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SET-VALUED PERTURBATION STABILITY OF LOCAL METRIC REGULARITY

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Abstract: This paper mainly discusses the stability of the metric regularity under set-valued perturbations which are not of the usual addition type. Under a condition which is not comparable with which was given in a previous result, we obtain a stability result of the local metric regularity in a different point of view. Furthermore, our result is stronger than an existing one. When the origin mapping is single-valued, we give a new way to prove the result using a fixed point theorem.

Key words: *local metric regularity, set-valued perturbation, stability, fixed point theorem*

Mathematics Subject Classification: *49J53, 49K40*

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1 Introduction

Let X, Y be metric spaces, $(\bar{x}, \bar{y}) \in X \times Y$ be such that $\bar{y} \in F(\bar{x})$ where $F : X \rightrightarrows Y$ is a set-valued mapping. It is well-known that F is said to be locally metrically regular at $\bar{x}$ for $\bar{y}$ with constant $\kappa > 0$ if there exists a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ such that

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)), \text{ for all } (x, y) \in U \times V. \tag{1.1}$$

where $d(x, C) := \inf_{u \in C} d(x, u)$ denotes the distance from x to a set C.

The topic of the perturbation stability of the metric regularity was widely discussed in the past few decades. That is, if a mapping $F : X \rightrightarrows Y$ is metrically regular, what conditions make another set-valued mapping $G : X \rightrightarrows Y$, which is 'close to' F in a sense, to be also metrically regular? The following theorem is a classic result about this question, which means that the local metric regularity (with constant $\kappa > 0$) is preserved under a Lipschitz continuous (with constant $\mu > 0$) perturbation if $\kappa\mu < 1$.

Theorem 1.1. *Let X be a Banach space and Y be a normed space. Let $F : X \rightrightarrows Y$ be a mapping, $(\bar{x}, \bar{y})$ be a point such that $\bar{y} \in F(\bar{x})$, and $h : X \to Y$ be a single-valued mapping. Assume that κ, μ and r are positive numbers with $\kappa\mu < 1$ and satisfy*

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)), \text{ for all } (x, y) \in B(\bar{x}, r) \times B(\bar{y}, r),$$

and

$$\|h(x) - h(x')\| \leq \mu \|x - x'\|, \text{ for all } x, x' \in B(\bar{x}, r).$$

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Then, for the mapping $G := F + h$, there exists $a > 0$ such that

$$d(x, G^{-1}(y)) \leq \frac{\kappa}{1-\kappa\mu} d(y, G(x)), \text{ for all } (x,y) \in B(\bar{x}, a) \times B(\bar{y}, a).$$

Plenty of results on the stability of the local metric regularity under single-valued perturbation can be found in [4], [5], [6], [8], [10] and references therein. A natural question is whether the preservation still holds under set-valued perturbations.

For the metric regularity, there are two types of perturbations. The first one is, as adopted in Theorem 1.1, the usual addition type. That is, for the mapping F and a perturbation mapping H (single-valued or set-valued), G is taken as $G = F + H$. In this perturbation type, a counter-example in [6] (Example 5I.1) shows that the preservation of the local metric regularity no longer holds in general. However, when adding the condition that the diameter of the perturbation mapping at the point under consideration is small enough, a stability result on the local metric regularity was proved in [1] (Theorem 3.2).

The second type of perturbation is not an obvious addition type, but to introduce a quantity measuring the closeness of the set-valued mappings F and G. For a point $x \in X$ and a constant $\varepsilon > 0$, the quantity is defined as follows (see [7]).

$$\sigma_{F,G}(x,\varepsilon) := \sup_{\eta \in G(x)} \inf_{\xi \in F(x)} \sup_{\substack{d(x',x) \leq \varepsilon \\ \xi' \in F(x')}} \inf_{\eta' \in G(x')} \|\eta - \xi + \xi' - \eta'\| . \tag{1.2}$$

The following stability result on the local metric regularity was obtained in [9], in which the set-valued perturbation is of the second type.

Theorem 1.2 (Theorem 3.2 in [9])**.** *Let X be a complete metric space and Y be a normed space. Let F, G be set-valued mappings with closed graphs, $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and $(\bar{x}, \bar{z}) \in \operatorname{gph} G$ be given points. Suppose that F is metrically regular at $\bar{x}$ for $\bar{y}$ with constant $\kappa > 0$ and that the following two conditions are satisfied:*

(i) There exist positive constants r, μ with $\mu \in (0, \kappa^{-1})$ such that

$$\sigma_{F,G}(x,\varepsilon) \leq \mu\varepsilon, \text{ whenever } x \in B(\bar{x}, r) \text{ and } \varepsilon \leq r. \tag{1.3}$$

(ii)

$$\lim_{x \to \bar{x}} \sup_{v \in F(x)} \inf_{w \in G(x)} \|(w - \bar{z}) - (v - \bar{y})\| = 0. \tag{1.4}$$

Then G is metrically regular at $\bar{x}$ for $\bar{z}$ with constant $\frac{\kappa}{1-\kappa\mu}$.

When both F and G are single-valued mappings, (1.3) reduces to the local Lipschitz continuity of $G - F$ around $\bar{x}$ while (1.4), which can be removed, reduces to the continuity of $G - F$ at $\bar{x}$.

Recently, along with the condition (1.3), a weaker stability was preserved for the local metric regularity as follows.

Theorem 1.3. *(Theorem 2 in [2]) Let X, Y be Banach spaces, $F, G : X \rightrightarrows Y$ be two set-valued mappings with closed graphs and $\bar{y} \in F(\bar{x})$. Consider some positive parameters κ, r and s such that*

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)), \text{ for all } (x,y) \in B(\bar{x}, r) \times B(\bar{y}, s).$$

Let $\mu > 0, \delta > 0$ and $\nu > 0$ satisfy

$$\kappa\mu < 1, \quad \frac{\kappa}{1-\kappa\mu}\nu < r, \quad \delta + (1+\kappa\mu)\nu < s.$$

In addition, assume that there is $\bar{z} \in G(\bar{x})$ with

$$\inf_{v\in F(x)} \sup_{w\in G(x)} \|(w-\bar{z}) - (v-\bar{y})\| \le \delta, \text{ for all } x \in B(\bar{x}, r). \tag{1.5}$$

If

$$\sigma_{F,G}(x,\varepsilon) \le \mu\varepsilon, \text{ whenever } x \in B(\bar{x}, r) \text{ and } \varepsilon \le r,$$

then one has

$$d(\bar{x}, G^{-1}(z)) \le \frac{\kappa}{1-\kappa\mu} d(z, G(\bar{x})), \text{ for all } z \in B(\bar{z}, \nu).$$

Remark 1.4. Here we draw a comparison between condition (1.5) and (1.4). We claim that even when the condition (1.5) is strengthened into

$$\lim_{x\to\bar{x}} \inf_{v\in F(x)} \sup_{w\in G(x)} \|(w-\bar{z}) - (v-\bar{y})\| = 0, \tag{1.6}$$

it is not comparable with (1.4) when F and G are both set-valued mappings, since in general, we cannot clear the size relation between quantities

$$\inf_{v\in F(x)} \sup_{w\in G(x)} \|(w-\bar{z}) - (v-\bar{y})\| \text{ and } \sup_{v\in F(x)} \inf_{w\in G(x)} \|(w-\bar{z}) - (v-\bar{y})\|.$$

Indeed, for any fixed x, although it holds obviously that

$$p_1(v) := \inf_{w\in G(x)} \|(w-\bar{z}) - (v-\bar{y})\| \le \sup_{w\in G(x)} \|(w-\bar{z}) - (v-\bar{y})\| := p_2(v),$$

it is indeterminate that which of $q_1 := \sup_{v\in F(x)} p_1(v)$ and $q_2 := \inf_{v\in F(x)} p_2(v)$ is larger. For instance,

(i) If $F(x) = [0,2],\ p_1(v) = v-3,\ p_2(v) = v$, then for any $v \in F(x)$ one has $p_1(v) \le p_2(v)$, but

$$q_1 = \sup_{v\in[0,2]} \{v-3\} = -1 < 0 = q_2 = \inf_{v\in[0,2]} \{v\};$$

(ii) If $F(x) = [0,2],\ p_1(v) = v-1,\ p_2(v) = v$, then for any $v \in F(x)$ one also has $p_1(v) \le p_2(v)$, but

$$q_1 = \sup_{v\in[0,2]} \{v-1\} = 1 > 0 = q_2 = \inf_{v\in[0,2]} \{v\}.$$

However, when G is a single-valued mapping, it is clear that condition (1.6) is weaker than (1.4) since for any x, one has

$$\inf_{v\in F(x)} \|(G(x)-\bar{z}) - (v-\bar{y})\| \le \sup_{v\in F(x)} \|(G(x)-\bar{z}) - (v-\bar{y})\|.$$

In this paper, under the same conditions in Theorem 1.3, we prove a result on the stability of the local metric regularity under the second type of set-valued perturbations. On the one hand, our result is obviously stronger than Theorem 1.3. On the other hand, as shown in Remark 1.4, since condition (1.5) is generally not comparable with (1.4), we present a set-valued perturbation stability result of the local metric regularity in a different point of view. Furthermore, by using a fixed point theorem, we prove the special case (the origin mapping is single-valued) of our main result.

Throughout this paper, the symbol $B(x,r)$ denotes the closed ball with center x and radius r in all spaces under consideration. The graph and the inverse of a set-valued mapping F are denoted by $\operatorname{gph} F := \{(x,y) \mid y \in F(x)\}$ and $F^{-1}(y) := \{x \mid y \in F(x)\}$, respectively.

2 Main Result

In this section, we introduce our main results.

Theorem 2.1. *Let X be a Banach space, Y be a normed space, $F, G : X \rightrightarrows Y$ be two set-valued mappings with closed graphs and $(\bar{x},\bar{y}) \in \operatorname{gph} F$. Consider positive constants κ and r such that*

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)), \text{ for all } (x,y) \in B(\bar{x},r) \times B(\bar{y},r). \tag{2.1}$$

Let $\mu > 0, \delta > 0$ and $\theta > 0$ satisfy that

$$\kappa\mu < 1, \qquad \max\left\{\frac{\kappa}{1-\kappa\mu}\theta, \delta + (1+\kappa\mu)\theta\right\} < r. \tag{2.2}$$

Suppose that there exists $\bar{z} \in G(\bar{x})$ such that

$$\inf_{v\in F(x)} \sup_{w\in G(x)} \|(w-\bar{z}) - (v-\bar{y})\| \leq \delta, \text{ for all } x \in B(\bar{x},r). \tag{2.3}$$

In addition, assume that

$$\sigma_{F,G}(x,\varepsilon) \leq \mu\varepsilon, \text{ whenever } x \in B(\bar{x},r) \text{ and } \varepsilon \leq r. \tag{2.4}$$

Then, there exists $r' > 0$ such that

$$d(x, G^{-1}(z)) \leq \frac{\kappa}{1-\kappa\mu} d(z, G(x)), \text{ for all } (x,z) \in B(\bar{x},\rho) \times B(\bar{z},\rho), \tag{2.5}$$

where $\rho = \min\left\{r', \frac{\theta\kappa}{1+\kappa-\kappa\mu}\right\}$.

Proof. From (2.2) we can pick $\kappa' > \kappa, \mu' > \mu$ and $r' > 0$ such that

$$\kappa'\mu' < 1, \quad \frac{\kappa}{1-\kappa'\mu'}\theta + r' < r, \qquad \delta + (1+\kappa'\mu')\theta < r. \tag{2.6}$$

The following proof will include two steps.
Step 1 We prove that the inequality

$$d(x, G^{-1}(z)) \leq \frac{\kappa}{1-\kappa\mu} d(z, G(x)) \tag{2.7}$$

holds for any $(x,z) \in B(\bar{x},r') \times B(\bar{z},\theta)$ with $d(z, G(x)) < \theta$.

Fix any $(x,z) \in B(\bar{x}, r') \times B(\bar{z}, \theta)$ with $d(z, G(x)) < \theta$. If $z \in G(x)$, we are done since both sides of the inequality (2.7) are zero in this case.

Now we consider the case that $z \notin G(x)$. Set $\lambda := d(z, G(x))$ and $x_0 := x$, then $0 < \lambda < \theta$, which implies that we can take sufficiently small $\alpha > 0$ satisfying $\lambda + \alpha < \theta$. Thus we can find $w_0 \in G(x) = G(x_0)$ such that

$$\|z - w_0\| < \lambda + \alpha < \theta.$$

Put $r_0 = \kappa(\lambda + \alpha) < \kappa\theta < r$, then from condition (2.4) we have

$$\inf_{\xi' \in F(x_0)} \sup_{\substack{\|x'-x_0\| \le r_0 \\ \eta \in F(x')}} \inf_{\eta' \in G(x')} \|w_0 - \xi' + \eta - \eta'\| \le \sigma_{F,G}(x_0, r_0) \le \mu r_0 < \mu' r_0.$$

Thus, there exists $v_0 \in F(x_0)$ such that

$$\sup_{\substack{\|x'-x_0\| \le r_0 \\ \eta \in F(x')}} \inf_{\eta' \in G(x')} \|w_0 - v_0 + \eta - \eta'\| < \mu' r_0. \tag{2.8}$$

Set $y_0 = z - w_0 + v_0$, it holds for any $\tau \in F(x_0)$ and $\tau' \in G(x_0)$ that

$$\begin{aligned} \|y_0 - \bar{y}\| &= \|z - \bar{z} + \bar{z} - w_0 + v_0 - \bar{y}\| \le \|z - \bar{z}\| + \|w_0 - v_0 + \bar{y} - \bar{z}\| \\ &\le \|z - \bar{z}\| + \|w_0 - v_0 + \tau - \tau'\| + \|(\tau' - \bar{z}) - (\tau - \bar{y})\|, \end{aligned}$$

and hence from (2.3) and (2.8), we have $y_0 \in B(\bar{y}, r)$ since

$$\begin{aligned} \|y_0 - \bar{y}\| \le \|z - \bar{z}\| &+ \sup_{\eta \in F(x_0)} \inf_{\eta' \in G(x_0)} \|w_0 - v_0 + \eta - \eta'\| \\ &+ \inf_{\eta \in F(x_0)} \sup_{\eta' \in G(x_0)} \|(\eta' - \bar{z}) - (\eta - \bar{y})\| \\ &\le \theta + \mu' r_0 + \delta < \theta + \mu' \kappa \theta + \delta < (1 + \kappa' \mu')\theta + \delta < r. \end{aligned}$$

It follows from the metric regularity of F at $\bar{x}$ for $\bar{y}$ with κ (condition (2.1)) that

$$d(x_0, F^{-1}(y_0)) \le \kappa d(y_0, F(x_0)) \le \kappa \|y_0 - v_0\| = \kappa \|z - w_0\| < \kappa(\lambda + \alpha),$$

which yields that there exists $x_1 \in F^{-1}(y_0)$ such that $\|x_1 - x_0\| < \kappa(\lambda + \alpha) = r_0$.

Since $y_0 \in F(x_1)$ and $\|x_1 - x_0\| < r_0$, it holds from (2.8) that

$$\begin{aligned} d(w_0 - v_0 + y_0, G(x_1)) &= \inf_{\eta' \in G(x_1)} \|w_0 - v_0 + y_0 - \eta'\| \\ &\le \sup_{\eta \in F(x_1)} \inf_{\eta' \in G(x_1)} \|w_0 - v_0 + \eta - \eta'\| \\ &\le \sup_{\substack{\|x'-x_0\| \le r_0 \\ \eta \in F(x')}} \inf_{\eta' \in G(x')} \|w_0 - v_0 + \eta - \eta'\| < \mu' r_0, \end{aligned}$$

thus we can find $w_1 \in G(x_1)$ such that

$$\|w_0 - v_0 + y_0 - w_1\| < \mu' r_0. \tag{2.9}$$

Let $r_1 := \kappa' \mu' r_0$, then $r_1 < r_0 < r$. From (2.4) and since

$$\|x_1 - \bar{x}\| \le \|x_1 - x_0\| + \|x_0 - \bar{x}\| < \kappa(\lambda + \alpha) + r' < \kappa\theta + r' < r, \tag{2.10}$$

we have $\sigma_{F,G}(x_1, r_1) \le \mu r_1 < \mu' r_1$. Noting that $w_1 \in G(x_1)$, we obtain

$$\inf_{\xi\in F(x_1)} \sup_{\substack{\|x'-x_1\|\le r_1\\ \eta\in F(x')}} \inf_{\eta'\in G(x')} \|w_1 - \xi + \eta - \eta'\| \le \mu r_1 < \mu' r_1,$$

which yields the existence of $v_1 \in F(x_1)$ satisfying

$$\sup_{\substack{\|x'-x_1\|\le r_1\\ \eta\in F(x')}} \inf_{\eta'\in G(x')} \|w_1 - v_1 + \eta - \eta'\| \le \mu r_1 < \mu' r_1,$$

and hence

$$\sup_{\eta\in F(x_1)} \inf_{\eta'\in G(x_1)} \|w_1 - v_1 + \eta - \eta'\| < \mu' r_1. \tag{2.11}$$

Set $y_1 = z - w_1 + v_1$, then for any $\tau \in F(x_1)$ and $\tau' \in G(x_1)$, from (2.3), (2.6) and (2.11) we have

$$\begin{aligned}
\|y_1 - \bar{y}\| &= \|z - \bar{z} + \bar{z} - w_1 + v_1 - \bar{y}\| \\
&\le \|z - \bar{z}\| + \|w_1 - v_1 + \tau - \tau'\| + \|(\tau' - \bar{z}) - (\tau - \bar{y})\| \\
&\le \|z - \bar{z}\| + \sup_{\eta\in F(x_1)} \inf_{\eta'\in G(x_1)} \|w_1 - v_1 + \eta - \eta'\| \\
&\quad + \inf_{\eta\in F(x_1)} \sup_{\eta'\in G(x_1)} \|(\eta' - \bar{z}) - (\eta - \bar{y})\| \\
&\le \theta + \mu' r_1 + \delta = \theta + \mu'\kappa'\mu' r_0 + \delta \\
&< \theta + \mu'\kappa(\lambda + \alpha) + \delta < (1 + \kappa'\mu')\theta + \delta < r,
\end{aligned}$$

which along with (2.10) implies $(x_1, y_1) \in B(\bar{x}, r) \times B(\bar{y}, r)$. From (2.1) and (2.9) and recalling $z = y_0 + w_0 - v_0$, we have

$$\begin{aligned}
d(x_1, F^{-1}(y_1)) &\le \kappa d(y_1, F(x_1)) \le \kappa \|y_1 - v_1\| = \kappa \|z - w_1\| \\
&= \kappa \|w_0 - v_0 + y_0 - w_1\| < \kappa\mu' r_0 < (\kappa'\mu') r_0,
\end{aligned}$$

and therefore there exists $x_2 \in F^{-1}(y_1)$ such that $\|x_2 - x_1\| < (\kappa'\mu') r_0$.

Now suppose that $x_0 = x, x_1, x_2, ..., x_n$ are given for $n \ge 2$ and that

$$v_0 \in F(x_0), v_1 \in F(x_1), ..., v_{n-1} \in F(x_{n-1}),$$

and

$$w_0 \in G(x_0), w_1 \in G(x_1), ..., w_{n-1} \in G(x_{n-1})$$

are found satisfying

(i) $x_{k+1} \in F^{-1}(y_k)$ for $y_k = z - w_k + v_k$ and $k \le n$;

(ii) $\|x_k - x_{k+1}\| < r_k$ with $r_k = (\kappa'\mu')^k r_0$;

(iii) $\|w_{k-1} - v_{k-1} + y_{k-1} - w_k\| < \mu' r_{k-1}$ hold for $k = 1, 2, ..., n-1$;

(iv) $\sup_{\substack{\|x'-x_k\|\le r_k\\ \eta\in F(x')}} \inf_{\eta'\in G(x')} \|w_k - v_k + \eta - \eta'\| < \mu' r_k$.

Since $\kappa'\mu' < 1$, thus for each n, we have

$$\|x_n - \bar{x}\| \le \|x_n - x_0\| + \|x - \bar{x}\| \le r' + \sum_{k=0}^{n-1} (\kappa'\mu')^k r_0 < r' + \frac{\kappa}{1-\kappa'\mu'}\theta < r, \tag{2.12}$$

which implies $x_n \in B(\bar{x}, r)$ for each $n > 0$. From (i) we have $y_{n-1} \in F(x_n)$, and then

$$\begin{aligned} d(w_{n-1} - v_{n-1} + y_{n-1}, G(x_n)) &= \inf_{\eta' \in G(x_n)} \|w_{n-1} - v_{n-1} + y_{n-1} - \eta'\| \\ &\le \sup_{\eta \in F(x_n)} \inf_{\eta' \in G(x_n)} \|w_{n-1} - v_{n-1} + \eta - \eta'\| . \end{aligned} \tag{2.13}$$

It holds from (ii) that $\|x_{n-1} - x_n\| < r_{n-1}$, and hence

$$\begin{aligned} &\sup_{\eta \in F(x_n)} \inf_{\eta' \in G(x_n)} \|w_{n-1} - v_{n-1} + \eta - \eta'\| \\ &\quad \le \sup_{\substack{\|x' - x_{n-1}\| \le r_{n-1} \\ \eta \in F(x')}} \inf_{\eta' \in G(x')} \|w_{n-1} - v_{n-1} + \eta - \eta'\| . \end{aligned} \tag{2.14}$$

Combining (2.13) and (2.14) and considering (iv), we have

$$d(w_{n-1} - v_{n-1} + y_{n-1}, G(x_n)) < \mu' r_{n-1}.$$

Thus there exists $w_n \in G(x_n)$ such that

$$\|w_{n-1} - v_{n-1} + y_{n-1} - w_n\| < \mu' r_{n-1}. \tag{2.15}$$

Put $r_n = (\kappa'\mu')^n r_0 < r$, then from (2.4) we obtain

$$\inf_{\xi \in F(x_n)} \sup_{\substack{\|x' - x_n\| \le r_n \\ \eta \in F(x')}} \inf_{\eta' \in G(x')} \|w_n - \xi + \eta - \eta'\| \le \sigma_{F,G}(x_n, r_n) \le \mu r_n < \mu' r_n,$$

which yields the existence of $v_n \in F(x_n)$ satisfying

$$\sup_{\substack{\|x' - x_n\| \le r_n \\ \eta \in F(x')}} \inf_{\eta' \in G(x')} \|w_n - v_n + \eta - \eta'\| < \mu' r_n,$$

and hence

$$\sup_{\eta \in F(x_n)} \inf_{\eta' \in G(x_n)} \|w_n - v_n + \eta - \eta'\| < \mu' r_n. \tag{2.16}$$

Set $y_n = z - w_n + v_n$, then for any $\tau \in F(x_n)$ and $\tau' \in G(x_n)$, we have

$$\begin{aligned} \|y_n - \bar{y}\| &= \|z - \bar{z} + \bar{z} - w_n + v_n - \bar{y}\| \\ &\le \|z - \bar{z}\| + \|w_n - v_n + \tau - \tau'\| + \|(\tau' - \bar{z}) - (\tau - \bar{y})\| \\ &\le \|z - \bar{z}\| + \sup_{\eta \in F(x_n)} \inf_{\eta' \in G(x_n)} \|w_n - v_n + \eta - \eta'\| \\ &\quad + \inf_{\eta \in F(x_n)} \sup_{\eta' \in G(x_n)} \|(\eta' - \bar{z}) - (\eta - \bar{y})\| \\ &\le \theta + \mu' r_n + \delta = \theta + \mu'(\kappa'\mu')^n r_0 + \delta \\ &< \theta + \mu'\kappa(\lambda + \alpha) + \delta < (1 + \kappa'\mu')\theta + \delta < r. \end{aligned}$$

Recalling $y_{n-1} = z - w_{n-1} + v_{n-1}$ and using the metric regularity of F, we obtain

$$\begin{aligned} d(x_n, F^{-1}(y_n)) &\le \kappa d(y_n, F(x_n)) \le \kappa \|y_n - v_n\| = \kappa \|z - w_n\| \\ &= \kappa \|w_{n-1} - v_{n-1} + y_{n-1} - w_n\| < \kappa\mu' r_{n-1} < (\kappa'\mu') r_{n-1}, \end{aligned}$$

hence there exists $x_{n+1} \in F^{-1}(y_n)$ such that

$$\|x_{n+1} - x_n\| < (\kappa'\mu') r_{n-1} = (\kappa'\mu')^n r_0 < r.$$

By induction, we conclude that (i)∼(iv) hold for all $k > 0$.

Given any nonnegative integer n, p, using the triangle inequality, we have

$$\|x_n - x_{n+p}\| \le \sum_{j=0}^{p-1} \|x_{n+j} - x_{n+j+1}\| \le \sum_{j=0}^{p-1} (\kappa'\mu')^{n+j} r_0 < (\kappa'\mu')^n \frac{\kappa}{1-\kappa'\mu'}(\lambda + \alpha), \quad (2.17)$$

which implies that $\{x_n\} \subset B(\bar{x}, r)$ is a Cauchy sequence and hence converges to some $x^* \in B(\bar{x}, r)$. By fixing $n = 0$ and letting $p \to \infty$ in (2.17), we obtain

$$\|x - x^*\| = \|x_0 - x^*\| \le \frac{\kappa}{1-\kappa'\mu'}(\lambda + \alpha).$$

Due to $\|z - w_n\| = \|w_{n-1} - v_{n-1} + y_{n-1} - w_n\| < \mu' r_{n-1}$ for all $n > 0$ and $r_{n-1} = (\kappa'\mu')^{n-1} r_0 \to 0$ (since $\kappa'\mu' < 1$), it holds that $w_n \to z$. Taking into account that $(x_n, w_n) \in \operatorname{gph} G$ and the closedness of $\operatorname{gph} G$, we get $z \in G(x^*)$. Consequently, we obtain

$$d(x, G^{-1}(z)) \le \|x - x^*\| \le \frac{\kappa}{1-\kappa'\mu'}(\lambda + \alpha). \quad (2.18)$$

By letting $\kappa' \to \kappa, \mu' \to \mu$ and $\alpha \to 0^+$ in (2.18), we have

$$d(x, G^{-1}(z)) \le \frac{\kappa}{1-\kappa\mu}\lambda = \frac{\kappa}{1-\kappa\mu} d(z, G(x)),$$

which completes Step 1 since (x, z) is chosen arbitrarily in $B(\bar{x}, r') \times B(\bar{z}, \theta)$ with $d(z, G(x)) < \theta$.

Step 2 In this step we prove that (2.5) holds for $\rho = \min\left\{r', \frac{\theta\kappa}{1+\kappa-\kappa\mu}\right\}$ where r' has been found in Step 1.

Fix any $(x, z) \in B(\bar{x}, \rho) \times B(\bar{z}, \rho)$. If $d(z, G(x)) < \theta$, then (2.5) holds immediately from the result we have got in Step 1 since $\rho \le \min\{r', \theta\}$.

Now we consider the case that $d(z, G(x)) \ge \theta$. Since

$$d(z, G(\bar{x})) \le \|z - \bar{z}\| \le \rho < \theta,$$

it can also be derived from the result in Step 1 that

$$d(\bar{x}, G^{-1}(z)) \le \frac{\kappa}{1-\kappa\mu} d(z, G(\bar{x})),$$

thus we get

$$\begin{aligned} d(x, G^{-1}(z)) &\le \|x - \bar{x}\| + d(\bar{x}, G^{-1}(z)) \le \frac{\kappa}{1-\kappa\mu} d(z, G(\bar{x})) + \|x - \bar{x}\| \\ &\le \frac{\kappa}{1-\kappa\mu}\|z - \bar{z}\| + \|x - \bar{x}\| \le (\frac{\kappa}{1-\kappa\mu} + 1)\rho = \frac{\kappa + 1 - \kappa\mu}{1-\kappa\mu}\rho \\ &\le \frac{\kappa + 1 - \kappa\mu}{1-\kappa\mu} \cdot \frac{\kappa\theta}{1+\kappa-\kappa\mu} = \frac{\kappa}{1-\kappa\mu}\theta \le \frac{\kappa}{1-\kappa\mu} d(z, G(x)). \end{aligned}$$

□

Example 2.2. Let set-valued mappings $F, G : R \rightrightarrows R$ be defined as

$$F(x) := [5x - \frac{1}{10}, 5x + \frac{1}{10}],$$

and

$$G(x) := [5x + \frac{1}{3} sinx - \frac{1}{10}, 5x + \frac{1}{3} sinx + \frac{1}{10}].$$

Note that $0 \in F(0)$ and $0 \in G(0)$. From the Robinson-Ursescu's Stability Theorem (See Theorem 2.83 in [3]), we have

$$d(x, F^{-1}(y)) \leq d(y, F(x)), \text{ for all } (x, y) \in (-\frac{1}{5}, \frac{1}{5}) \times (-\frac{1}{5}, \frac{1}{5}).$$

Further, it can be checked that

$$\sigma_{F,G}(x, \varepsilon) \leq \frac{\varepsilon}{3}, \text{ whenever } x \in (-\frac{1}{5}, \frac{1}{5}) \text{ and } \varepsilon \leq \frac{1}{5}.$$

and

$$\inf_{v \in F(x)} \sup_{w \in G(x)} \|v - w\| \leq \frac{1}{6}, \text{ for all } x \in (-\frac{1}{5}, \frac{1}{5}).$$

Thus, by applying Theorem 2.1 with

$$\bar{x} = \bar{y} = \bar{z} = 0, \ r = \frac{1}{5}, \ \delta = \frac{1}{6}, \ \kappa = 1, \ \mu = \frac{1}{3}, \ \theta = \frac{1}{45},$$

we can conclude that there exists $r' > 0$ such that

$$d(x, G^{-1}(z)) \leq \frac{2}{3} d(z, G(x)), \text{ for all } (x, z) \in (-\rho, \rho) \times (-\rho, \rho),$$

where $\rho = \min\left\{r', \frac{1}{75}\right\}$.

Without a doubt, the proof presented before is sufficient for the result when the mapping F is a single-valued mapping and G has finite values. However, utilizing a fixed point theorem obtained in [11], we introduce a different way to prove the result when F is a continuous single-valued mapping.

Theorem 2.3 (Theorem 1 in [11])**.** *Let $\delta > 0, \theta \in (0, 1), \bar{x} \in X$ and $\Phi : B(\bar{x}, \delta) \to 2^X \backslash \{\emptyset\}$ satisfy the following properties:*

(i) For each $\eta \in (0, \delta)$, the intersection of $\operatorname{gph} \Phi$ *with $B(\bar{x}, \eta) \times B(\bar{x}, \eta)$ is closed;*

(ii)

$$d(\bar{x}, \Phi(\bar{x})) < (1 - \theta)\delta, \tag{2.19}$$

$$d(x, \Phi(x)) \leq \theta d(x, \Phi^{-1}(x)), \ \textit{for all } x \in B(\bar{x}, \delta). \tag{2.20}$$

Then for any $\beta > 0$, there exists $z \in B(\bar{x}, \delta)$ such that $z \in \Phi(z)$ and

$$d(z, \bar{x}) \leq \frac{1 + \beta}{1 - \theta} d(\bar{x}, \Phi(\bar{x})).$$

Theorem 2.4. *Let X be a Banach space, Y be a normed space, $f : X \to Y$ be a continuous single-valued mapping and $G : X \rightrightarrows Y$ be a set-valued mapping with a closed graph and a finite value. Consider positive constants κ and r such that*

$$d(x, f^{-1}(y)) \leq \kappa \|y - f(x)\|, \text{ for all } (x, y) \in B(\bar{x}, r) \times B(f(\bar{x}), r). \tag{2.21}$$

Let $\mu > 0, \delta > 0$ and $\theta > 0$ satisfy

$$\kappa\mu < 1, \qquad \max\left\{\frac{2\kappa}{1-\kappa\mu}\theta, \delta + \theta\right\} < r. \tag{2.22}$$

Suppose that there exists $\bar{z} \in G(\bar{x})$ such that

$$\sup_{w \in G(x)} \|(w - \bar{z}) - (f(x) - f(\bar{x}))\| \leq \delta, \text{ for all } x \in B(\bar{x}, r). \tag{2.23}$$

In addition, assume that for any $x \in B(\bar{x}, r)$ and $\varepsilon \leq r$, it holds

$$\sup_{\xi \in G(x)} \sup_{\|x' - x\| \leq \varepsilon} \inf_{\eta' \in G(x')} \|\xi - f(x) + f(x') - \eta'\| \leq \mu\varepsilon. \tag{2.24}$$

Then, there exists $r' > 0$ such that

$$d(x, G^{-1}(z)) \leq \frac{\kappa}{1-\kappa\mu} d(z, G(x)), \text{ for all } (x, z) \in B(\bar{x}, \rho) \times B(\bar{z}, \rho), \tag{2.25}$$

where $\rho = \min\left\{r', \frac{\theta\kappa}{1+\kappa-\kappa\mu}\right\}$.

Proof. From (2.22) we can pick $\kappa' > \kappa, \mu' > \mu$ and $r' > 0$ such that

$$\kappa'\mu' < 1, \qquad \frac{2\kappa}{1-\kappa'\mu'}\theta + r' < r. \tag{2.26}$$

Similar to the proof of Theorem 2.1, we only need to show that the inequality

$$d(x, G^{-1}(z)) \leq \frac{\kappa}{1-\kappa\mu} d(z, G(x)) \tag{2.27}$$

holds for any $(x, z) \in B(\bar{x}, r') \times B(\bar{z}, \theta)$ with $d(z, G(x)) < \theta$.

Fix $x \in B(\bar{x}, r')$ and $z \in B(\bar{z}, \theta)$ with $d(z, G(x)) < \theta$. If $z \in G(x)$, we are done since both sides of the inequality (2.27) are zero in this case.

Now we consider the case that $z \notin G(x)$. Set $\lambda := d(z, G(x))$, then $0 < \lambda < \theta$, which implies that we can take sufficiently small $\alpha > 0$ satisfying $\lambda + \alpha < \theta$. Thus we can find $w \in G(x)$ such that

$$\|z - w\| < \lambda + \alpha < \theta.$$

Set $y = z - w + f(x)$, then it holds from (2.22) and (2.23) that

$$\begin{aligned}\|y - f(\bar{x})\| &= \|z - \bar{z} + \bar{z} - w + f(x) - f(\bar{x})\| \\ &\leq \|z - \bar{z}\| + \|(w - \bar{z}) - (f(x) - f(\bar{x}))\| \\ &\leq \|z - \bar{z}\| + \sup_{w \in G(x)} \|(w - \bar{z}) - (f(x) - f(\bar{x}))\| \leq \theta + \delta < r,\end{aligned}$$

and hence we have $(x, y) \in B(\bar{x}, r) \times B(f(\bar{x}), r)$. Thus from (2.21) and (2.26) we obtain

$$d(x, f^{-1}(y)) = d(x, f^{-1}(z - w + f(x)))$$

$$\leq \kappa \|z - w\| < \kappa(\lambda + \alpha) < \kappa\theta < (1 - \kappa'\mu')(\frac{r - r'}{2}). \tag{2.28}$$

Define

$$\Phi : B(x, \frac{r - r'}{2}) \ni x' \mapsto \bigcup_{w' \in G(x')} f^{-1}(z - w' + f(x')). \tag{2.29}$$

The continuity of f yields the closedness of $\Phi(x')$ for each $x' \in B(x, \frac{r-r'}{2})$. From (2.28) and noting that $w \in G(x)$, we get

$$\begin{aligned} d(x, \Phi(x)) &= d(x, \bigcup_{w' \in G(x)} f^{-1}(z - w' + f(x))) \\ &\leq d(x, f^{-1}(z - w + f(x))) < (1 - \kappa'\mu')(\frac{r - r'}{2}). \end{aligned} \tag{2.30}$$

Pick $x' \in B(x, \frac{r-r'}{2})$. For an arbitrary $t > 0$, we can find $\tilde{x} \in \Phi^{-1}(x')$ such that

$$h := \|x' - \tilde{x}\| < d(x', \Phi^{-1}(x')) + t, \tag{2.31}$$

then from (2.29) we know

$$x' \in \Phi(\tilde{x}) = \bigcup_{w' \in G(\tilde{x})} f^{-1}(z - w' + f(\tilde{x})), \tag{2.32}$$

and $\tilde{x} \in B(x, \frac{r-r'}{2})$, and hence

$$h = \|x' - \tilde{x}\| \leq \|x' - x\| + \|x - \tilde{x}\| \leq \frac{r - r'}{2} + \frac{r - r'}{2} < r.$$

Furthermore, we have $\tilde{x} \in B(\bar{x}, r)$ since

$$\|\tilde{x} - \bar{x}\| \leq \|\tilde{x} - x\| + \|x - \bar{x}\| \leq \frac{r - r'}{2} + r' < r.$$

Then it holds from (2.24) that

$$\begin{aligned} &\sup_{\xi \in G(\tilde{x})} \inf_{\eta' \in G(x')} \|\xi - f(\tilde{x}) + f(x') - \eta'\| \\ &\quad \leq \sup_{\xi \in G(\tilde{x})} \sup_{\|x'' - \tilde{x}\| \leq h} \inf_{\eta' \in G(x'')} \|\xi - f(\tilde{x}) + f(x'') - \eta'\| \leq \mu h. \end{aligned} \tag{2.33}$$

Fix any $\hat{w} \in G(x')$, by (2.23) and noting that

$$\|x' - \bar{x}\| \leq \|x' - x\| + \|x - \bar{x}\| \leq \frac{r - r'}{2} + r' < r,$$

we get

$$\|z - \hat{w} + f(x') - f(\bar{x})\| \leq \|z - \bar{z}\| + \|\bar{z} - \hat{w} + f(x') - f(\bar{x})\| \leq \theta + \delta < r.$$

We also have from (2.32) that there exists $\tilde{w} \in G(\tilde{x})$ such that

$$x' \in f^{-1}(z - \tilde{w} + f(\tilde{x})),$$

and hence

$$z = \tilde{w} - f(\tilde{x}) + f(x'). \tag{2.34}$$

Then, by (2.21) and (2.34) we obtain

$$\begin{aligned} d(x', \Phi(x')) &= d(x', \bigcup_{w' \in G(x')} f^{-1}(z - w' + f(x'))) \le d(x', f^{-1}(z - \hat{w} + f(x'))) \\ &\le \kappa \|z - \hat{w}\| = \kappa \|\tilde{w} - f(\tilde{x}) + f(x') - \hat{w}\| . \end{aligned}$$

Since $\hat{w}$ can be chosen arbitrarily in $G(x')$, it holds from (2.31) and (2.33) that

$$\begin{aligned} d(x', \Phi(x')) &\le \kappa \inf_{\eta' \in G(x')} \|\tilde{w} - f(\tilde{x}) + f(x') - \eta'\| \\ &\le \kappa \sup_{\xi \in G(\tilde{x})} \inf_{\eta' \in G(x')} \|\xi - f(\tilde{x}) + f(x') - \eta'\| \\ &\le \kappa\mu h < \kappa'\mu'(d(x', \Phi^{-1}(x')) + t). \end{aligned} \tag{2.35}$$

Then we have

$$d(x', \Phi(x')) \le \kappa'\mu' d(x', \Phi^{-1}(x'))$$

by letting $t \to 0^+$ in (2.35).

By Theorem 2.3 we know that for any $\beta > 0$, there exists $x^* \in B(x, \frac{r-r'}{2})$ such that $x^* \in \Phi(x^*)$ and

$$\|x - x^*\| \le \frac{1+\beta}{1-\kappa'\mu'} d(x, \Phi(x)). \tag{2.36}$$

By the definition of Φ, we can find $w^* \in G(x^*)$ such that

$$x^* \in f^{-1}(z - w^* + f(x^*)),$$

which yields $z = w^* \in G(x^*)$ and hence $x^* \in G^{-1}(z)$.
Consequently, from (2.28) and (2.36) we obtain

$$\begin{aligned} d(x, G^{-1}(z)) &\le \|x - x^*\| \le \frac{1+\beta}{1-\kappa'\mu'} d(x, \Phi(x)) \\ &= \frac{1+\beta}{1-\kappa'\mu'} d(x, \bigcup_{w' \in G(x)} f^{-1}(z - w' + f(x))) \\ &\le \frac{1+\beta}{1-\kappa'\mu'} d(x, f^{-1}(z - w + f(x))) \\ &\le \frac{(1+\beta)\kappa}{1-\kappa'\mu'} (d(z, G(x)) + \alpha). \end{aligned} \tag{2.37}$$

By letting $\kappa' \to \kappa, \mu' \to \mu, \alpha \to 0^+$, and $\beta \to 0^+$ in (2.37), we obtain (2.27) and complete the proof. □

Example 2.5. Let single-valued mapping $f : R \to R$ and set-valued mapping $G : R \rightrightarrows R$ be defined as

$$f(x) := x^2 - 2x \quad \text{and} \quad G(x) := \left\{ x^2 - 2x + \frac{1}{2}|x|, x^2 - 2x + \frac{1}{8} \right\}.$$

We can prove

$$d(x, f^{-1}(y)) \le \frac{3}{5}|y - x^2 + 2x|, \text{ for all } (x, y) \in (-\frac{1}{6}, \frac{1}{6}) \times (-\frac{1}{6}, \frac{1}{6}).$$

It can also be checked that

$$\sup_{\xi\in G(x)} \sup_{\|x'-x\|\le\varepsilon} \inf_{\eta'\in G(x')} \|\xi - f(x) + f(x') - \eta'\| \le \frac{\varepsilon}{4}, \text{ for all } x \in (-\frac{1}{6}, \frac{1}{6}) \text{ and } \varepsilon \le \frac{1}{6},$$

and

$$\sup_{w\in G(x)} \|w - f(x)\| \le \frac{1}{7}, \text{ for all } x \in (-\frac{1}{6}, \frac{1}{6}).$$

Then, by applying Theorem 2.4 with

$$\bar{x} = \bar{z} = 0,\ r = \frac{1}{6},\ \kappa = \frac{3}{5},\ \mu = \frac{1}{2},\ \delta = \frac{1}{7},\ \theta = \frac{1}{48},$$

we have that there exists $r' > 0$ such that

$$d(x, G^{-1}(z)) \le \frac{6}{7} d(z, G(x)), \text{ for all } (x, y) \in (-\rho, \rho) \times (-\rho, \rho),$$

where $\rho = \min\left\{r', \frac{1}{104}\right\}$.

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