_1$.

Using the idea of Lemma 3.2, we can solve the split common null point problem for two generalized resolvents of maximal monotone operators with generalized projections in two Banach spaces. The following was proved by TTakahashi [28]

Theorem 3.4 ([28]). *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let A and B be maximal monotone operators of E into 2^{E^*} and F into 2^{F^*} such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$, respectively. Let Q^A_λ and Q^B_μ be the generalized resolvents of A for $\lambda > 0$ and B for $\mu > 0$, respectively. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n T^*(J_F T x_n - J_F Q^B_{\mu_n} T x_n)\big), \\ y_n = Q^A_{\lambda_n} z_n, \\ C_n = \{z \in E : 2\langle x_n - z, J_E x_n - J_E z_n\rangle \geq r_n \phi_F(Tx_n, Q^B_{\mu_n} T x_n)\}, \\ D_n = \{z \in E : \langle y_n - z, J_E z_n - J_E y_n\rangle \geq 0\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E x_1 - J_E x_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\lambda_n\}, \{\mu_n\} \subset (0, \infty)$ satisfy that for some $a, b \in \mathbb{R}$,

$$0 < a \leq r_n \leq \frac{1}{\|T\|^2} \text{ and } 0 < b \leq \lambda_n, \mu_n \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, where $z_0 = \Pi_{A^{-1}0 \cap T^{-1}(B^{-1}0)} x_1$.

The following is the hybrid method of solving the split common null point problem for metric resolvents and generalized resolvents of maximal monotone operators with generalized projections in two Banach spaces. The following was proved by Takahashi [30].

Theorem 3.5 ([30]). *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let $A, B \subset E \times E^*$ be maximal monotone operators and let $G \subset F \times F^*$ be a maximal monotone operator. Let $J^A_\mu = (I + \mu J_E^{-1} A)^{-1}$ be the metric resolvent of A for all $\mu > 0$, let $Q^B_\lambda = (J_E + \lambda B)^{-1} J_E$ be the generalized resolvent of B for all $\lambda > 0$ and let $Q^G_\eta = (J_F + \eta G)^{-1} J$ be the generalized resolvent of G for all $\eta > 0$. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that*

$$\Omega = A^{-1}0 \cap B^{-1}0 \cap T^{-1}(G^{-1}0) \neq \emptyset.$$

Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n T^*(J_F T x_n - J_F Q^G_{\eta_n} T x_n)\big), \\ y_n = J^A_{\mu_n} z_n, \\ u_n = Q^B_{\lambda_n} y_n, \\ B_n = \{z \in E : 2\langle y_n - z, J_E y_n - J_E u_n\rangle \geq \phi_E(y_n, u_n) + \phi_E(u_n, y_n)\}, \\ C_n = \{z \in E : 2\langle x_n - z, J_E x_n - J_E z_n\rangle \geq r_n \phi_F(Tx_n, Q^G_{\eta_n} T x_n)\}, \\ D_n = \{z \in E : \langle z_n - z, J_E(z_n - y_n)\rangle \geq \|z_n - y_n\|^2\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E x_1 - J_E x_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{B_n \cap C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\}, \{\lambda_n\}, \{\mu_n\}, \{\eta_n\} \subset (0, \infty)$ and $a, b \in \mathbb{R}$ satisfy the following inequalities:

$$0 < a \leq r_n \leq \frac{1}{\|T\|^2} \quad \text{and} \quad b \leq \lambda_n, \mu_n, \eta_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ *converges strongly to a point* $z_0 \in \Omega$, *where* $z_0 = \Pi_\Omega x_1$.

Finally, using the hybrid method, we solve the split common null point problem for generalized resolvents and metric resolvents of maximal onotone operators with metric projections in two Banach spaces. The following was proved by Takahashi [30].

Theorem 3.6 ([30])**.** *Let* E *and* F *be uniformly convex and uniformly smooth Banach spaces and let* J_E *and* J_F *be the duality mappings on* E *and* F, *respectively. Let* $A, B \subset E \times E^*$ *be maximal monotone operators and let* $G \subset F \times F^*$ *be a maximal monotone operator. Let* $Q^A_\mu = (J_E + \mu A)^{-1} J_E$ *be the generalized resolvent of* A *for all* $\mu > 0$, *let* $J^B_\lambda = (I + \lambda J_E^{-1} B)^{-1}$ *be the metric resolvent of* B *for all* $\lambda > 0$ *and let* $J^G_\eta = (I + \eta J_F^{-1} G)^{-1}$ *be the metric resolvent of* G *for all* $\eta > 0$. *Let* $T : E \to F$ *be a bounded linear operator such that* $T \neq 0$ *and let* T^* *be the adjoint operator of* T. *Suppose that*

$$\Omega = A^{-1}0 \cap B^{-1}0 \cap T^{-1}(G^{-1}0) \neq \emptyset.$$

Let $x_1 \in E$ *and let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = x_n - r_n J_E^{-1} T^* J_F (T x_n - J^G_{\eta_n} T x_n), \\ y_n = Q^A_{\mu_n} z_n, \\ u_n = J^B_{\lambda_n} y_n, \\ B_n = \{z \in E : \langle y_n - z, J(y_n - u_n)\rangle \geq \|y_n - u_n\|^2\}, \\ C_n = \{z \in E : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0\}, \\ D_n = \{z \in E : 2\langle z_n - z, J_E z_n - J_E y_n\rangle \geq \phi_E(z_n, y_n)\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{B_n \cap C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\}, \{\lambda_n\}, \{\mu_n\}, \{\eta_n\} \subset (0, \infty)$ *and* $a, b \in \mathbb{R}$ *satisfy the following:*

$$0 < a \leq r_n \leq \frac{1}{\|T\|^2} \quad \text{and} \quad b \leq \lambda_n, \mu_n, \eta_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ *converges strongly to a point* $w_1 \in \Omega$, *where* $w_1 = P_\Omega x_1$.

4 Four Results under the Shrinking Projection Method

Using the shrinking projection method by Takahashi, Takeuchi and Kubota [31], we can solve the split common null point problem for two metric resolvents of maximal monotone operators with metric projections in two Banach spaces. The following theorem was proved by Takahashi and Takahashi [**?**].

Theorem 4.1 ([?])**.** *Let* E *and* F *be uniformly convex and smooth Banach spaces and let* J_E *and* J_F *be the duality mappings on* E *and* F, *respectively. Let* A *and* B *be maximal monotone operators of* E *into* 2^{E^*} *and* F *into* 2^{F^*} *such that* $A^{-1}0 \neq \emptyset$ *and* $B^{-1}0 \neq \emptyset$, *respectively. Let* J^A_λ *and* J^B_μ *be the metric resolvents of* A *for* $\lambda > 0$ *and* B *for* $\mu > 0$, *respectively. Let* $T : E \to F$ *be a bounded linear operator such that* $T \neq 0$ *and let* T^* *be the adjoint operator of* T. *Suppose that* $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. *Let* $x_1 \in E$ *and let* $C_1 = E$.

Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = x_n - \eta_n J_E^{-1} T^* J_F (Tx_n - J_{\mu_n}^B Tx_n), \\ y_n = J_{\lambda_n}^A z_n, \\ C_{n+1} = \{z \in C_n : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0 \\ \qquad\qquad \text{and} \quad \langle y_n - z, J_E(z_n - y_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\eta_n\}, \{\lambda_n\}, \{\mu_n\} \subset (0, \infty)$ satisfy the following conditions such that for some $a, b, c \in \mathbb{R}$,

$$0 < a \leq \eta_n \|T\|^2 \leq b < 1 \text{ and } 0 < c \leq \lambda_n, \mu_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $w_1 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, where $w_1 = P_{A^{-1}0 \cap T^{-1}(B^{-1}0)} x_1$.

Next, using the shrinking projection method, we can solve the split common null point problem for two generalized resolvents of maximal monotone operators with generalized projections in two Banach spaces. The following theorem was proved by Takahashi and Takahashi [20].

Theorem 4.2 ([20])**.** *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let A and B be maximal monotone operators of E into 2^{E^*} and F into 2^{F^*} such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$, respectively. Let Q_λ^A and Q_μ^B be the generalized resolvents of A for $\lambda > 0$ and B for $\mu > 0$, respectively. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $x_1 \in E$ and let $C_1 = E$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n T^*(J_F Tx_n - J_F Q_{\mu_n}^B Tx_n)\big), \\ y_n = Q_{\lambda_n}^A z_n, \\ C_{n+1} = \{z \in C_n : 2\langle x_n - z, J_E x_n - J_E z_n\rangle \geq r_n \phi_F(Tx_n, Q_{\mu_n}^B Tx_n) \\ \qquad\qquad \text{and} \quad \langle y_n - z, J_E z_n - J_E y_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\}, \{\lambda_n\}, \{\mu_n\} \subset (0, \infty)$ satisfy the following conditions such that for some $a, b \in \mathbb{R}$,

$$0 < a \leq r_n \leq \frac{1}{\|T\|^2} \text{ and } 0 < b \leq \lambda_n, \mu_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $w_1 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, where $w_1 = \Pi_{A^{-1}0 \cap T^{-1}(B^{-1}0)} x_1$.

The following is the shrinking projection method of solving the split common null point problem for metric resolvents and generalized resolvents of maximal monotone operators with generalized projections in two Banach spaces. The following was proved by Takahashi [29].

Theorem 4.3 ([29])**.** *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let $A, B \subset E \times E^*$ be maximal monotone operators and let $G \subset F \times F^*$ be a maximal monotone operator. Let $J_\mu^A = (I + \mu J_E^{-1} A)^{-1}$ be the metric resolvent of A for all $\mu > 0$, let $Q_\lambda^B = (J_E + \lambda B)^{-1} J_E$ be* *the generalized resolvent of B for all $\lambda > 0$ and let $Q_\eta^G = (J_F + \eta G)^{-1}J$ be the generalized resolvent of G for all $\eta > 0$. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that*

$$\Omega = A^{-1}0 \cap B^{-1}0 \cap T^{-1}(G^{-1}0) \neq \emptyset.$$

For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n T^*(J_F T x_n - J_F Q_{\eta_n}^G T x_n)\big), \\ y_n = J_{\mu_n}^A z_n, \\ u_n = Q_{\lambda_n}^B y_n, \\ C_{n+1} = \Big\{ z \in C_n : 2\langle y_n - z, J_E y_n - J_E u_n\rangle \geq \phi_E(y_n, u_n) + \phi_E(u_n, y_n), \\ \qquad\qquad \langle z_n - z, J_E(z_n - y_n)\rangle \geq \|z_n - y_n\|^2 \\ \qquad\qquad\quad \text{and} \quad 2\langle x_n - z, J_E x_n - J_E z_n\rangle \geq r_n \phi_F(Tx_n, Q_{\eta_n}^G T x_n)\Big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1 \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\}, \{\lambda_n\}, \{\mu_n\}, \{\eta_n\} \subset (0,\infty)$ and $a, b \in \mathbb{R}$ satisfy the following:

$$0 < a \leq r_n \leq \frac{1}{\|T\|^2} \quad \text{and} \quad 0 < b \leq \lambda_n, \mu_n, \eta_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \Omega$, where $z_0 = \Pi_\Omega x_1$.

Finally, using the shrinking method, we solve the split common null point problem for generalized resolvents and metric resolvents of maximal onotone operators with metric projections in two Banach spaces. The following was proved by Takahashi [29].

Theorem 4.4 ([29]). *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let $A, B \subset E \times E^*$ be maximal monotone operators and let $G \subset F \times F^*$ be a maximal monotone operator. Let $Q_\mu^A = (J_E + \mu A)^{-1}J_E$ be the generalized resolvent of A for all $\mu > 0$, let $J_\lambda^B = (I + \lambda J_E^{-1}B)^{-1}$ be the metric resolvent of B for all $\lambda > 0$ and let $J_\eta^G = (I + \eta J_F^{-1}G)^{-1}$ be the metric resolvent of G for all $\eta > 0$. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that*

$$\Omega = A^{-1}0 \cap B^{-1}0 \cap T^{-1}(G^{-1}0) \neq \emptyset.$$

For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = x_n - r_n J_E^{-1} T^* J_F (T x_n - J_{\eta_n}^G T x_n), \\ y_n = Q_{\mu_n}^A z_n, \\ u_n = J_{\lambda_n}^B y_n, \\ C_{n+1} = \Big\{ z \in C_n : \langle y_n - z, J(y_n - u_n)\rangle \geq \|y_n - u_n\|^2, \\ \qquad\qquad 2\langle z_n - z, J_E z_n - J_E y_n\rangle \geq \phi_E(z_n, y_n) \\ \qquad\qquad\qquad \text{and} \quad \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0 \ \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\}, \{\lambda_n\}, \{\mu_n\}, \{\eta_n\} \subset (0,\infty)$ and $a, b \in \mathbb{R}$ satisfy the following inequalities

$$0 < a \leq r_n \leq \frac{1}{\|T\|^2} \quad \text{and} \quad 0 < b \leq \lambda_n, \mu_n, \eta_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $w_1 \in \Omega$, where $w_1 = P_\Omega x_1$.

5 Applications

In this section, using Theorems 3.5 and 3.6, we get new strong convergence theorems which are connected with the split feasibility problem and the split common null point problem in two Banach spaces. Let E be a Banach space and let $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. Define the subdifferential of f as follows:

$$\partial f(x) = \{x^* \in E^* : f(y) \geq \langle y - x, x^* \rangle + f(x),\ \forall y \in E\}$$

for all $x \in E$. Then we know that ∂f is a maximal monotone operator; see [17] for more details. Let C be a nonempty, closed and convex subset of E and let i_C be the indicator function, that is,

$$i_C = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Then we have that ∂i_C is a maximal monotone operator and the generalized resolvent $Q_r = \Pi_C$ for all $r > 0$, where Π_C is the generalized projection of E onto C. In fact, for any $x \in E$ and $r > 0$, we have from Lemma 2.4 that

$$\begin{aligned} z = Q_r x &\Leftrightarrow Jz + r\partial i_C(z) \ni Jx \\ &\Leftrightarrow Jx - Jz \in r\partial i_C(z) \\ &\Leftrightarrow i_C(y) \geq \left\langle y - z, \frac{Jx - Jz}{r} \right\rangle + i_C(z),\ \forall y \in E \\ &\Leftrightarrow 0 \geq \langle y - z, Jx - Jz \rangle,\ \forall y \in C \\ &\Leftrightarrow z = \arg\min_{y \in C} \phi(y, x) \\ &\Leftrightarrow z = \Pi_C. \end{aligned}$$

Furthermore, the metric resolvent $J_r = P_C$ for all $r > 0$, where P_C is the metric projection of E onto C. In fact, for any $x \in E$ and $r > 0$, we have that

$$\begin{aligned} z = J_r x &\Leftrightarrow J(z - x) + r\partial i_C(z) \ni 0 \\ &\Leftrightarrow J(x - z) \in r\partial i_C(z) \\ &\Leftrightarrow i_C(y) \geq \langle y - z, \frac{J(x - z)}{r} \rangle + i_C(z),\ \forall y \in E \\ &\Leftrightarrow 0 \geq \langle y - z, J(x - z) \rangle,\ \forall y \in C \\ &\Leftrightarrow z = P_C x. \end{aligned}$$

As a direct consequence of Theorem 3.5, we have the following theorem for finding a solution of the split common null point problem in two Banach spaces.

Theorem 5.1 ([30]). *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let $A, B \subset E \times E^*$ be maximal monotone operators and let $G \subset F \times F^*$ be a maximal monotone operator. Let $J_\mu^A = (I + \mu J_E^{-1} A)^{-1}$ be the metric resolvent of A for all $\mu > 0$, let $Q_\lambda^B = (J_E + \lambda B)^{-1} J_E$ be the generalized resolvent of B for all $\lambda > 0$ and let $Q_\eta^G = (J_F + \eta G)^{-1} J$ be the generalized resolvent of G for all $\eta > 0$. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that*

$$\Omega = A^{-1}0 \cap B^{-1}0 \cap T^{-1}(G^{-1}0) \neq \emptyset.$$

Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n T^*(J_F T x_n - J_F Q_\eta^G T x_n)\big), \\ y_n = J_\mu^A z_n, \\ u_n = Q_\lambda^B y_n, \\ B_n = \{z \in E : 2\langle y_n - z, J_E y_n - J_E u_n\rangle \geq \phi_E(y_n, u_n) + \phi_E(u_n, y_n)\}, \\ C_n = \{z \in E : 2\langle x_n - z, J_E x_n - J_E z_n\rangle \geq r_n \phi_F(T x_n, Q_\eta^G T x_n)\}, \\ D_n = \{z \in E : \langle z_n - z, J_E(z_n - y_n)\rangle \geq \|z_n - y_n\|^2\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E x_1 - J_E x_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{B_n \cap C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $a \in \mathbb{R}$ satisfy the following inequalities:

$$0 < a \leq r_n \leq \frac{1}{\|T\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \Omega$, where $z_0 = \Pi_\Omega x_1$.

Next, using Theorem 3.5, we have the following theorem for finding a solution of the split feasibility problem in two Banach spaces.

Theorem 5.2 ([30]). *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let C and D be nonempty, closed and convex subsets of E and let H be a nonempty, closed and convex subset of F. Let P_C be the metric projection of E onto C, let Π_C be the generalized projection of E onto C. and let Π_H be the generalized projection of F onto H. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that*

$$\Omega = C \cap D \cap T^{-1} H \neq \emptyset.$$

Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n T^*(J_F T x_n - J_F \Pi_H T x_n)\big), \\ y_n = P_C z_n, \\ u_n = \Pi_D y_n, \\ B_n = \{z \in E : 2\langle y_n - z, J_E y_n - J_E u_n\rangle \geq \phi_E(y_n, u_n) + \phi_E(u_n, y_n)\}, \\ C_n = \{z \in E : 2\langle x_n - z, J_E x_n - J_E z_n\rangle \geq r_n \phi_F(T x_n, \Pi_H T x_n)\}, \\ D_n = \{z \in E : \langle z_n - z, J_E(z_n - y_n)\rangle \geq \|z_n - y_n\|^2\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E x_1 - J_E x_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{B_n \cap C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $a \in \mathbb{R}$ satisfy the following inequalities:

$$0 < a \leq r_n \leq \frac{1}{\|T\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \Omega$, where $z_0 = \Pi_\Omega x_1$.

Proof. We have that $Q_{\eta_n}^G = \Pi_H$, $J_{\mu_n}^A = P_C$ and $Q_{\lambda_n}^B y_n = Pi_D$ in Theorem 3.5. Therefore, we have the desired result from Theorem 3.5. □

Similarly, using Theorem 3.6, we have the following strong convergence theorems for the split common null point problem and the split feasibility problem in two Banach spaces.

Theorem 5.3 ([30]). *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let $A, B \subset E \times E^*$ be maximal monotone operators and let $G \subset F \times F^*$ be a maximal monotone operator. Let $Q_\mu^A = (J_E + \mu A)^{-1} J_E$ be the generalized resolvent of A for all $\mu > 0$, let $J_\lambda^B = (I + \lambda J_E^{-1} B)^{-1}$ be the metric resolvent of B for all $\lambda > 0$ and let $J_\eta^G = (I + \eta J_F^{-1} G)^{-1}$ be the metric resolvent of G for all $\eta > 0$. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that*

$$\Omega = A^{-1}0 \cap B^{-1}0 \cap T^{-1}(G^{-1}0) \neq \emptyset.$$

Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = x_n - r_n J_E^{-1} T^* J_F (T x_n - J_\eta^G T x_n), \\ y_n = Q_\mu^A z_n, \\ u_n = J_\lambda^B y_n, \\ B_n = \{z \in E : \langle y_n - z, J(y_n - u_n) \rangle \geq \|y_n - u_n\|^2\}, \\ C_n = \{z \in E : \langle z_n - z, J_E(x_n - z_n) \rangle \geq 0\}, \\ D_n = \{z \in E : 2\langle z_n - z, J_E z_n - J_E y_n \rangle \geq \phi_E(z_n, y_n)\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E(x_1 - x_n) \rangle \geq 0\}, \\ x_{n+1} = P_{B_n \cap C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $a \in \mathbb{R}$ satisfy the following:

$$0 < a \leq r_n \leq \frac{1}{\|T\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $w_1 \in \Omega$, where $w_1 = P_\Omega x_1$.

Theorem 5.4 ([30]). *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let C and D be nonempty, closed and convex subsets of E and let H be a nonempty, closed and convex subset of F. Let Π_C be the generalized projection of E onto C, let P_D be the metric projection of E onto D and let P_H be the metric projection of F onto H. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that*

$$\Omega = C \cap D \cap T^{-1}H \neq \emptyset.$$

Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = x_n - r_n J_E^{-1} T^* J_F (T x_n - P_H T x_n), \\ y_n = \Pi_C z_n, \\ u_n = P_D y_n, \\ B_n = \{z \in E : \langle y_n - z, J(y_n - u_n) \rangle \geq \|y_n - u_n\|^2\}, \\ C_n = \{z \in E : \langle z_n - z, J_E(x_n - z_n) \rangle \geq 0\}, \\ D_n = \{z \in E : 2\langle z_n - z, J_E z_n - J_E y_n \rangle \geq \phi_E(z_n, y_n)\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E(x_1 - x_n) \rangle \geq 0\}, \\ x_{n+1} = P_{B_n \cap C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0,\infty)$ *and* $a \in \mathbb{R}$ *satisfy the following:*

$$0 < a \le r_n \le \frac{1}{\|T\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ *converges strongly to a point* $w_1 \in \Omega$, *where* $w_1 = P_\Omega x_1$.

As a direct consequence of Theorem 4.3, we can also prove the following theorem for finding a solution of the split common null point problem in two Banach spaces.

Theorem 5.5 ([29]). *Let* E *and* F *be uniformly convex and uniformly smooth Banach spaces and let* J_E *and* J_F *be the duality mappings on* E *and* F, *respectively. Let* $A, B \subset E \times E^*$ *be maximal monotone operators and let* $G \subset F \times F^*$ *be a maximal monotone operator. Let* $J_\mu^A = (I + \mu J_E^{-1} A)^{-1}$ *be the metric resolvent of* A *for all* $\mu > 0$, *let* $Q_\lambda^B = (J_E + \lambda B)^{-1} J_E$ *be the generalized resolvent of* B *for all* $\lambda > 0$ *and let* $Q_\eta^G = (J_F + \eta G)^{-1} J$ *be the generalized resolvent of* G *for all* $\eta > 0$. *Let* $T : E \to F$ *be a bounded linear operator such that* $T \ne 0$ *and let* T^* *be the adjoint operator of* T. *Suppose that*

$$\Omega = A^{-1}0 \cap B^{-1}0 \cap T^{-1}(G^{-1}0) \ne \emptyset.$$

For $x_1 \in E$ *and* $C_1 = E$, *let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n T^*(J_F T x_n - J_F Q_\eta^G T x_n)\big), \\ y_n = J_\mu^A z_n, \\ u_n = Q_\lambda^B y_n, \\ C_{n+1} = \Big\{ z \in C_n : 2\langle y_n - z, J_E y_n - J_E u_n\rangle \ge \phi_E(y_n, u_n) + \phi_E(u_n, y_n), \\ \qquad\qquad \langle z_n - z, J_E(z_n - y_n)\rangle \ge \|z_n - y_n\|^2 \\ \qquad\qquad\quad \text{and} \quad 2\langle x_n - z, J_E x_n - J_E z_n\rangle \ge r_n \phi_F(Tx_n, Q_{\eta_n}^G T x_n) \Big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1 \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0,\infty)$ *and* $a \in \mathbb{R}$ *satisfy the following:*

$$0 < a \le r_n \le \frac{1}{\|T\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ *converges strongly to a point* $z_0 \in \Omega$, *where* $z_0 = \Pi_\Omega x_1$.

Next, using Theorem 4.3, we have the following theorem for finding a solution of the split feasibility problem in two Banach spaces.

Theorem 5.6 ([29]). *Let* E *and* F *be uniformly convex and uniformly smooth Banach spaces and let* J_E *and* J_F *be the duality mappings on* E *and* F, *respectively. Let* C *and* D *be nonempty, closed and convex subsets of* E *and let* H *be a nonempty, closed and convex subset of* F. *Let* P_C *be the metric projection of* E *onto* C, *let* Π_C *be the generalized projection of* E *onto* C. *and let* Π_H *be the generalized projection of* F *onto* H. *Let* $T : E \to F$ *be a bounded linear operator such that* $T \ne 0$ *and let* T^* *be the adjoint operator of* T. *Suppose that*

$$\Omega = C \cap D \cap T^{-1}H \ne \emptyset.$$

For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n T^*(J_F T x_n - J_F \Pi_H T x_n)\big), \\ y_n = P_C z_n, \\ u_n = \Pi_D y_n, \\ C_{n+1} = \Big\{ z \in C_n : 2\langle y_n - z, J_E y_n - J_E u_n\rangle \geq \phi_E(y_n, u_n) + \phi_E(u_n, y_n), \\ \qquad \langle z_n - z, J_E(z_n - y_n)\rangle \geq \|z_n - y_n\|^2 \\ \qquad \text{and} \quad 2\langle x_n - z, J_E x_n - J_E z_n\rangle \geq r_n \phi_F(T x_n, \Pi_H T x_n) \Big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1 \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $a \in \mathbb{R}$ satisfy the following:

$$0 < a \leq r_n \leq \frac{1}{\|T\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \Omega$, where $z_0 = \Pi_\Omega x_1$.

Proof. We have that $Q^G_{\eta_n} = \Pi_H$, $J^A_{\mu_n} = P_C$ and $Q^B_{\lambda_n} y_n = Pi_D$ in Theorem 4.3. Therefore, we have the desired result from Theorem 4.3. □

Similarly, using Theorem 4.4, we have the following strong convergence theorems for the split common null point problem and the split feasibility problem in two Banach spaces.

Theorem 5.7 ([29]). *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let $A, B \subset E \times E^*$ be maximal monotone operators and let $G \subset F \times F^*$ be a maximal monotone operator. Let $Q^A_\mu = (J_E + \mu A)^{-1} J_E$ be the generalized resolvent of A for all $\mu > 0$, let $J^B_\lambda = (I + \lambda J_E^{-1} B)^{-1}$ be the metric resolvent of B for all $\lambda > 0$ and let $J^G_\eta = (I + \eta J_F^{-1} G)^{-1}$ be the metric resolvent of G for all $\eta > 0$. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that*

$$\Omega = A^{-1}0 \cap B^{-1}0 \cap T^{-1}(G^{-1}0) \neq \emptyset.$$

For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = x_n - r_n J_E^{-1} T^* J_F(T x_n - J^G_\eta T x_n), \\ y_n = Q^A_\mu z_n, \\ u_n = J^B_\lambda y_n, \\ C_{n+1} = \Big\{ z \in C_n : \langle y_n - z, J(y_n - u_n)\rangle \geq \|y_n - u_n\|^2, \\ \qquad 2\langle z_n - z, J_E z_n - J_E y_n\rangle \geq \phi_E(z_n, y_n) \\ \qquad \text{and} \quad \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0 \ \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $a \in \mathbb{R}$ satisfy the following inequalities

$$0 < a \leq r_n \leq \frac{1}{\|T\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $w_1 \in \Omega$, where $w_1 = P_\Omega x_1$.

Theorem 5.8 ([29]). *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let C and D be nonempty, closed and convex subsets of E and let H be a nonempty, closed and convex subset of F. Let Π_C be the generalized projection of E onto C, let P_D be the metric projection of E onto D and let P_H be the metric projection of F onto H. Let $T: E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that*

$$\Omega = C \cap D \cap T^{-1}H \neq \emptyset.$$

For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = x_n - r_n J_E^{-1} T^* J_F(Tx_n - P_H Tx_n), \\ y_n = \Pi_C z_n, \\ u_n = P_D y_n, \\ C_{n+1} = \Big\{ z \in C_n : \langle y_n - z, J(y_n - u_n)\rangle \geq \|y_n - u_n\|^2, \\ \qquad\qquad 2\langle z_n - z, J_E z_n - J_E y_n\rangle \geq \phi_E(z_n, y_n) \\ \qquad\qquad\qquad \text{and} \quad \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0 \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $a \in \mathbb{R}$ satisfy the following inequalities

$$0 < a \leq r_n \leq \frac{1}{\|T\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $w_1 \in \Omega$, where $w_1 = P_\Omega x_1$.

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