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AN EFFICIENT METHOD FOR NON-CONVEX QCQP PROBLEMS

Naser Osmanpour and Mohammad Keyanpour*

Abstract: In this paper we consider a quadratically constrained quadratic programming problem which contains convex and non-convex constraints and non-convex objective function. To obtain a global solution, we transform the original problem to a linear cone programming problem. If the solution of the new problem satisfies the constraints, it is the global optimal solution, otherwise a lower bound of the optimal value is obtained. We reduced the problem by adding a sequence of ellipsoid constraints to a family of problems whose optimal solutions converge to the global optimal solution. The convergence of the proposed method is investigated. Numerical examples are given to show the applicability of the proposed method.

Key words: *non-convex quadratically constrained quadratic programming, adaptive ellipsoid based method, global optimization, semidefinite programming*

Mathematics Subject Classification: *90C20, 90C22, 90C25, 90C26*

1 Introduction

In this paper, we study the quadratically constrained quadratic programming (QCQP) problems in the following form:

$$\text{(QCQP)} \qquad \begin{array}{ll} \min_{x\in\mathbb{R}^n} f(x) & \\ h_i(x) \le 0 & i = 1, 2, \dots, m_1 \\ g_j(x) \le 0 & j = 1, 2, \dots, m_2, \end{array} \tag{1.1}$$

where $f(x) = x^TQ_0x + 2{p_0}^T x + r_0$ is a non-convex function, $h_i(x) = x^TQ_ix + 2{p_i}^T x + r_i$ for $i = 1, 2, \dots, m_1$ are convex constraints and $g_j(x) = x^T\overline{Q}_jx + 2{\overline{p}_j}^T x + \overline{r}_j$ for $j = 1, 2, \dots, m_2$ are non-convex constraints. Q_0, Q_i and $\overline{Q}_j$ are $n \times n$ real symmetric matrices, p_i and $\overline{p}_j$ are real vectors in $\mathbb{R}^n$ and $\mathrm{r}_i, \overline{\mathrm{r}}_\mathrm{j} \in \mathbb{R}$ are real numbers. The QCQP problem has applications in wireless communications and networking [11], radar [24], and signal processing [16]. Some important subclasses of this problem are 0-1 quadratic programming problem [20], generalized trust region problem [8], Celis-Dennis-Tapia problem [3] and Max-Cut problem [20].

*Corresponding author

It is well known that the problem (1.1), in general, is NP-hard [9] and solving this problem is an open problem. In this article we will consider the case in which Q_0 may not be positive semidefinite and $m_2 \neq 0$.

A non-convex QCQP problem as a linear programming problem with an additional reverse convex constraint is first studied in [14, 24]. Ye and Zhang in [29, 30] studied a non-convex QCQP problem with two quadratic constraints and used semidefinite programming (SDP) techniques. In [6, 10], problem (1.1) is reformulated as a copositive programming problem. To our best knowledge, there is not any directly method for solving this copositive programming problem. Yuill et al. in [31] introduced an iterative method, called CCCP (convex-concave procedure) method, for solving problem (1.1). This method solves D.C. (difference of convex functions) problem by solving a sequence of convex problems. Lipp and Boyd extended CCCP method to be initialized without a feasible point and generalized the method to solve problems including vector inequalities [18]. It is noted that CCCP method is locally convergent [28].

Lu et al. in [21] reformulated QCQP problem as a linear conic programming and by investigating relationship between Lagrange multipliers and related linear conic programming problem provided a global optimality condition. Shi and Jin proved that if the Hessian of the corresponding Lagrangian is copositive over a set, then KKT solution is a global optimal solution [32].

In [5], Deng et al. extended the idea of Lu and gave a new method called AE (*Adaptive Ellipsoid-based*) method. They developed a conic formulation which leads to an approximation to problem (1.1) and then they give a better approximation of optimal solution using Reformulation-LinearizationTechnique (RLT). But AE method is suitable only for a narrow class of problem (1.1) where all constraints are convex. Recently in [15], a method is proposed to solve the problem (1.1) where $m_1 \geq 0$ and $m_2 = 1$. The idea of this method is to exchange the one non-convex constraint with the objective function so that the problem can be solved by AE method.

In this paper, we present an extension of AE method to solve the problem (1.1). This extension leads to a computational procedure for solving a general non-convex QCQP where the objective function and some of constraints are non-convex. First, problem (1.1) is approximated by a relaxed SDP problem. If the optimal solution of the new problem is inside the feasible region, then it can be considered as an optimal solution. Otherwise by adding a sequence of ellipsoid constraints to the problem, the optimal solution of the problem is achieved.

The rest of this paper is organized as follows: In Section 2, we give details of the CCCP and AE methods. Section 3 contains details of extension of AE method and new convergence proof. Computational results are given in Section 4, and finally conclusions are in Section 5.

Notation. The following notations will be used in our work.

S^n	the set of all $n \times n$ symmetric matrices,
S^n_+	the set of all $n \times n$ symmetric positive semidefinite matrices,
Y_{ij}	the (i, j)th entry of matrix Y,
$M \bullet N$	the inner product two matrices M and N,
$A \succeq 0$	denoted A is positive semidefinite matrix,
$int(C)$	the interior of the set C.

Morever, we define $M^+ \succeq 0$ and $M^- \succeq 0$ of $M = M^+ - M^- \in S^n$ as

$$M^+ = \sum_{\{i:\lambda_i(M)\geq 0\}} \lambda_i(M) q_i q_i^T, \quad M^- = -\sum_{\{i:\lambda_i(M)\leq 0\}} \lambda_i(M) q_i q_i^T,$$

where q_i is an eigenvector of M corresponding to the eigenvalue $\lambda_i(M)$.

2 Preliminaries

This section provides a review of some methods needed for the other sections.

2.1 The convex-concave procedure (CCCP)

The CCCP method [31] is an iterative procedure that solves D.C problems via a sequence of convex programing. Let the current point $\overline{x}$ be given. At first we decompose

$$\overline{Q}_j = \overline{Q}_j^+ - \overline{Q}_j^- \text{ and } Q_0 = Q_0^+ - Q_0^-, \tag{2.1}$$

that $Q_0^+, Q_0^-, \overline{Q}_j^+$ and $\overline{Q}_j^-$ are positive semidefinite matrices, for $j = 1, 2, \ldots, m_2$. So the problem (1.1) is converted into D.C problem as following:

$$\begin{array}{ll} \min_{x\in\mathbb{R}^n} x^T Q_0^+ x + 2{p_0}^T x + r_0 - x^T Q_0^- x & \\ x^T Q_i x + 2{p_i}^T x + r_i \leq 0, & i = 1, 2, \ldots, m_1 \\ x^T \overline{Q}_j^+ x + 2{\overline{p}_j}^T x + \overline{r}_j - x^T \overline{Q}_j^- x \leq 0, & j = 1, 2, \ldots, m_2. \end{array} \tag{2.2}$$

From the first-order convexity condition [4] of $x^T \overline{Q}_j^- x$ and $x^T Q_0^- x$ at a given feasible point $\overline{x}$, we have

$$x^T \overline{Q}_j^- x \geq -(\overline{x})^T \overline{Q}_j^- \overline{x} + 2x^T \overline{Q}_j^- \overline{x}$$

$$x^T Q_0^- x \geq -(\overline{x})^T Q_0^- \overline{x} + 2x^T Q_0^- \overline{x}.$$

So problem (2.2) is approximated as follows:

$$\begin{array}{lll} \min_{x\in\mathbb{R}^n} x^T Q_0^+ x + 2{p_0}^T x + r_0 + (\overline{x})^T Q_0^- \overline{x} - 2x^T Q_0^- \overline{x} & & \\ x^T Q_i x + 2{p_i}^T x + r_i \leq 0 & & i = 1, 2, \ldots, m_1 \\ x^T \overline{Q}_j^+ x + 2{\overline{p}_j}^T x + \overline{r}_j + (\overline{x})^T \overline{Q}_j^- \overline{x} - 2x^T \overline{Q}_j^- \overline{x} & \leq 0 & j = 1, 2, \ldots, m_2. \end{array} \tag{2.3}$$

It is easy to show that the feasible region of problem (2.3) is convex and can be solved efficiently (i.e., in polynomial-time) [2, 4]. The algorithm of CCCP method is as follows:

Algorithm 1 (CCCP)

- **Initialization:** Take the initial feasible solution $x_0 \in \mathbb{R}^n$, $k = 0$ and $\varepsilon > 0$.
- **Step 1:** Construct problem (2.3) at the current point x_k.
- **Step 2:** Solve problem (2.3) to find the new solution x_{k+1}.
- **Step 3:** If $\|x_k - x_{k+1}\| < \varepsilon$, then terminate, otherwise set $k = k + 1$ and go to Step 1.

Example 2.1. Consider the following problem

$$\begin{aligned} &\min(x_1+0.5)^2+(x_2-0.5)^2 \\ &x_1^2+x_2^2 \le 9, \\ &-x_1^2+x_2^2 \le -1. \end{aligned} \tag{2.4}$$

The global optimal solution and the optimal value are $x^* = (-1.0524, 0.3279)$ and 0.3347, respectively. By applying Algorithm 1, for several different initial point x_0, we have:

- For $x_0 = (0, 0)$, the algorithm failed.
- For $x_0 = (1, 0)$, the output of algorithm is the local optimal solution $x^1 = (-1.0200, 0.2008)$ and optimal value 2.3998.
- For $x_0 = (-1, 0)$, the output of algorithm is the global optimal solution $x^2 = (-1.0524, 0.3279)$ and optimal value 0.3347.

This concludes that the CCCP method depends on the initial solution.

$\boxed{2.2}$ AE method

In this section we discuss AE method, proposed by Deng et al. in [5]. This method can only solve a narrow class of problem (1.1) where all constraints are convex, i.e., $m_2 = 0$. So the problem (1.1) can be rewritten as follows.

$$\min\nolimits_{x\in C} f(x) \tag{2.5}$$

where C is a convex set as follows:

$$C = \{x \in \mathbb{R}^n |\ h_i(x) \le 0,\ i = 1, 2, \dots, m_1\}$$

The AE method is based on the cone of nonnegative quadratic functions that has been proposed by Sturm and Zhang [29]. As shown in [29], problem (2.5) is equivalent to the following linear conic programming problem:

$$(\mathrm{CP}) \qquad \begin{aligned} &\min H_0 \bullet Y \\ &Y_{11} = 1 \\ &Y \in D_C^*, \end{aligned} \tag{2.6}$$

where $D_C^* = cone\{Y \in S^{n+1} |\ Y = \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^T \ for\ some\ x \in C\}$ and $H_0 = [r_0\ p_0^T; p_0\ Q_0]$. Since the desired LMI representation of D_C^* does not exist, so the cone D_C^* is untractable. Therefore the cone D_C^* must be approximated by tractable cones. The tractable cone D_F and its dual cone D_F^* over the set $F \subseteq \mathbb{R}^n$ are introduced in [29], and proved that $S_+^{n+1} \subseteq D_F \subseteq D_C$ and $D_C^* \subseteq D_F^* \subseteq S_+^{n+1}$. By substituting the tractable cone D_F^* in problem (2.6) instead of D_C^*, a tighter lower bound is obtained.

Let $\mathcal{F} = \left\{\mathcal{F}_e^1, \dots, \mathcal{F}_e^l\right\}$ be a collection of full-dimensional ellipsoids,

$$\mathcal{F}_e^t = \left\{x \in \mathbb{R}^n \,\middle|\, x^T Q_e^t x + 2\left(p_e^t\right)^T x + r_e^t \le 0\right\}, \tag{2.7}$$

where $Q_e^t \in S_+^n$, $p_e^t \in \mathbb{R}^n$ and $r_e^t \in \mathbb{R}$, for $t = 1, \ldots, l$ such that $\{\mathcal{F}_e^t\}_{t=1}^l$ is an ellipsoidal cover of C, i.e.,

$$C \subseteq F = \bigcup_{t=1}^{l} \mathcal{F}_e^t. \tag{2.8}$$

Each cone $D_{\mathcal{F}_e^t}$ has an LMI representation [23]. According to [29], the cone D_F and its dual cone D_F^* over set F are defined as follows:

$$D_F = \left\{ U \in S^{n+1} \,\middle|\, \begin{bmatrix} 1 \\ x \end{bmatrix}^T U \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0 \;\; for\ all\ x \in F \right\} \tag{2.9}$$

$$D_F^* = cone \left\{ Y \in S^{n+1} \,\middle|\, Y = \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^T \;\; for\ some\ x \in F \right\}. \tag{2.10}$$

Since $D_C^* \subseteq D_F^*$, the problem (2.6) can be relaxed as the following problem

$$\text{(RCP)} \qquad \begin{aligned} \min\ & H_0 \bullet Y \\ & Y_{11} = 1 \\ & Y \in D_F^*. \end{aligned} \tag{2.11}$$

Theorem 2.2 ([5])**.** *Let sets $\mathcal{F}_e^t$, F, D_F and D_F^* be defined as in* (2.7)–(2.10)*. Then the following statements are equivalent:*

- *For any matrix $Y \in S^{n+1}$, we have $Y \in D_F^*$.*
- *For $t = 1, \ldots, l$*

$$Y = Y^1 + Y^2 + \cdots + Y^l, \;\; \begin{bmatrix} r_e^t & (p_e^t)^T \\ p_e^t & Q_e^t \end{bmatrix} \bullet Y^t \leq 0, \;\; Y^t \in S_+^{n+1}.$$

Using Theorem (2.2), problem (2.11) is converted as the following problem:

$$\text{(RCP)} \qquad \begin{aligned} \min\ & H_0 \bullet Y \\ & Y = Y^1 + Y^2 + \cdots + Y^l, \qquad Y_{11} = 1 \\ & \begin{bmatrix} r_e^t & (p_e^t)^T \\ p_e^t & Q_e^t \end{bmatrix} \bullet Y^t \leq 0, \quad Y^t \in S_+^{n+1}, \quad t = 1, 2, \ldots, l. \end{aligned} \tag{2.12}$$

The Reformulation-Linearization Technique (RLT) is applied to the problem (2.12) by adding the constraints $\begin{bmatrix} r_i & p_i^T \\ p_i & Q_i \end{bmatrix} \bullet Y^t \leq 0$ for $i = 1, 2, \ldots, m_1$ and $t = 1, 2, \ldots, l$. So problem (RCP-RLT) is as follows:

$$\text{(RCP-RLT)} \qquad \begin{aligned} l^* = \min\ & H_0 \bullet Y \\ & Y = Y^1 + Y^2 + \cdots + Y^l, \qquad Y_{11} = 1 \\ & \begin{bmatrix} r_i & p_i^T \\ p_i & Q_i \end{bmatrix} \bullet Y^t \leq 0, \quad i = 1, 2, \ldots, m_1, \quad t = 1, 2, \ldots, l \\ & \begin{bmatrix} r_e^t & (p_e^t)^T \\ p_e^t & Q_e^t \end{bmatrix} \bullet Y^t \leq 0, \quad Y^t \in S_+^{n+1}, \quad t = 1, 2, \ldots, l. \end{aligned} \tag{2.13}$$

Problem (2.13) is a conic programming problem. Let $\left(Y^*, (Y^1)^*, \dots, (Y^l)^*\right)$ be an optimal solution of problem (2.13). The next theorem provides a relationship between optimal solution of problems (2.5), (2.6), (2.12) and (2.13).

Theorem 2.3 ([5]). *Let F and $\mathcal{F}_e^t$ be defined as in* (2.9) *and* (2.7) *respectively and $V(P)$, $V(CP)$, $V(RCP)$ and $V(RCP\text{-}RLT)$ denote the optimal values of problems* (2.5), (2.6), (2.12) *and* (2.13) *respectively. If $C \subseteq \mathrm{F}$, then*

$$V(RCP) \leq V(RCP\text{-}RLT) \leq V(CP) = V(P).$$

Theorem 2.4 ([5]). *If $int(\mathcal{F}_e^t \cap C) \neq \emptyset$, for $t = 1, \dots, l$, then problem (RCP-RLT) is strongly feasible.*

Now the question arises how to find ellipsoids (2.7)? To answer this question, assume that a rectangle set $T = [u, v] = \{x \in \mathbb{R}^n \,|u_i \leq x_i \leq v_i\}$ is given. The corresponding ellipsoid $\mathcal{F}_{\mathrm{e}}^{\mathrm{T}}$ generated by T is considered as follows [5]:

$$\mathcal{F}_e^T = \left\{ x \in \mathbb{R}^n \left| \sum_{i=1}^{n} \frac{\left(2x_i - v_i - u_i\right)^2}{\left(v_i - u_i\right)^2} \leq n \right.\right\}. \tag{2.14}$$

To obtain initial rectangle $T_1 = \left[u^1, v^1\right] = \left[u_1^1, v_1^1\right] \times \cdots \times \left[u_n^1, v_n^1\right]$, we solve the following problems:

$$u_{i1}^1 = \min\nolimits_{x \in C} x_{i1} \tag{2.15}$$

and

$$v_{i1}^1 = \max\nolimits_{x \in C} x_{i1} \ , \tag{2.16}$$

for $i1 = 1, 2, \dots, n$. It is obvious that $C \subseteq T_t$. The initial ellipsoid $\mathcal{F}_{\mathrm{e}}^1$ is generated from T_1 as (2.14).

Let the rectangle sets $\{T_t\}$ and the ellipsoids $\{\mathcal{F}_e^t\}$ generated by the rectangle sets be detected. Let $T = \bigcup_{t=1}^l T_t$ and the set $\mathfrak{T} = \bigcup_{t=1}^l \{T_t\}$ be such that $C \subseteq \mathfrak{T}$. Then the set $F = \bigcup_{t=1}^l \mathcal{F}_e^t$, is an ellipsoid cover of C. The ellipsoids $\{\mathcal{F}_e^t\}$ will need to be an efficient arrangement to cover C. Consider the following decomposition

$$Y^* = \sum_{i:(Y^i)^* \neq 0} \sum_{s=1}^{n_i} \alpha_{is} \begin{bmatrix} 1 \\ x^{is} \end{bmatrix} \begin{bmatrix} 1 \\ x^{is} \end{bmatrix}^T, \tag{2.17}$$

that $n_i \in \{1, \dots, n+1\}$, $\alpha_{is} > 0$, $x^{is} \in \mathcal{F}_e^j$ and $\sum_{i:(Y^i)^* \neq 0} \sum_{s=1}^{n_i} \alpha_{is} = 1$. The solution x_{sen} is the most sensitive point if

$$x_{sen} = \underset{\{x^{is}:(Y^i)^* \neq 0; s=1,2,\dots,n_i\}}{\arg\min} \left\{ \left(x^{is}\right)^T Q_0 x^{is} + 2p_0^T x^{is} + r_0 \right\} \tag{2.18}$$

and the ellipsoid $\mathcal{F}_e^t$ is called the most sensitive ellipsoid, if $x_{sen} \in \mathcal{F}_e^t$ [5].

Theorem 2.5 ([5]). *Assume Y^* is the optimal solution to problem (RCP-RLT) with the most sensitive point x_{sen}, then*

$$\begin{bmatrix} 1 \\ x_{sen} \end{bmatrix} \begin{bmatrix} 1 \\ x_{sen} \end{bmatrix}^T \bullet H_0 \leq V(P).$$

Moreover, if $x_{sen} \in C$, then the matrix $\begin{bmatrix} 1 \\ x_{sen} \end{bmatrix} \begin{bmatrix} 1 \\ x_{sen} \end{bmatrix}^T$ *and x_{sen} are optimal solution for problems* (2.6) *and* (2.5) *respectively.*

Therefore, if $x_{sen} \in C$, then x_{sen} is an optimal solution of problem (2.5). Otherwise $f(x_{sen})$ is a lower bound of problem (2.5). To get a better lower bound, the ellipsoid cover F must be refined. The most sensitive point x_{sen} and the most sensitive ellipsoid $\mathcal{F}_e^t$ in F are used to determine which rectangle sets need to be refined. Let $id = arg\ max_{\{i=1,\dots,n\}} \{v_i^t - u_i^t\}$, then $T_t = [u^t, v^t]$ is splited by half as $T_{t1} = \left[u^{t1}, v^{t1}\right]$ and $T_{t2} = \left[u^{t2}, v^{t2}\right]$, where $u^{t1} = u^t$, $v^{t2} = v^t$, $v_i^{t1} = v_i^t$, $u_i^{t2} = u_i^t$, for $i \neq id$, and $v_{id}^{t1} = u_{id}^{t2} = \frac{u_{id}^t + v_{id}^t}{2}$. Two ellipsoids $\mathcal{F}_e^{t1}$ and $\mathcal{F}_e^{t2}$ generated from T_{t1} and T_{t2} are considered as (2.7). If $int(T_{ti} \cap C) = \emptyset$, the rectangle set T_{ti} is eliminated for either $i = 1$ *or* 2. To specify which rectangle set must be eliminated, the following problems are considered:

$$\begin{array}{c} \varphi = \min_{x \in C} x_{id} \\ u^t \leq x \leq v^t \end{array} \tag{2.19}$$

and

$$\begin{array}{c} \psi = \max_{x \in C} x_{id} \\ u^t \leq x \leq v^t \end{array} \tag{2.20}$$

The problems (2.19) and (2.20) are convex and can be solved efficiently [2,4].

The rectangle sets in $\mathfrak{T}$ are changed in the following way:

$$\mathfrak{T} = \mathfrak{T} \backslash \{T_t\} \cup \{T_{t2}\}, \qquad \text{if} \quad \varphi > \frac{u_{id}^t + v_{id}^t}{2} \tag{2.21}$$

$$\mathfrak{T} = \mathfrak{T} \backslash \{T_t\} \cup \{T_{t1}\}, \qquad \text{if} \quad \psi < \frac{u_{id}^t + v_{id}^t}{2} \tag{2.22}$$

$$\mathfrak{T} = \mathfrak{T} \backslash \{T_t\} \cup \{T_{t1}\} \cup \{T_{t2}\}, \qquad \text{otherwise} \tag{2.23}$$

Theorem 2.6 ([5])**.** *If the rectangle sets in $\mathfrak{T}$ are constructed according to* (2.21)–(2.23)*, then*

$$\forall T_t \in \mathfrak{T}: \ int(T_t \cap C) \neq \emptyset.$$

If $x_{sen} \notin C$ then the measure of infeasibility of the current sensitive point x_{sen} is defined as the optimal value of the following problem:

$$\min_{x \in C} \|x - x_{sen}\|_\infty \ . \tag{2.24}$$

Problem (2.24) is convex and can be solved efficiently. Let $\overline{x}$ be an optimal solution of problem (2.24). If $\|\overline{x} - x_{sen}\|_\infty \leq \varepsilon$, $\overline{x}$ is as an ε-optimal solution of problem (2.5), that is an approximate optimal solution when ε is sufficiently small.

Algorithm of AE method is as follows:

Algorithm 2. AE method

- **Initialization**: Given an tolerance $\varepsilon > 0$ and set $low = -\infty$ and $upp = +\infty$.

- **Step 1**: Solve problems (2.15) and (2.16) for $i = 1, 2, \ldots, n$ to get an initial rectangle set T_1 and the corresponding ellipsoid $\mathcal{F}_e^1$. Set $\mathfrak{T} = \{T_1\}$ and $F = \{\mathcal{F}_e^1\}$.
- **Step 2**: Solve problem (2.13) with the approximation cone D_F^*. Take $low = \max\{low, l^*\}$.
- **Step 3**: Decompose Y^* according to (2.17) and obtain the most sensitive point x_{sen} and the most sensitive ellipsoid $\mathcal{F}_e^t \in F$ with (2.18). If $x_{sen} \in C$, x_{sen} is the optimal solution and stop.
- **Step 4**: Set $id = \arg\max_{\{i=1,\ldots,n\}} \{v_i^t - u_i^t\}$ and generate ellipsoids $\mathcal{F}_e^{t1}$ and $\mathcal{F}_e^{t2}$.
- **Step 5**: Solve problems (2.19) and (2.20) to obtain the optimal values $\overline{\varphi}$ and $\overline{\psi}$, respectively.

  – If $\varphi > \frac{u_{id}^t + v_{id}^t}{2}$, set $F = F \backslash \{\mathcal{F}_e^t\} \cup \left\{\mathcal{F}_e^{t2}\right\}$ and $\mathfrak{T} = \mathfrak{T} \backslash \{T_t\} \cup \left\{\left[u^{t2}, v^{t2}\right]\right\}$.
  – If $\psi < \frac{u_{id}^t + v_{id}^t}{2}$, set $F = F \backslash \{\mathcal{F}_e^t\} \cup \left\{\mathcal{F}_e^{t1}\right\}$ and $\mathfrak{T} = \mathfrak{T} \backslash \{T_t\} \cup \left\{\left[u^{t1}, v^{t1}\right]\right\}$.
  – Otherwise $F = F \backslash \{\mathcal{F}_e^t\} \cup \left\{\mathcal{F}_e^{t1}\right\} \cup \left\{\mathcal{F}_e^{t2}\right\}$ and $\mathfrak{T} = \mathfrak{T} \backslash \{T_t\} \cup \left\{\left[u^{t1}, v^{t1}\right]\right\} \cup \left\{\left[u^{t2}, v^{t2}\right]\right\}$.

- **Step 6**: Solve problem (2.24) to obtain the solution $\overline{x}$. Set $upp = \min\{p(\overline{x}), upp\}$.
- **Step 7**: If $|upp - low| < \varepsilon$, stop and return $\tilde{x} = \overline{x}$ as an ε-optimal solution. Otherwise, take $k = k + 1$ and go to *Step 2*.

$\boxed{3}$ Proposed Method

In this section, we extend AE method, we called it EAE (extended Adaptive Ellipsoid-based) method, to solve problem (1.1) and we prove the convergence of the proposed method. We rewrite problem (1.1) as follows:

$$\min_{x \in \Lambda} f(x) \tag{3.1}$$

where $\Lambda = C \cap N$ and the sets C and N are as follows:

$$C = \{x \in \mathbb{R}^n |\ h_i(x) \leq 0,\ i = 1, 2, \ldots, m_1\}$$

$$N = \{x \in \mathbb{R}^n |\ g_j(x) \leq 0,\ j = 1, 2, \ldots, m_2\}.$$

Λ is a non-convex set, and we make the following assumptions in the rest of the paper.

Assumption 3.1. The feasible set of problem (1.1) is nonempty (i.e., $\Lambda \neq \emptyset$).

Assumption 3.2. The problem (1.1) contains at least one strictly convex constraint, i.e., there exists i such that the function $h_i(x)$ be strictly convex.

3.1 EAE method

In this subsection, we modify some steps of the AE method to find a global solution of the non-convex problem (1.1).

* _Initial rectangle._ According to Assumption 3.2, the feasible region of the problem (1.1) is bounded. So, this assumption guarantees the existence of an initial rectangle. Similar to AE method, to get an initial rectangle, we consider the following problems:

$$u_{i1}^1 = \min_{x\in\Lambda} x_{i1} \tag{3.2}$$

and

$$v_{i1}^1 = \max_{x\in\Lambda} x_{i1} \quad , \tag{3.3}$$

for $i1 = 1, 2, \ldots, n$. Since Λ is non-convex, we cannot solve problems (3.2) and (3.3). We use the SDP relaxation of problems (3.2) and (3.3) as follows:

$$\begin{array}{rl} \overline{u}_{i1}^1 = \min_{x\in\mathbb{R}^n} x_{i1} & \\ Q_i \bullet X + 2{p_i}^T x + r_i \leq 0 & i = 1, 2, \ldots, m_1 \\ \overline{Q}_j \bullet X + 2{\overline{p}_j}^T x + \overline{r}_j \leq 0 & j = 1, 2, \ldots, m_2 \\ X - xx^T \succeq 0 & \end{array} \tag{3.4}$$

and

$$\begin{array}{rl} \overline{v}_{i1}^1 = \max_{x\in\mathbb{R}^n} x_{i1} & \\ Q_i \bullet X + 2{p_i}^T x + r_i \leq 0 & i = 1, 2, \ldots, m_1 \\ \overline{Q}_j \bullet X + 2{\overline{p}_j}^T x + \overline{r}_j \leq 0 & j = 1, 2, \ldots, m_2 \\ X - xx^T \succeq 0. & \end{array} \tag{3.5}$$

Now, we take $\overline{T}_1 = \left[\overline{u}^1, \overline{v}^1\right] = \left[\overline{u}_1^1, \overline{v}_1^1\right] \times \cdots \times \left[\overline{u}_n^1, \overline{v}_n^1\right]$ as an approximate of the initial rectangle set $T_1 = \left[u^1, v^1\right]$. Since problems (3.4) and (3.5) are SDP relaxation problems corresponding to problems (2.15) and (2.16) respectively, we have $\overline{u}_{i1}^1 \leq u_{i1}^1 \leq v_{i1}^1 \leq \overline{v}_{i1}^1$ (for $i1 = 1, \ldots, n$), thus $\mathrm{T}_1 \subseteq \overline{\mathrm{T}}_1$ and rectangle $\overline{\mathrm{T}}_1$ covers the feasible region Λ. In addition, we take the ellipsoid $\overline{\mathcal{F}}_{\mathrm{e}}^1$ generated from $\overline{\mathrm{T}}_1$ by (2.14) as the initial ellipsoid covers F of Λ.

* _Solving problem (RCP-RLT)_. Let the ellipsoid $\overline{\mathcal{F}}_{\mathrm{e}}^{\mathrm{t}}$ be generated by $\overline{\mathrm{T}}_{\mathrm{t}}$ such that $\Lambda \subseteq F = \bigcup_{t=1}^{l} \overline{\mathcal{F}}_e^t$. Therefore problem (2.13) is unchanged and by adding the RLT constraints $\begin{bmatrix} r_i & p_i^T \\ p_i & Q_i \end{bmatrix} \bullet Y^t \leq 0$ and $\begin{bmatrix} \overline{r}_j & \overline{p}_j^T \\ \overline{p}_j & \overline{Q}_j \end{bmatrix} \bullet Y^t \leq 0$ (for $i = 1, 2, \ldots, m_1$, $j = 1, 2, \ldots, m_2$ and $t = 1, 2, \ldots, l$) to problem (2.13), we have

$$\text{(RCP-RLT)} \qquad \begin{array}{l} l^* = \min H_0 \bullet Y \\ \qquad Y = Y^1 + Y^2 + \cdots + Y^l, \qquad Y_{11} = 1 \\ \begin{bmatrix} r_i & p_i^T \\ p_i & Q_i \end{bmatrix} \bullet Y^t \leq 0, \quad i = 1, 2, \ldots, m_1, \quad t = 1, 2, \ldots, l \\ \begin{bmatrix} \overline{r}_j & \overline{p}_j^T \\ \overline{p}_j & \overline{Q}_j \end{bmatrix} \bullet Y^t \leq 0, \quad j = 1, 2, \ldots, m_2, \quad t = 1, 2, \ldots, l \\ \begin{bmatrix} r_e^t & (p_e^t)^T \\ p_e^t & Q_e^t \end{bmatrix} \bullet Y^t \leq 0, \quad Y^t \in S_+^{n+1}, \quad t = 1, 2, \ldots, l. \end{array} \tag{3.6}$$

Problem (3.6) is a linear conic programming problem that can be solved effective. Let $\left(Y^*, (Y^1)^*, \ldots, (Y^l)^*\right)$ be an optimal solution of problem (3.6), then we have the following theorem.

Theorem 3.1. *Let $V(QCQP)$ and $V(RCP\text{-}RLT)$ denote the optimal values of problems* (1.1) *and* (3.6) *respectively. If* $\Lambda \subseteq \mathrm{F}$, *then*

$$V(RCP\text{-}RLT) \leq V(QCQP).$$

Proof. Let $x \in \Lambda$. There exists some $t_0 \in \{1, \ldots, l\}$ such that $x \in \mathcal{F}_e^{t_0}$. Take $Y^{t0} = \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^T$ and $Y^t = 0$ for $t \neq t0$. So $\left(Y, Y^1, \ldots, Y^l\right)$ is a feasible solution to problem (3.6), where $Y = \sum_{i=1}^{l} Y^i$. Thus $V(\text{RCP-RLT}) \leq V(QCQP)$ and the proof is comleted. ☐

Theorem 3.2. *Let $\overline{\mathcal{F}}_{\mathrm{e}}^{\mathrm{t}}$ be generated from $\overline{T}_t$ and $int(\overline{\mathcal{F}}_{\mathrm{e}}^{\mathrm{t}} \cap \Lambda) \neq \emptyset$ for $t = 1, \ldots, l$, then problem* (3.6) *is strongly feasible.*

Proof. Since for any $t = 1, \ldots, l$, $int(\overline{\mathcal{F}}_{\mathrm{e}}^{\mathrm{t}} \cap \Lambda) \neq \emptyset$, thus there exists a point $\hat{x}^t \in int(\overline{\mathcal{F}}_e^t \cap \Lambda)$. So

$$(\hat{x}^t)^T Q_i \hat{x}^t + 2{p_i}^T \hat{x}^t + r_i < 0, \qquad i = 1, 2, \ldots, m_1$$

$$(\hat{x}^t)^T \overline{Q}_j \hat{x}^t + 2{\overline{p}_j}^T \hat{x}^t + \overline{r}_j < 0, \qquad j = 1, 2, \ldots, m_2$$

$$(\hat{x}^t)^T Q_e^t \hat{x}^t + 2{p_e^t}^T \hat{x}^t + r_e^t < 0,$$

for $t = 1, \ldots, l$. Let us define $Y^t = \begin{bmatrix} 1 \\ \hat{x}^t \end{bmatrix} \begin{bmatrix} 1 \\ \hat{x}^t \end{bmatrix}^T$ for $t = 1, \ldots, l$. It is easy to check that $Y^t \in S_{++}^{n+1}$ and

$$\begin{bmatrix} r_i & p_i^T \\ p_i & Q_i \end{bmatrix} \bullet Y^t < 0, \quad \begin{bmatrix} \overline{r}_j & \overline{p}_j^T \\ \overline{p}_j & \overline{Q}_j \end{bmatrix} \bullet Y^t < 0, \begin{bmatrix} r_e^t & (p_e^t)^T \\ p_e^t & Q_e^t \end{bmatrix} \bullet Y^t < 0.$$

Thus $\left(Y, Y^1, \ldots, Y^l\right)$ is strongly feasible for problem *(RCP-RLT)* where $Y = Y^1 + \cdots + Y^l$. ☐

* <u>Matrix decomposition</u>. Now, to find the solution of the problem (3.6), we use matrix decomposition (2.17) for the optimal solution $\left(Y^*, (Y^1)^*, \ldots, (Y^l)^*\right)$ of problem (3.6).
* <u>*Selection most sensitive ellipsoid and change the rectangle set*</u>. We remember that if $x_{sen} \in \Lambda$, then x_{sen} is an optimal solution of problem (1.1), otherwise, $f(x_{sen})$ is a lower bound of problem (1.1). Analogous to AE method, to obtain the tighter lower bound, we use the most sensitive ellipsoid $\overline{\mathcal{F}}_e^t$ containing x_{sen} to determine that which ones need to be refined. Let $id = \arg\max_{\{i=1,\ldots,n\}} \left\{\overline{v}_i^t - \overline{u}_i^t\right\}$, then $\overline{\mathrm{T}}_{\mathrm{t}}$ is splitted by $\overline{T}_{t1} = \left[\overline{u}^{t1}, \overline{v}^{t1}\right]$ and $\overline{T}_{t2} = \left[\overline{u}^{t2}, \overline{v}^{t2}\right]$, where $\overline{u}^{t1} = \overline{u}^t$, $\overline{v}^{t2} = \overline{v}^t$, $\overline{v}_i^{t1} = \overline{v}_i^t$, $\overline{u}_i^{t2} = \overline{u}_i^t$, for

$i \neq id$, and $\overline{v}_{id}^{t1} = \overline{u}_{id}^{t2} = \frac{\overline{u}_{id}^{t} + \overline{v}_{id}^{t}}{2}$. Two ellipsoids $\overline{\mathcal{F}}_{\mathrm{e}}^{\mathrm{t1}}$ and $\overline{\mathcal{F}}_{\mathrm{e}}^{\mathrm{t2}}$ are generated by $\overline{\mathrm{T}}_{\mathrm{t1}}$ and $\overline{\mathrm{T}}_{\mathrm{t2}}$ according to

$$\overline{\mathcal{F}}_e^{t1} = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} \frac{\left(2x_i - \overline{v}_i^{t1} - \overline{u}_i^{t1}\right)^2}{\left(\overline{v}_i^{t1} - \overline{u}_i^{t1}\right)^2} \leq n \right\} \tag{3.7}$$

and

$$\overline{\mathcal{F}}_e^{t2} = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} \frac{\left(2x_i - \overline{v}_i^{t2} - \overline{u}_i^{t2}\right)^2}{\left(\overline{v}_i^{t2} - \overline{u}_i^{t2}\right)^2} \leq n \right\}. \tag{3.8}$$

In order to determine which rectangle set should be eliminated, we have to solve the following problems:

$$\begin{array}{c} \varphi = \min_{x \in \Lambda} x_{id} \\ \overline{u}^t \leq x \leq \overline{v}^t \end{array} \tag{3.9}$$

and

$$\begin{array}{c} \psi = \max_{x \in \Lambda} x_{id} \\ \overline{u}^t \leq x \leq \overline{v}^t. \end{array} \tag{3.10}$$

Since problems (3.9) and (3.10) are non-convex, we consider the relaxation SDP form of them as follows:

$$\begin{array}{c} \overline{\varphi} = \min_{x \in \mathbb{R}^n} x_{id} \\ Q_i \bullet X + 2{p_i}^T x + r_i \leq 0 \quad i = 1, 2, \ldots, m_1 \\ \overline{Q}_j \bullet X + 2{\overline{p}_j}^T x + \overline{r}_j \leq 0 \quad j = 1, 2, \ldots, m_2 \\ X - xx^T \succeq 0 \\ \overline{u}^t \leq x \leq \overline{v}^t. \end{array} \tag{3.11}$$

and

$$\begin{array}{c} \overline{\psi} = \max_{x \in \mathbb{R}^n} x_{id} \\ Q_i \bullet X + 2{p_i}^T x + r_i \leq 0 \quad i = 1, 2, \ldots, m_1 \\ \overline{Q}_j \bullet X + 2{\overline{p}_j}^T x + \overline{r}_j \leq 0 \quad j = 1, 2, \ldots, m_2 \\ X - xx^T \succeq 0 \\ \overline{u}^t \leq x \leq \overline{v}^t. \end{array} \tag{3.12}$$

Problems (3.11) and (3.12) are convex programming problems. It is obvious that the relationship between optimal value of problems (3.9)–(3.12) is as follows:

$$\overline{\varphi} \leq \varphi \leq \psi \leq \overline{\psi} \tag{3.13}$$

Lemma 3.3. *Let* $int(\overline{T}_t \cap \Lambda) \neq \emptyset$ *, then we have*

1. *if* $\overline{\varphi} > \frac{\overline{u}_{id}^t + \overline{v}_{id}^t}{2}$ *then* $int\left(\overline{T}_{t1} \cap \Lambda\right) = \emptyset$ *.*
2. *if* $\overline{\psi} < \frac{\overline{u}_{id}^t + \overline{v}_{id}^t}{2}$ *then* $int\left(\overline{T}_{t2} \cap \Lambda\right) = \emptyset$*.*

Proof. 1. Suppose, contrary to claim, that $int(\overline{T}_{t1} \cap \Lambda) \neq \emptyset$. Therefore there is a point x^{t1} that $x^{t1} \in \overline{T}_{t1} \cap \Lambda$. Since φ is the optimal value of problem (3.9), we have

$$\varphi \leq x_{id}^{t1} \leq \frac{\overline{u}_{id}^t + \overline{v}_{id}^t}{2} < \overline{\varphi}$$

that contradicts to (3.13).

2. Its proof is similar to part (1).

$\square$

The sets $\mathfrak{T}$ is updated as follows:

$$\mathfrak{T}=\mathfrak{T}\backslash\left\{\overline{T}_t\right\}\cup\left\{\overline{T}_{t2}\right\}, \qquad \text{if} \quad \overline{\varphi}>\frac{\overline{u}^t_{id}+\overline{v}^t_{id}}{2} \tag{3.14}$$

$$\mathfrak{T}=\mathfrak{T}\backslash\left\{\overline{T}_t\right\}\cup\left\{\overline{T}_{t1}\right\}, \qquad \text{if} \quad \overline{\psi}<\frac{\overline{u}^t_{id}+\overline{v}^t_{id}}{2} \tag{3.15}$$

$$\mathfrak{T}=\mathfrak{T}\backslash\left\{\overline{T}_t\right\}\cup\left\{\overline{T}_{t1}\right\}\cup\left\{\overline{T}_{t2}\right\}, \qquad \text{otherwise} \tag{3.16}$$

Theorem 3.4. *Let $int(\overline{T}_t\cap\Lambda)\neq\emptyset$ for each rectangle set $\overline{\mathrm{T}}_{\mathrm{t}}$ in $\mathfrak{T}$. If the rectangle sets are added into $\mathfrak{T}$ according to* (3.14)–(3.16)*, then $int(\overline{T}_{t1}\cap\Lambda)\neq\emptyset$ or $int(\overline{T}_{t2}\cap\Lambda)\neq\emptyset$.*

Proof. According to (3.14)–(3.16), we have three following cases:

- If $\overline{\varphi}>\frac{\overline{u}^t_{id}+\overline{v}^t_{id}}{2}$, then $\overline{T}_{t1}\cap\Lambda$ has no interior and can not add to $\mathfrak{T}$. We claim that $\overline{T}_t\cap\Lambda=\overline{T}_{t2}\cap\Lambda$. To see this, assume there is a point x^{t1} such that $x^{t1}\in\left(\overline{T}_t\backslash\overline{T}_{t2}\right)\cap\Lambda$, then $x^{t1}_{id}\leq\frac{\overline{u}^t_{id}+\overline{v}^t_{id}}{2}<\overline{\varphi}$. Since x^{t1} is a feasible solution for problem (3.11), thus $\overline{\varphi}$ is not optimal value for (3.11), which contradicts to $\overline{\varphi}>\frac{\overline{u}^t_{id}+\overline{v}^t_{id}}{2}$.
- If $\overline{\psi}<\frac{\overline{u}^t_{id}+\overline{v}^t_{id}}{2}$, then the proof is similar.
- If $\overline{\varphi}\leq\frac{\overline{u}^t_{id}+\overline{v}^t_{id}}{2}\leq\overline{\psi}$, let $x_{\min}$ and $x_{\max}$ be the optimal solution of problems (3.11) and (3.12), respectively, and also $x^{int}\in int(\overline{T}_t\cap\Lambda)$. Then

  $$x^{t1}=\lambda x^{int}+(1-\lambda)x_{\min}\in int(\overline{T}_{t1}\cap\Lambda)$$

  or

  $$x^{t2}=\lambda x^{int}+(1-\lambda)x_{\max}\in int(\overline{T}_{t2}\cap\Lambda),$$

  for some $\lambda\in(0,1)$.

So proof is completed. $\square$

* <u>*Solving problem* (2.24)</u>. Similar to AE method, if $x_{sen}\notin\Lambda$ then by solving the following problem, we can check whether x_{sen} is close enough to the feasible region Λ:

$$\min\nolimits_{x\in\Lambda}\|x-x_{sen}\|_{\infty}\ . \tag{3.17}$$

Since the feasible region Λ is non-convex, we apply the CCCP method [31] for solving problem (3.17). First, we decompose

$$\overline{Q}_j=\overline{Q}^+_j-\overline{Q}^-_j\ ,$$

That $\overline{Q}_j^+, \overline{Q}_j^- \succcurlyeq 0$, for $j = 1, 2, \ldots, m_2$. Let $\overline{x}$ be a feasible solution for problem (1.1). From the first-order convexity condition [4] of $x^T\overline{Q}_j^- x$ at the feasible solution $\overline{x}$, we have

$$x^T\overline{Q}_j^- x \geq -(\overline{x})^T\overline{Q}_j^-\overline{x} + 2x^T\overline{Q}_j^-\overline{x}.$$

Therefore the problem (3.17) is approximated to the following problem:

$$\begin{array}{rl}\min_{x\in\mathbb{R}^n} \|x - x_{sen}\|_\infty & \\ x^TQ_ix + 2{p_i}^Tx + r_i \leq 0 & i = 1, 2, \ldots, m_1 \\ x^T\overline{Q}_j^+ x + 2{\overline{p}_j}^Tx + \overline{r}_j + (\overline{x})^T\overline{Q}_j^-\overline{x} - 2x^T\overline{Q}_j^-\overline{x} \ \leq 0 & j = 1, 2, \ldots, m_2.\end{array} \tag{3.18}$$

Problem (3.18) is convex and assume that $\overline{x}_{new}$ is an optimal solution of problem (3.18). If $\|\overline{x}_{new} - x_{sen}\|_\infty \leq \varepsilon$, we consider $\overline{x}_{new}$ as an ε -optimal solution of problem (1.1).

The algorithm of proposed method is as follows:

Algorithm 3. (EAE)

- **Initialization**: Given $\varepsilon > 0$. Let $low = -\infty$, $upp = +\infty$ and $x^0 \in \Lambda$.
- **Step 1**: Solve problems (3.4) and (3.5) to get $\overline{T}_1$ and $\overline{\mathcal{F}}_e^1$ and set $\mathfrak{T} = \{\overline{T}_1\}$, $F = \{\overline{\mathcal{F}}_e^1\}$, $\overline{\mathfrak{T}} = \emptyset$ and $k = 1$.
- **Step 2**: Solve problem (3.6) and set $low = \max\{low, l^*\}$.
- **Step 3**: Decompose Y^* according to (2.17) and obtain the most sensitive point x_{sen} , the most sensitive ellipsoid $\overline{\mathcal{F}}_e^t$ and the corresponding rectangle $\overline{T}_t$. Set $y^k = x_{sen}$. If $x_{sen} \in \Lambda$, then stop and x_{sen} is an optimal solution and take $x^k = y^k = x_{sen}$.
- **Step 4**: Set $id = \arg\max_{\{i=1,\ldots,n\}}\left\{\overline{v}_i^t - \overline{u}_i^t\right\}$. Generate ellipsoids $\overline{\mathcal{F}}_e^{t1}$ and $\overline{\mathcal{F}}_e^{t2}$ according to (3.7) and (3.8), respectively.
- **Step 5**: Calculate $\overline{\varphi}$ and $\overline{\psi}$ from solving problems (3.11) and (3.12), respectively.
  - If $\overline{\varphi} > \frac{u_{id}^t + v_{id}^t}{2}$, set $F = F\backslash\{\mathcal{F}_e^t\} \cup \{\overline{\mathcal{F}}_e^{t2}\}$, $\mathfrak{T} = \mathfrak{T}\backslash\{\overline{T}_t\} \cup \left\{\left[\overline{u}^{t2}, \overline{v}^{t2}\right]\right\}$ and $\overline{\mathfrak{T}} = \overline{\mathfrak{T}} \cup \left\{\left[\overline{u}^{t1}, \overline{v}^{t1}\right]\right\}$.
  - If $\overline{\psi} < \frac{\overline{u}_{id}^t + \overline{v}_{id}^t}{2}$, set $F = F\backslash\{\mathcal{F}_e^t\} \cup \{\overline{\mathcal{F}}_e^{t1}\}$, $\mathfrak{T} = \mathfrak{T}\backslash\{\overline{T}_t\} \cup \left\{\left[\overline{u}^{t1}, \overline{v}^{t1}\right]\right\}$ and $\overline{\mathfrak{T}} = \overline{\mathfrak{T}} \cup \left\{\left[\overline{u}^{t2}, \overline{v}^{t2}\right]\right\}$.
  - Otherwise $F = F\backslash\{\mathcal{F}_e^t\}\cup\{\overline{\mathcal{F}}_e^{t1}\}\cup\{\overline{\mathcal{F}}_e^{t2}\}$ and $\mathfrak{T} = \mathfrak{T}\backslash\{\overline{T}_t\}\cup\left\{\left[\overline{u}^{t1}, v^{t1}\right]\right\}\cup\left\{\left[\overline{u}^{t2}, v^{t2}\right]\right\}$.
- **Step 6**: Solve problem (3.18) at the current solution x^{k-1} to obtain a new solution x^k and set $upp = \min\{f(x^k), upp\}$.

- **Step 7**: If $|upp - low| < \varepsilon$, stop and the output is x^k. Otherwise, set $k = k+1$ and go to *Step 2*.

Remark 3.5. From Step 3, we conclude that at most one additional rectangle is added to the current rectangle cover at the end of each iteration. Therefore, the complexity of problem (RCP-RLT) does not increase drastically.

Remark 3.6. The set $\overline{\mathfrak{T}}$ is the set of deleted rectangles and is used for the proof in Lemma (3.7). It is easy to verify that the total volume of all rectangle sets in $\mathfrak{T}$ and $\overline{\mathfrak{T}}$ always equals to the volume of the initial rectangle set $\overline{T}_1$.

3.2 Convergence of the EAE method

In this subsection, we prove the convergence of the proposed method. Also, we show that the algorithm will terminate in finite steps. The following lemmas are useful in the proof of convergence of EAE method.

Lemma 3.7. *Let $\varepsilon > 0$, $\overline{T}_1 = [\overline{u}^1, \overline{v}^1] \in \mathbb{R}^n$ be the initial rectangle and δ_0 be the longest edge of the initial rectangle, i.e. $\delta_0 = \max_{1\le i\le n}\{\overline{v}_i^1 - \overline{u}_i^1\}$. Then, after $\left(\left\lceil \frac{\delta_0\sqrt{n}}{\varepsilon}\right\rceil\right)^n$ iterations of Algorithm 3, there exists at least some rectangle $[\overline{u}^t, \overline{v}^t]$ such that $\|\overline{v}^t - \overline{u}^t\|_\infty \le \frac{\varepsilon}{\sqrt{n}}$.*

Proof. First, we prove the lemma for the case $n = 1$. Let $\overline{T}_1 = [\overline{u}^1, \overline{v}^1] \subseteq \mathbb{R}$, so $\delta_0 = \overline{v}^1 - \overline{u}^1$. We claim that after $k \ge 1$ iterations, there exists at least one subinterval $[\overline{u}^{t_k}, \overline{v}^{t_k}] \subseteq \overline{T}_1$ satisfying $\overline{v}^{t_k} - \overline{u}^{t_k} \le \delta$, where $\delta = \frac{\delta_0}{k}$. Suppose, contrary to the claim, that after k iterations, the length of all subintervals in $\mathfrak{T} \cup \overline{\mathfrak{T}}$ is grether than δ. Let δ_i denote the length of i-th subinterval, for $i = 1, \ldots, k$. So

$$\sum_{i=1}^{k} \delta_i > \sum_{i=1}^{k} \frac{\delta_0}{k} = \delta_0.$$

It contradicts to $\sum_{i=1}^{k} \delta_i = \delta_0$. Thus the claim is proved.

It is necessary to show that, after $k = \left\lceil \frac{\delta_0}{\varepsilon}\right\rceil$ iterations, there is at least one subinterval $[\overline{u}^{t_k}, \overline{v}^{t_k}]$ such that $\overline{v}^{t_k} - \overline{u}^{t_k} \le \varepsilon$. This fact is concluded from the following inequality

$$\frac{\delta_0}{k} \le \varepsilon.$$

Let $\overline{T}_1 = [\overline{u}^1, \overline{v}^1] = [\overline{u}_1^1, \overline{v}_1^1] \times [\overline{u}_2^1, \overline{v}_2^1] \times \cdots \times [\overline{u}_n^1, \overline{v}_n^1] \subseteq \mathbb{R}^n$ and $\delta_0 = \max_{1\le i\le n}\{v_i^1 - u_i^1\}$. So, after $\left(\left\lceil \frac{\delta_0\sqrt{n}}{\varepsilon}\right\rceil\right)^n$ iterations, there is at least one rectangle $[\overline{u}^{t_k}, \overline{v}^{t_k}] \subseteq \overline{T}_1$ such that $\|\overline{v}^{t_k} - \overline{u}^{t_k}\|_\infty \le \frac{\varepsilon}{\sqrt{n}}$. $\square$

Let x^k be obtained by solving problem (3.18) and y^k be the most sensitive point at iteration k. We have the following lemma.

Lemma 3.8. *For any given instance of problem (1.1) and $\varepsilon > 0$, there exists an N_ε-th iteration such that $\|x^k - y^k\|_\infty \le \varepsilon$ for $k \ge N_\varepsilon$.*

Proof. If at some iteration k, Algorithm 3 obtains a solution y^k such that $y^k \in \Lambda$, then $x^k = y^k$ and the lemma holds. Otherwise, let $B_\varepsilon(\Delta) = \{y \in \mathbb{R}^n | \|x - y\|_\infty \leq \varepsilon, \ \ \exists x \in \Lambda\}$. It is necessary to show that at some iteration, there exists a solution y^k such that $y^k \in B_\varepsilon(\Delta)$. We denote the longest edge of the initial rectangle $\overline{T}_1$ by δ_0, i.e. $\delta_0 = \max_{1 \leq i \leq n}\{v_i^1 - u_i^1\}$. From (2.14), it is clear that the length of i-th half axis of ellipsoid $\mathcal{F}_e^T$ is equal to $\sqrt{n}(\overline{v}_i^{t_k} - \overline{u}_i^{t_k})/2$. According to Lemma (3.7), after $\left(\left\lceil \frac{\delta_0\sqrt{n}}{2\varepsilon} \right\rceil\right)^n$ iterations, there is at least one rectangle $[\overline{u}^{t_k}, \overline{v}^{t_k}]$ such that $\|\overline{v}^{t_k} - \overline{u}^{t_k}\|_\infty \leq \frac{\varepsilon}{\sqrt{n}}$. Assume that this rectangle was generated at N_1 iterations. Thus, for $x^k, y^k \in \mathcal{F}_e^T$, we have

$$\|x^k - y^k\|_\infty \leq \sqrt{n}\|\overline{v}^{t_k} - \overline{u}^{t_k}\|_\infty \leq \varepsilon.$$

So, $y^k \in B_\varepsilon(\Delta)$ and the proof is completed. □

The following theorem proves convergence of the proposed method.

Theorem 3.9. *Let z^* be an optimal solution for problem* (1.1) *and the sequences $\left\{x^k\right\}_{k=1}^\infty$ and $\left\{y^k\right\}_{k=1}^\infty$ be generated by Algorithm 3, then:*

1. $lim_{k\to\infty} f(x^k) = lim_{k\to\infty} f(y^k) = f(z^*)$.
2. *The solution z^* is a global optimal solution for problem* (1.1).

Proof. 1. From Theorem (3.1) and the problem (3.18), it is obvious that

$$\forall k \geq 1, \qquad f(y^k) \leq f(z^*) \leq f(x^k). \tag{3.19}$$

According to Lemma (3.8) and this fact that the objective function $f(x)$ is continuous, we conclude that

$$lim_{k\to\infty} |f(x^k) - f(y^k)| = 0, \tag{3.20}$$

It follows from (3.19) and (3.20) that

$$lim_{k\to\infty} f(x^k) = lim_{k\to\infty} f(y^k) = f(z^*).$$

2. Contrary to claim, we assume that z^* is not a global optimal solution. So, there is a feasible solution $\overline{z} \in \Lambda$ such that

$$f(\overline{z}) < f(z^*).$$

According to (3.19), we conclude

$$f(\overline{z}) < f(y^k), \tag{3.21}$$

for some $k \geq 1$. Since $\{f(y^k)\}$ are the optimal values of problem (3.6), (3.21) contradicts to Theorem (3.1).

□

4 Numerical Results

In this section, numerical results are given to validate the theoretical results obtained in previous sections. The proposed algorithm is run on an Intel Core i7 PC and 8G memory under Windows 7 and MATLAB R2014a. Also, all convex programming problems given in this paper are solved with CVX solver [12]. The parameter ε is chosen 10^{-5}.

$\boxed{4.1}$ Numerical examples

In this subsection we give three examples to demonstrate how the proposed algorithm works.

Example 4.1. Consider the problem (1.1) with the following date [29]:

$$Q_0 = \begin{bmatrix} -2 & 10 & 2 \\ 10 & 4 & 1 \\ 2 & 1 & -7 \end{bmatrix}, \ p_0 = \begin{bmatrix} -12 \\ -6 \\ 56 \end{bmatrix}, \ r_0 = 0,$$

$$Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \ p = \begin{bmatrix} -2 \\ 0 \\ -16 \end{bmatrix}, \ r = 64,$$

$$\overline{Q} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 8 \end{bmatrix}, \ \overline{p} = \begin{bmatrix} 0 \\ -2 \\ -64 \end{bmatrix}, \ \overline{r} = 256.$$

The optimal solution of this problem is $x^* = (0, 0, 8)$ and the optimal objective value is 224. In Table 1 and Figure 1, we see that the gap between the upper bound and lower bound is less than 0.0001 after 27 iterations. So the algorithm is stopped and we obtain the optimal solution as a good approximation.

Table 1: The upper and lower bounds of Example 4.1.

Iter	1	5	10	15	20	25	27
Upper bound	225.1463	224.6415	224.0227	224.0002	224.0001	224.0001	224.0000
Lower bound	222.8841	223.5040	223.9286	223.9738	223.9951	223.9961	223.9999

Example 4.2. Consider the following problem [30]

$$\begin{aligned} \min\ & -x_1^2 + x_2^2 + x_1 \\ & x_1^2 - x_2^2 \leq 4 \\ & (x_1 + x_2)^2 + x_2^2 - 2x_1 \leq 0. \end{aligned}$$

Solving this problem is similar to Example 4.1. The optimal solution and the optimal value are $x^* = (2, 0)$ and -2 respectively. Since the gap between the upper bound and lower bound is less than ε at 22nd iteration (Table 2 and Figure 2), so the EAE algorithm is stopped.

Table 2: The upper and lower bounds of Example 4.2.

Iter	1	5	10	15	20	22
Upper bound	-0.5899	-1.7841	-1.9989	-2.0000	-2.0000	-2.0000
Lower bound	-2.7639	-2.0722	-2.0034	-2.0006	-2.0001	-2.0000

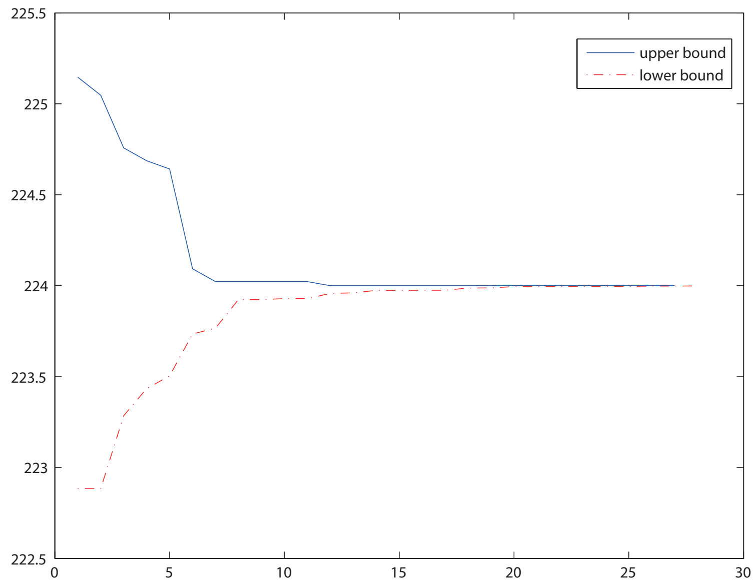

Figure 1: The upper and lower bounds of Example 4.1

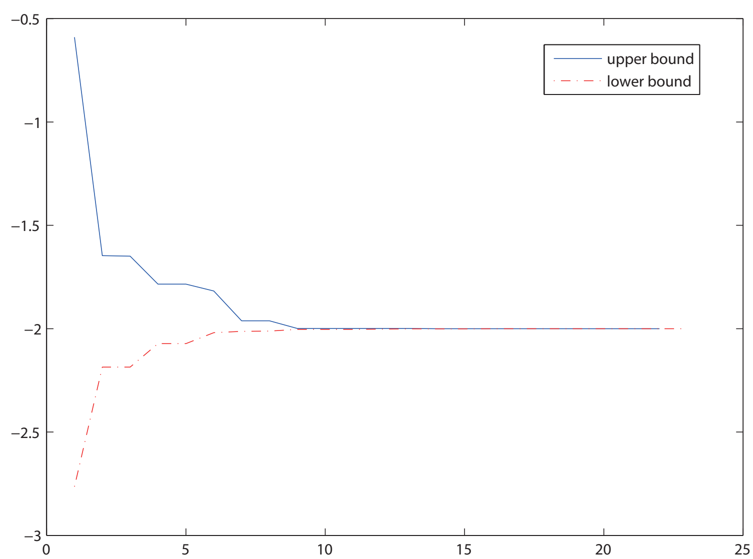

Figure 2: The upper and lower bounds of Example 4.2

4.2 Random tests

The test problems are considered as follows [33]:

$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} & x^TQ_0x+2(p_0)^Tx+r_0 \\ & x^TQ_ix+2(p_i)^Tx+r_i\leq 0 \quad i=1,2,\ldots,m_1 \\ & x^TQ_jx+2(p_j)^Tx+r_j\leq 0 \quad j=m_1+1,\ldots,m_1+m_2. \end{array} \tag{4.1}$$

where the k first constraints (i.e., for $i=1,\ldots,m_k$) are convex and the objective function and other constraints (i.e., for $j=m_1+1,\ldots,m_1+m_2$) are non-convex. The test problems are generated using the following code:
for $j=0,1,\ldots,m_1+m_2$

- $r_j=-6+5\ rand$
- $p_j=-50+50\ rand(n,1);$
- for i=1:3
  - $\omega_i=-1+2\ rand(n,1);\ W_i^j=I_{n\times n}-2\frac{\omega_i\omega_i^T}{\|\omega\|^2};$

  end {for}
- $O_j=W_1^jW_2^jW_3^j;$
- If $1\leq j\leq m_1$ (for convex constraints)
  - $v=50\ rand(n,1);$
  - $D_j=diag(v);$
  - $Q_j=O_jD_j(O_j)^T$
- else (for objective function and non-convex constraints)
  - $n1=n/2;$
  - $n2=n-n1;$
  - $v1=-50+50\ rand(n1,1);$
  - $o1=zeros(n2,1);$
  - $D_j^-=diag(v1,o1);$
  - $Q_j^-=-O_jD_j^-(O_j)^T$
  - $v2=50\ rand(n2,1);$
  - $o2=zeros(n1,1);$
  - $D_j^+=diag(o2,v2);$
  - $Q_j^+=O_jD_j^+(O_j)^T$

end {if}

end {for}
Let $Q_0 = Q_0^+ - Q_0^-$ and $Q_j = Q_j^+ - Q_j^-$, for $j = m_1 + 1, \ldots, m_1 + m_2$. Since $r_i, r_j \le 0$, $x = 0 \in \mathbb{R}^n$ is an initial feasible solution for problem (4.1).

The CPU time for solving 50 test problems for each cases n =20,30,40,50,60,100 and $m_1 = m_2$ =5,10,15,20 by CCCP [18] and EAE methods, are summarized by using performance profile [7] in Figure 3. Performance profile plots the function

$$\pi_s(\alpha) = \frac{1}{|P|} \left|\{p \in P : \log_2(r_{p,s}) \le \alpha\}\right|, \tag{4.2}$$

where P denotes the set of test problems and $r_{p,s}$ denotes the ratio the amount of CPU time needed to solve problem p with method s with the least amount of CPU time needed to solve problem p .

Figure 3 shows that the CPU time of EAE method is less than CCCP method and the EAE method is successful in more than 85% of problems in each cases (a)-(f). In addition EAE method can solve all problems with $n = 60, 100$. In CCCP method, the objective function and feasible region of the original problem are approximated to convex ones. So for problems involving more non-convex functions, we expect CCCP method works worse.

5 Conclusions

In this paper, we have proposed an extension method to solve a non-convex QCQP problem which contains non-convex objective function and non-convex constraints. For solving this problem, we extended the AE method and combine it with CCCP method. At each iteration of the proposed method, two solutions are generated. One of them is outside the feasible region and found by solving problem (3.6). Another solution is inside and calculated by solving problem (3.18). If the solution of problem (3.6) is in feasible region, this solution is a global optimal solution. Otherwise by adding a sequence of ellipsoid constraints to problem (3.6), distance of two solutions of problems (3.6) and (3.18) is convergent to zero. Therefore they are convergent to the optimal solution. The convergence of EAE method is investigated. Numerical results show that EAE method can be successfully applied to problems (1.1).

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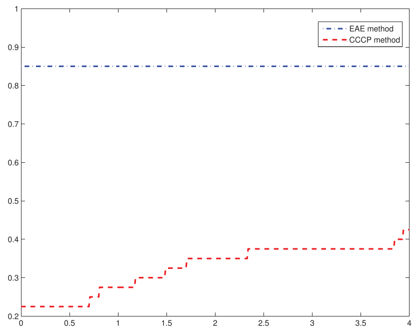
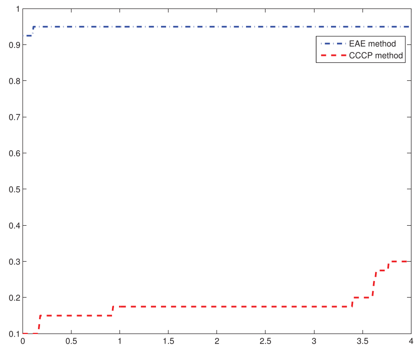
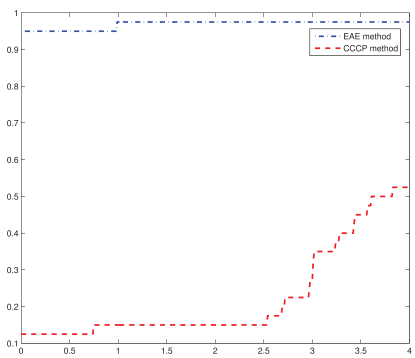


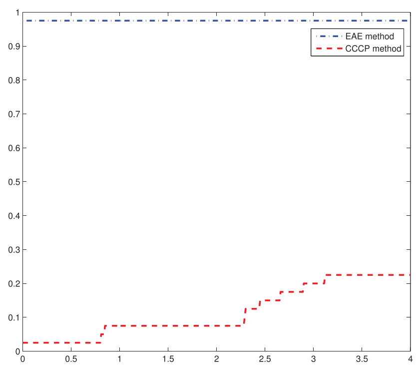
(a) n=20



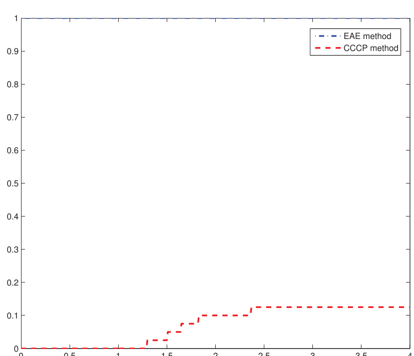
(b) n=30



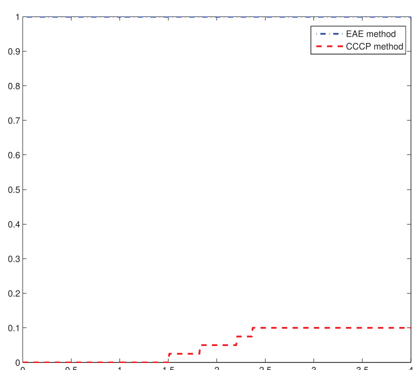
(c) n=40



(d) n=50



(e) n=60



(f) n=100

Figure 3: Performance profile for CPU time

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N. Osmanpour
Faculty of Mathematical Sciences
University of Guilan,Rasht, Iran
E-mail address: naser.osmanpour@gmail.com

M. Keyanpour
Department of Applied Mathematics, Faculty of Mathematical Sciences
University of Guilan, Rasht, Iran.
Center of Excellence for Mathematical Modeling
Optimization and Combinatorial Computing (MMOCC)
University of Guilan, Rasht, Iran
E-mail address: kianpour@guilan.ac.ir