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GENERALIZED SHADOW PRICES IN CONVEX PROGRAMMING*

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Abstract: In this paper we are concerned with the existence of shadow prices for a convex optimization problem. We propose the generalized shadow price to capture the case where resources are required proportionally. We establish a new line of analysis for asserting the existence of the generalized shadow price without resorting to the constraint qualifications. In particular, we propose a sufficient condition which is weaker than classical constraint qualifications, to guarantee the existence of the generalized shadow price. We also build a close connection of the generalized shadow prices to Lagrange multipliers by proposing the shadow price mapping. It maps any given compact set to a particular type of Lagrange multiplier which conveys the sensitivity information as the generalized shadow price. Based on the shadow price mapping, more Lagrange multipliers with special properties are identified as the generalized shadow price.

Key words: *shadow price, convex programming, multiple Lagrange multipliers*

Mathematics Subject Classification: *90C25, 90C30, 90C31*

1 Introduction

The shadow price (SP) is identified as the marginal cost of a minimization problem, or interpreted as the marginal utility of a social welfare maximization problem [23]. It has a strong connection with Lagrange multipliers (hereafter referred to as, LM) in classical economic theory [31]. This connection is strengthened with the emerging theory of linear and nonlinear programming. [17] interprets SP for the ith constraint to be the amount by which the optimal value is improved if the right-hand side of the ith constraint is increased by one, which is exactly LM or the dual variable in linear programming models. In nonlinear programming models, SP is defined as the partial derivative of the optimal value function with respect to the right-hand side parameter of an equality constraint, and it also can be represented by LM [31]. However, in order to invest on a part of an economic system to achieve maximum improvement of the objective, not only the right-hand side of all constraints should be investigated, but also all modifiable specifications in constraints need to be processed [8]. With respect to nonlinear programming models [21] study SPs when the

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perturbation are not only on the right-hand side of the constraints, while with integer and mixed integer linear programming models, [11] computes the SP which is caused by the changes in the coefficients matrix.

Furthermore, the development of constraint qualifications consolidates the relationships between SP and LM in mathematical programming models [3, 4]. In particular, several conditions (e.g., the linear independence constraint qualification [7] and the strict Mangasarian-Fromovitz constraint qualification [28]) guarantee the uniqueness of LM and also the equivalence of LM and SP [29, 15]). In this sense, a central folk wisdom of SP in classical economic theory is that SP is equivalent to LM (see the comments in [1]). This equivalence, however, fails to hold when LM is not unique, since the graph of the value function is not necessarily smooth when the set of LMs is not a singleton [16]. It is therefore conceivable that multiple LMs may lead to incorrect computation of SP in economic models [26]. This observation raises the natural question of what kind of sensitivity information do LMs convey when they are not unique [7].

The first step in this direction is taken by [2, 1, 25], who propose two types of SPs (the buying SP and the selling SP) in linear programming models. The buying and selling shadow prices in convex programming models are developed by [25] under the assumption of the Slater constraint qualification. The series works by Gauvin propose general sensitivity results on the optimal value function when the related problem is nonconvex [18, 19, 20, 21, 22]. A significant step forward for the derivation of SP in the multiple LMs' case is taken by [6], which show that an LM with the minimum Euclidean norm is exactly SP. The minimum norm shadow price is extremely useful in nonconvex optimization models since it builds the equivalence of SP to a particular LM, while buying / selling SPs fail to satisfy the property [18]. Moreover, the minimum norm Lagrange multiplier can be computed efficiently by first - order algorithms [32]. However, the economic significance of the minimum norm LM is obscure which limits its practical value in the management areas. It is also noted that the existence of the minimum norm LM is guaranteed only if the tangent cone of the abstract constraint (i.e., the domain of the decision variables) is convex [7].

The concept of buying / selling SP and the minimum norm SP are based on the marginal analysis of the value function. Instead of the traditional marginal analysis, [9] and [27] developed the concept of average shadow price (hereafter referred to as, ASP) by an average analysis. The ASP coincides with the marginal shadow price in convex programming and suggests a priori information for decision - making problems about resources. It is particularly useful when the abstract constraint of decision variables is discrete (in which case the minimum norm SP may not exist), and therefore it is widely applied in integer programming and mixed integer programming problems [27, 10, 30]. In [8], the ASP is applied to determine the best ways for future investments to improve the profit (i.e.,"remove the bottleneck" of a particular resource [8]). However, the main limitation of using ASP to identify "bottlenecks" is that it is only applicable if objective and constraints are linear [8, 11].

Although the theory of shadow prices are well developed in recent years, there are still following relevant issues needed to be further addressed.

(1) **Existence of SPs.** The shadow price, by its definition, has an intimate connection with the differentiability of the value function[16]. The classical line of analysis on the

For further details of the "bottlenecks", please refer to [8] differentiability of the value function are mainly focused on the parametric mathematical programming problem [13, 14], and the differential properties are usually obtained under relatively strong assumptions, which are unlikely to be guaranteed in some real - world problems. However, the shadow price only provides the sensitivity information of a special parametric mathematical programming problem, in which the perturbations are only occurred on the right - hand side of the constraints. Therefore it is conceivable that some weaker conditions can be established to guarantee the existence of SP (also the differentiability of the value function) [25, 24]. [6] propose a weaker condition for the existence of shadow price, which however, fails to capture the case of the zero - multiplier. According to [6], the zero - multiplier is the minimum norm multiplier, and hence is a shadow price. However, as would become clear below, the zero - multiplier may fail to express the meaning of shadow price in some situations. Therefore, it is natural to understand which condition lies on the basis to ascertain the existence of shadow prices.

(2) **Generalized shadow price.** In real - world applications, it is not sufficient to consider the shadow price of a particular resource. For example, the ingredients required for pharmaceutical manufacturing should be input proportionally. This observation raises the natural question that what is the shadow price of a set of resources when they are required to be input proportionally, i.e., the generalized shadow price.

Inspired by the idea of average shadow prices, in this research we first extend the notion of shadow prices to the generalized shadow price, which is defined as the directional derivative of the value function. Then we propose the sufficient condition for the existence of shadow price without assuming any constraint qualifications. Our proposed condition is weaker than the classical constraint qualifications. Finally, we introduce the shadow price mapping as a tool to build the relationship between the set of Lagrange multipliers and the set of shadow prices. Based on the shadow price mapping, a unified framework is proposed to analyze the property of the set of shadow price. Our framework offers advantages that the Lagrange multipliers with nice theoretical properties can be identified as the shadow prices.

Structure of the Paper

In Section 2, we review the definition and properties of two classical shadow prices. In Section 3, we propose the notion of generalized shadow price, and the sufficient condition guaranteeing the existence of generalized shadow price. In Section 4, we analyze the relation of the our proposed sufficient condition and the classical constraint qualifications. In Section 5, we introduce the shadow price mapping as a tool to build the relationship between the shadow prices and Lagrange multipliers. In addition, based on the shadow price mapping, more Lagrange multipliers with nice theoretical properties are identified as shadow prices. In Section 6, we use an illustrative example to show the power of the generalized shadow price and the shadow price mapping. Finally, Section 7 contains some concluding remarks.

2 Overview of the Shadow Price

Consider the following cost minimization problem:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & h_i(x) = 0, i = 1, \ldots, m, \\ & g_j(x) \leq 0, j = 1, \ldots, r, \\ & x \in \mathbf{X}. \end{array} \tag{2.1}$$

Throughout this paper, we assume that $f : \mathbb{R}^n \to (-\infty, \infty]$, $g_j : \mathbb{R}^n \to (-\infty, \infty]$ $j = 1, \ldots, r$ are proper, convex and continuously differentiable functions, $h_i : \mathbb{R}^n \to (-\infty, \infty]$, $i = 1, \ldots, m$ are affine functions, and $\mathbf{X}$ is a nonempty closed convex set. Furthermore, we assume that the optimal value of Problem (2.1) is finite, i.e.,

$$-\infty < f_{opt} < \infty. \tag{2.2}$$

The objective function f represents the cost of the economic system, and each $h_i(x) = 0, i = 1, \ldots, m$, and $g_j(x) \leq 0, j = 1, \ldots, r$ is viewed as a restriction of the availability of the ith (or jth) resource. In this sense, the objective of Problem (2.1) aims to determine the best input of each resource to minimize the overall system cost subject to the given resource constraints.

Let

$$\mathbf{X}^* = \{x \in \mathbf{X} | f(x) = f_{opt}, h_i(x) = 0, i = 1, \ldots, m, g_j(x) \leq 0, j = 1, \ldots, r\}$$

be the optimal solution set, and assume that $\mathbf{X}^* \neq \emptyset$. Suppose $x^* \in \mathbf{X}^*$ is a minimum of Problem (2.1), and let $A(x^*)$ be the set of indices of the active constraints, i.e., those constraints are satisfied as equations at x^*,

$$A(x^*) = \{i | g_i(x^*) = 0\}.$$

A vector $d \in \mathbb{R}^n$ is said to be a tangent of $\mathbf{X}$ at x^* if either $d = 0$ or there exists a sequence $\{x_k\}$ in $\mathbf{X}$ such that $x_k \neq x^*$ for all k and

$$x_k \to x^*, \frac{x_k - x^*}{\|x_k - x^*\|} \to \frac{d}{\|d\|}.$$

The set of all tangents of $\mathbf{X}$ at x^* is called the tangent cone of $\mathbf{X}$ at x^*, and is denoted by $T_{\mathbf{X}}(x^*)$.

The Lagrange dual problem of Problem (2.1) is defined as follows:

$$\max_{\lambda \in \mathbb{R}^m, \mu \geq 0} \min_{x \in \mathbf{X}} \{L(x, \lambda, \mu)\}, \tag{2.3}$$

where $L : \mathbb{R}^{n+m+r} \mapsto \mathbb{R}$ is the Lagrange function of Problem (2.1)

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{r} \mu_j g_j(x).$$

Furthermore, let

$$S(u, v) = \{x \in \mathbf{X} \,| h_i(x) = u_i, i = 1, \ldots, m, g_j(x) \leq v_j, j = 1, \ldots, r, \}$$

be the set of feasible solutions at the level of (u, v). The value function $\nu(u, v)$, which denotes the optimal cost given the parameters (u, v), is defined by

$$\nu(u,v) = \begin{cases} \inf_{x \in S(u,v)} f(x), & \text{if } S(u,v) \neq \emptyset, \\ \infty, & \text{if } S(u,v) = \emptyset. \end{cases}$$

It is clear that $\nu(0, 0) = f_{opt}$.

In Problem (2.1), the shadow price of the ith resource p_i is defined as the partial derivative of the marginal function at u [33],

$$p_i = \frac{\partial \nu(u,v)}{\partial u_i}, i = 1, \ldots, m,$$

and

$$p_j = \frac{\partial \nu(u,v)}{\partial v_i}, j = 1, \ldots, r.$$

This definition is valid, provided assumptions are satisfied to guarantee the existence of the above partial derivatives. Typically, these assumptions include the linear independence of the active constraint gradients [e.g., the linear independence constraint qualification (LICQ) [7]]. Furthermore, LICQ also asserts the equivalence of shadow prices and Lagrange multipliers.

A Lagrange multiplier $(\lambda^*, \mu^*) \in \mathbb{R}^{m+r}$ of Problem (2.1) at x^* is a kind of vector satisfies the following conditions,

$$\left(\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla h_i(x^*) + \sum_{j=1}^{r} \mu_j^* \nabla g_j(x^*) \right)^T y \geq 0, \forall y \in T_{\mathbf{X}}(x^*) \tag{2.4}$$

$$\mu^* \geq 0, \tag{2.5}$$

$$\mu_i^* = 0, i \notin A(x^*), \tag{2.6}$$

where $T_{\mathbf{X}}(x^*)$ denotes the tangent cone of $\mathbf{X}$ at x^*. Note that if $\mathbf{X} = \mathbb{R}^n$, then **Eq**. (2.4) is simplified as

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla h_i(x^*) + \sum_{j=1}^{r} \mu_j^* \nabla g_j(x^*) = 0.$$

For Problem (2.1), the Lagrange multiplier (λ^*, μ^*) also satisfies the following condition

$$f_{opt} = \inf_{x \in \mathbf{X}} L(x, \lambda^*, \mu^*), \tag{2.7}$$

and can be referred to as the "geometric multiplier" [7]. Therefore if there exists a Lagrange multiplier (λ^*, μ^*), then strong duality holds for Problem (2.1) and (2.3) [7], that is,

$$f_{opt} = q(\lambda^*, \mu^*) = q_{opt}, \tag{2.8}$$

where $q(\lambda, \mu) = \inf_{x \in \mathbf{X}} L(x, \lambda, \mu)$ is the dual function of Problem (2.1). For ease of reference, we denote by $M(x^*)$ the set of all Lagrange multipliers at the optimal solution x^*,

$$M(x^*) = \{(\lambda, \mu) | (\lambda, \mu) \text{ satisfies conditions (2.4), (2.5) and (2.6)}\}.$$

However, the derivatives in the above definition of shadow price may not exist, and to address this issue, [20] proposes two types of shadow prices by using the Dini partial derivatives [see also in [1]].

Definition 2.1. When $\nu(0,0)$ is finite, the **buying shadow price** of the resources i and j are

$$p_i^+(0,0) = \varliminf_{t\to 0^+} \frac{\nu(0,0)-\nu(0+te_i,0)}{t},$$

and

$$p_j^+(0,0) = \varliminf_{t\to 0^+} \frac{\nu(0,0)-\nu(0,0+te_j)}{t}.$$

Similarly, the **selling shadow price** of the resources i and j are

$$p_i^-(0,0) = \varlimsup_{t\to 0^-} \frac{\nu(0,0)-\nu(0+te_i,0)}{t},$$

and

$$p_j^-(0,0) = \varlimsup_{t\to 0^-} \frac{\nu(0,0)-\nu(0,0+te_j)}{t}.$$

[20] shows that the buying / selling shadow prices of resource i (or j) are equal to the negative of the smallest and largest ith (or jth) entry of the corresponding Lagrange multipliers respectively, when the Mangasarian - Fromovitz constraint qualification is satisfied and the feasible solution set $S(u,v)$ is uniformly compact near $(0,0)$; that is,

$$\begin{aligned} p_i^+(0,0) &= \min_{(\lambda,\mu)\in M(x^*)} \lambda_i, \\ p_j^+(0,0) &= \min_{(\lambda,\mu)\in M(x^*)} \mu_j, \\ p_i^-(0,0) &= \max_{(\lambda,\mu)\in M(x^*)} \lambda_i, \\ p_j^-(0,0) &= \max_{(\lambda,\mu)\in M(x^*)} \mu_j. \end{aligned} \tag{2.9}$$

The buying / selling shadow prices have limited connection with the real - world application, since the resources are usually required proportionally. For example, the ingredients required for pharmaceutical manufacturing should be input proportionally. Therefore, the need arises to extend the buying / selling prices to a more generalized formulation. [6] show that the minimum Euclidean norm Lagrange multiplier (MNLM) is informative, and thus is a kind of shadow price. For ease of reference, we referred to this kind of shadow price as the MNLM shadow price (i.e., p_{MNLM}). p_{MNLM} expresses the rate of cost reduction per unit constraint violation, along the maximum reduction direction d^{MNLM} [7]. It extends the notion of shadow prices to directional derivatives of the value function ν along the direction $d^{\text{MNLM}} \in \mathbb{R}^{m+r}$, in the sense that

$$\begin{aligned} p_{\text{MNLM}} &= \|(\lambda^{\text{MNLM}}, \mu^{\text{MNLM}})\| \\ &= \lim_{t\to 0^+} \tfrac{\nu(0)-\nu(td^{\text{MNLM}})}{t}, \end{aligned} \tag{2.10}$$

where $(\lambda^{\text{MNLM}}, \mu^{\text{MNLM}})$ is the MNLM. In view of equation (2.10), p_{MNLM} is not the shadow price of some particular resource, it express the rate of reduction per unit increment of all $m+r$ resources in proportion with weights $d_1^{\text{MNLM}} : d_2^{\text{MNLM}} : \cdots : d_{m+r}^{\text{MNLM}}$. We note that these weights are determined by $f, h_i, g_j, i = 1,\ldots,m, j = 1,\ldots,r$ and $\mathbf{X}$ in Problem (2.1), while can not be selected by decision makers to their own wills.

In the managerial application, [9] point out that a manager may get additional profit by buying a certain reasonable additional amount of resource i, even when the market price of the resource is greater than the buying shadow price. Therefore, a new type of SP - the ASP, is proposed to overcome the limitation the buying / selling SPs in nonconvex models (especially in the integer or mixed integer linear programming models). The ASP measures the contribution of resources along the direction $d = (d_1, d_2) \in \mathbb{R}^{m+r}$ in an average sense, and is defined as

$$p^{ASP}(d) = \sup_{t>0} \frac{\nu(0,0) - \nu(0 + td_1, 0 + td_2)}{t},$$

In particular, $p^{ASP}(e_i)$ refers to the ASP of a particular resource i. It is noted that in Model (2.1), the average shadow price $p^{ASP}(e_i)$ coincides with the buying shadow price $p_i^+(0,0)$.

Based on the work of [6], [7] and [9], we extend the definition of shadow price to the generalized shadow price by using directional derivative of the value function ν.

Definition 2.2. Given a direction vector $d \in \mathbb{R}^{m+r}$, a real valued $p(d)$ is said to be the *generalized shadow price of the direction d, or simply generalized shadow price* if the directional derivative of the value function ν at the point 0 in the direction d exists, and

$$p(d) = \lim_{t \to 0^+} \frac{\nu(0) - \nu(td)}{t}. \tag{2.11}$$

The set of all generalized shadow prices is denoted by

$$SP = \{p(d) | d \in \mathbb{R}^{m+r}\}.$$

For ease of reference, we refer to the generalized shadow price as GSP for short. We provide some orientation by summarizing the main attributes of the GSP.

(1) Clearly, under the assumption that $p(d)$ exists for any d, then by Definition 2.2, the buying / selling shadow prices $[p_i^+$ and $p_i^-]$, and the MNLM shadow price p_{MNLM} are three special GSPs of three particular directions, e_i, $-e_i$ and d^{MNLM}, respectively. Note that $p_i^- = p(-e_i)$.

(2) The salient property of the GSP is that it is not only consistent with the classical sensitivity interpretation of Lagrange multipliers, but also it offers full control for decision makers on selecting the weights to input resources. To get some insight, assume that there are no inequality constraints, $\mathbf{X} = \mathbb{R}^n$ and the equality constraint gradients at the local minimum x^* are linearly independent. This assumption is usually denoted as the LICQ under which there exists a unique Lagrange multiplier λ^* at x^*. Furthermore, for each perturbation vector $u \in \mathbb{R}^m$, the following perturbed problem

$$\begin{aligned} \nu(u) = \quad & \min \quad f(x) \\ & \text{s.t.} \quad h_i(x) = u_i, i = 1, \ldots, m \end{aligned}$$

always has a unique optimal solution $x(u)$, and the sensitivity property of the value function ν at $u = 0$ can be represented as

$$p(d) = \nu'(0, d) = \lambda^{*T} d, d \in \mathbb{R}^m.$$

We note that the preceding formula of $p(d)$ is consistent with the result given in [15]. Furthermore, when we select $d = e_i$, then $\lambda_i^* = p(e_i)$, which interprets the rate of cost reduction when the ith constraint is violated. This is consistent with the notion of the classical shadow price. On the other hand, when we choose $\tilde{d} = (\omega_1, \ldots, \omega_m)^T$, where ω_i represents the weight for increasing resource i. Then $p(\tilde{d})$ denotes the rate of cost reduction when all resources are increased proportionally with weights $\omega_1, \ldots, \omega_m$.

(3) We note that assuming $0 \in \mathbf{int\,dom}(\nu)$, then by the property of the directional derivative [5], the GSP $p(d)$ is convex with respect to d.

(4) More importantly, as would become clear below, the GSP set has an intimate connection with the set of Lagrange multipliers, so that each GSP can be represented by a corresponding Lagrange multiplier, and computed by primal and dual algorithms.

3 Sufficient Conditions for the Existence of Shadow Price

In this section, we discuss the existence of the GSPs. [7] prove that the existence of the minimum norm shadow price p_{MNLM} is guaranteed, provided that there exists at least one Lagrange multiplier vector. Since the constraint qualifications ascertain the existence of Lagrange multipliers, then it follows that there exists shadow prices as long as $\mathbf{X}$ is convex and some constraint qualifications are satisfied at x^*. However, as it would become clear below, not all constraint qualifications assert the existence of GSPs. The reason is that the zero multiplier case is excluded from consideration in analysis of [7]. This can be seen by considering the following example where there are no GSPs, although the set of Lagrange multipliers is nonempty.

$$\begin{array}{ll} \min & x_1 + x_2 \\ \text{s.t.} & x_1^2 = 0 \\ & x \in \mathbf{X} = \{(x_1, x_2) | x_1 \geq 0, x_2 \geq 0\}. \end{array} \tag{3.1}$$

The minimum of Problem (3.1) is $(x_1^*, x_2^*) = (0, 0)$, and the set of Lagrange multipliers is

$$\{\lambda | \lambda \in \mathbb{R}\}.$$

Hence, $\lambda^* = 0$ is the minimum norm Lagrange multiplier of Problem (3.1). However, notice that the value function of Problem (3.1) is:

$$\begin{aligned} \nu(u) &= \min\{x_1 + x_2 | x_1^2 = u, (x_1, x_2) \in \mathbf{X}\} \\ &= \left\{ \begin{array}{ll} \sqrt{u}, & \text{if } u \geq 0, \\ \infty, & \text{if } u < 0 \end{array} \right. , \end{aligned}$$

then by Definition 2.2, for any nonzero direction d (i.e., $d = \alpha$ or $d = -\alpha$ with $\alpha > 0$), the GSPs

$$\begin{aligned} p(\alpha) &= \lim_{t \to 0^+} \tfrac{\nu(0) - \nu(t\alpha)}{t}, \\ p(-\alpha) &= \lim_{t \to 0^+} \tfrac{\nu(0) - \nu(-t\alpha)}{t}, \end{aligned}$$

do not exist. Hence it is not generally true that the existence of Lagrange multipliers ascertains the existence of GSPs. Naturally, the need arises to develop conditions that guaranteeing the existence of GSPs. In Problem (3.1), although the value function ν has nice properties, (e.g. ν is proper, closed and convex), it however fails to satisfy the condition of $0 \in \mathbf{int}(\mathbf{dom}(\nu))$. Hence it is conceivable that the condition $0 \in \mathbf{int}(\mathbf{dom}(\nu))$ lies at the heart of guaranteeing the existence of the GSP. In fact, if $0 \in \mathbf{int}(\mathbf{dom}(\nu))$, then for any direction $d \in \mathbb{R}^{\mathbf{m+r}}$, the quotient

$$\frac{\nu(0) - \nu(0 + td)}{t}$$

is well-defined, which then combined with the convexity of the value function ν, yields the existence of direction derivatives of ν. The following proposition presents the sufficient condition guaranteeing the existence of the GSP.

Proposition 3.1. *Let $\nu(u, v)$ be the value function of Problem (2.1). Assume that $0 \in \mathbf{int}(\mathbf{dom}(\nu))$. Then there exists the generalized shadow price $p(d)$.*

In order to prove Proposition 3.1, we need the following lemma [5].

Lemma 3.2. *Let $f : \mathbb{R}^n \mapsto (-\infty, \infty]$ be a proper convex function and let $x \in \mathbf{int}(\mathbf{dom}(f))$. Then for any $d \in \mathbb{R}^n$, the directional derivative $f'(x; d)$ exists.*

Proof. **Proof of Proposition 3.1** We will use Lemma 3.2 to prove Proposition 3.1. We first note that by the assumption $0 \in \mathbf{int}(\mathbf{dom}(\nu))$, the value function $\nu(u, v)$ is proper in a neighbourhood N of the origin point satisfying $N \subseteq \mathbf{int}(\mathbf{dom}(\nu))$. Then we will show that the value function $\nu(u, v)$ of Problem (2.1) is convex. Consider the function

$$F(x, u, v) = \begin{cases} \sup\limits_{\lambda \in \mathbb{R}^m, \mu \geq 0} \{L(x, \lambda, \mu) - \lambda^T u - \mu^T v\}, & \text{if } x \in \mathbf{X}, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.2}$$

It is clear that

$$\nu(u, v) = \inf_{x \in \mathbf{X}} F(x, u, v). \tag{3.3}$$

Our proof will revolve around the function $F(x, u, v)$.

Since $f, g_i, i = 1, \ldots, m$ are proper and convex and $h_j, j = 1, \ldots, r$ are affine, it follows that

$$L(x, \lambda, \mu) - \lambda^T u - \mu^T v$$

is proper and convex with respect to x. Thus $F(x, u, v)$ is also proper and convex in the sense that the supreme of a class of convex functions is also convex. Suppose that $(u_1, v_1), (u_2, v_2) \in \mathbb{R}^{m+r}$, and $\beta \in (0, 1)$. Then by **Eq**. (3.3), there exist sequences $\{x_1^k\}$ and $\{x_2^k\}$ in $\mathbf{X}$ such that $\lim\limits_{k\to\infty} F(x_1^k, u_1, v_1) = \nu(u_1, v_1)$ and $\lim\limits_{k\to\infty} F(x_2^k, u_2, v_2) = \nu(u_2, v_2)$. Therefore, by **Eq**. (3.3) and the convexity of F, we have

$$\begin{aligned} \nu(\beta(u_1, v_1) + (1-\beta)(u_2, v_2)) &= \inf_{x \in \mathbf{X}} \{F(x, \beta u_1 + (1-\beta)u_2, \beta v_1 + (1-\beta)v_2)\} \\ &\leq F(\beta x_1^k + (1-\beta)x_2^k, \beta u_1 + (1-\beta)u_2, \beta v_1 + (1-\beta)v_2) \\ &\leq \beta F(x_1^k, u_1, v_1) + (1-\beta)F(x_2^k, u_2, v_2). \end{aligned}$$

Taking the limit in the above inequality, yields

$$\nu(\beta(u_1, v_1) + (1-\beta)(u_2, v_2)) \leq \beta\nu(u_1, v_1) + (1-\beta)\nu(u_2, v_2),$$

which indicates that the value function $\nu(u, v)$ is convex.

Thus we have $\nu(u,v)$ is proper and convex on N, combining $0 \in \mathbf{int}(\mathbf{dom}(\nu))$, and in view of Lemma 3.2 we obtain that for any $d \in \mathbb{R}^{m+r}$, the GSP

$$p(d) = \lim_{t \to 0^+} \frac{\nu(0) - \nu(td)}{t}$$

exists. □

For ease of reference, we refer to the condition $0 \in \mathbf{int}(\mathbf{dom}(\nu))$ as the *shadow price existence condition* (SPE for short). We will show that the set of GSPs is closely related to the set of Lagrange multipliers under the SPE condition. The following proposition clarifies the relationship between the set of GSPs and the set of LMs.

Proposition 3.3. *Let $\partial\nu(0,0)$ be the subdifferential of the value function at the origin. Assume that the SPE condition holds. Then*

(1) the set of Lagrange multipliers $M(x^*)$ is equivalent to $-\partial\nu(0,0)$;

(2) the generalized shadow price $p(d)$ of direction d is equal to

$$-\min_{g \in M(x^*)} g^T d.$$

In order to prove Proposition 3.3, we need the following lemma of "max formula"[5]:

Lemma 3.4. ***(max formula.)****Let $f : \mathbb{R}^n \mapsto (-\infty, \infty]$ be a proper convex function, and $\partial f(x)$ be the subdifferential of function f at the point x. Then for any $x \in \mathbf{int}(\mathbf{dom}(f))$ and each $d \in \mathbb{R}^n$, the direction derivative of f at x in the direction d is*

$$\begin{aligned} f'(x;d) &= \lim_{t \to 0^+} \tfrac{f(x+td)-f(x)}{t}, \\ &= \max_{g \in \partial f(x)} g^T d. \end{aligned}$$

Proof. **Proof of Proposition 3.3**

(1) We first note that for any $\lambda \in \mathbb{R}^m, \mu \geq 0$, the dual function $q(\lambda,\mu)$ of Problem (2.1) is

$$\begin{aligned} q(\lambda,\mu) &= \inf_{x \in \mathbf{X}} \{f(x) + \lambda^T h(x) + \mu^T g(x)\} \\ &= \inf_{(u,v) \in \mathbb{R}^{\mathbf{m+r}}} \inf_{\substack{h(x)=u,\, g(x) \leq v, \\ x \in \mathbf{X},}} \{f(x) + \lambda^T h(x) + \mu^T g(x)\} \\ &= \inf_{(u,v) \in \mathbb{R}^{\mathbf{m+r}}} \{\nu(u,v) + \lambda^T u + \mu^T v\}. \end{aligned}$$

Suppose that (λ^*, μ^*) is a Lagrange multiplier, then by strong duality we have $q(\lambda^*,\mu^*) = f_{opt} = \nu(0,0)$ [c.f. **Eq**. (2.8)], and it follows that

$$\nu(0,0) \leq \nu(u,v) + (\lambda^*)^T u + (\mu^*)^T v, \forall (u,v) \in \mathbb{R}^{\mathbf{m+r}}. \tag{3.4}$$

Therefore by the definition of subdifferentials, we have $(\lambda^*, \mu^*) \in -\partial\nu(0,0)$.

Conversely, suppose ν is proper and **Eq**. (3.4) holds for some (λ, μ). Then since ν is monotonically nonincreasing with respect to the component of v, it follows that $\mu^* \geq 0$. Moreover, we have

$$\begin{aligned} f_{opt} &= \nu(0,0) \\ &\leq \inf_{u,v}\{\nu(u,v) + \lambda^T u + \mu^T v\} \\ &= q(\lambda, \mu) \\ &\leq q_{opt} \\ &\leq f_{opt}, \end{aligned}$$

it follows that (λ, μ) is a Lagrange multiplier.

(2) We recall from the proof of Proposition 3.1 that the value function ν is proper and convex. In view of the SPE condition, combining Lemma 3.4 and the definition of the generalized shadow price, we obtain for any $d \in \mathbb{R}^m$

$$\begin{aligned} p(d) &= \nu'((0,0); d) \\ &= \max_{g \in \partial\nu(0,0)} g^T d \\ &= -\min_{g \in M(x^*)} g^T d, \end{aligned}$$

which proves the second part of Proposition 3.3.

□

We note that the equivalence of the Lagrange multiplier set $M(x^*)$ and the subdifferential $-\partial\nu(0,0)$ is also proved in [15], under the assumption that the Slater constraint qualification holds. As would become clear in Section 4, the Slater constraint qualification implies the SPE condition, and therefore Proposition 3.3 can be seen as a generalization of the result in [15].

4 Relations between the Shadow Price Existence and Constraint Qualifications

In the preceding section, we propose the SPE condition to ascertain the existence of shadow price. The SPE condition, however, is hard to verify in practice. Furthermore, the classical line of development of the shadow price theory revolves around the constraint qualifications. Therefore, naturally the need arises to understand the relation of the SPE condition and the constraint qualifications. In this section, we discuss the relation of the SPE condition with the following four classical constraint qualifications.

(1) (**CQ1. Linear Independence Constraint Qualification**) $\mathbf{X} = \mathbb{R}^{\mathbf{n}}$ and x^* satisfies **LICQ**, i.e., the equality constraint gradients $\nabla h_i(x^*), i = 1, \ldots, m$, and the active inequality constraint gradients $\nabla g_j(x^*)$, $j \in A(x^*)$, are linearly independent.

(2) (**CQ2. Mangasarian Fromovitz Constraint Qualification**) $\mathbf{X} = \mathbb{R}^{\mathbf{n}}$, the equality constraint gradients $\nabla h_i(x^*), i = 1, \ldots, m$, are linearly independent, and there exists a $y \in \mathbb{R}^{\mathbf{n}}$ such that

$$\begin{aligned} \nabla h_i(x^*) y = 0, i = 1, \ldots, m, \\ \nabla g_j(x^*) y < 0, j \in A(x^*). \end{aligned} \tag{4.1}$$

(3) (**CQ3. Slater Constraint Qualification**) There are no equality constraints. $\mathbf{X} = \mathbb{R}^{\mathbf{n}}$, the inequality constraints $g_j(x), j = 1, \ldots, r$ are convex and there exists a feasible vector $\bar{x} \in \mathbf{X}$ satisfying:

$$\begin{aligned} &g_j(\bar{x}) < 0, j \in A(x^*), \\ &g_j(\bar{x}) \leq 0, j \notin A(x^*). \end{aligned} \tag{4.2}$$

(4) (**CQ4**) There are no inequality constraints. All the equality constraints $h_i(x), i = 1, \ldots, m$ are affine, and $\mathbf{X} = \mathbb{R}^{\mathbf{n}}$.

We will show that the SPE condition can be derived from all these constraint qualifications except for **CQ4**. This implies that the constraint qualifications are sufficient to guarantee the existence of the GSP. Compared with classical constraint qualifications, the SPE condition is a weaker condition, and it lies at the heart of guaranteeing the existence of the GSP. As is well - known, either **CQ1** or **CQ3** implies **CQ2**, hence we only prove that the SPE condition can be derived from **CQ2**.

We first introduce a lemma from [7] to show that the value function ν is lower semicontinuous at $(u, v) = (0, 0)$ under either of the preceding four constraint qualifications.

Lemma 4.1. *Assume that $f_{opt} < \infty$, the set $\mathbf{X}$ is convex, and f and $g_j, = 1, \ldots, r$ are convex over $\mathbf{X}$, $h_i, i = 1, \ldots, m$ are affine over $\mathbf{X}$. Then, there is no duality gap if and only if ν is lower semicontinuous at $(u, v) = (0, 0)$.*

We note that by assumptions on Problem (2.1) [c.f. f,g_j and $\mathbf{X}$ are convex, h_i are affine], **CQ1** to **CQ4** ascertain the strong duality of Problem (2.1) and (2.3). Then combining **Eq**. (2.2) and Lemma 4.1, the value function ν is lower semicontinuous at $(u, v) = (0, 0)$, implying $\nu(u, v) \geq \nu(0, 0) = f_{opt}$ for any (u, v) in the neighbourhood of the origin. Hence in order to show **CQ1,CQ2** and **CQ3** will lead to the SPE condition, it is sufficient to prove that the following perturbed problem

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & h_i(x) = u_i, i = 1, \ldots, m \\ & g_j(x) \leq v_j, j = 1, \ldots, r \\ & x \in \mathbf{X} \end{aligned} \tag{4.3}$$

is feasible.

To prove **CQ2** is sufficient to guarantee the SPE condition, we need the following lemma which can be viewed as the enhanced form of the implicit function theorem [12].

Lemma 4.2. *Assume that $g : \mathbb{R}^{\mathbf{n}} \mapsto \mathbb{R}^{\mathbf{m}}$ is a continuously differentiable function, and $\nabla g_1(x_0), \ldots, \nabla g_m(x_0)$ are linear independent, then*

(1) $\forall h \in \mathbb{R}^{\mathbf{n}}$, there exists a $t_0 > 0$ such that $\forall -t_0 \leq t \leq t_0$, there exists a unique continuous function $x = x(t), t \in (-t_0, t_0)$, satisfying $x(0) = x_0$ and $x'(0) = h$,

(2) $g(x(t)) = g(x_0) + t \sum_{i=1}^{m} h_i \nabla g_i(x_0)$, for all $t \in (-t_0, t_0)$.

Proposition 4.3. *Suppose that **CQ2** holds, then $0 \in \mathbf{int}(\mathbf{dom}(\nu))$.*

Proof. **Proof of Proposition 4.3** Let $Dh(x^*) = \begin{pmatrix} \nabla h_1(x^*), & \cdots & ,\nabla h_m(x^*) \end{pmatrix}$, and

$$\begin{aligned} \alpha &= \max_{j \notin A(x^*)} g_j(x^*), \\ \delta_1 &= \min_{j \in A(x^*)} \{|\nabla g_j(x^*) y|\}, \\ \delta_2 &= \max_{j \notin A(x^*)} \{|\nabla g_j(x^*) y|\}, \end{aligned} \tag{4.4}$$

where $y \in \mathbb{R}^n$ satisfies **Eq**. (4.1). For $j = 1, \dots, r$, given any vector $\xi \in \mathbb{R}^m$ satisfying $\|\xi\| \le 1$, the following problem

$$\begin{aligned} \max_z \quad & |\nabla g_j(x^*)^T z| \\ \text{s.t.} \quad & Dh(x^*)^T z = \xi, \end{aligned}$$

has an optimal solution z_j^*, because of the linear independent assumption. Let

$$M_j = |\nabla g_j(x^*)^T z_j^*|, M = \max_{j=1,\dots,r} \{M_j\},$$

$$\tilde{j} = \arg \max_{j=1,\dots,r} \{M_j\}, z = \frac{\delta_1}{2M} z_{\tilde{j}}^*.$$

Then for any j we have

$$|\nabla g_j(x^*)^T z_{\tilde{j}}| \le M, \tag{4.5}$$

$$|\nabla g_j(x^*)^T z| \le \frac{\delta_1}{2}, \tag{4.6}$$

$$Dh(x^*) z = \eta, \tag{4.7}$$

with some η satisfying $\|\eta\| \le \frac{\delta_1}{2M}$.

Moreover, since $\nabla h_i(x^*), i = 1, \dots, m$ are linear independent, then by Lemma 4.2, there exists t_0 such that $\forall -t_0 \le t \le t_0$, there exists a continuous function $x(t)$ such that

$$\begin{aligned} x(0) &= x^*, \\ x'(0) &= y + z, \end{aligned}$$

and combing **Eqs**. (4.4), (4.1), (4.6) and (4.7), we obtain

$$h(x(t)) = h(x^*) + tDh(x^*)(y + z) = t\eta, \tag{4.8}$$

and for all $j \in A(x^*)$,

$$\begin{aligned} g_j(x(t)) \quad &= g_j(x^*) + t\nabla g_j(x^*)(y + z) + o(t) \\ &\le t(-\delta_1 + \nabla g_j(x^*) z) + o(t) \\ &\le -\tfrac{\delta_1 t}{2} + o(t) \\ &\le -\tfrac{\delta_1 t}{4}. \end{aligned} \tag{4.9}$$

Moreover, noticing that $\alpha = \max_{j \notin A(x^*)} g_j(x^*)$ [c.f. **Eq**. (4.4)]. Then for all $j \notin A(x^*)$, we have

$$\begin{aligned} g_j(x(t)) \quad &= g_j(x^*) + t\nabla g_j(x^*)(y + z) + o(t) \\ &\le \alpha + t(\delta_2 + \tfrac{\delta_1}{2}) + o(t) \\ &\le \alpha + t(\delta_2 + \tfrac{3\delta_1}{4}). \end{aligned} \tag{4.10}$$

Setting $t_1 = \min\{t_0, -\frac{\alpha}{2(\delta_2+\frac{3}{4}\delta_1)}\}$, we have $\forall t \in [\frac{t_1}{2}, t_1]$, function $x(t)$ satisfies

$$\begin{gathered} h(x(t)) = tu, \|u\| \leq \tfrac{\delta_1}{2M}, \\ g_j(x(t)) \leq -\tfrac{\delta t}{4} \leq -\tfrac{\delta t_1}{8}, \forall j \in A(x^*), \\ g_j(x(t)) \leq \alpha + t(\delta_2 + \tfrac{3\delta_1}{4}) \leq \tfrac{\alpha}{2}, \forall j \notin A(x^*). \end{gathered}$$

Therefore, let $\rho = \min\{\frac{\delta t_1}{8}, \frac{\delta t_1}{4M}, -\frac{\alpha}{2}\}$. Then for all $(u, v) \in B(0, \rho)$, there exists $\tilde{t} \in [\frac{t_1}{2}, t_1]$ such that

$$\begin{gathered} h_i(x(\tilde{t})) = u, \\ g_j(x(\tilde{t})) \leq -\tfrac{\delta t_1}{8} \leq v_j, \forall j \in A(x^*), \\ g_j(x(\tilde{t})) \leq \tfrac{\alpha}{2} \leq v_j, \forall j \notin A(x^*). \end{gathered}$$

Therefore, similar to the proof in preceding proposition, this implies that for all $(u, v) \in B(0, \rho)$, the perturbed problem [c.f. Problem (4.3)] is feasible, which proves the desired result. ☐

Proposition 4.4. *Suppose that **CQ1** or **CQ3** holds, then* $0 \in \mathbf{int}(\mathbf{dom}(\nu))$.

From Proposition 4.4, we can immediately obtain the following two corollaries which coincides with the results in [24] and [25].

Corollary 4.5. *Suppose there are no equality constraints in Problem (2.1). Assume the Slater constraint qualification holds at some feasible point $\bar{x}$. Then there exists the directional derivative of the value function ν at $v = 0$ in the direction $\alpha \in \mathbb{R}^r$ and is given by*

$$\nu'(0, \alpha) = \max_{\mu \in M(x^*)} \{(-\mu^T \alpha)\}.$$

Corollary 4.6. *Suppose there are no equality constraints in Problem (2.1). Assume the Slater constraint qualification holds at some feasible point $\bar{x}$. Moreover, let the set of Lagrange multipliers $M(x^*)$ be compact. Then it holds*

$$p_i^+ = \nu'(0, e_i) = -\min_{\mu \in M(x^*)} \mu_i,$$

$$p_i^- = -\nu'(0, -e_i) = -\max_{\mu \in M(x^*)} \mu_i.$$

We have shown that either of the first three constraint qualifications [c.f. **CQ1**, **CQ2** and **CQ3**] guarantees the SPE condition. However, **CQ4** is not sufficient for the SPE condition. As a example, consider the following problem

$$\begin{array}{ll} \min & f(x_1, x_2) \\ \text{s.t.} & x_1 + 2x_2 - 1 = 0 \\ & 2x_1 + 4x_2 - 2 = 0 \end{array}$$

Assume that there exists a perturbation $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ on the righthand side of the two equality constraints with $u_2 \neq 2u_1$. Then the perturbed problem is inconsistent, and thus $\nu(u_1, u_2) = \infty$, which implies the SPE condition fails to be satisfied.

[11] proposes the generalized average shadow price which corresponds to the perturbation on the coefficients matrix of constraints. It is naturally to ask whether our proposed GSP

can be applied to convex programming models when the perturbations are not only on the right-hand side.

Consider the following perturbed convex programming problem

$$\begin{array}{ll} \min_x & f(x) \\ \text{s.t.} & h_i(x,y) = 0, i = 1, \ldots, m, \\ & g_j(x,y) \le 0, j = 1, \ldots, r, \\ & x \in \mathbf{X}, \end{array} \tag{4.11}$$

where $y \in \mathbb{R}^{m+r}$ is a perturbation vector, $f : \mathbb{R}^n \to (-\infty, \infty]$, $g_j : \mathbb{R}^n \times \mathbb{R}^{m+r} \to (-\infty, \infty]$ $j = 1, \ldots, r$ are proper, convex and continuously differentiable functions, $h_i : \mathbb{R}^n \times \mathbb{R}^{m+r} \to (-\infty, \infty]$, $i = 1, \ldots, m$ are affine functions, and $\mathbf{X}$ is a nonempty closed convex set. Let

$$S(y) = \{x \in \mathbf{X} \,|\, h_i(x,y) = 0, i = 1, \ldots, m, g_j(x,y) \le 0, j = 1, \ldots, r, \}$$

be the set of feasible solutions at the level of y. The value function $\nu(y)$, which denotes the optimal cost given the parameters y, is defined by

$$\nu(y) = \begin{cases} \inf\limits_{x \in S(y)} f(x), & \text{if } S(y) \ne \emptyset, \\ \infty, & \text{if } S(y) = \emptyset. \end{cases}$$

Similarly to Proposition 3.1, it is not difficult to show the SPE condition ($0 \in \mathbf{int\,dom}(\nu)$) guarantees the existence of GSP of Problem (4.11). However, we can not build the equivalence of the Lagrange multipliers' set ($M(x^*)$) and $-\partial\nu(0)$, since **Eq**. (3.4) may not hold without any further assumptions, even if the constraints are linear. Consider the following example.

$$\begin{array}{ll} \min & x \\ \text{s.t.} & -x \le 0, \\ & x \le 1. \end{array} \tag{4.12}$$

The optimal solution is $x^* = 0$ and the set of Lagrange multipliers is $M(x^*) = \{(\lambda_1, \lambda_2) | \lambda_1 - \lambda_2 = 1, \lambda_1 \ge 0, \lambda_2 \ge 0\}$, which indicates that λ_1 and λ_2 can not be both zero valued. Suppose the coefficient matrix of constraints are perturbed as follows,

$$\begin{array}{ll} \min_x & x \\ \text{s.t.} & (-1 + \Delta_1)x \le 0, \\ & (1 + \Delta_2)x \le 1, \end{array} \tag{4.13}$$

where Δ_1 and Δ_2 are sufficiently small scalars. Then for any direction $d \in \mathbb{R}^2$, the GSP of Problem (4.12) is $p(d) = \lim_{t \to 0^+} \frac{\nu(0) - \nu(td)}{t} = 0$, which cannot be expressed by the inner product of any Lagrange multipliers and the direction d.

It is also noted that [21] shows the equivalence of $M(x^*)$ with $-\partial\nu(0)$ under the assumption of **CQ.1** and the uniformly compactness of the solution set $S(y)$ near $y = 0$.

5 The Shadow Price Mapping

Traditional treatments of the shadow price theory revolve around identifying special Lagrange multipliers with particular properties [7]. In this line of analysis, the Lagrange multipliers are treated separately. In this section, by proposing the shadow price mapping, we develop a unified framework to analyze the properties of the shadow price set. The line of our framework is new, and offers some advantages. For example, more Lagrange multiplies of nice theoretical properties can be identified as shadow price in our framework.

The idea of the shadow price mapping are primarily motivated as follows. Recall from Proposition 3.3 in Section 3 that the shadow price

$$p(d) = -\min_{g\in M(x^*)} g^T d,$$

provided the SPE condition holds. Notice that the SPE condition also guarantees the nonemptiness and compactness of the subdifferential $\partial\nu(0,0)$ [5], then by Proposition (3.3), we obtain that the Lagrange multiplier set $M(x^*)$ is nonempty and compact. Therefore, for any compact set D in $\mathbb{R}^{m+r}$, the following minmax equality holds

$$-\max_{d\in D}\min_{g\in M(x^*)} g^T d = -\min_{g\in M(x^*)}\max_{d\in D} g^T d. \tag{5.1}$$

Since the left - hand side of **Eq**. (5.1) is equal to $\min_{d\in D} p(d)$, implying a kind of shadow price of a particular direction d. On the other hand, assume that d^* is the optimal solution of $\max_{d\in D} g^T d$, then the right - hand side of **Eq**. (5.1) can be simplified as

$$-\min_{g\in M(x^*)} g^T d^*,$$

implying a particular Lagrange multiplier. In this sense, by **Eq**. (5.1), we can build a close relationship between the shadow prices and the Lagrange multipliers. The rest part of this section revolves around the saddle point problem [c.f. **Eq**. (5.1)].

First, we introduce the notion of the shadow price mapping.

Definition 5.1. Assume the SPE condition holds. Let $M(x^*)$ be the set of Lagrange multipliers at x^*, SP be the set of GSPs in $\mathbb{R}^{m+r}$, $\mathcal{D}$ be the set of compact sets. Given a compact set $D \in \mathcal{D}$, the mapping $\mathbf{T_{SP}}$

$$\begin{aligned}\mathbf{T_{SP}}:\ \ & \mathfrak{D} \mapsto SP\\ & D \mapsto \mathbf{T_{SP}}(D) = \min_{d\in D} p(d),\end{aligned}$$

is called the shadow price mapping.

By the compactness of $D \in \mathfrak{D}$ and $M(x^*)$, there exists a saddle point of the saddle point problem [c.f. **Eq**. (5.1)], and it follows that the shadow price mapping $\mathbf{T_{SP}}$ is well defined. Furthermore, the shadow price mapping $\mathbf{T_{SP}}$ is surjective, since $\mathbf{T_{SP}}(\{\{d\}|d\in\mathbb{R}^{m+r}\}) = SP$, where $\{d\}$ denotes a singleton set.

The main potential advantage of the shadow price mapping is that it provides a way to identify more Lagrange multipliers as GSPs. Let us consider several different GSPs.

(1) **The minimum norm shadow price.** In **Eq**. (5.1), by setting $D = \{d|\|d\| \le 1\}$, we obtain

$$\begin{aligned}\mathbf{T_{SP}}(D) &= \min_{\{d|\|d\|\le 1\}} p(d)\\ &= -\min_{g\in M(x^*)}\max_{\{d|\|d\|\le 1\}} g^T d.\end{aligned} \tag{5.2}$$

Since $g^T d \leq \|g\|\|d\|$, and the inequality holds as a equality when $d = \frac{g}{\|g\|}$, thus **Eq**. (5.2) equals to

$$-\min_{g\in M(x^*)} \|g\|,$$

implying that the minimum norm Lagrange multiplier is a type of shadow price $p(d)$, where $d = \arg \max_{\{d|\|d\|\leq 1\}} p(d)$. We denote this type of shadow price as the minimum norm shadow price.

(2) **Buying / Selling Shadow Price.** In **Eq**. (5.1), by setting $D = \{e_i\}$, $i = 1, \ldots, m+r$, we obtain

$$\begin{aligned} \mathbf{T_{SP}}(\{e_i\}) &= \min p(e_i) \\ &= -\min_{g\in M(x^*)} g_i, \end{aligned} \tag{5.3}$$

implying that the Lagrange multiplier with the minimum ith entry is a type of shadow price, namely the buying shadow price of the ith resource [c.f. **Eq**. (2.9)]. Similarly, the selling shadow price of the ith resource can be obtained by setting $D = \{-e_i\}$.

(3) **Minimum and Maximum Value Shadow Price.** In **Eq**. (5.1), by setting $D = \{e\}$, we obtain

$$\begin{aligned} \mathbf{T_{SP}}(D) &= \min p(e) \\ &= -\min_{g\in M(x^*)} \sum_{i=1}^{m+r} g_i, \end{aligned} \tag{5.4}$$

implying that the Lagrange multiplier with the minimum sum of entries is a type of shadow price $p(e)$. We denote this type of shadow price as the minimum value shadow price. Similarly, by setting $D = \{d|d = -e\}$, the Lagrange multiplier with the maximum sum of entries is also a type of shadow price, and is denoted as the maximum value shadow price.

(4) **Minimum p-Norm Shadow Price.** In **Eq**. (5.1), by setting $D = \{d|\|d\|_p \leq 1\}$ (here $\|d\|_p = (\sum_{i=1}^{m+r} d_i^p)^{\frac{1}{p}}$ denotes the p-norm of the vector d), we obtain

$$\begin{aligned} \mathbf{T_{SP}}(D) &= \min_{\{d|\|d\|_p\leq 1\}} p(d) \\ &= -\min_{g\in M(x^*)} \max_{\{d|\|d\|_p\leq 1\}} g^T d. \end{aligned} \tag{5.5}$$

Since $g^T d \leq \|g\|_q\|d\|_p$, where $\frac{1}{p} + \frac{1}{q} = 1$, and the inequality holds as a equality when $d = \frac{g}{\|g\|_q}$, thus **Eq**. (5.5) equals to

$$-\min_{g\in M(x^*)} \|g\|_q,$$

implying that the Lagrange multiplier with the minimum q-norm is a type of shadow price. In particular, by setting $p = 1$, we obtain

$$\begin{aligned} \min_{\{d|\|d\|_1\leq 1\}} p(d) &= -\min_{g\in M(X^*)} \|g\|_\infty \\ &= -\min_{g\in M(X^*)} \max_{i=1,\ldots,r+m} |g_i|, \end{aligned}$$

implying that the Lagrange multiplier whose largest absolute value of entries is the minimum, is a type of shadow price. Moreover, by setting $p = \infty$, we obtain

$$\begin{aligned}\min_{\{d \mid \|d\|_\infty \le 1\}} p(d) &= -\min_{g \in M(X^*)} \|g\|_1 \\ &= -\min_{g \in M(X^*)} \sum_{i=1}^{r+m} |g_i|,\end{aligned}$$

implying that the Lagrange multiplier with the minimum sum of absolute value of entries, is a type of shadow price.

(5) **Maximum Largest Absolute Value Shadow Price.** Let us now consider the unit simplex set $D = \Delta_{m+r} = \{d \mid \sum_{i=1}^{m+r} d_i = 1, d_i \ge 0, i = 1, \ldots, m+r\}$. We will show that the Lagrange multiplier with maximum entry is a type of shadow price. To prove this desired result, we need the following lemma [5].

Lemma 5.2. *Let $f : \mathbb{R}^n \mapsto (-\infty, \infty]$ be a proper convex function. Suppose that $\mathbf{ri}(\Delta_n) \bigcap \mathbf{ri}\,\mathbf{dom}(f) \ne \emptyset$. Then $x^* \in \Delta_n$ is an optimal solution of*

$$\min\{f(x) \mid x \in \Delta_n\}$$

if and only if there exists $g \in \partial f(x^*)$ and $\mu \in \mathbb{R}$ for which

$$g_i = \begin{cases} = \mu, & x_i^* > 0, \\ \ge \mu, & x_i^* = 0. \end{cases}$$

In **Eq**. (5.1), by setting $D = \{d \mid d \in \Delta_{m+r}\}$, we obtain

$$\begin{aligned}\mathbf{T_{SP}}(D) &= \min_{\{d \mid d \in \Delta_{m+r}\}} p(d) \\ &= -\min_{g \in M(X^*)} \max_{\{d \mid d \in \Delta_{m+r}\}} g^T d.\end{aligned} \tag{5.6}$$

By Lemma 5.2, the optimal solution of

$$\max_{\{d \mid d \in \Delta_{m+r}\}} g^T d \tag{5.7}$$

is

$$g^* = \begin{cases} \mu, & d_i^* > 0, \\ \ge \mu, & d_i^* = 0. \end{cases} \tag{5.8}$$

It can be inferred that $\mu = -\max_{i=1,\ldots,m+r} \{|g_i|\}$, since otherwise **Eq**. (5.8) will be violated. Notice that $d^* \in \Delta_{m+r}$, hence the optimal value of Problem (5.7) is $g^{*T} d^* = -\max_{i=1,\ldots,m+r} \{|g_i|\}$ Combining this **Eqs**. (5.6) and (5.8), we obtain that

$$\begin{aligned}\min_{\{d \mid d \in \Delta_{m+r}\}} p(d) &= -\min_{g \in M(X^*)} \{-\max_{i=1,\ldots,m+r} |g_i|\} \\ &= \max_{g \in M(X^*)} \max_{i=1,\ldots,r+m} |g_i|,\end{aligned}$$

implying that the Lagrange multiplier whose largest absolute value of entries is the maximum, is a type of shadow price.

6 Illustrative Example

In this section, we will use an illustrative example to show the power of GSP and the shadow price mapping. It will be shown that incorrect decisions would be made if policy makers only consider the buying / selling shadow price. Consider the following example.

Medicare is a small pharmaceutical manufacturing company that manufactures medicines I and II. To produce 1 unit of I, Medicare needs to invest 3 units of material A and 2 units of material B, while to produce 1 unit of II, Medicare needs to invest 2 units of material A and 4 units of material B. Assume that Medicare has available $\frac{7}{2}$ units of material A and 5 units of material B. Let x_1 and x_2 denote the quantity of medicines I and II manufactured. If in addition, the production cost of I and II can be expressed as $(x_1-1)^2+(x_2-2)^2$. Then the optimal production of medicines I and II can be obtained by solving the following quadratic programming:

$$\begin{array}{ll} \min & (x_1-1)^2+(x_2-2)^2 \\ \text{s.t.} & g_1(x)=3x_1+2x_2-\frac{7}{2}\leq 0 \\ & g_2(x)=2x_1+4x_2-5\leq 0 \\ & x\in\mathbf{X}=\{(x_1,x_2)|x_1\geq 0,x_2\geq 0\}. \end{array} \tag{6.1}$$

The optimal solution of Problem (6.1) is $x^*=(\frac{1}{2},1)^T$, and the corresponding Lagrange multiplier set is $M_\mu=\{(\mu_1,\mu_2)|\mu_1+2\mu_2\geq 0,\mu_1\geq 0,\mu_2\geq 0\}$.

Assume that the two materials A and B can either be produced by Medicare or be purchased from the supplier. If Medicare choose to purchase A and B from the suppliers, the prices for purchasing A and B are $\frac{1}{200}$ and $\frac{1}{100}$, respectively. Otherwise, Medicare can only produce A and B by themselves. Suppose in addition, the materials A and B are produced proportionally by Medicare, i.e., 1 unit of raw material can simultaneously produce 1 unit of A and $\frac{1}{3}$ unit of B, and the cost of 1 unit of raw material is $\frac{1}{10}$. Then Medicare must determine whether to expand their production of medicines.

Since the gradients of the two constraints at the optimal solution $\nabla g_1(x^*)=(3,2)^T$ and $\nabla g_2(x^*)=(2,4)^T$ are linear independent, it follows that for any small perturbation (u_1,u_2) on the right - hand side of the constraints, there exists a continuous function $x(u)$ of the following system

$$\left\{\begin{array}{l} g_1(x)=3x_1+2x_2-3.5=u_1 \\ g_2(x)=2x_1+4x_2-5=u_2 \end{array}\right. ,$$

satisfying $x(0)=x^*$. Notice that x^* is positive, then by the continuity of $x(u)$, we have $x(u)\in\mathbf{X}$. Therefore, combining Lemma 4.1, the SPE condition holds for Problem (6.1).

We first consider the case of purchasing A and B from the Supplier. By using the shadow price mapping with $d_1=(1,0)^T$ and $d_2=(0,1)^T$, we have

$$\mathbf{T_{SP}}(d_1)=-\min_{(\mu_1,\mu_2)\in M_\mu}\mu_1=0$$

and

$$\mathbf{T_{SP}}(d_2)=-\min_{(\mu_1,\mu_2)\in M_\mu}\mu_2=0$$

implying that the manufacturing cost of medicines I and II will not decrease if an extra unit of A or B are purchased from suppliers.

We then consider the case of producing A and B by Medicare. By using the shadow price mapping with $d_3 = (1, \frac{1}{3})^T$ we have

$$\mathbf{T_{SP}}(d_3) = -\min_{(\mu_1,\mu_2)\in M_\mu} \mu_1 + \frac{1}{3}\mu_2,$$

yields $\mathbf{T_{SP}}(d) = -\frac{1}{6}$. It implies that if the material A and B are increased proportionally with $d = (1, \frac{1}{3})$, the production cost of medicines I and II can be reduced by $\frac{1}{6}$. Recall that the cost of producing 1 unit of A and $\frac{1}{3}$ unit of B is $\frac{1}{10}$, therefore Medicare should expand their production level by producing the material A and B themselves.

7 Conclusion

The major insights from our analysis are

(1) We extend the notion of shadow prices to the generalized shadow price, which is defined as the directional derivative of the value function.

(2) We propose the sufficient condition to guarantee the existence of shadow price from a new point of view. Our proposed condition is weaker than the classical constraint qualifications.

(3) The shadow price mapping is proposed to build a close relationship between the set of Lagrange multipliers and the set of shadow prices.

(4) Based on the shadow price mapping, a unified framework is proposed to analyze the property of the set of shadow price. Our framework offers advantages that the Lagrange multipliers with nice theoretical properties can be identified as the shadow prices.

Future studies can be focused on identifying and computing generalized shadow prices when the perturbations of the optimization model are not only on the right - hand side of the constraints.

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