Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 17, No. 1, pp. 99–113, 2021

A STRONG CONVERGENCE THEOREM IN BANACH SPACES BY A NEW SHRINKING PROJECTION METHOD FOR TWO DEMIMETRIC MAPPINGS IN A BANACH SPACE

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Abstract: In this paper, using a new shrinking projection method, we prove a strong convergence theorem for two demimetric mappings in a Banach space. Using this result, we get well-known and new strong convergence theorems in Hilbert spaces and Banach spaces.

Key words: fixed point, metric projection, metric resolvent, shrinking projection method, duality mappinng

Mathematics Subject Classification: 47H05, 47H09

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1 Introduction

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. For a mapping $U : C \to H$, we denote by $F(U)$ the set of fixed points of U. Let k be a real number with $0 \le k < 1$. A mapping $U : C \to H$ is called a k-strict pseud-contraction [3] if

$$\|Ux - Uy\|^2 \le \|x - y\|^2 + k\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. If U is a k-strict pseud-contraction and $F(U) \neq \emptyset$, then we have that, for $x \in C$ and $q \in F(U)$,

$$\|Ux - q\|^2 \le \|x - q\|^2 + k\|x - Ux\|^2.$$

From this, we have that

$$\|Ux - x\|^2 + \|x - q\|^2 + 2\langle Ux - x, x - q\rangle \le \|x - q\|^2 + k\|x - Ux\|^2.$$

Therefore, we have that

$$2\langle x - Ux, x - q\rangle \ge (1 - k)\|x - Ux\|^2 \tag{1.1}$$

for all $x \in C$ and $q \in F(U)$. A mapping $U : C \to H$ is called generalized hybrid [6] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1 - \alpha)\|x - Uy\|^2 \le \beta\|Ux - y\|^2 + (1 - \beta)\|x - y\|^2$$

*The second author was partially supported by Grant-in-Aid for Scientific Research No. 20K03660 from Japan Society for the Promotion of Science.

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers. Such a mapping U is called $(\alpha,\ \beta)$-generalized hybrid. If U is $(\alpha,\ \beta)$-generalized hybrid and $F(U) \neq \emptyset$, then we have that, for $x \in C$ and $q \in F(U)$,

$$\alpha\|q - Ux\|^2 + (1-\alpha)\|q - Ux\|^2 \leq \beta\|q - x\|^2 + (1-\beta)\|q - x\|^2$$

and hence $\|Ux - q\|^2 \leq \|x - q\|^2$. From this, we have that

$$2\langle x - q, x - Ux\rangle \geq \|x - Ux\|^2. \tag{1.2}$$

On the other hand, there exists such a mapping in a Banach space. Let E be a smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. If J_λ is the metric resolvent of B for $\lambda > 0$, then we have from [1, 16] that, for any $x \in E$ and $q \in B^{-1}0$,

$$\langle J_\lambda x - q, J(x - J_\lambda x)\rangle \geq 0.$$

Then we get $\langle J_\lambda x - x + x - q, J(x - J_\lambda x)\rangle \geq 0$ and hence

$$\begin{aligned} \langle x - q, J(x - J_\lambda x)\rangle &\geq \langle x - J_\lambda x, J(x - J_\lambda x)\rangle \\ &= \|x - J_\lambda x\|^2, \end{aligned} \tag{1.3}$$

where J is the duality mapping on E. Motivated by (1.1) (1.2) and (1.3) Takahashi [20] defined a nonlinear mapping as follows: Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let η be a real number with $\eta \in (-\infty, 1)$. A mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called η-demimetric if, for any $x \in C$ and $q \in F(U)$,

$$2\langle x - q, J(x - Ux)\rangle \geq (1-\eta)\|x - Ux\|^2. \tag{1.4}$$

According to this definition, we have that a k-strict pseud-contraction U with $F(U) \neq \emptyset$ is k-demimetric, a generalized hybrid mapping U with $F(U) \neq \emptyset$ is 0-demimetric and the metric resolvent J_λ with $B^{-1}0 \neq \emptyset$ is (-1)-demimetric. On the other hand, we know the following strong convergence theorem by the shrinking projection method which was introduced by Takahashi, Takeuchi and Kubota [21] for finding a fixed point of a nonexpansive mapping in a Hilbert space.

Theorem 1.1 ([21]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let T be a nonexpansive mapping of C into H. Assume that $F(T) \neq \emptyset$. Let $x_1 \in C$ and $C_1 = C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = (1-\lambda_n)x_n + \lambda_n T x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a \in \mathbb{R}$ and $\{\lambda_n\} \subset (0, \infty)$ satisfy the following:

$$0 < a \leq \lambda_n \leq 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to a point $z_0 \in F(T)$, where $z_0 = P_{F(T)} x_1$ and $P_{F(T)}$ is the metric projection of H onto $F(T)$.

In this paper, using a new shrinking projection method, we establish a strong convergence theorem for finding a common element of the set of zero points of a maximal monotone operator and the set of common fixed points of two demimetric mappings in a Banach space. Moreover we apply our result to obtain well-known and new strong convergence theorems in a Hilbert space and a Banach space.

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon \right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, i.e., $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$; see [4, 11].

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [15] and [16]. Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \le \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the metric projection of E onto C.

Lemma 2.1 ([15]). *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = P_C x_1$;

(2) $\langle z - y, J(x_1 - z)\rangle \geq 0, \quad \forall y \in C$.

Let E be a Banach space and let B be a mapping of E into 2^{E^*}. The effective domain of B is denoted by $dom(B)$, that is, $dom(B) = \{x \in E : Bx \neq \emptyset\}$. A multi-valued mapping B on E is said to be monotone if $\langle x - y, u^* - v^*\rangle \geq 0$ for all $x, y \in dom(B)$, $u^* \in Bx$, and $v^* \in By$. A monotone operator B on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [2]; see also [16, Theorem 3.5.4].

Theorem 2.2 ([2]). *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let B be a monotone operator of E into 2^{E^*}. Then B is maximal if and only if for any $r > 0$,*

$$R(J + rB) = E^*,$$

where $R(J + rB)$ is the range of $J + rB$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let B be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rBx_r.$$

This equation has a unique solution x_r. In fact, for $x \in E$, define

$$Gy = B(y + x) \quad \forall y \in E.$$

Since $0 \in E^* = R(J + rG)$ for all $r > 0$, there exists $w \in D(G)$ such that

$$0 \in Jw + rGw = Jw + B(w + x).$$

Putting $x_r = w + x$, we have $0 \in J(x_r - x) + rBx_r$. We show that such a solution x_r is unique. Take $z_1, z_2 \in D(B)$ such that $0 \in J(z_1 - x) + rBz_1$ and $0 \in J(z_2 - x) + rBz_2$. We have $-\frac{1}{r}J(z_1 - x) \in Bz_1$ and $-\frac{1}{r}J(z_2 - x) \in Bz_2$. Since B and J are monotone, we have

$$\begin{aligned} 0 &\leq \left\langle z_1 - z_2, -\frac{1}{r}J(z_1 - x) + \frac{1}{r}J(z_2 - x)\right\rangle \\ &= -\frac{1}{r}\langle z_1 - x - (z_2 - x) J(z_1 - x) - J(z_2 - x)\rangle \leq 0 \end{aligned}$$

and hence

$$\langle z_1 - x - (z_2 - x) J(z_1 - x) - J(z_2 - x)\rangle = 0.$$

Since E is strictly convex, we have $z_1 - x = z_2 - x$ and hence $z_1 = z_2$. We define J_r by $x_r = J_r x$. Such a J_r is denoted by

$$J_r = (I + rJ^{-1}A)^{-1}$$

and is called the metric resolvent of B. For $r > 0$, the Yosida approximation $A_r : E \to E^*$ is defined by

$$A_r x = \frac{J(x - J_r x)}{r}, \quad \forall x \in E.$$

We have that $A_r x \in BJ_r x$ for all $x \in E$. The set of null points of B is defined by $B^{-1}0 = \{z \in E : 0 \in Bz\}$. We know that $B^{-1}0$ is closed and convex; see [16].

Let E be a smooth Banach space and let J be the duality mapping on E. Let η be a real number with $\eta \in (-\infty, 1)$. Then a mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called η-demimetric [20] if it satisfies (1.4) that is, for any $x \in C$ and $q \in F(U)$,

$$2\langle x - q, J(x - Ux)\rangle \geq (1 - \eta)\|x - Ux\|^2, \tag{2.2}$$

where $F(U)$ is the set of fixed points of U.

Examples.

(1) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \leq k < 1$. If U is a k-strict pseud-contraction and $F(U) \neq \emptyset$, then U is k-demimetric; see [20].

(2) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. If U is generalized hybrid and $F(U) \neq \emptyset$, then U is 0-demimetric; see [20]. Notice that the class of generalized hybrid mappings covers several well-known classes of mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is nonspreading [7, 8] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [17] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [5].

(3) Let E be a strictly convex, reflexive and smooth Banach space and let C be a nonempty, closed and convex subset of E. Let P_C be the metric projection of E onto C. Then P_C is (-1)-demimetric. In fact, since P_C is the metric projection of E onto C, we have that, for any $x \in E$ and $q \in C$,

$$\langle P_C x - q, J(x - P_C x)\rangle \geq 0.$$

Then we get

$$\langle P_C x - x + x - q, J(x - P_C x)\rangle \geq 0$$

and hence

$$\begin{aligned}\langle x - q, J(x - P_C x)\rangle &\geq \langle x - P_C x, J(x - P_C x)\rangle \\ &= \|x - P_C x\|^2.\end{aligned}$$

This means that P_C is (-1)-demimetric; see [20].

(4) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let $\lambda > 0$. Then the metric resolvent J_λ is (-1)-demimetric; see [20].

The following lemma which was proved by Takahashi [20] is important and crucial in the proof of our main result.

Lemma 2.3 ([20]). *Let E be a smooth and strictly convex Banach space and let C be a nonempty, closed and convex subset of E. Let η be a real number with $\eta \in (-\infty, 1)$. Let U be a η-demimetric mapping of C into E. Then $F(U)$ is closed and convex.*

For a sequence $\{C_n\}$ of nonempty, closed and convex subsets of a Banach space E, define $\text{s-Li}_n C_n$ and $\text{w-Ls}_n C_n$ as follows: $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in \text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies

$$C_0 = \underset{n}{\text{s-Li}}\, C_n = \underset{n}{\text{w-Ls}}\, C_n, \tag{2.3}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [10] and we write $C_0 = \text{M-lim}_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [10]. The following lemma was proved by Tsukada [26].

Lemma 2.4 ([26]). *Let E be a uniformly convex Banach space. Let $\{C_n\}$ be a sequence of nonempty, closed and convex subsets of E. If $C_0 = \text{M-lim}_{n\to\infty} C_n$ exists and nonempty, then for each $x \in E$, $\{P_{C_n} x\}$ converges strongly to $P_{C_0} x$, where P_{C_n} and P_{C_0} are the metric projections of E onto C_n and C_0, respectively.*

3 Main result

In this section, using a new shrinking projection method, we prove a strong convergence theorem for finding a common element of the set of zero points of a maximal monotone operator and the set of common fixed points of two demimetric mappings in a Banach space. For the proof of the theorem, we use the ideas of [13, 14, 19]. Let E be a Banach space and let D be a nonempty, closed and convex subset of E. A mapping $U : D \to E$ is called demiclosed if for a sequence $\{x_n\}$ in D such that $x_n \rightharpoonup p$ and $x_n - Ux_n \to 0$, $p = Up$ holds.

Theorem 3.1. *Let E be a uniformly convex and smooth Banach space and let C be a nonempty, closed and convex subset of E. Let $A \subset E \times E^*$ be a maximal monotone operator and let $J_r = (I + rJ^{-1}A)^{-1}$ be the metric resolvent of A for all $r > 0$. Let $\eta, \tau \in (-\infty, 1)$ and let S and T be η and τ-demimetric mappings from C into itself, respectively, such that they are demiclosed. Suppose that*

$$\Omega = F(S) \cap F(T) \cap A^{-1}0 \neq \emptyset.$$

For $x_1 \in C$ and $C_1 = C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} u_n = J_{r_n} z_n, \\ z_n = \beta_n v_n + (1-\beta_n) T v_n, \\ v_n = \alpha_n x_n + (1-\alpha_n) S x_n, \\ C_{n+1} = \Big\{ z \in C_n : \langle z_n - z, J(z_n - u_n) \rangle \geq \|z_n - u_n\|^2, \\ \qquad\qquad 2\langle v_n - z, J(v_n - z_n) \rangle \geq (1-\tau)\|v_n - z_n\|^2 \\ \qquad\qquad \text{and} \quad 2\langle x_n - z, J(x_n - v_n) \rangle \geq (1-\eta)\|x_n - v_n\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where J is the duality mapping on E, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $1 - \alpha_n \geq b > 0$ and $1 - \beta_n \geq c > 0$ for some $b, c \in (0,1)$, then $\{x_n\}$ converges strongly to $P_\Omega x_1$, where P_Ω is the metric projection of E onto Ω.

Proof. It follows that C_n are closed and convex for all $n \in \mathbb{N}$. We show that $\Omega \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that $\Omega \subset C_1 = C$. Suppose that $\Omega \subset C_k$ for some $k \in \mathbb{N}$. To show $\Omega \subset C_{k+1}$, let us show that $\langle z_k - z, J(z_k - u_k) \rangle \geq \|z_k - u_k\|^2$,

$$\langle v_k - z, J(v_k - z_k) \rangle \geq (1-\tau)\|v_k - z_k\|^2$$

and $2\langle x_k - z, J(x_k - v_k) \rangle \geq (1-\eta)\|x_k - v_k\|^2$ for all $z \in \Omega$. Let $z \in \Omega$. Since J_{r_k} is the metric resolvent, we have from [1, 16] that

$$\langle J_{r_k} z_k - z, J(z_k - J_{r_k} z_k) \rangle \geq 0$$

for all $z \in \Omega \subset A^{-1}0$. From this, we get that

$$\langle J_{r_k} z_k - z_k + z_k - z, J(z_k - J_{r_k} z_k) \rangle \geq 0$$

and hence

$$\langle z_k - z, J(z_k - J_{r_k} z_k) \rangle \geq \|z_k - J_{r_k} z_k\|^2.$$

This implies that

$$\langle z_k - z, J(z_k - u_k) \rangle \geq \|z_k - u_k\|^2.$$

Since T is τ-demimetric, we also have that for any $z \in \Omega$,

$$\begin{aligned} 2\langle v_k - z, \ J(v_k - z_k) \rangle &= 2(1-\beta_k)\langle v_k - z, J(v_k - T v_k) \rangle \\ &\geq (1-\beta_k)(1-\tau)\|v_k - T v_k\|^2 \\ &\geq (1-\beta_k)^2(1-\tau)\|v_k - T v_k\|^2 \\ &= (1-\tau)\|v_k - z_k\|^2. \end{aligned}$$

Similarly, we have that

$$2\langle x_k - z, J(x_k - v_k) \rangle \geq (1-\eta)\|x_k - v_k\|^2.$$

Then $\Omega \subset C_{k+1}$. We have by mathematical induction that $\Omega \subset C_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

We have that $F(S)$ and $F(T)$ are closed and convex from Lemma 2.3. We also have that $A^{-1}0$ is closed and convex. Thus Ω is nonempty, closed and convex. Then there exists $w_1 \in \Omega$ such that $w_1 = P_\Omega x_1$. From $x_n = P_{C_n} x_1$, we have that

$$\|x_1 - x_n\| \leq \|x_1 - y\|$$

for all $y \in C_n$. Since $w_1 \in \Omega \subset C_n$, we have that

$$\|x_1 - x_n\| \leq \|x_1 - w_1\|. \tag{3.1}$$

Let $C_0 = \bigcap_{n=1}^\infty C_n$. Since $\emptyset \neq \Omega \subset C_0$, we have that C_0 is nonempty. Since $C_0 =$ M-$\lim_{n\to\infty} C_n$ and $x_n = P_{C_n} x_1$ for all $n \in \mathbb{N}$, by Lemma 2.4 we have that

$$x_n \to z_0 = P_{C_0} x_1. \tag{3.2}$$

We have from $x_{n+1} \in C_{n+1}$ that

$$2\langle x_n - x_{n+1}, J(x_n - v_n)\rangle \geq (1-\eta)\|x_n - v_n\|^2$$

and hence

$$2\|x_n - x_{n+1}\| \geq (1-\eta)\|x_n - v_n\|.$$

Since $\|x_n - x_{n+1}\| \to 0$ from (3.2) we get that $x_n - v_n \to 0$. On the other hand, from

$$\|x_n - v_n\| = (1-\alpha_n)\|x_n - Sx_n\| \geq b\|x_n - Sx_n\|,$$

we have that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{3.3}$$

Furthermore, we have from $x_{n+1} \in C_{n+1}$ that

$$2\langle v_n - x_{n+1}, J(v_n - z_n)\rangle \geq (1-\tau)\|v_n - z_n\|^2.$$

From this, we have that

$$2\|v_n - x_{n+1}\| \geq (1-\tau)\|v_n - z_n\|$$

and hence

$$2\|v_n - x_n + x_n - x_{n+1}\| \geq (1-\tau)\|v_n - z_n\|.$$

From $\|v_n - x_n\| \to 0$ and $\|x_n - x_{n+1}\| \to 0$, we have that $\lim_{n\to\infty} \|v_n - z_n\| = 0$. From

$$\|v_n - z_n\| = (1-\beta_n)\|v_n - Tv_n\| \geq c\|v_n - Tv_n\|,$$

we get that

$$\lim_{n\to\infty} \|v_n - Tv_n\| = 0. \tag{3.4}$$

We also have from $x_{n+1} \in C_{n+1}$ that

$$\langle z_n - x_{n+1}, J(z_n - u_n)\rangle \geq \|z_n - u_n\|^2$$

and hence

$$\|z_n - x_{n+1}\| \geq \|z_n - u_n\|.$$

From $\|z_n - x_{n+1}\| \leq \|z_n - v_n\| + \|v_n - x_n\| + \|x_n - x_{n+1}\|$, $z_n - v_n \to 0$, $v_n - x_n \to 0$ and $x_n - x_{n+1} \to 0$, we have $\|z_n - x_{n+1}\| \to 0$. Then we get that

$$\lim_{n\to\infty} \|z_n - u_n\| = 0$$

and hence

$$\lim_{n\to\infty} \|z_n - J_{r_n} z_n\| = 0. \tag{3.5}$$

Since $x_n \to z_0$ and S is demiclosed, we have from (3.3) that $z_0 \in F(S)$. Similarly, from $x_n - v_n \to 0$, we get $v_n \to z_0$. Since T is demiclosed, we have from (3.4) that $z_0 \in F(T)$. We show $z_0 \in A^{-1}0$. From $r_n \geq a$ and (3.5) we have

$$\lim_{n\to\infty} \frac{1}{r_n} \|J(z_n - J_{r_n} z_n)\| = 0.$$

Therefore, we have

$$\lim_{n\to\infty} \|A_{r_n} z_n\| = \lim_{n\to\infty} \frac{1}{r_n} \|J(z_n - J_{r_n} z_n)\| = 0. \tag{3.6}$$

For $(p, p^*) \in A$, from the monotonicity of A, we have

$$\langle p - J_{r_n} z_n, p^* - A_{r_n} z_n \rangle \geq 0 \tag{3.7}$$

for all $n \in \mathbb{N}$. From $v_n - z_n \to 0$ and $v_n \to z_0$, we get $z_n \to z_0$. Furthermore, from (3.5) we have $J_{r_n} z_n \to 0$. From $J_{r_n} z_n \to 0$, (3.7) and (3.6), we get $\langle p - z_0, p^* \rangle \geq 0$. From the maximality of A, we have $z_0 \in A^{-1}0$. Therefore, we have $z_0 \in \Omega$.

From $w_1 = P_\Omega x_1$, $z_0 \in \Omega$ and (3.1) we have that

$$\|x_1 - w_1\| \leq \|x_1 - z_0\| = \lim_{n\to\infty} \|x_1 - x_n\| \leq \|x_1 - w_1\|.$$

Then we get that $\|x_1 - w_1\| = \|x_1 - z_0\|$ and hence $z_0 = w_1$. Therefore, we have $x_n \to z_0 = w_1$. This completes the proof. □

4 Applications

In this section, using Theorem 3.1, we get well-known and new strong convergence theorems in Hilbert spaces and Banach spaces. We know the following result obtained by Marino and Xu [9]; see also [23].

Lemma 4.1 ([9]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let k be a real number with $0 \leq k < 1$. Let $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

We also know the following result from Kocourek, Takahashi and Yao [6]; see also [24].

Lemma 4.2 ([6]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $U : C \to H$ be generalized hybrid. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Using Theorem 3.1 and Lemmas 4.1 and 4.2, we have the following theorem.

Theorem 4.3. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $A \subset H \times H$ be a maximal monotone operator and let $J_r = (I + rA)^{-1}$ for all $r > 0$. Let k be a real number with $k \in [0, 1)$. Let $S : C \to C$ be a nonexpansive mapping with $F(S) \neq \emptyset$ and let $T : C \to C$ be a k-strict pseud-contraction such that $F(T) \neq \emptyset$. Suppose that $\Omega = F(S) \cap F(T) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and $C_1 = C$*

$$\begin{cases} u_n = J_{r_n} z_n, \\ z_n = \beta_n v_n + (1 - \beta_n) T v_n, \\ v_n = S x_n, \\ C_{n*1} = \Big\{ z \in C_n : \langle z_n - z, z_n - u_n \rangle \geq \|z_n - u_n\|^2, \\ \qquad\qquad 2\langle v_n - z, v_n - z_n \rangle \geq (1 - k)\|v_n - z_n\|^2 \\ \qquad\qquad\qquad \text{and } \ 2\langle x_n - z, x_n - v_n \rangle \geq \|x_n - v_n\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $1 - \beta_n \geq c > 0$ for some $c \in (0, 1)$, then $\{x_n\}$ converges strongly to $P_\Omega x_1$, where P_Ω is the metric projection of H onto Ω.

Proof. Since T is a k-strict pseud-contraction of C into itself such that $F(T) \neq \emptyset$, from (1) in Examples, T is k-demimetric. Furthermore, from Lemma 4.1, T is demiclosed. Furthermore, we know that a nonexpansive maping S is 0-demimetric and demiclosed. We also know that the resolvent J_r of A for $r > 0$ is (-1)-demimetric and demiclosed. Therefore, we have the desired result from Theorem 3.1. □

The following is a strong convergence theorem for nonexpansive mappings and generalized hybrid mappings in a Hilbert space.

Theorem 4.4. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $A \subset H \times H$ be a maximal monotone operator and let $J_r = (I + rA)^{-1}$ for all $r > 0$. Let $S : C \to C$ be a nonexpansive mapping with $F(S) \neq \emptyset$ and let $T : C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Suppose that $\Omega = F(S) \cap F(T) \cap A^{-1}0 \neq \emptyset$. For $x_1 \in C$ and $C_1 = C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u_n = J_{r_n} z_n, \\ z_n = T v_n, \\ v_n = S x_n, \\ C_{n+1} = \Big\{ z \in C_n : \langle z_n - z, z_n - u_n \rangle \geq \|z_n - u_n\|^2, \\ \qquad\qquad 2\langle v_n - z, v_n - z_n \rangle \geq \|v_n - z_n\|^2 \\ \qquad\qquad\qquad \text{and } \ 2\langle x_n - z, x_n - v_n \rangle \geq \|x_n - v_n\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then $\{x_n\}$ converges strongly to $P_\Omega x_1$, where P_Ω is the metric projection of H onto Ω.

Proof. Since T is a generalized hybrid mapping of C into itself such that $F(T) \neq \emptyset$, from (2) in Examples, T is 0-demimetric. Furthermore, from Lemma 4.2, T is demiclosed. A nonexpansive maping S and the resolvent J_r are as in the proof of Theorem 4.3. Therefore, we have the desired result from Theorem 3.1. □

Let E be a Banach space and let $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. Define the subdifferential of f as follows:

$$\partial f(x) = \{x^* \in E^* : f(y) \geq \langle y - x, x^* \rangle + f(x),\ \forall y \in E\}$$

for each $x \in E$. Then, we know that ∂f is a maximal monotone operator; see [12] for more details. The following is a strong convergence theorem for three metric projections in a Banach space.

Theorem 4.5. *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping on E. Let B, C and D be nonempty, closed and convex subsets of E. Let P_B, P_C and P_D be the metric projections of E onto B, C and D, respectively. Suppose that $\Omega = B \cap C \cap D \neq \emptyset$. For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u_n = P_B z_n, \\ z_n = P_C v_n, \\ v_n = P_D x_n, \\ C_{n+1} = \Big\{ z \in C_n : \langle z_n - z, J(z_n - u_n) \rangle \geq \|z_n - u_n\|^2, \\ \qquad\qquad \langle v_n - z, J(v_n - z_n) \rangle \geq \|v_n - z_n\|^2 \} \\ \qquad\qquad\qquad \text{and } \ \langle x_n - z, J(x_n - v_n) \rangle \geq \|x_n - v_n\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}. \end{cases}$$

Then $\{x_n\}$ converges strongly to a point $P_\Omega x_1$, where P_Ω is the metric projection of E onto Ω.

Proof. Set $A = \partial i_B$ in Theorem 3.1, where i_B is the indicator function, that is,

$$i_B = \begin{cases} 0, & x \in B, \\ \infty, & x \notin B. \end{cases}$$

Then, we have that ∂i_B is a maximal monotone operator and $J_r = P_B$ for $r > 0$. In fact, for any $x \in E$ and $r > 0$, we have that

$$\begin{aligned} z = J_r x &\Leftrightarrow J(z - x) + r\partial i_B(z) \ni 0 \\ &\Leftrightarrow J(x - z) \in r\partial i_B(z) \\ &\Leftrightarrow i_B(y) \geq \langle y - z, \frac{J(x - z)}{r} \rangle + i_B(z),\ \forall y \in E \\ &\Leftrightarrow 0 \geq \langle y - z, J(x - z) \rangle,\ \forall y \in B \\ &\Leftrightarrow z = P_B x. \end{aligned}$$

Since P_B is the metric projection of E onto B, from (3) in Examples, P_B is (-1)-demimetric. We also have that if $\{x_n\}$ is a sequence in E such that $x_n \rightharpoonup p$ and $x_n - P_B x_n \to 0$, then

$p = P_B p$. In fact, assume that $x_n \rightharpoonup p$ and $x_n - P_B x_n \to 0$. It is clear that $P_B x_n \rightharpoonup p$ and $\|J(x_n - P_B x_n)\| = \|x_n - P_B x_n\| \to 0$. Since P_B is the metric projection of E onto B, we have that

$$\langle P_B x_n - P_B p, J(x_n - P_B x_n) - J(p - P_B p)\rangle \geq 0.$$

This implies that $-\|p - P_B p\|^2 = \langle p - P_B p, -J(p - P_B p)\rangle \geq 0$ and hence $p = P_B p$. Similarly, P_C and P_D are (-1)-demimetric and demiclosed. Therefore, we have the desired result from Theorem 3.1. □

The following is a strong convergence theorem for three metric resolvents in a Banach space.

Theorem 4.6. *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping on E. Let A, B and G be maximal monotone operators of $E \times E^*$ and let $J_r = (I + rJ^{-1}A)^{-1}$, $Q_\lambda = (I + \lambda J^{-1}B)^{-1}$ and $R_\mu = (I + \mu J^{-1}G)^{-1}$, for all $r > 0$, $\lambda > 0$ and $\mu > 0$, respectively. Suppose that*

$$\Omega = A^{-1}0 \cap B^{-1}0 \cap G^{-1}0 \neq \emptyset.$$

For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} u_n = J_r z_n, \\ z_n = Q_\lambda v_n, \\ v_n = R_\mu x_n, \\ C_{n+1} = \Big\{ z \in C_n : \langle z_n - z, J(z_n - u_n)\rangle \geq \|z_n - u_n\|^2, \\ \qquad\qquad \langle v_n - z, J(v_n - z_n)\rangle \geq \|v_n - z_n\|^2 \\ \qquad\qquad\qquad \text{and } \ \langle x_n - z, J(x_n - v_n)\rangle \geq \|x_n - v_n\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}. \end{cases}$$

Then $\{x_n\}$ converges strongly to a point $P_\Omega x_1$, where P_Ω is the metric projection of E onto Ω.

Proof. Since Q_λ is the metric resolvent of B for $\lambda > 0$, from (4) in Examples, Q_λ is (-1)-demimetric. We also have that if $\{x_n\}$ is a sequence in E such that $x_n \rightharpoonup p$ and $x_n - Q_\lambda x_n \to 0$, then $p = Q_\lambda p$. In fact, assume that $x_n \rightharpoonup p$ and $x_n - Q_\lambda x_n \to 0$. It is clear that $Q_\lambda x_n \rightharpoonup p$ and $\|J(x_n - Q_\lambda x_n)\| = \|x_n - Q_\lambda x_n\| \to 0$. Since Q_λ is the metric resolvent of B, we have from [1] that

$$\langle Q_\lambda x_n - Q_\lambda p, J(x_n - Q_\lambda x_n) - J(p - Q_\lambda p)\rangle \geq 0.$$

This implies that $-\|p - Q_\lambda p\|^2 = \langle p - Q_\lambda p, -J(p - Q_\lambda p)\rangle \geq 0$ and hence $p = Q_\lambda p$. Similarly, J_r and R_μ are (-1)-demimetric and demiclosed. Therefore, we have the desired result from Theorem 3.1. □

Acknowledgments

The authors would like to thank the associate editor and the anonymous reviewers for their valuable suggestions which have helped to improve the presentation.

References

[1] K. Aoyama, F. Kohsaka and W. Takahashi, Three generalizations of firmly nonexpansive mappings: Their relations and continuous properties, *J. Nonlinear Convex Anal.* **10** (2009) 131–147.

[2] F. E. Browder, Nonlinear maximal monotone operators in Banach spaces, *Math. Ann.* 175 (1968) 89–113.

[3] F.E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, *J. Math. Anal. Appl.* 20 (1967) 197–228.

[4] I. Cioranescu, Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Kluwer, Dordrecht, 1990.

[5] T. Igarashi, W. Takahashi and K.Tanaka, Weak convergence theorems for nonspreading mappings and equilibrium problems, in: *Nonlinear Analysis and Optimization*, S. Akashi, W. Takahashi and T. Tanaka (eds.) Yokohama Publishers, Yokohama, 2008, pp. 75–85.

[6] P. Kocourek, W. Takahashi and J.-C. Yao, Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert space, *Taiwanese J. Math.* 14 (2010) 2497–2511.

[7] F. Kosaka and W. Takahashi, Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces, *SIAM. J. Optim.* 19 (2008) 824-835.

[8] F. Kosaka and W. Takahashi, Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces, *Arch. Math. (Basel)* 91 (2008) 166-177.

[9] G. Marino and H.-K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, *J. Math. Anal. Appl.* 329 (2007) 336–346.

[10] U. Mosco, Convergence of convex sets and of solutions of variational inequalities, *Adv. Math.* 3 (1969) 510–585.

[11] S. Reich, Book Review: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, *Bull. Amer. Math. Soc.* 26 (1992) 367–370.

[12] R.T. Rockafellar, On the maximal monotonicity of subdifferential mappings, *Pacific J. Math.* 33 (1970) 209–216.

[13] S. Takahashi and W. Takahashi, The split common null point problem and the shrinking projection method in Banach spaces, *Optimization* 65 (2016) 281–287.

[14] S. Takahashi and W. Takahashi, The split common null point problem and the shrinking projection method in two Banach spaces, *Linear Nonlinear Anal.* 1 (2015) 297–304.

[15] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[16] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama, 2000 (Japanese).

[17] W. Takahashi, Fixed point theorems for new nonlinear mappings in a Hilbert space, *J. Nonlinear Convex Anal.* 11 (2010) 79–88.

[18] W. Takahashi, The split common null point problem in two Banach spaces, *J. Nonlinear Convex Anal.* 16 (2015) 2343–2350.

[19] W. Takahashi, The split common fixed point problem and strong convergence theorems by hybrid methods in two Banach spaces, *J. Nonlinear Convex Anal.* 17 (2016) 1051–1067.

[20] W. Takahashi, The split common fixed point problem and the shrinking projection method in Banach spaces, *J. Convex Anal.* 24 (2017) 1015–1028.

[21] W. Takahashi, Y. Takeuchi and R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.* 341 (2008) 276–286.

[22] W. Takahashi, H.-K. Xu and J.-C. Yao, Iterative methods for generalized split feasibility problems in Hilbert spaces, *Set-Valued Var. Anal.* 23 (2015) 205–221.

[23] W. Takahashi, N.-C. Wong and J.-C. Yao, *Weak and strong mean convergence theorems for extended hybrid mappings in Hilbert spaces*, *J. Nonlinear Convex Anal.* 12 (2011) 553–575.

[24] W. Takahashi, J.-C. Yao and P. Kocourek, Weak and strong convergence theorems for generalized hybrid nonself-mappings in Hilbert spaces, *J. Nonlinear Convex Anal.* 11 (2010) 567–586.

[25] H.-K. Xu, A variable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem, *Inverse Problems* 22 (2006) 2021–2034.

[26] M. Tsukada, Convergence of best approximations in a smooth Banach space, *J. Approx. Theory* 40 (1984) 301–309.

Manuscript received 15 April 2020
revised 17 April
accepted for publication 28 February 2021

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