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$l^{p,q}$-SINGULAR VALUES OF A PARTIALLY SYMMETRIC RECTANGULAR TENSOR*

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Abstract: Let $\mathcal{A}$ be a real partially symmetric rectangular tensor. In order to judge the positive definiteness of $\mathcal{A}$, an $l^{p,q}$-singular value inclusion set with parameters is first constructed. Subsequently, by selecting appropriate parameters, the optimal singular value inclusion interval is derived, which provides a sufficient condition for the positive definiteness of $\mathcal{A}$. Secondly, lower and upper bounds for the $l^{p,q}$-spectral radius of a nonnegative rectangular tensor are given. Thirdly, the relationship between $l^{2,2}$-singular values/vectors of $\mathcal{A}$ and Z-eigenpairs of the lifting square tensor of $\mathcal{A}$ is derived, which provides an alternative method to find all $l^{2,2}$-singular values/vectors of $\mathcal{A}$. Moreover, the relationship between $l^{p,q}$-singular values/vectors of $\mathcal{A}$ and generalized eigenvalues/eigenvectors of the lifting square tensor of $\mathcal{A}$ and the lifting square tensor of the identity rectangular tensor is derived, which provides an alternative method to find all $l^{p,q}$-singular values/vectors of $\mathcal{A}$. Finally, numerical examples are given to verify the theoretical results.

Key words: *rectangular tensors, nonnegative tensors, $l^{k,s}$-singular values, $l^{p,q}$-singular values, positive definiteness*

Mathematics Subject Classification: *15A18, 15A42, 15A69* .

1 Introduction

Real rectangular tensors arise from the strong ellipticity condition problem in solid mechanics [9, 17] and the entanglement problem in quantum physics [4, 6]. The definition of singular values of a real rectangular tensor is introduced by Lim [10] and Chang et al. [1]. Recently, Ling and Qi [11] extended the concept of singular values of a rectangular tensor in [1] to $l^{k,s}$-singular value of a rectangular tensor and yielded many properties on $l^{k,s}$-singular values. Subsequently, Yao et al. [19] made further research on $l^{k,s}$-singular values of a rectangular tensor. Now, let us recall the concept of $l^{k,s}$-singular values of a real rectangular tensor.

Let p, q, m and n be positive integers, $m, n \geq 2$, $[n] := \{1, 2, \ldots, n\}$, $\mathbb{C}$ (resp. $\mathbb{R}$) be the set of all complex (resp. real) numbers, $\mathbb{R}^n$ (resp. $\mathbb{R}^n_+$) be the set of all dimension n real (resp. nonnegative) vectors. We call $\mathcal{A} = (a_{i_1\cdots i_p j_1\cdots j_q})$ a real (p, q)-th order $m \times n$ dimensional rectangular tensor, denoted by $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$, if

$$a_{i_1\cdots i_p j_1\cdots j_q} \in \mathbb{R}, \quad i_1, \ldots, i_p \in [m], \quad j_1, \ldots, j_q \in [n].$$

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For simplicity, we call $\mathcal{A}$ a real rectangular tensor. If all entries of $\mathcal{A}$ are nonnegative numbers, then $\mathcal{A}$ is called a nonnegative rectangular tensor and it is represented by $\mathcal{A} \in \mathbb{R}_+^{[p;q;m;n]}$. Furthermore, $\mathcal{A}$ is called a real partially symmetric rectangular tensor, if

$$a_{\pi(i_1,\ldots,i_p)\tau(j_1,\ldots,j_q)} = a_{i_1\cdots i_p j_1\cdots j_q}, \quad \forall\ \pi \in S_p,\ \forall\ \tau \in S_q,$$

where S_p (resp. S_q) is the permutation group of p (resp. q) indices and π (resp. τ) is any permutation of indices among $i_1,\ldots,i_p$ (resp. $j_1,\ldots,j_q$).

For any vector $z = (z_1, z_2, \ldots, z_n)^\top \in \mathbb{R}^n$ and any positive integer k, denote

$$\begin{aligned} z^{[k]} :=& (z_1^k, \ldots, z_n^k)^\top, \\ \|z\|_k :=& (|z_1|^k + \cdots + |z_n|^k)^{1/k}, \\ \varphi_k^{(n)}(z) :=& (\mathrm{sign}(z_1)|z_1|^k, \ldots, \mathrm{sign}(z_n)|z_n|^k)^\top, \end{aligned}$$

where

$$\mathrm{sign}(z) = \begin{cases} 1, & z > 0, \\ 0, & z = 0, \\ -1, & z < 0. \end{cases}$$

Let $x = (x_1, x_2, \ldots, x_m)^\top \in \mathbb{R}^m$, $y = (y_1, y_2, \ldots, y_n)^\top \in \mathbb{R}^n$, $\mathcal{A}x^{p-1}y^q$ be a vector in $\mathbb{R}^m$, whose ith component is

$$(\mathcal{A}x^{p-1}y^q)_i = \sum_{i_2,\ldots,i_p \in [m], j_1,\ldots,j_q \in [n]} a_{ii_2\cdots i_p j_1\cdots j_q} x_{i_2}\cdots x_{i_p} y_{j_1}\cdots y_{j_q},$$

and $\mathcal{A}x^p y^{q-1}$ be a vector in $\mathbb{R}^n$, whose jth component is

$$(\mathcal{A}x^p y^{q-1})_j = \sum_{i_1,\ldots,i_p \in [m], j_2,\ldots,j_q \in [n]} a_{i_1\cdots i_p j j_2\cdots j_q} x_{i_1}\cdots x_{i_p} y_{j_2}\cdots y_{j_q}.$$

Definition 1.1 ([11, Definition 2.1])**.** Let $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$. For the given integers $k, s \in \{2, \ldots, p+q, \ldots\}$, if $(\lambda, x, y) \in \mathbb{R} \times (\mathbb{R}^m \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\})$ is a solution of the system

$$\begin{cases} \mathcal{A}x^{p-1}y^q = \lambda\varphi_{k-1}^{(m)}(x), & (1.1) \\ \mathcal{A}x^p y^{q-1} = \lambda\varphi_{s-1}^{(n)}(y), & (1.2) \\ \|x\|_k = \|y\|_s = 1, & (1.3) \end{cases}$$

then λ is called an $l^{k,s}$-singular value of $\mathcal{A}$ and (x, y) is called a pair of $l^{k,s}$-singular vectors of $\mathcal{A}$ associated with λ.

For the existence of the $l^{k,s}$-singular values/vectors pair of $\mathcal{A}$, Ling and Qi in [11, Theorem 2.1] showed the fact:

Theorem 1.2 ([11, Theorem 2.1])**.** *Let $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$ be partially symmetric. Then for every $k, s \in \{2, \ldots, p+q, \ldots\}$, its $l^{k,s}$-singular values and singular vectors pair always exist.*

Based on Theorem 1.2, we in this paper assume that $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$ always is a real partially symmetric rectangular tensor, denote by $\sigma(\mathcal{A})$ the set of all $l^{k,s}$-singular values of $\mathcal{A}$, and call $\rho_{k,s}(\mathcal{A})$ the $l^{k,s}$-spectral radius of $\mathcal{A}$ [11, Definition 3.1] if it is the largest absolute $l^{k,s}$-singular values of $\mathcal{A}$, i.e.,

$$\rho_{k,s}(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

Also in [11], Ling and Qi obtained bounds for the $l^{k,s}$-spectral radius of $\mathcal{A}$ as follows:

Lemma 1.3 ([11, Corollary 3.3]). *Let $\mathcal{A} \in \mathbb{R}_+^{[p;q;m;n]}$ be partially symmetric. If $k \le p$ and $s \le q$, then*

$$\rho_{k,s}(\mathcal{A}) \le \max_{i\in[m],j\in[n]} \left\{R_i(\mathcal{A}), C_j(\mathcal{A})\right\},$$

where

$$\begin{aligned} R_i(\mathcal{A}) &= \sum_{i_2,\ldots,i_p\in[m],j_1,\ldots,j_q\in[n]} |a_{ii_2\cdots i_p j_1\cdots j_q}|, \\ C_j(\mathcal{A}) &= \sum_{i_1,\ldots,i_p\in[m],j_2,\ldots,j_q\in[n]} |a_{i_1\cdots i_p j j_2\cdots j_q}|. \end{aligned} \tag{1.4}$$

Due to the diversity of selection of p, q, k and s, many scholars have studied the properties of $l^{k,s}$-singular values when these parameters are taken as special values. For example, when $k = s = p + q$, such $l^{k,s}$-singular values of rectangular tensors are introduced by Chang et al. and called singular value of rectangular tensors in [1]. Subsequently, properties, lower and upper bounds of spectral radius and inclusion sets for singular values of rectangular tensors are studied in [15, 18, 21, 22, 23, 24]. When $k = p$ and $s = q$, such $l^{k,s}$-singular values of rectangular tensors are called V-singular values and studied in [7]. In this paper, such $l^{k,s}$-singular values are called $l^{p,q}$-singular values as $k = p$ and $s = q$.

Given a partially symmetric rectangular tensor $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$, it determines a multivariate polynomial

$$f(x,y) = \mathcal{A}x^p y^q = \sum_{i_1,\ldots,i_p\in[m],j_1,\ldots,j_q\in[n]} a_{i_1\cdots i_p j_1\cdots j_q} x_{i_1}\cdots x_{i_p} y_{j_1}\cdots y_{j_q}. \tag{1.5}$$

When both p and q are even, if $f(x, y) > 0$ for all $x \in \mathbb{R}^m \setminus \{0\}$ and $y \in \mathbb{R}^n \setminus \{0\}$, then we say that $\mathcal{A}$ is positive definite. When $\mathcal{A}$ is the elasticity tensor, which is a real rectangular tensor with $p = q = 2$ and $m = n = 2$ or 3, the strong ellipticity condition holds if and only if $\mathcal{A}$ is positive definite [14]. Since the strong ellipticity condition plays an important role in nonlinear elasticity and materials, positive definiteness of such a partially symmetric tensor has a sound application background. When k and s are even and $k, s \ge 2$, Yao et al. [19] proposed the following method to judge the positive definiteness of $\mathcal{A}$ by using its $l^{k,s}$-singular values.

Theorem 1.4 ([19, Theorem 2]). *Let $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$ be partially symmetric with p and q being even, k and s be even and $k, s \ge 2$. Then $\mathcal{A}$ is positive definite if and only if all of its $l^{k,s}$-singular values are positive.*

There is another way to judge the positive definiteness of $\mathcal{A}$: One can try to construct a set which includes all $l^{k,s}$-singular values of $\mathcal{A}$ in the complex plane, and furthermore if the set lies in the right half complex plane, then we can conclude that all $l^{k,s}$-singular values of $\mathcal{A}$ are positive and, consequently, $\mathcal{A}$ is positive definite.

Although many researchers have constructed such sets [15, 20, 21, 22, 23, 24], unfortunately, all these sets contain the origin, and hence they cannot be used to judge the positive definiteness of a real partially symmetric rectangular tensor. Then, a question is naturally raised: How to construct a singular value inclusion set that can be used to judge the positive definiteness of a real partially symmetric rectangular tensor? We focus on this issue in this paper.

The rest is arranged as follows. In Section 2, we construct an $l^{p,q}$-singular value inclusion set with parameter vectors α and β to locate all $l^{p,q}$-singular values of a real rectangular tensor. Subsequently, by selecting appropriate parameter vectors α and β, we obtain the optimal singular value inclusion interval and use it to judge the positive definiteness of a real partially symmetric rectangular tensor. In other words, as an application of the set, we present a sufficient condition of the positive definiteness of a real partially symmetric rectangular tensor. As another application of the set, we obtain an upper bound of the $l^{p,q}$-spectral radius of a nonnegative rectangular tensor in Section 3. Also in Section 3, we present a lower bound of the $l^{p,q}$-spectral radius. In Section 4, we focus on calculation of all $l^{p,q}$-singular values/vectors of a real rectangular tensor $\mathcal{A}$, derive the relationship between all $l^{2,2}$-singular values/vectors of $\mathcal{A}$ and Z-eigenpairs of the lifting square tensor $\mathcal{C}_{\mathcal{A}}$ of $\mathcal{A}$, and derive the relationship between all $l^{p,q}$-singular values/vectors of $\mathcal{A}$ and generalized eigenvalues/eigenvectors of $\mathcal{C}_{\mathcal{A}}$ and the lifting square tensor $\mathcal{C}_{\mathcal{I}}$ of the identity rectangular tensor $\mathcal{I}$, which provides an alternative method to find all $l^{p,q}$-singular values/vectors of $\mathcal{A}$. In Section 5, we use two examples to verify the theoretical results. In the end, we give some conclusions to end this paper.

2 Locations for $l^{p,q}$-Singular Values of a Real Rectangular Tensor with p and q Even

Taking $k = p$ and $s = q$ in Definition 1.1, then (1.1), (1.2) and (1.3) reduce to the following equations:

$$\begin{cases} \mathcal{A}x^{p-1}y^q = \lambda\varphi_{p-1}^{(m)}(x), & (2.1)\\ \mathcal{A}x^p y^{q-1} = \lambda\varphi_{q-1}^{(n)}(y), & (2.2)\\ \|x\|_p = \|y\|_q = 1. & (2.3)\end{cases}$$

Let both p and q be even, $x = (x_1, x_2, \ldots, x_m)^\top \in \mathbb{R}^m$ and $y = (y_1, y_2, \ldots, y_n)^\top \in \mathbb{R}^n$. For any given x_i, $i \in [m]$, if $x_i > 0$, then $\mathrm{sign}(x_i) = 1$ and hence $\mathrm{sign}(x_i)|x_i|^{p-1} = x_i^{p-1}$; if $x_i < 0$, then $\mathrm{sign}(x_i) = -1$ and hence $\mathrm{sign}(x_i)|x_i|^{p-1} = (-1)(-x_i)^{p-1} = x_i^{p-1}$; and if $x_i = 0$, then $\mathrm{sign}(x_i)|x_i|^{p-1} = x_i^{p-1}$. Consequently, $\mathrm{sign}(x_i)|x_i|^{p-1} = x_i^{p-1}$ for any $x_i \in \mathbb{R}$, $i \in [m]$, which implies that $\varphi_{p-1}^{(m)}(x) = x^{[p-1]}$. Similarly, it follows that $\varphi_{q-1}^{(n)}(y) = y^{[q-1]}$. Then (2.1), (2.2) and (2.3) are equivalent to

$$\begin{cases} \mathcal{A}x^{p-1}y^q = \lambda x^{[p-1]}, & (2.4)\\ \mathcal{A}x^p y^{q-1} = \lambda y^{[q-1]}, & (2.5)\\ x_1^p + \cdots + x_m^p = 1, & (2.6)\\ y_1^q + \cdots + y_n^q = 1, & (2.7)\end{cases}$$

and then we call λ an $l^{p,q}$-singular value of $\mathcal{A}$ and (x, y) a pair of $l^{p,q}$-singular vectors of $\mathcal{A}$ associated with λ. Here, $\sigma(\mathcal{A})$ is the set of all $l^{p,q}$-singular values of $\mathcal{A}$.

Now, we construct a set with parameter vectors α and β to locate all $l^{p,q}$-singular values of a real rectangular tensor.

Theorem 2.1. *Let $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$ with both p and q even, $\alpha = (\alpha_1, \ldots, \alpha_m)^\top \in \mathbb{R}^m$ and $\beta = (\beta_1, \ldots, \beta_n)^\top \in \mathbb{R}^n$. Then*

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}, \alpha, \beta) := \widetilde{\Gamma}(\mathcal{A}, \alpha) \cap \widehat{\Gamma}(\mathcal{A}, \beta), \tag{2.8}$$

where

$$\widetilde{\Gamma}(\mathcal{A}, \alpha) := \bigcup_{i\in[m]} \widetilde{\Gamma}_i(\mathcal{A}, \alpha_i), \quad \widehat{\Gamma}(\mathcal{A}, \beta) := \bigcup_{j\in[n]} \widehat{\Gamma}_j(\mathcal{A}, \beta_j),$$

$$\widetilde{\Gamma}_i(\mathcal{A},\alpha_i) := \left\{ z \in \mathbb{R} : |z-\alpha_i| \le \sum_{t\in[n]} |a_{i\cdots it\cdots t}-\alpha_i| + r_i(\mathcal{A}) \right\},$$
$$\widehat{\Gamma}_j(\mathcal{A},\beta_j) := \left\{ z \in \mathbb{R} : |z-\beta_j| \le \sum_{t\in[m]} |a_{t\cdots tj\cdots j}-\beta_j| + c_j(\mathcal{A}) \right\},$$
$$r_i(\mathcal{A}) := R_i(\mathcal{A}) - \sum_{t\in[n]} |a_{i\cdots it\cdots t}|, \quad c_j(\mathcal{A}) := C_j(\mathcal{A}) - \sum_{t\in[m]} |a_{t\cdots tj\cdots j}|, \quad i\in[m],\ j\in[n], \tag{2.9}$$

and $R_i(\mathcal{A})$ and $C_j(\mathcal{A})$ are defined in (1.4).

Proof. Let $\lambda \in \sigma(\mathcal{A})$, $x = (x_1, x_2, \ldots, x_m)^\top \in \mathbb{R}^m \setminus \{0\}$ and $y = (y_1, y_2, \ldots, y_n)^\top \in \mathbb{R}^n \setminus \{0\}$ be an $l^{p,q}$-singular vectors pair of $\mathcal{A}$ associated with λ. Let $|x_g| = \max_{i\in[m]}\{|x_i|\}$ and $|y_h| = \max_{j\in[n]}\{|y_j|\}$. Then $0 < |x_g| \le 1$ and $0 < |y_h| \le 1$. For any given real number α_g, by the g-th equation of (2.4), i.e.,

$$\lambda x_g^{p-1} = \sum_{i_2,\ldots,i_p\in[m],j_1,\ldots,j_q\in[n]} a_{gi_2\cdots i_pj_1\cdots j_q} x_{i_2}\cdots x_{i_p} y_{j_1}\cdots y_{j_q},$$

and (2.7), we have

$$\begin{aligned}
& (\lambda-\alpha_g)x_g^{p-1} \\
= {} & \lambda x_g^{p-1} - \alpha_g x_g^{p-1}(y_1^q + \cdots + y_n^q) && (2.10)\\
= {} & \sum_{i_2,\ldots,i_p\in[m],j_1,\ldots,j_q\in[n]} a_{gi_2\cdots i_pj_1\cdots j_q} x_{i_2}\cdots x_{i_p} y_{j_1}\cdots y_{j_q} - \alpha_g x_g^{p-1}(y_1^q + \cdots + y_n^q) \\
= {} & \sum_{\substack{i_2,\ldots,i_p\in[m],j_1,\ldots,j_q\in[n],\\ (i_2,\ldots,i_p,j_1,\ldots,j_q)\neq(g,\ldots,g,1,\ldots,1),\ldots,(g,\ldots,g,n,\ldots,n)}} a_{gi_2\cdots i_pj_1\cdots j_q} x_{i_2}\cdots x_{i_p} y_{j_1}\cdots y_{j_q} \\
& + (a_{g\cdots g1\cdots 1}-\alpha_g)x_g^{p-1}y_1^q + \cdots + (a_{g\cdots gn\cdots n}-\alpha_g)x_g^{p-1}y_n^q. && (2.11)
\end{aligned}$$

By (2.7) and q being even, we have $0 \le |y_j| \le 1$ for each $j \in [n]$. Taking modulus in (2.11) and using the triangle inequality, we have

$$\begin{aligned}
& |\lambda-\alpha_g||x_g|^{p-1} \\
\le {} & \sum_{\substack{i_2,\ldots,i_p\in[m],j_1,\ldots,j_q\in[n],\\ (i_2,\ldots,i_p,j_1,\ldots,j_q)\neq(g,\ldots,g,1,\ldots,1),\ldots,(g,\ldots,g,n,\ldots,n)}} |a_{gi_2\cdots i_pj_1\cdots j_q}||x_{i_2}|\cdots|x_{i_p}||y_{j_1}|\cdots|y_{j_q}| \\
& + |a_{g\cdots g1\cdots 1}-\alpha_g||x_g|^{p-1}|y_1|^q + \cdots + |a_{g\cdots gn\cdots n}-\alpha_g||x_g|^{p-1}|y_n|^q \\
\le {} & \sum_{\substack{i_2,\ldots,i_p\in[m],j_1,\ldots,j_q\in[n],\\ (i_2,\ldots,i_p,j_1,\ldots,j_q)\neq(g,\ldots,g,1,\ldots,1),\ldots,(g,\ldots,g,n,\ldots,n)}} |a_{gi_2\cdots i_pj_1\cdots j_q}||x_g|^{p-1} \\
& + |a_{g\cdots g1\cdots 1}-\alpha_g||x_g|^{p-1} + \cdots + |a_{g\cdots gn\cdots n}-\alpha_g||x_g|^{p-1} \\
= {} & \Big(R_g(\mathcal{A}) - \sum_{t\in[n]}|a_{g\cdots gt\cdots t}|\Big)|x_g|^{p-1} + \sum_{t\in[n]}|a_{g\cdots gt\cdots t}-\alpha_g||x_g|^{p-1},
\end{aligned}$$

which implies that

$$|\lambda-\alpha_g| \le \sum_{t\in[n]}|a_{g\cdots gt\cdots t}-\alpha_g| + R_g(\mathcal{A}) - \sum_{t\in[n]}|a_{g\cdots gt\cdots t}|,$$

and, consequently,

$$\lambda \in \widetilde{\Gamma}_g(\mathcal{A},\alpha_g) \subseteq \bigcup_{i\in[m]} \widetilde{\Gamma}_i(\mathcal{A},\alpha_i) = \widetilde{\Gamma}(\mathcal{A},\alpha). \tag{2.12}$$

For any given real number β_h, by the h-th equation of (2.5), i.e.,

$$\lambda y_h^{q-1} = \sum_{i_1,\ldots,i_p\in[m],j_2,\ldots,j_q\in[n]} a_{i_1\cdots i_p h j_2\cdots j_q} x_{i_1}\cdots x_{i_p} y_{j_2}\cdots y_{j_q},$$

and (2.6), we have

$$\begin{aligned}
& (\lambda-\beta_h)y_h^{q-1} \\
= \; & \lambda y_h^{q-1} - \beta_h(x_1^p+\cdots+x_m^p)y_h^{q-1} \qquad (2.13)\\
= \; & \sum_{i_1,\ldots,i_p\in[m],j_2,\ldots,j_q\in[n]} a_{i_1\cdots i_p h j_2\cdots j_q} x_{i_1}\cdots x_{i_p} y_{j_2}\cdots y_{j_q} - \beta_h(x_1^p+\cdots+x_m^p)y_h^{q-1} \\
= \; & \sum_{\substack{i_1,\ldots,i_p\in[m],j_2,\ldots,j_q\in[n],\\ (i_1,\ldots,i_p,j_2,\ldots,j_q)\neq(1,\ldots,1,h,\ldots,h),\cdots,(m,\ldots,m,h,\ldots,h)}} a_{i_1\ldots i_p h j_2\ldots j_q} x_{i_1}\cdots x_{i_p} y_{j_2}\cdots y_{j_q} \\
& +(a_{1\cdots 1h\cdots h}-\beta_h)x_1^p y_h^{q-1} + \cdots + (a_{m\cdots mh\cdots h}-\beta_h)x_m^p y_h^{q-1}. \qquad (2.14)
\end{aligned}$$

By (2.6) and p being even, we have $0 \le |x_i| \le 1$ for each $i \in [m]$. Taking modulus in (2.14) and using the triangle inequality, we have

$$\begin{aligned}
& |\lambda-\beta_h||y_h|^{q-1} \\
\le \; & \sum_{\substack{i_1,\ldots,i_p\in[m],j_2,\ldots,j_q\in[n],\\ (i_1,\ldots,i_p,j_2,\ldots,j_q)\neq(1,\ldots,1,h,\ldots,h),\ldots,(m,\ldots,m,h,\ldots,h)}} |a_{i_1\cdots i_p h j_2\cdots j_q}||x_{i_1}|\cdots|x_{i_p}||y_{j_2}|\cdots|y_{j_q}| \\
& + |a_{1\cdots 1h\cdots h}-\beta_h||x_1|^p|y_h|^{q-1} + \cdots + |a_{m\cdots mh\cdots h}-\beta_h||x_m|^p|y_h|^{q-1} \\
\le \; & \sum_{\substack{i_1,\ldots,i_p\in[m],j_2,\ldots,j_q\in[n],\\ (i_1,\ldots,i_p,j_2,\ldots,j_q)\neq(1,\ldots,1,h,\ldots,h),\ldots,(m,\ldots,m,h,\ldots,h)}} |a_{i_1\cdots i_p h j_2\cdots j_q}||y_h|^{q-1} \\
& + |a_{1\cdots 1h\cdots h}-\beta_h||y_h|^{q-1} + \cdots + |a_{m\cdots mh\cdots h}-\beta_h||y_h|^{q-1} \\
= & \Big(C_h(\mathcal{A}) - \sum_{t\in[m]} |a_{t\cdots th\cdots h}|\Big)|y_h|^{q-1} + \sum_{t\in[m]} |a_{t\cdots th\cdots h}-\beta_h||y_h|^{q-1},
\end{aligned}$$

which implies that

$$|\lambda-\beta_h| \le \sum_{t\in[m]} |a_{t\cdots th\cdots h}-\beta_h| + C_h(\mathcal{A}) - \sum_{t\in[m]} |a_{t\cdots th\cdots h}|,$$

and, consequently,

$$\lambda \in \widehat{\Gamma}_h(\mathcal{A},\beta_h) \subseteq \bigcup_{j\in[n]} \widehat{\Gamma}_j(\mathcal{A},\beta_j) = \widehat{\Gamma}(\mathcal{A},\beta). \tag{2.15}$$

Combining (2.12) and (2.15), we have $\lambda \in [\widetilde{\Gamma}(\mathcal{A},\alpha) \cap \widehat{\Gamma}(\mathcal{A},\beta)]$, i.e., $\lambda \in \Gamma(\mathcal{A},\alpha,\beta)$, which implies that the conclusion (2.8) follows. $\square$

Next, we consider a problem: How to choose appropriate parameter vectors α and β to optimize the $l^{p,q}$-singular value inclusion interval in Theorem 2.1? Before giving the optimal inclusion interval for $\Gamma(\mathcal{A}, \alpha, \beta)$ in Theorem 2.1, two lemmas are given by taking $a = 1$ in Lemmas 4.1 and 4.2 of [16].

Lemma 2.2. *Let*

$$f(x) = x - \sum_{i \in [n]} |x - b_i| - c$$

be a real valued function about x, where $b_i \in \mathbb{R}$, $b_1 \leq b_2 \leq \cdots \leq b_n$ and $c \in \mathbb{R}$.

(a) *If n is odd, then*

$$\max_{x \in \mathbb{R}} f(x) = \sum_{i=1}^{\frac{n+1}{2}} b_i - \sum_{i=\frac{n+3}{2}}^{n} b_i - c,$$

and this takes place for every $x \in [b_{\frac{n+1}{2}}, b_{\frac{n+3}{2}}]$ if $b_{\frac{n+1}{2}} \neq b_{\frac{n+3}{2}}$, and only for $x = b_{\frac{n+1}{2}}$ if $b_{\frac{n+1}{2}} = b_{\frac{n+3}{2}}$. Note that let $[b_{\frac{n+1}{2}}, b_{\frac{n+3}{2}}]$ be $[b_{\frac{n+1}{2}}, +\infty)$ if $b_{\frac{n+3}{2}}$ does not exist.

(b) *If n is even, then*

$$\max_{x \in \mathbb{R}} f(x) = \sum_{i=1}^{\frac{n}{2}} b_i - \sum_{i=\frac{n}{2}+2}^{n} b_i - c,$$

and this maximum is reached when $x = b_{\frac{n}{2}+1}$.

Lemma 2.3. *Let*

$$g(x) = x + \sum_{i \in [n]} |x - b_i| + c$$

be a real valued function about x, where $b_i \in \mathbb{R}$, $b_1 \leq b_2 \leq \cdots \leq b_n$ with $n \geq 2$, and $c \in \mathbb{R}$.

(a) *If n is odd, then*

$$\min_{x \in \mathbb{R}} g(x) = \sum_{i=\frac{n+1}{2}}^{n} b_i - \sum_{i=1}^{\frac{n-1}{2}} b_i + c,$$

and this takes place for every $x \in [b_{\frac{n-1}{2}}, b_{\frac{n+1}{2}}]$ if $b_{\frac{n-1}{2}} \neq b_{\frac{n+1}{2}}$, and only for $x = b_{\frac{n-1}{2}}$ if $b_{\frac{n-1}{2}} = b_{\frac{n+1}{2}}$.

(b) *If n is even, then*

$$\min_{x \in \mathbb{R}} g(x) = \sum_{i=\frac{n}{2}+1}^{n} b_i - \sum_{i=1}^{\frac{n}{2}-1} b_i + c,$$

and this minimum is reached when $x = b_{\frac{n}{2}}$.

Theorem 2.4. *Let $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$ with both p and q even. Then*

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) := \widetilde{\Gamma}(\mathcal{A}) \cap \widehat{\Gamma}(\mathcal{A}), \tag{2.16}$$

where

$$\widetilde{\Gamma}(\mathcal{A}) := \bigcup_{i\in[m]} \big(\widetilde{\Gamma}_i(\mathcal{A}) := [\tilde{l}_i, \tilde{u}_i]\big), \quad \widehat{\Gamma}(\mathcal{A}) := \bigcup_{j\in[n]} \big(\widehat{\Gamma}_j(\mathcal{A}) := [\hat{l}_j, \hat{u}_j]\big),$$

and $\tilde{l}_i$, $\tilde{u}_i$, $\hat{l}_j$ and $\hat{u}_j$ are taken by the following methods:

(a) *if n is odd, then*

$$\tilde{l}_i = \sum_{t=1}^{\frac{n+1}{2}} b_{i,t} - \sum_{t=\frac{n+3}{2}}^{n} b_{i,t} - r_i(\mathcal{A}), \quad \tilde{u}_i = \sum_{t=\frac{n+1}{2}}^{n} b_{i,t} - \sum_{t=1}^{\frac{n-1}{2}} b_{i,t} + r_i(\mathcal{A});$$

(b) *if n is even, then*

$$\tilde{l}_i = \sum_{t=1}^{\frac{n}{2}} b_{i,t} - \sum_{t=\frac{n}{2}+2}^{n} b_{i,t} - r_i(\mathcal{A}), \quad \tilde{u}_i = \sum_{t=\frac{n}{2}+1}^{n} b_{i,t} - \sum_{t=1}^{\frac{n}{2}-1} b_{i,t} + r_i(\mathcal{A});$$

(c) *if m is odd, then*

$$\hat{l}_j = \sum_{t=1}^{\frac{m+1}{2}} d_{t,j} - \sum_{t=\frac{m+3}{2}}^{m} d_{t,j} - c_j(\mathcal{A}), \quad \hat{u}_j = \sum_{t=\frac{m+1}{2}}^{m} d_{t,j} - \sum_{t=1}^{\frac{m-1}{2}} d_{t,j} + c_j(\mathcal{A});$$

(d) *if m is even, then*

$$\hat{l}_j = \sum_{t=1}^{\frac{m}{2}} d_{t,j} - \sum_{t=\frac{m}{2}+2}^{m} d_{t,j} - c_j(\mathcal{A}), \quad \hat{u}_j = \sum_{t=\frac{m}{2}+1}^{m} d_{t,j} - \sum_{t=1}^{\frac{m}{2}-1} d_{t,j} + c_j(\mathcal{A}).$$

Here, for each $i \in [m]$, $b_{i,1} \le b_{i,2} \le \cdots \le b_{i,n}$ is an arrangement in non-decreasing order of $a_{i\cdots it\cdots t}$ for $t \in [n]$; for each $j \in [n]$, $d_{1,j} \le d_{2,j} \le \cdots \le d_{m,j}$ is an arrangement in non-decreasing order of $a_{t\cdots tj\cdots j}$ for $t \in [m]$; and $r_i(\mathcal{A})$ and $c_j(\mathcal{A})$ are defined in (2.9).

Proof. Let $\lambda \in \sigma(\mathcal{A})$. By Theorem 2.1, we have $\lambda \in \Gamma(\mathcal{A}, \alpha, \beta)$, which implies that there exists an index $i \in [m]$ and an index $j \in [n]$ such that $\lambda \in \widetilde{\Gamma}_i(\mathcal{A}, \alpha_i)$ and $\lambda \in \widehat{\Gamma}_j(\mathcal{A}, \beta_j)$, that is,

$$|\lambda - \alpha_i| \le \sum_{t\in[n]} |a_{i\cdots it\cdots t} - \alpha_i| + r_i(\mathcal{A}), \quad \text{i.e.,} \quad \lambda \in [\tilde{f}(\alpha_i), \tilde{g}(\alpha_i)], \tag{2.17}$$

and

$$|\lambda - \beta_j| \le \sum_{t\in[m]} |a_{t\cdots tj\cdots j} - \beta_j| + c_j(\mathcal{A}), \quad \text{i.e.,} \quad \lambda \in [\hat{f}(\beta_j), \hat{g}(\beta_j)], \tag{2.18}$$

where

$$\begin{aligned}
\tilde{f}(\alpha_i) &= \alpha_i - \sum_{t\in[n]} |\alpha_i - a_{i\cdots it\cdots t}| - r_i(\mathcal{A}) = \alpha_i - \sum_{t\in[n]} |\alpha_i - b_{i,t}| - r_i(\mathcal{A}),\\
\tilde{g}(\alpha_i) &= \alpha_i + \sum_{t\in[n]} |\alpha_i - a_{i\cdots it\cdots t}| + r_i(\mathcal{A}) = \alpha_i + \sum_{t\in[n]} |\alpha_i - b_{i,t}| + r_i(\mathcal{A}),\\
\hat{f}(\beta_j) &= \beta_j - \sum_{t\in[m]} |\beta_j - a_{t\cdots tj\cdots j}| - c_j(\mathcal{A}) = \beta_j - \sum_{t\in[m]} |\beta_j - d_{t,j}| - c_j(\mathcal{A}),\\
\hat{g}(\beta_j) &= \beta_j + \sum_{t\in[m]} |\beta_j - a_{t\cdots tj\cdots j}| + c_j(\mathcal{A}) = \beta_j + \sum_{t\in[m]} |\beta_j - d_{t,j}| + c_j(\mathcal{A}).
\end{aligned}$$

Next, we consider a question: How to choose parameters α_i and β_j to minimize the inclusion intervals $[\tilde{f}(\alpha_i), \tilde{g}(\alpha_i)]$ in (2.17) and $[\hat{f}(\alpha_i), \hat{g}(\alpha_i)]$ in (2.18)?

(a) Assume that n is odd. By Lemma 2.2, we have

$$\max_{\alpha_i\in\mathbb{R}} \tilde{f}(\alpha_i) = \sum_{t=1}^{\frac{n+1}{2}} b_{i,t} - \sum_{t=\frac{n+3}{2}}^{n} b_{i,t} - r_i(\mathcal{A}), \tag{2.19}$$

and this maximum is reached for any $\alpha_i \in [b_{i,\frac{n+1}{2}}, b_{i,\frac{n+3}{2}}]$. By Lemma 2.3, we have

$$\min_{\alpha_i\in\mathbb{R}} \tilde{g}(\alpha_i) = \sum_{t=\frac{n+1}{2}}^{n} b_{i,t} - \sum_{t=1}^{\frac{n-1}{2}} b_{i,t} + r_i(\mathcal{A}), \tag{2.20}$$

and this minimum is reached for any $\alpha_i \in [b_{i,\frac{n-1}{2}}, b_{i,\frac{n+1}{2}}]$. Taking $\alpha_i = b_{i,\frac{n+1}{2}}$ in (2.17) and using (2.19) and (2.20), we have

$$\lambda \in \left[\sum_{t=1}^{\frac{n+1}{2}} b_{i,t} - \sum_{t=\frac{n+3}{2}}^{n} b_{i,t} - r_i(\mathcal{A}),\ \sum_{t=\frac{n+1}{2}}^{n} b_{i,t} - \sum_{t=1}^{\frac{n-1}{2}} b_{i,t} + r_i(\mathcal{A})\right],$$

i.e., $\lambda \in [\tilde{l}_i, \tilde{u}_i]$, which implies that $\lambda \in \widetilde{\Gamma}_i(\mathcal{A}) \subseteq \widetilde{\Gamma}(\mathcal{A})$ and, consequently, $\sigma(\mathcal{A}) \subseteq \widetilde{\Gamma}(\mathcal{A})$.

(b) Assume that n is even. By Lemma 2.2, we have

$$\max_{\alpha_i\in\mathbb{R}} \tilde{f}(\alpha_i) = \tilde{f}(b_{i,\frac{n}{2}+1}) = \sum_{t=1}^{\frac{n}{2}} b_{i,t} - \sum_{t=\frac{n}{2}+2}^{n} b_{i,t} - r_i(\mathcal{A}) \geq \tilde{f}(b_{i,\frac{n}{2}}). \tag{2.21}$$

By Lemma 2.3, we have

$$\min_{\alpha_i\in\mathbb{R}} \tilde{g}(\alpha_i) = \tilde{g}(b_{i,\frac{n}{2}}) = \sum_{t=\frac{n}{2}+1}^{n} b_{i,t} - \sum_{t=1}^{\frac{n}{2}-1} b_{i,t} + r_i(\mathcal{A}) \leq \tilde{g}(b_{i,\frac{n}{2}+1}). \tag{2.22}$$

Taking $\alpha_i = b_{i,\frac{n}{2}}$ and $\alpha_i = b_{i,\frac{n}{2}+1}$ in (2.17), respectively, we have

$$\lambda \in \left[\tilde{f}(b_{i,\frac{n}{2}}), \tilde{g}(b_{i,\frac{n}{2}})\right] \quad \text{and} \quad \lambda \in \left[\tilde{f}(b_{i,\frac{n}{2}+1}), \tilde{g}(b_{i,\frac{n}{2}+1})\right].$$

By (2.21), (2.22) and the existence of λ, we have $\lambda \in \left[\tilde{f}(b_{i,\frac{n}{2}+1}), \tilde{g}(b_{i,\frac{n}{2}})\right]$, i.e., $\lambda \in [\tilde{l}_i, \tilde{u}_i]$, which implies that $\lambda \in \widetilde{\Gamma}_i(\mathcal{A}) \subseteq \widetilde{\Gamma}(\mathcal{A})$ and, consequently, $\sigma(\mathcal{A}) \subseteq \widetilde{\Gamma}(\mathcal{A})$.

(c) Assume that m is odd. For the interval (2.18), by using the similar method as (a) to derive the maximum of $\hat{f}(\beta_j)$ and the minimum of $\hat{g}(\beta_j)$, we have

$$\lambda \in \left[\sum_{t=1}^{\frac{m+1}{2}} d_{t,j} - \sum_{t=\frac{m+3}{2}}^{m} d_{t,j} - c_j(\mathcal{A}),\ \sum_{t=\frac{m+1}{2}}^{m} d_{t,j} - \sum_{t=1}^{\frac{m-1}{2}} d_{t,j} + c_j(\mathcal{A})\right],$$

i.e., $\lambda \in [\hat{l}_j, \hat{u}_j]$, which implies that $\lambda \in \widehat{\Gamma}_j(\mathcal{A}) \subseteq \widehat{\Gamma}(\mathcal{A})$, and, consequently, $\sigma(\mathcal{A}) \subseteq \widehat{\Gamma}(\mathcal{A})$.

(d) Assume that m is even. Similar to the proof of (b), we have

$$\lambda \in \left[\sum_{t=1}^{\frac{m}{2}} d_{t,j} - \sum_{t=\frac{m}{2}+2}^{m} d_{t,j} - c_j(\mathcal{A}),\ \sum_{t=\frac{m}{2}+1}^{m} d_{t,j} - \sum_{t=1}^{\frac{m}{2}-1} d_{t,j} + c_j(\mathcal{A})\right],$$

i.e., $\lambda \in [\hat{l}_j, \hat{u}_j]$, which implies that $\lambda \in \widehat{\Gamma}_j(\mathcal{A}) \subseteq \widehat{\Gamma}(\mathcal{A})$, and, consequently, $\sigma(\mathcal{A}) \subseteq \widehat{\Gamma}(\mathcal{A})$.

In combination with (a), (b), (c) and (d), (2.16) follows. □

Based on the interval $\Gamma(\mathcal{A})$ in Theorem 2.4, a sufficient condition for the positive definiteness of a partially symmetric rectangular tensor is derived.

Theorem 2.5. *Let $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$ with both p and q even, and λ be an $l^{p,q}$-singular value of $\mathcal{A}$. If*

$$\min_{i\in[m]} \tilde{l}_i > 0 \quad or \quad \min_{j\in[n]} \hat{l}_j > 0, \tag{2.23}$$

where $\tilde{l}_i$ and $\hat{l}_j$ are defined in Theorem 2.4, then $\lambda > 0$. Furthermore, if $\mathcal{A}$ is also partially symmetric, then $\mathcal{A}$ is positive definite, consequently, $f(x)$ defined in (1.5) is positive definite.

Proof. Suppose on the contrary that $\lambda \leq 0$. By Theorem 2.4, we have $\lambda \in \Gamma(\mathcal{A})$, which implies that there is an $i_0 \in [m]$ and a $j_0 \in [n]$ such that $\lambda \in [\tilde{l}_{i_0}, \tilde{u}_{i_0}]$ and $\lambda \in [\hat{l}_{j_0}, \hat{u}_{j_0}]$, which conflicts with the assumption $\lambda \leq 0$ from (2.23). Hence, $\lambda > 0$. By Theorem 1.4, the conclusion follows. □

3 Bounds for the $l^{p,q}$-Spectral Radius of Nonnegative Rectangular Tensors

In this section, we present a lower bound and an upper bound for the $l^{p,q}$-spectral radius $\rho_{p,q}(\mathcal{A})$ of a nonnegative rectangular tensor $\mathcal{A}$, and prove that the upper bound is smaller than that in Lemma 1.3, that is, Corollary 3.3 in [11].

Lemma 3.1 ([11, Theorem 3.1])**.** *Let $\mathcal{A} \in \mathbb{R}_+^{[p;q;m;n]}$ be partially symmetric. If there exist $\omega > 0$, $x \in \mathbb{R}_+^m \setminus \{0\}$ and $y \in \mathbb{R}_+^n \setminus \{0\}$ such that*

$$\mathcal{A}x^{p-1}y^q \geq \omega x^{[k-1]}, \quad \mathcal{A}x^p y^{q-1} \geq \omega y^{[s-1]},$$

where $k, s \in \{2, \ldots, p+q, \ldots\}$, then

$$\rho_{k,s}(\mathcal{A}) \geq \omega \max\left\{\frac{\|y\|_s^{s-q}}{\|x\|_k^p}, \frac{\|x\|_k^{k-p}}{\|y\|_s^q}\right\}.$$

Lemma 3.2 ([12, Lemma 2.1]). *Consider the real function of the real variable*

$$\phi(x) = \sum_{i\in[n]} |x - b_i|,$$

for which $b_1 \le b_2 \cdots \le b_n$ *are real numbers.*

(i) *If* n *is odd, then*

$$\min_{x\in\mathbb{R}} \phi(x) = (b_n + \cdots + b_{\frac{n+3}{2}}) - (b_{\frac{n-1}{2}} + \cdots + b_1).$$

This minimum is reached when $x = b_{\frac{n+1}{2}}$.

(ii) *If* n *is even, then*

$$\min_{x\in\mathbb{R}} \phi(x) = (b_n + \cdots + b_{\frac{n}{2}+1}) - (b_{\frac{n}{2}} + \cdots + b_1).$$

This takes place for every $x \in [b_{\frac{n}{2}}, b_{\frac{n}{2}+1}]$ *if* $b_{\frac{n}{2}} \neq b_{\frac{n}{2}+1}$ *and only for* $x = b_{\frac{n}{2}}$ *if* $b_{\frac{n}{2}} = b_{\frac{n}{2}+1}$.

Theorem 3.3. *Let* $\mathcal{A} \in \mathbb{R}_+^{[p;q;m;n]}$ *be partially symmetric with both* p *and* q *even. Then*

$$\max\left\{\frac{1}{m}, \frac{1}{n}\right\} \min_{i\in[m],j\in[n]} \{R_i(\mathcal{A}), C_j(\mathcal{A})\} \le \rho_{p,q}(\mathcal{A}) \le \rho^*(\mathcal{A}), \tag{3.1}$$

where

$$\rho^*(\mathcal{A}) = \begin{cases} \min\{\eta_1, \eta_3\}, & \text{if } m \text{ and } n \text{ are odd,} \\ \min\{\eta_2, \eta_4\}, & \text{if } m \text{ and } n \text{ are even,} \\ \min\{\eta_1, \eta_4\}, & \text{if } m \text{ is even and } n \text{ is odd,} \\ \min\{\eta_2, \eta_3\}, & \text{if } m \text{ is odd and } n \text{ is even,} \end{cases}$$

$$\eta_1 = \max_{i\in[m]} \left\{ \sum_{t=\frac{n+3}{2}}^{n+1} \hat{b}_{i,t} - \sum_{t=1}^{\frac{n+1}{2}} \hat{b}_{i,t} + r_i(\mathcal{A}) \right\}, \qquad \eta_2 = \max_{i\in[m]} \left\{ \sum_{t=\frac{n}{2}+2}^{n+1} \hat{b}_{i,t} - \sum_{t=1}^{\frac{n}{2}} \hat{b}_{i,t} + r_i(\mathcal{A}) \right\},$$
$$\eta_3 = \max_{j\in[n]} \left\{ \sum_{t=\frac{m+3}{2}}^{m+1} \hat{d}_{t,j} - \sum_{t=1}^{\frac{m+1}{2}} \hat{d}_{t,j} + c_j(\mathcal{A}) \right\}, \quad \eta_4 = \max_{j\in[n]} \left\{ \sum_{t=\frac{m}{2}+2}^{m+1} \hat{d}_{t,j} - \sum_{t=1}^{\frac{m}{2}} \hat{d}_{t,j} + c_j(\mathcal{A}) \right\},$$

and $R_i(\mathcal{A})$ *and* $C_j(\mathcal{A})$ *are defined in (1.4),* $r_i(\mathcal{A})$ *and* $c_j(\mathcal{A})$ *are defined in (2.9). Furthermore,* $\rho^*(\mathcal{A}) \le \max_{i\in[m],j\in[n]} \{R_i(\mathcal{A}), C_j(\mathcal{A})\}$.

Here, for each $i \in [m]$, $\hat{b}_{i,1} \le \hat{b}_{i,2} \le \cdots \le \hat{b}_{i,n+1}$ *is an arrangement in non-decreasing order of 0 and* $a_{i\cdots it\cdots t}$ *for* $t \in [n]$, *and for each* $j \in [n]$, $\hat{d}_{1,j} \le \hat{d}_{2,j} \le \cdots \le \hat{d}_{m+1,j}$ *is an arrangement in non-decreasing order of 0 and* $a_{t\cdots tj\cdots j}$ *for* $t \in [m]$.

Proof. Let $\omega = \min_{i\in[m],j\in[n]} \{R_i(\mathcal{A}), C_j(\mathcal{A})\}$. If $\omega > 0$, taking $x = (1,\ldots,1)^\top \in \mathbb{R}_+^m$ and $y = (1,\ldots,1)^\top \in \mathbb{R}_+^n$, it follows that

$$\begin{cases} \mathcal{A}x^{p-1}y^q = (R_1(\mathcal{A}),\ldots,R_m(\mathcal{A}))^\top \ge \omega x^{[p-1]}, \\ \mathcal{A}x^p y^{q-1} = (C_1(\mathcal{A}),\ldots,C_n(\mathcal{A}))^\top \ge \omega y^{[q-1]}. \end{cases}$$

By Lemma 3.1, we have

$$\rho_{p,q}(\mathcal{A}) \ge \omega \max\left\{\frac{1}{\|x\|_p^p}, \frac{1}{\|y\|_q^q}\right\} = \omega \max\left\{\frac{1}{m}, \frac{1}{n}\right\}. \tag{3.2}$$

If $\omega = 0$, then (3.2) also holds. Hence, the left inequality in (3.1) follows.

Next, we prove the second inequality in (3.1). Let λ be the $l^{p,q}$-singular value with $|\lambda| = \rho_{p,q}(\mathcal{A})$. By Theorem 2.1, we have $\lambda \in \widetilde{\Gamma}(\mathcal{A},\alpha) \cap \widehat{\Gamma}(\mathcal{A},\beta)$, that is, there exists an $i \in [m]$ and a $j \in [n]$ such that $\lambda \in \widetilde{\Gamma}_i(\mathcal{A},\alpha_i)$ and $\lambda \in \widehat{\Gamma}_j(\mathcal{A},\beta_j)$, i.e.,

$$|\lambda - \alpha_i| \leq \sum_{t\in[n]} |a_{i\cdots it\cdots t} - \alpha_i| + r_i(\mathcal{A})$$

and

$$|\lambda - \beta_j| \leq \sum_{t\in[m]} |a_{t\cdots tj\cdots j} - \beta_j| + c_j(\mathcal{A}),$$

which implies that

$$\rho_{p,q}(\mathcal{A}) = |\lambda| \leq |\alpha_i - 0| + \sum_{t\in[n]} |\alpha_i - a_{i\cdots it\cdots t}| + r_i(\mathcal{A}) = \sum_{t\in[n+1]} |\alpha_i - \hat{b}_{i,t}| + r_i(\mathcal{A}) \tag{3.3}$$

and

$$\rho_{p,q}(\mathcal{A}) = |\lambda| \leq |\beta_j - 0| + \sum_{t\in[m]} |\beta_j - a_{t\cdots tj\cdots j}| + c_j(\mathcal{A}) = \sum_{t\in[m+1]} |\beta_j - \hat{d}_{t,j}| + c_j(\mathcal{A}). \tag{3.4}$$

If n is odd, then $n+1$ is even, and by (3.3) and Lemma 3.2, we have

$$\rho_{p,q}(\mathcal{A}) \leq \sum_{t=\frac{n+3}{2}}^{n+1} \hat{b}_{i,t} - \sum_{t=1}^{\frac{n+1}{2}} \hat{b}_{i,t} + r_i(\mathcal{A}) \leq \max_{i\in[m]} \left\{ \sum_{t=\frac{n+3}{2}}^{n+1} \hat{b}_{i,t} - \sum_{t=1}^{\frac{n+1}{2}} \hat{b}_{i,t} + r_i(\mathcal{A}) \right\} = \eta_1.$$

If n is even, then $n+1$ is odd, and by (3.3) and Lemma 3.2, we have

$$\rho_{p,q}(\mathcal{A}) \leq \sum_{t=\frac{n}{2}+2}^{n+1} \hat{b}_{i,t} - \sum_{t=1}^{\frac{n}{2}} \hat{b}_{i,t} + r_i(\mathcal{A}) \leq \max_{i\in[m]} \left\{ \sum_{t=\frac{n}{2}+2}^{n+1} \hat{b}_{i,t} - \sum_{t=1}^{\frac{n}{2}} \hat{b}_{i,t} + r_i(\mathcal{A}) \right\} = \eta_2.$$

If m is odd, then $m+1$ is even, and by (3.4) and Lemma 3.2, we have

$$\rho_{p,q}(\mathcal{A}) \leq \sum_{t=\frac{m+3}{2}}^{m+1} \hat{d}_{t,j} - \sum_{t=1}^{\frac{m+1}{2}} \hat{d}_{t,j} + c_j(\mathcal{A}) \leq \max_{j\in[n]} \left\{ \sum_{t=\frac{m+3}{2}}^{m+1} \hat{d}_{t,j} - \sum_{t=1}^{\frac{m+1}{2}} \hat{d}_{t,j} + c_j(\mathcal{A}) \right\} = \eta_3.$$

If m is even, then $m+1$ is odd, and by (3.4) and Lemma 3.2, we have

$$\rho_{p,q}(\mathcal{A}) \leq \sum_{t=\frac{m}{2}+2}^{m+1} \hat{d}_{t,j} - \sum_{t=1}^{\frac{m}{2}} \hat{d}_{t,j} + c_j(\mathcal{A}) \leq \max_{j\in[n]} \left\{ \sum_{t=\frac{m}{2}+2}^{m+1} \hat{d}_{t,j} - \sum_{t=1}^{\frac{m}{2}} \hat{d}_{t,j} + c_j(\mathcal{A}) \right\} = \eta_4.$$

Apparently, if m and n are odd, then $\rho_{p,q}(\mathcal{A}) \leq \min\{\eta_1, \eta_3\}$; if m and n are even, then $\rho_{p,q}(\mathcal{A}) \leq \min\{\eta_2, \eta_4\}$; if m is even and n is odd, then $\rho_{p,q}(\mathcal{A}) \leq \min\{\eta_1, \eta_4\}$; if m is odd and n is even, then $\rho_{p,q}(\mathcal{A}) \leq \min\{\eta_2, \eta_3\}$.

Finally, we prove that $\rho^*(\mathcal{A}) \leq \max_{i\in[m],j\in[n]}\{R_i(\mathcal{A}), C_j(\mathcal{A})\}$. By $\mathcal{A} \in \mathbb{R}_+^{[p;q;m;n]}$, we have $a_{i_1\cdots i_p j_1\cdots j_q} \geq 0$ for $i_1,\dots,i_p \in [m], j_1,\dots,j_q \in [n]$. Hence,

$$\eta_1 \leq \max_{i\in[m]}\left\{\sum_{t=\frac{n+3}{2}}^{n+1}\hat{b}_{i,t} + \sum_{t=1}^{\frac{n+1}{2}}\hat{b}_{i,t} + r_i(\mathcal{A})\right\} = \max_{i\in[m]} R_i(\mathcal{A}) \leq \max_{i\in[m],j\in[n]}\{R_i(\mathcal{A}), C_j(\mathcal{A})\},$$

$$\eta_2 \leq \max_{i\in[m]}\left\{\sum_{t=\frac{n}{2}+2}^{n+1}\hat{b}_{i,t} + \sum_{t=1}^{\frac{n}{2}}\hat{b}_{i,t} + r_i(\mathcal{A})\right\} \leq \max_{i\in[m]} R_i(\mathcal{A}) \leq \max_{i\in[m],j\in[n]}\{R_i(\mathcal{A}), C_j(\mathcal{A})\},$$

$$\eta_3 \leq \max_{j\in[n]}\left\{\sum_{t=\frac{m+3}{2}}^{m+1}\hat{d}_{t,j} + \sum_{t=1}^{\frac{m+1}{2}}\hat{d}_{t,j} + c_j(\mathcal{A})\right\} = \max_{j\in[n]} C_j(\mathcal{A}) \leq \max_{i\in[m],j\in[n]}\{R_i(\mathcal{A}), C_j(\mathcal{A})\},$$

$$\eta_4 \leq \max_{j\in[n]}\left\{\sum_{t=\frac{m}{2}+2}^{m+1}\hat{d}_{t,j} + \sum_{t=1}^{\frac{m}{2}}\hat{d}_{t,j} + c_j(\mathcal{A})\right\} \leq \max_{j\in[n]} C_j(\mathcal{A}) \leq \max_{i\in[m],j\in[n]}\{R_i(\mathcal{A}), C_j(\mathcal{A})\},$$

and, consequently, the conclusion $\rho^*(\mathcal{A}) \leq \max_{i\in[m],j\in[n]}\{R_i(\mathcal{A}), C_j(\mathcal{A})\}$ follows. □

4 Calculation of $l^{p,q}$-Singular Values via the Lifting Square Tensors

In this section, we considered a question: How to calculate all $l^{p,q}$-singular values of a given real rectangular tensor $\mathcal{A}$? We first derive the relationship between the $l^{2,2}$-singular values of $\mathcal{A}$ and the Z-eigenvalues of its lifting square tensor $\mathcal{C}_\mathcal{A}$, which provide a way to find all $l^{2,2}$-singular values of $\mathcal{A}$. Subsequently, we derive the relationship between the $l^{p,q}$-singular values of $\mathcal{A}$ and the generalized eigenvalues of $\mathcal{C}_\mathcal{A}$ and $\mathcal{C}_\mathcal{I}$, which provide a way to find all $l^{p,q}$-singular values of $\mathcal{A}$. The idea of converting the singular value problem to an eigenvalue problem comes from Chen, Qi, Yang and Yang's work in [2, pp. 3725], in which the concept of the lifting square tensor $\mathcal{C}_\mathcal{A}$ of a real rectangular tensor $\mathcal{A}$ is introduced.

For a rectangular tensor $\mathcal{A} = (a_{i_1\cdots i_p j_1\cdots j_q}) \in \mathbb{R}^{[p;q;m;n]}$, its lifting square tensor $\mathcal{C}_\mathcal{A} = (c_{t_1t_2\cdots t_{p+q}})$ is an order $p+q$ dimension $m+n$ tensor which is defined as follows:

$$c_{t_1t_2\cdots t_{p+q}}$$
$$= \begin{cases} a_{t_1,\cdots,t_p,t_{p+1}-m,\cdots,t_{p+q}-m}, & \text{if } 1 \leq t_1,\dots,t_p \leq m,\ m+1 \leq t_{p+1},\dots,t_{p+q} \leq m+n, \\ a_{t_{q+1},\cdots,t_{q+p},t_1-m,\cdots,t_q-m}, & \text{if } m+1 \leq t_1,\dots,t_q \leq m+n,\ 1 \leq t_{q+1},\dots,t_{q+p} \leq m, \\ 0, & \textit{otherwise}. \end{cases}$$

Let $x = (x_1,\dots,x_m)^\top \in \mathbb{R}^m$, $y = (y_1,\dots,y_n)^\top \in \mathbb{R}^n$ and $z = (x^\top, y^\top)^\top \in \mathbb{R}^{m+n}$. Then

$$\mathcal{C}_\mathcal{A} z^{p+q-1} = \begin{pmatrix} \mathcal{A}x^{p-1}y^q \\ \mathcal{A}x^p y^{q-1} \end{pmatrix}. \tag{4.1}$$

Now, let us recall the concept of an order m dimension n square tensor $\mathcal{B}$ and the definition of Z-eigenvalues of $\mathcal{B}$, which is introduced by Qi in [13]. We call $\mathcal{B}$ a real order m dimension n square tensor and denote by $\mathcal{B} = (b_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$, if $b_{i_1i_2\cdots i_m} \in \mathbb{R}$ for $i_1,\dots,i_m \in [n]$.

Definition 4.1 ([13])**.** Let $\mathcal{B} = (b_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$. If there are $\lambda \in \mathbb{R}$ and a nonzero vector $x \in \mathbb{R}^n \setminus \{0\}$ such that

$$\mathcal{B}x^{m-1} = \lambda x \quad \text{and} \quad x^\top x = 1,$$

where $\mathcal{B}x^{m-1} \in \mathbb{R}^n$, whose ith component is

$$(\mathcal{B}x^{m-1})_i = \sum_{i_2,\ldots,i_m \in [n]} b_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m},$$

then λ is called a Z-eigenvalue of $\mathcal{B}$ and x is called a Z-eigenvector of $\mathcal{B}$ associated with λ. For simplicity, we call (λ, x) a Z-eigenpair of $\mathcal{B}$.

$\boxed{4.1}$ Calculation of $l^{2,2}$-Singular Values via a Lifting Square Tensor

Taking $k = s = 2$ in Definition 1.1 and using $\operatorname{sign}(a)|a| = a$ for any $a \in \mathbb{R}$, then (1.1), (1.2) and (1.3) are equivalent to the following system

$$\begin{cases} \mathcal{A}x^{p-1}y^q = \lambda x, \\ \mathcal{A}x^p y^{q-1} = \lambda y, \\ \|x\|_2 = \|y\|_2 = 1, \end{cases}$$

and then we call λ an $l^{2,2}$-singular value of $\mathcal{A}$ and (x, y) a pair of $l^{2,2}$-singular vectors of $\mathcal{A}$ associated with λ.

Next, the relationship between the $l^{2,2}$-singular values/vectors of $\mathcal{A}$ and the Z-eigenvalues/ vectors of its lifting square tensor $\mathcal{C}_\mathcal{A}$ is given.

Theorem 4.2. *Let $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$ be partially symmetric.*

(a) *If λ is an $l^{2,2}$-singular value of $\mathcal{A}$ with corresponding singular vectors pair (x, y), then $\lambda/\sqrt{2}^{p+q-2}$ is the Z-eigenvalue of $\mathcal{C}_\mathcal{A}$ and $z = (x^\top/\sqrt{2}, y^\top/\sqrt{2})^\top$ is its Z-eigenvector.*

(b) *If λ $(\neq 0)$ is a Z-eigenvalue of $\mathcal{C}_\mathcal{A}$ with corresponding Z-eigenvector $z = (z_1, \ldots, z_m, z_{m+1}, \ldots, z_{m+n})^\top$, then $\sqrt{2}^{p+q-2}\lambda$ is the $l^{2,2}$-singular value of $\mathcal{A}$ with corresponding singular vectors pair $(\sqrt{2}z_x, \sqrt{2}z_y)$, where $z_x = (z_1, \ldots, z_m)^\top$ and $z_y = (z_{m+1}, \ldots, z_{m+n})^\top$.*

(c) *Assume that 0 is a Z-eigenvalue of $\mathcal{C}_\mathcal{A}$ with corresponding Z-eigenvector $z = (z_1, \ldots, z_m, z_{m+1}, \ldots, z_{m+n})^\top$. Let $z_x = (z_1, \ldots, z_m)^\top$ and $z_y = (z_{m+1}, \ldots, z_{m+n})^\top$. If $z_x \neq 0$ and $z_y \neq 0$, then 0 is an $l^{2,2}$-singular values of $\mathcal{A}$ with corresponding singular vector pair $(z_x/\|z_x\|_2, z_y/\|z_y\|_2)$. If $z_x = 0$ or $z_y = 0$, then 0 is not an $l^{2,2}$-singular value of $\mathcal{A}$.*

Proof. (a) Let λ be an $l^{2,2}$-singular value of $\mathcal{A}$ with corresponding singular vectors pair (x, y). Then $\mathcal{A}x^{p-1}y^q = \lambda x$, $\mathcal{A}x^p y^{q-1} = \lambda y$ and $\|x\|_2 = \|y\|_2 = 1$. Let $z = (x^\top/\sqrt{2}, y^\top/\sqrt{2})^\top$. Then $\|z\|_2 = 1$. By (4.1), we have

$$\begin{aligned} \mathcal{C}_\mathcal{A} z^{p+q-1} &= \begin{pmatrix} \frac{\mathcal{A}x^{p-1}y^q}{\sqrt{2}^{p+q-1}} \\ \frac{\mathcal{A}x^p y^{q-1}}{\sqrt{2}^{p+q-1}} \end{pmatrix} = \begin{pmatrix} \frac{\lambda x}{\sqrt{2}^{p+q-1}} \\ \frac{\lambda y}{\sqrt{2}^{p+q-1}} \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{\sqrt{2}^{p+q-2}} \frac{x}{\sqrt{2}} \\ \frac{\lambda}{\sqrt{2}^{p+q-2}} \frac{y}{\sqrt{2}} \end{pmatrix} \\ &= \frac{\lambda}{\sqrt{2}^{p+q-2}} \begin{pmatrix} \frac{x}{\sqrt{2}} \\ \frac{y}{\sqrt{2}} \end{pmatrix} = \frac{\lambda}{\sqrt{2}^{p+q-2}} z, \end{aligned}$$

which implies that $\lambda/\sqrt{2}^{p+q-2}$ is a Z-eigenvalue of $\mathcal{C}_{\mathcal{A}}$ and z is its Z-eigenvector.

(b) Let $\lambda\ (\neq 0)$ be a Z-eigenvalue of $\mathcal{C}_{\mathcal{A}}$ and $z = (z_1, \ldots, z_m, z_{m+1}, \ldots, z_{m+n})^\top \neq 0$ be its a Z-eigenvector. Let $z_x = (z_1, \ldots, z_m)^\top$ and $z_y = (z_{m+1}, \ldots, z_{m+n})^\top$. Then $z = (z_x^\top, z_y^\top)^\top$. By (4.1), we have

$$\lambda \begin{pmatrix} z_x \\ z_y \end{pmatrix} = \lambda z = \mathcal{C}_{\mathcal{A}} z^{p+q-1} = \begin{pmatrix} \mathcal{A} z_x^{p-1} z_y^q \\ \mathcal{A} z_x^p z_y^{q-1} \end{pmatrix}. \tag{4.2}$$

Now, we prove the fact: $z_x \neq 0$ and $z_y \neq 0$. Suppose that $z_y = 0$ (Similarly, we can also assume that $z_x = 0$. Here, we omit the proof for this case). By $z \neq 0$, we have $z_x \neq 0$, which implies that there is an $i \in [m]$ such that $z_i \neq 0$. By $\lambda \neq 0$, $z_y = 0$ and

$$\begin{aligned}
\lambda z_i &= (\mathcal{C}_{\mathcal{A}} z^{p+q-1})_i \\
&= \sum_{t_2,\ldots,t_{p+q} \in [m+n]} c_{i t_2 \cdots t_p t_{p+1} \cdots t_{p+q}} z_{t_2} \cdots z_{t_p} z_{t_{p+1}} \cdots z_{t_{p+q}} \\
&= \sum_{\substack{1 \le t_2,\ldots,t_p \le m, \\ m+1 \le t_{p+1},\ldots,t_{p+q} \le m+n}} a_{i,t_2,\cdots,t_p,t_{p+1}-m,\cdots,t_{p+q}-m} z_{t_2} \cdots z_{t_p} z_{t_{p+1}} \cdots z_{t_{p+q}} \\
&= 0,
\end{aligned}$$

we have $z_i = 0$, which conflicts with that $z_i \neq 0$. Hence, both z_x and z_y must be not zero.

Next, we prove that $\|z_x\|_2 = \|z_y\|_2 = 1/\sqrt{2}$. For any $g \in [m]$, by

$$\begin{aligned}
\lambda z_g &= (\mathcal{C}_{\mathcal{A}} z^{p+q-1})_g \\
&= \sum_{t_2,\ldots,t_{p+q} \in [m+n]} c_{g t_2 \cdots t_p t_{p+1} \cdots t_{p+q}} z_{t_2} \cdots z_{t_p} z_{t_{p+1}} \cdots z_{t_{p+q}} \\
&= \sum_{\substack{1 \le t_2,\ldots,t_p \le m, \\ m+1 \le t_{p+1},\ldots,t_{p+q} \le m+n}} a_{g,t_2,\cdots,t_p,t_{p+1}-m,\cdots,t_{p+q}-m} z_{t_2} \cdots z_{t_p} z_{t_{p+1}} \cdots z_{t_{p+q}},
\end{aligned}$$

we have

$$\lambda z_g^2 = \sum_{\substack{1 \le t_2,\ldots,t_p \le m, \\ m+1 \le t_{p+1},\ldots,t_{p+q} \le m+n}} a_{g,t_2,\cdots,t_p,t_{p+1}-m,\cdots,t_{p+q}-m} z_g z_{t_2} \cdots z_{t_p} z_{t_{p+1}} \cdots z_{t_{p+q}},$$

and, consequently,

$$\begin{aligned}
&\lambda(z_1^2 + \cdots + z_m^2) \\
&= \sum_{g \in [m]} \sum_{\substack{1 \le t_2,\ldots,t_p \le m, \\ m+1 \le t_{p+1},\ldots,t_{p+q} \le m+n}} a_{g,t_2,\cdots,t_p,t_{p+1}-m,\cdots,t_{p+q}-m} z_g z_{t_2} \cdots z_{t_p} z_{t_{p+1}} \cdots z_{t_{p+q}} \\
&= \sum_{\substack{1 \le t_1,t_2,\ldots,t_p \le m, \\ m+1 \le t_{p+1},\ldots,t_{p+q} \le m+n}} a_{t_1,t_2,\cdots,t_p,t_{p+1}-m,\cdots,t_{p+q}-m} z_{t_1} z_{t_2} \cdots z_{t_p} z_{t_{p+1}} \cdots z_{t_{p+q}} \\
&= \sum_{\substack{1 \le t_1,t_2,\ldots,t_p \le m, \\ 1 \le t_{p+1}-m,\ldots,t_{p+q}-m \le n}} a_{t_1,t_2,\cdots,t_p,t_{p+1}-m,\cdots,t_{p+q}-m} (z_x)_{t_1} (z_x)_{t_2} \\
&\qquad\qquad \cdots (z_x)_{t_p} (z_y)_{t_{p+1}-m} \cdots (z_y)_{t_{p+q}-m} \\
&= \sum_{\substack{1 \le i_1,\ldots,i_p \le m, \\ 1 \le j_1,\ldots,j_q \le n}} a_{i_1 \cdots i_p j_1 \cdots j_q} (z_x)_{i_1} \cdots (z_x)_{i_p} (z_y)_{j_1} \cdots (z_y)_{j_q}.
\end{aligned} \tag{4.3}$$

For any $m+1 \le h \le m+n$, by

$$\begin{aligned}\lambda z_h &= (\mathcal{C}_{\mathcal{A}} z^{p+q-1})_h \\ &= \sum_{t_2,\dots,t_{p+q}\in[m+n]} c_{ht_2\cdots t_q t_{q+1}\cdots t_{q+p}} z_{t_2}\cdots z_{t_q} z_{t_{q+1}}\cdots z_{t_{q+p}} \\ &= \sum_{\substack{m+1\le t_2,\dots,t_q\le m+n,\\ 1\le t_{q+1},\dots,t_{q+p}\le m}} a_{t_{q+1},\cdots,t_{q+p},h-m,t_2-m,\cdots,t_q-m} z_{t_2}\cdots z_{t_q} z_{t_{q+1}}\cdots z_{t_{q+p}},\end{aligned}$$

we have

$$\lambda z_h^2 = \sum_{\substack{m+1\le t_2,\dots,t_q\le m+n,\\ 1\le t_{q+1},\dots,t_{q+p}\le m}} a_{t_{q+1},\cdots,t_{q+p},h-m,t_2-m,\cdots,t_q-m} z_h z_{t_2}\cdots z_{t_q} z_{t_{q+1}}\cdots z_{t_{q+p}},$$

and, consequently,

$$\begin{aligned}&\lambda(z_{m+1}^2+\cdots+z_{m+n}^2) \\ &= \sum_{m+1\le h\le m+n} \sum_{\substack{m+1\le t_2,\dots,t_q\le m+n,\\ 1\le t_{q+1},\dots,t_{q+p}\le m}} a_{t_{q+1},\cdots,t_{q+p},h-m,t_2-m,\cdots,t_q-m} z_h z_{t_2}\cdots z_{t_q} z_{t_{q+1}}\cdots z_{t_{q+p}} \\ &= \sum_{\substack{m+1\le t_1,t_2,\dots,t_q\le m+n,\\ 1\le t_{q+1},\dots,t_{q+p}\le m}} a_{t_{q+1},\cdots,t_{q+p},t_1-m,t_2-m,\cdots,t_q-m} z_{t_1} z_{t_2}\cdots z_{t_q} z_{t_{q+1}}\cdots z_{t_{q+p}} \\ &= \sum_{\substack{1\le t_1-m,t_2-m,\dots,t_q-m\le n,\\ 1\le t_{q+1},\dots,t_{q+p}\le m}} a_{t_{q+1},\cdots,t_{q+p},t_1-m,t_2-m,\cdots,t_q-m} (z_y)_{t_1-m}(z_y)_{t_2-m} \\ &\qquad\qquad \cdots (z_y)_{t_q-m}(z_x)_{t_{q+1}}\cdots(z_x)_{t_{q+p}} \\ &= \sum_{\substack{1\le i_1,\dots,i_p\le m,\\ 1\le j_1,\dots,j_q\le n}} a_{i_1\cdots i_p j_1\cdots j_q}(z_x)_{i_1}\cdots(z_x)_{i_p}(z_y)_{j_1}\cdots(z_y)_{j_q}. \end{aligned} \tag{4.4}$$

From (4.3), (4.4) and $\lambda \neq 0$, we have $z_1^2+\cdots+z_m^2 = z_{m+1}^2+\cdots+z_{m+n}^2$. Furthermore, by $\|z\|_2 = 1$, we have $z_1^2+\cdots+z_m^2 = z_{m+1}^2+\cdots+z_{m+n}^2 = 1/2$, i.e., $\|z_x\|_2 = \|z_y\|_2 = 1/\sqrt{2}$.

Let $x = z_x/\|z_x\|_2$ and $y = z_y/\|z_y\|_2$. Then $x = \sqrt{2}z_x$, $y = \sqrt{2}z_y$ and $\|x\|_2 = \|y\|_2 = 1$. By (4.2), we have

$$\begin{aligned}\begin{pmatrix} \mathcal{A}x^{p-1}y^q \\ \mathcal{A}x^p y^{q-1}\end{pmatrix} &= \begin{pmatrix} \sqrt{2}^{p+q-1}\mathcal{A}z_x^{p-1}z_y^q \\ \sqrt{2}^{p+q-1}\mathcal{A}z_x^p z_y^{q-1}\end{pmatrix} = \sqrt{2}^{p+q-1}\begin{pmatrix} \mathcal{A}z_x^{p-1}z_y^q \\ \mathcal{A}z_x^p z_y^{q-1}\end{pmatrix} = \sqrt{2}^{p+q-1}\mathcal{C}_{\mathcal{A}}z^{p+q-1} \\ &= \sqrt{2}^{p+q-1}\lambda z = \sqrt{2}^{p+q-2}\lambda\begin{pmatrix}\sqrt{2}z_x \\ \sqrt{2}z_y\end{pmatrix} = \sqrt{2}^{p+q-2}\lambda\begin{pmatrix} x \\ y\end{pmatrix},\end{aligned}$$

which implies that $\sqrt{2}^{p+q-2}\lambda$ is an $l^{2,2}$-singular values of $\mathcal{A}$ with the singular vector pair (x, y).

(c) Let $\lambda = 0$ be a Z-eigenvalue of $\mathcal{C}_{\mathcal{A}}$ and $z = (z_1,\dots,z_m,z_{m+1},\dots,z_{m+n})^\top \neq 0$ be its a Z-eigenvector. Let $z_x = (z_1,\dots,z_m)^\top$ and $z_y = (z_{m+1},\dots,z_{m+n})^\top$. Then $z = (z_x^\top, z_y^\top)^\top$ and (4.2) also holds. Suppose that $z_x \neq 0$ and $z_y \neq 0$. Let $x = z_x/\|z_x\|_2$ and $y = z_y/\|z_y\|_2$. Then $\|x\|_2 = \|y\|_2 = 1$ and

$$\begin{pmatrix} \mathcal{A}x^{p-1}y^q \\ \mathcal{A}x^p y^{q-1}\end{pmatrix} = \begin{pmatrix} \frac{\mathcal{A}z_x^{p-1}z_y^q}{\|z_x\|_2^{p-1}\|z_y\|_2^q} \\ \frac{\mathcal{A}z_x^p y^{q-1}}{\|z_x\|_2^p\|z_y\|_2^{q-1}}\end{pmatrix} = \begin{pmatrix} \frac{\lambda z_x}{\|z_x\|_2^{p-1}\|z_y\|_2^q} \\ \frac{\lambda z_x}{\|z_x\|_2^p\|z_y\|_2^{q-1}}\end{pmatrix} = \begin{pmatrix} 0 \\ 0\end{pmatrix} = 0\begin{pmatrix} x \\ y\end{pmatrix}$$

show that 0 is an $l^{2,2}$-singular value of $\mathcal{A}$ with corresponding singular vectors pair (x, y).

Suppose that either z_x or z_y is a zero vector. If $z_x = 0$, then by $z \neq 0$, we have $z_y \neq 0$. From (4.2), it can be seen that if (x, y) is a singular vector pair of $\mathcal{A}$ associated with the singular value 0, then $\|x\|_2 = \|y\|_2 = 1$, $x = \alpha z_x$ and $y = \beta z_y$, where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. By $\|z_x\|_2 = 0$, we have $\|x\|_2 = 0$, which implies that 0 cannot be a singular value of $\mathcal{A}$. Similarly, one can prove that 0 cannot be a singular value of $\mathcal{A}$ if $z_y = 0$. Hence, the proof is completed. □

Based on Theorem 4.2, one can find all $l^{2,2}$-singular values of a rectangular tensor $\mathcal{A}$ by calculating all Z-eigenvalues of its lifting square tensor $\mathcal{C}_{\mathcal{A}}$.

4.2 Calculation of $l^{p,q}$-Singular Values via Two Lifting Square Tensors

Let $\mathcal{I} = (e_{i_1\cdots i_p j_1\cdots j_q}) \in \mathbb{R}^{[p;q;m;n]}$ be the identity rectangular tensor whose entries are defined as follows:

$$e_{i_1\cdots i_p j_1\cdots j_q} = \begin{cases} 1, & i_1 = \cdots = i_p,\ j_1 = \cdots = j_q, \\ 0, & otherwise. \end{cases}$$

It is easy to verify that if both p and q are even, then

$$\mathcal{I}x^{p-1}y^q = x^{[p-1]} \quad \text{and} \quad \mathcal{I}x^p y^{q-1} = y^{[q-1]} \tag{4.5}$$

for any $x \in \mathbb{R}^m$ with $\|x\|_p = 1$ and $y \in \mathbb{R}^n$ with $\|y\|_q = 1$.

Similarly, the lifting square tensor $\mathcal{C}_{\mathcal{I}} = (c_{t_1 t_2 \cdots t_{p+q}})$ of $\mathcal{I}$ is an order $p+q$ dimension $m+n$ real tensor which is defined as follows:

$$c_{t_1 t_2 \cdots t_{p+q}} = \begin{cases} 1, & \text{if } 1 \leq t_1 = \cdots = t_p \leq m,\ m+1 \leq t_{p+1} = \cdots = t_{p+q} \leq m+n, \\ 1, & \text{if } m+1 \leq t_1 = \cdots = t_q \leq m+n,\ 1 \leq t_{q+1} = \cdots = t_{q+p} \leq m, \\ 0, & otherwise. \end{cases}$$

Let $x = (x_1, \ldots, x_m)^\top \in \mathbb{R}^m$, $y = (y_1, \ldots, y_n)^\top \in \mathbb{R}^n$ and $z = (x^\top, y^\top)^\top \in \mathbb{R}^{m+n}$. Then

$$\mathcal{C}_{\mathcal{I}} z^{p+q-1} = \begin{pmatrix} \mathcal{I}x^{p-1}y^q \\ \mathcal{I}x^p y^{q-1} \end{pmatrix}. \tag{4.6}$$

The determinant $\det(\mathcal{A})$ of an order m dimension n tensor $\mathcal{A}$ is the resultant [3] of the system of homogeneous equations $\mathcal{A}x^{m-1} = 0$, which is the unique polynomial on the entries of $\mathcal{A}$ satisfying that $\det(\mathcal{A}) = 0$ if and only if $\mathcal{A}x^{m-1} = 0$ has a nonzero solution. In view of this, we call $\mathcal{A}$ a singular tensor if $\det(\mathcal{A}) = 0$ and a nonsingular tensor if $\det(\mathcal{A}) \neq 0$. From (4.6), it is easy to verify that $\det(\mathcal{C}_{\mathcal{I}}) = 0$ only when both x and y are zero vectors. Hence, $\det(\mathcal{C}_{\mathcal{I}}) \neq 0$.

Next, let us recall the generalized eigenvalue problem of tensor pairs which is introduced by Ding and Wei in [5]. Let $\mathbb{C}_{1,2}$ be the projective plane in which $(\alpha_1, \beta_1) \in \mathbb{C} \times \mathbb{C}$ and $(\alpha_2, \beta_2) \in \mathbb{C} \times \mathbb{C}$ are regarded as the same point, if there is a nonzero scalar $t \in \mathbb{C}$ such that $(\alpha_1, \beta_1) = (t\alpha_2, t\beta_2)$. Let $\mathcal{A}$ and $\mathcal{B}$ be two order m dimension n complex tensors. We call $\{\mathcal{A}, \mathcal{B}\}$ a regular tensor pair if $\det(\beta\mathcal{A} - \alpha\mathcal{B}) \neq 0$ for some $(\alpha, \beta) \in \mathbb{C}_{1,2}$, and call $\{\mathcal{A}, \mathcal{B}\}$ a singular tensor pair if $\det(\beta\mathcal{A} - \alpha\mathcal{B}) = 0$ for all $(\alpha, \beta) \in \mathbb{C}_{1,2}$.

Let $\{\mathcal{A}, \mathcal{B}\}$ be a regular tensor pair. If there are $(\alpha, \beta) \in \mathbb{C}_{1,2}$ and $x \in \mathbb{C}^n \setminus \{0\}$ such that

$$\beta\mathcal{A}x^{m-1} = \alpha\mathcal{B}x^{m-1},$$

then (α,β) is called an eigenvalue of $\{\mathcal{A},\mathcal{B}\}$ and x is called an eigenvector associated with (α,β). It is proved in [8, Theorem 3.1] that when $\mathcal{B}$ is nonsingular, i.e., $\det(\mathcal{B})\neq 0$, there is not a vector $x\in\mathbb{C}^n\setminus\{0\}$ such that $\mathcal{B}x^{m-1}=0$. This implies that $\beta\neq 0$ if (α,β) is an eigenvalue of $\{\mathcal{A},\mathcal{B}\}$. Hence, when $\det(\mathcal{B})\neq 0$, $\lambda=\alpha/\beta\in\mathbb{C}$ is called an eigenvalue of $\{\mathcal{A},\mathcal{B}\}$, and

$$\lambda(\mathcal{A},\mathcal{B})=\{\lambda\in\mathbb{C}:\det(\mathcal{A}-\lambda\mathcal{B})=0\}$$

is called the *spectrum*, i.e., the set of all eigenvalues, of $\{\mathcal{A},\mathcal{B}\}$. Furthermore, if $\lambda\in\mathbb{R}$ and $x\in\mathbb{R}^n\setminus\{0\}$, then λ is called an H-eigenvalue of $\{\mathcal{A},\mathcal{B}\}$ and x is called its corresponding H-eigenvector [5].

Theorem 4.3. *Let $\mathcal{A}\in\mathbb{R}^{[p;q;m;n]}$ be partially symmetric with both p and q even.*

(a) *If λ is an $l^{p,q}$-singular value of $\mathcal{A}$ with corresponding singular vectors pair (x,y), then λ is an H-eigenvalue of the regular tensor pair $\{\mathcal{C}_\mathcal{A},\mathcal{C}_\mathcal{I}\}$ and $z=(x^\top,y^\top)^\top$ is its corresponding H-eigenvector.*

(b) *Assume that λ is an H-eigenvalue of the regular tensor pair $\{\mathcal{C}_\mathcal{A},\mathcal{C}_\mathcal{I}\}$ with corresponding H-eigenvector $z=(z_1,\ldots,z_m,z_{m+1},\ldots,z_{m+n})^\top$. If $z_x:=(z_1,\ldots,z_m)^\top\neq 0$ and $z_y:=(z_{m+1},\ldots,z_{m+n})^\top\neq 0$, then λ is an $l^{p,q}$-singular value of $\mathcal{A}$ with corresponding singular vectors pair $(z_x/\|z_x\|_p, z_y/\|z_y\|_q)$. If $z_x=0$ or $z_y=0$, then λ is not an $l^{p,q}$-singular value of $\mathcal{A}$.*

Proof. (a) If λ is an $l^{p,q}$-singular value of $\mathcal{A}$ with corresponding singular vectors pair (x,y), then $x\neq 0$, $y\neq 0$, and hence $z=(x^\top,y^\top)^\top\neq 0$. By (4.1), (4.5) and (4.6), we have

$$\mathcal{C}_\mathcal{A}z^{p+q-1}=\begin{pmatrix}\mathcal{A}x^{p-1}y^q\\ \mathcal{A}x^py^{q-1}\end{pmatrix}=\begin{pmatrix}\lambda x^{[p-1]}\\ \lambda y^{[q-1]}\end{pmatrix}=\begin{pmatrix}\lambda\mathcal{I}x^{p-1}y^q\\ \lambda\mathcal{I}x^py^{q-1}\end{pmatrix}=\lambda\,\mathcal{C}_\mathcal{I}z^{p+q-1},$$

which implies that λ is an H-eigenvalue of the regular tensor pair $\{\mathcal{C}_\mathcal{A},\mathcal{C}_\mathcal{I}\}$ and $z=(x^\top,y^\top)^\top$ is an H-eigenvector associated with λ. Here, $\{\mathcal{C}_\mathcal{A},\mathcal{C}_\mathcal{I}\}$ is a regular tensor pair because $\det(\mathcal{C}_\mathcal{I})=\det(0\mathcal{C}_\mathcal{A}-(-1)\mathcal{C}_\mathcal{I})\neq 0$ for $(0,-1)\in\mathbb{C}_{1,2}$.

(b) Let λ be an H-eigenvalue of $\{\mathcal{C}_\mathcal{A},\mathcal{C}_\mathcal{I}\}$ and $z=(z_x^\top,z_y^\top)^\top$ be its corresponding H-eigenvector, where $z_x=(z_1,\ldots,z_m)^\top$ and $z_y=(z_{m+1},\ldots,z_{m+n})^\top$. By (4.1), (4.5) and (4.6), we have

$$\begin{pmatrix}\mathcal{A}z_x^{p-1}z_y^q\\ \mathcal{A}z_x^pz_y^{q-1}\end{pmatrix}=\mathcal{C}_\mathcal{A}z^{p+q-1}=\lambda\mathcal{C}_\mathcal{I}z^{p+q-1}=\lambda\begin{pmatrix}\mathcal{I}z_x^{p-1}z_y^q\\ \mathcal{I}z_x^pz_y^{q-1}\end{pmatrix}=\begin{pmatrix}\lambda z_x^{p-1}\\ \lambda z_y^{q-1}\end{pmatrix}. \tag{4.7}$$

Assume that $z_x\neq 0$ and $z_y\neq 0$. Let $x=z_x/\|z_x\|_p$ and $y=z_y/\|z_y\|_q$. Then $\|x\|_p=1$ and $\|y\|_q=1$. Furthermore, by (4.5) and (4.7), we have

$$\begin{aligned}\begin{pmatrix}\mathcal{A}x^{p-1}y^q\\ \mathcal{A}x^py^{q-1}\end{pmatrix}&=\begin{pmatrix}\frac{\mathcal{A}z_x^{p-1}z_y^q}{\|z_x\|_p^{p-1}\|z_y\|_q^q}\\ \frac{\mathcal{A}z_x^pz_y^{q-1}}{\|z_x\|_p^p\|z_y\|_q^{q-1}}\end{pmatrix}=\begin{pmatrix}\frac{\lambda\mathcal{I}z_x^{p-1}z_y^q}{\|z_x\|_p^{p-1}\|z_y\|_q^q}\\ \frac{\lambda\mathcal{I}z_x^pz_y^{q-1}}{\|z_x\|_p^p\|z_y\|_q^{q-1}}\end{pmatrix}\\ &=\begin{pmatrix}\lambda\mathcal{I}x^{p-1}y^q\\ \lambda\mathcal{I}x^py^{q-1}\end{pmatrix}=\begin{pmatrix}\lambda x^{[p-1]}\\ \lambda y^{[q-1]}\end{pmatrix},\end{aligned}$$

which implies that λ is an $l^{p,q}$-singular value of $\mathcal{A}$ with the singular vectors pair $(z_x/\|z_x\|_p, z_y/\|z_y\|_q)$.

Assume that either z_x or z_y is a zero vector. If $z_x = 0$, then by $z \neq 0$, we have $z_y \neq 0$. From (4.7), it can be seen that if (x, y) is an $l^{p,q}$-singular vectors pair of $\mathcal{A}$ associated with λ, then $\|x\|_p = \|y\|_q = 1$, $x = \eta_1 z_x$ and $y = \eta_2 z_y$, where $\eta_1 \in \mathbb{R}$ and $\eta_2 \in \mathbb{R}$. By $z_x = 0$, we have $\|x\|_p = 0$, which implies that λ cannot be an $l^{p,q}$-singular value of $\mathcal{A}$. Similarly, one can prove that λ cannot be a singular value of $\mathcal{A}$ if $z_y = 0$. Hence, the proof is completed. □

Based on Theorem 4.3, one can find all $l^{p,q}$-singular values of a rectangular tensor $\mathcal{A}$ by calculating all H-eigenvalues of its lifting square tensor pair $\{\mathcal{C}_\mathcal{A}, \mathcal{C}_\mathcal{I}\}$.

5 Numerical Examples

In this section, two numerical examples are given to verify the theoretical results.

Example 5.1. Let $\mathcal{A} = (a_{i_1 i_2 j_1 j_2}) \in \mathbb{R}^{[2;2;2;2]}$ be a partially symmetric rectangular tensor with entries defined as follows:

$$a_{1111} = a_{2222} = 10,\ a_{1112} = a_{1121} = -1,\ a_{1122} = a_{2211} = 9,\ a_{1211} = a_{2111} = -1,$$
$$a_{1212} = a_{1221} = a_{2112} = a_{2121} = -2,\ a_{1222} = a_{2122} = -1,\ a_{2212} = a_{2221} = -1.$$

Obviously, $p = q = m = n = 2$.

I. Localization for all $l^{2,2}$-singular values of $\mathcal{A}$.

We first consider the localization of all $l^{2,2}$-singular values of $\mathcal{A}$. By Theorem 2.4, we have

$$\tilde{l}_1 = \tilde{l}_2 = \hat{l}_1 = \hat{l}_2 = 1 \text{ and } \tilde{u}_1 = \tilde{u}_2 = \hat{u}_1 = \hat{u}_2 = 18, \tag{5.1}$$

and hence

$$\Gamma(\mathcal{A}) = [1, 18].$$

II. Secondly, the positive definiteness of $\mathcal{A}$ is considered.

By (5.1) and Theorem 2.5, one can judge that $\mathcal{A}$ is positive definite.

III. Finally, we find all $l^{2,2}$-singular values of $\mathcal{A}$.

By computation, all entries of the lifting square tensor $\mathcal{C}_\mathcal{A} = (c_{ijkl}) \in \mathbb{R}^{[4,4]}$ are as follows:

$$\begin{aligned}
&c_{1133} = c_{2244} = c_{3311} = c_{4422} = 10,\\
&c_{1134} = c_{1143} = c_{1233} = c_{1244} = c_{2133} = c_{2144} = c_{2234} = c_{2243}\\
&\qquad = c_{3312} = c_{3321} = c_{3411} = c_{3422} = c_{4311} = c_{4322} = c_{4412} = c_{4421} = -1,\\
&c_{1144} = c_{2233} = c_{3322} = c_{4411} = 9,\\
&c_{1234} = c_{1243} = c_{2134} = c_{2143} = c_{3412} = c_{3421} = c_{4312} = c_{4321} = -2,
\end{aligned}$$

and other $c_{ijkl} = 0$. Calculating all Z-eigenvalues of $\mathcal{C}_\mathcal{A}$ by using *zeig* from the MATLAB toolbox 'TenEig', we get 80 Z-eigenvalues counting multiplicity and their corresponding Z-eigenvectors. The 80 Z-eigenvalues are 0 (multiplicity 48), 2.7500 (multiplicity 4), 4.7500 (multiplicity 4), 4.8333 (multiplicity 8), 5.2000 (multiplicity 8) and 5.7500 (multiplicity 8). All different Z-eigenvalues and their parts of Z-eigenvectors of $\mathcal{C}_\mathcal{A}$ are listed in Table 1:

Table 1
Z-eigenvalues λ_z and their parts of Z-eigenvectors $z = (z_1, z_2, z_3, z_4)^\top$ of $\mathcal{C}_\mathcal{A}$.

λ_z	z_1	z_2	z_3	z_4
0	0.0040	1.0000	0	0
2.7500	0.5000	0.5000	0.5000	0.5000
4.7500	0.5000	−0.5000	0.5000	−0.5000
4.8333	0.6606	−0.2523	0.2523	−0.6606
5.2000	0.1445	−0.6922	0.1445	−0.6922
5.7500	0.5000	0.5000	0.5000	−0.5000

Table 1 shows that $z = (0.0040, 1.0000, 0, 0)^\top$ is a Z-eigenvector associated with the Z-eigenvalue $\lambda_z = 0$. Let $z_x = (z_1, z_2)^\top$ and $z_y = (z_3, z_4)^\top$. Then $z_x = (0.0040, 1.0000)^\top \neq 0$ and $z_y = (0, 0)^\top = 0$. In fact, all Z-eigenvectors $z = (z_1, z_2, z_3, z_4)^\top$ associated with the Z-eigenvalue 0 have the characteristic: either $z_x = 0$ or $z_y = 0$. By Theorem 4.2, it follows that 0 is not an $l^{2,2}$-singular value of $\mathcal{A}$.

Let λ be an $l^{2,2}$-singular value of $\mathcal{A}$ and (x, y) be its a singular vectors pair. By Theorem 4.2, $\lambda = 2\lambda_z$, $x = \sqrt{2} z_x$ and $y = \sqrt{2} z_y$. From this, we can get all $l^{2,2}$-singular values of $\mathcal{A}$ and their corresponding singular vectors pairs. All different $l^{2,2}$-singular values of $\mathcal{A}$ and parts of their singular vector pairs (corresponding to those data in Table 1) are listed in Table 2:

Table 2
All $l^{2,2}$-singular values λ and their parts of singular vectors pairs (x, y) of $\mathcal{A}$.

λ	x_1	x_2	y_1	y_2
5.5000	0.7071	0.7071	0.7071	0.7071
9.5000	0.7071	−0.7071	0.7071	−0.7071
9.6667	0.9342	−0.3568	0.3568	−0.9342
10.4000	0.2043	−0.9789	0.2043	−0.9789
11.5000	0.7071	0.7071	0.7071	−0.7071

Table 2 shows that all different $l^{2,2}$-singular values of $\mathcal{A}$ are 5.5000, 9.5000, 9.6667, 10.4000, 11.5000, which verifies Theorem 2.4, that is, $\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$.

Example 5.2. Consider the nonnegative rectangular tensor $\mathcal{A} = (a_{i_1 i_2 j_1 j_2}) \in \mathbb{R}_+^{[2;2;2;2]}$, where

$$a_{1111} = a_{2222} = 2,\ a_{1112} = a_{1121} = 1,\ a_{1122} = a_{2211} = 2,\ a_{1211} = a_{2111} = 1,$$
$$a_{1212} = a_{1221} = a_{2112} = a_{2121} = 1,\ a_{1222} = a_{2122} = 1,\ a_{2212} = a_{2221} = 1.$$

I. Calculation of all $l^{2,2}$-singular values of $\mathcal{A}$.

By computation, all entries of the lifting square tensor $\mathcal{C}_\mathcal{A} = (c_{ijkl}) \in \mathbb{R}^{[4,4]}$ are as follows:

$$\begin{aligned} & c_{1133} = c_{1144} = c_{2233} = c_{2244} = c_{3311} = c_{3322} = c_{4411} = c_{4422} = 2, \\ & c_{1134} = c_{1143} = c_{1233} = c_{1234} = c_{1243} = c_{1244} = c_{2133} = c_{2134} = c_{2143} = c_{2144} = c_{2234} = c_{2243} \\ & \quad = c_{3312} = c_{3321} = c_{3411} = c_{3412} = c_{3421} = c_{3422} = c_{4311} = c_{4312} = c_{4321} = c_{4322} = c_{4412} \\ & \quad = c_{4421} = 1. \end{aligned}$$

and other $c_{ijkl} = 0$. Calculating all Z-eigenvalues of $\mathcal{C}_\mathcal{A}$ by using *zeig* from the MATLAB toolbox 'TenEig', we get 48 Z-eigenvalues counting multiplicity, which are 0 (multiplicity 32), 0.5000 (multiplicity 12) and 2.5000 (multiplicity 4). All different Z-eigenvalues and their parts of Z-eigenvectors of $\mathcal{C}_\mathcal{A}$ are listed in Table 3:

Table 3

Z-eigenvalues λ_z and their parts of Z-eigenvectors $z = (z_1, z_2, z_3, z_4)^\top$ of $\mathcal{C}_{\mathcal{A}}$.

λ_z	z_1	z_2	z_3	z_4
0	0.7071	−0.7071	0	0
0.5000	0.5000	−0.5000	0.5000	−0.5000
2.5000	0.5000	0.5000	0.5000	0.5000

Table 3 shows that $z = (0.7071, -0.7071, 0, 0)^\top$ is a Z-eigenvector associated with the Z-eigenvalue $\lambda_z = 0$. Let $z_x = (z_1, z_2)^\top$ and $z_y = (z_3, z_4)^\top$. Then $z_x = (0.7071, -0.7071)^\top \neq 0$ and $z_y = (0, 0)^\top = 0$. In fact, all Z-eigenvectors $z = (z_1, z_2, z_3, z_4)^\top$ associated with the Z-eigenvalue 0 have the characteristic: either $z_x = 0$ or $z_y = 0$. By Theorem 4.2, it follows that 0 is not an $l^{2,2}$-singular value of $\mathcal{A}$.

Let λ be an $l^{2,2}$-singular value of $\mathcal{A}$ and (x, y) be its a singular vectors pair. By Theorem 4.2, $\lambda = 2\lambda_z$, $x = \sqrt{2}z_x$ and $y = \sqrt{2}z_y$. From this, we can get all $l^{2,2}$-singular values of $\mathcal{A}$ and their corresponding singular vectors pairs. All different $l^{2,2}$-singular values of $\mathcal{A}$ and parts of their singular vector pairs (corresponding to those data in Table 3) are listed in Table 4:

Table 4

All $l^{2,2}$-singular values λ and their parts of singular vectors pairs (x, y) of $\mathcal{A}$.

λ	x_1	x_2	y_1	y_2
1.0000	0.7071	−0.7071	0.7071	−0.7071
5.0000	0.7071	0.7071	0.7071	0.7071

Table 4 shows that by calculating all Z-eigenvalues of its lifting square tensor $\mathcal{C}_{\mathcal{A}} \in \mathbb{R}^{[4,4]}$, we find all different $l^{2,2}$-singular values of $\mathcal{A}$, they are 1.0000 and 5.0000.

II. Bounds for the $l^{2,2}$-spectral radius of $\mathcal{A}$.

By Lemma 1.3, i.e., Corollary 3.3 of [11], we have

$$\rho_{2,2}(\mathcal{A}) \leq 10.$$

By Theorem 3.3, we have

$$5 \leq \rho_{2,2}(\mathcal{A}) \leq 8,$$

which shows that the upper bound is smaller than that in Corollary 3.3 of [11] and that the lower bound can reach the exact value of $l^{2,2}$-spectral radius of $\mathcal{A}$ in some case.

6 Conclusion

In this paper, we first in Theorem 2.1 constructed an $l^{p,q}$-singular value inclusion interval $\Gamma(\mathcal{A}, \alpha, \beta)$ with two parameter vectors α and β for a real rectangular tensor $\mathcal{A}$. Subsequently, by selecting appropriate parameters α and β, we derived the optimal singular value inclusion interval $\Gamma(\mathcal{A})$ in Theorem 2.4, which provides a sufficient condition for the positive definiteness of a real partially symmetric rectangular tensor in Theorem 2.5. Based on the intervals in Theorem 2.1 and Theorem 3.1 of [11], we in Theorem 3.3 gave the lower and upper bounds for the $l^{p,q}$-spectral radius $\rho_{p,q}(\mathcal{A})$ of a nonnegative rectangular tensor $\mathcal{A}$. In order to find all $l^{2,2}$-singular values/vectors of $\mathcal{A}$, we in Theorem 4.2 derived the relationship between $l^{2,2}$-singular values/vectors of $\mathcal{A}$ and Z-eigenpairs of its lifting square tensor $\mathcal{C}_{\mathcal{A}}$ and used the relationship to find all $l^{2,2}$-singular values/vectors of $\mathcal{A}$, which is verified to be feasible by Example 5.1. Similarly, in order to find all $l^{p,q}$-singular values/vectors of $\mathcal{A}$, we converted the $l^{p,q}$-singular value problem of $\mathcal{A}$ to generalized eigenvalue problem of $\mathcal{C}_{\mathcal{A}}$ and $\mathcal{C}_{\mathcal{I}}$, and in Theorem 4.3 derived the relationship between $l^{p,q}$-singular values/vectors of $\mathcal{A}$ and H-eigenvalues/eigenvectors of its lifting square tensor pair $\{\mathcal{C}_{\mathcal{A}}, \mathcal{C}_{\mathcal{I}}\}$, which provides an alternative method to find all $l^{p,q}$-singular values/vectors of $\mathcal{A}$.

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