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THE LAGRANGE PROBLEM FOR DIFFERENTIAL INCLUSIONS WITH BOUNDARY VALUE CONDITIONS AND DUALITY

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Abstract: The present article studies the duality of the Lagrange problem of optimal control theory with the boundary value constraints given by second-order polyhedral differential inclusions. Our primary aim is to establish results of duality for a boundary value problem with second-order differential inclusions. As a supplementary problem, we consider differential problems and formulate sufficient conditions of optimality, including particular transversality conditions incorporating the Euler-Lagrange type inclusions. After constructing the dual problem for second-order polyhedral differential inclusions, we prove that the adjoint Euler-Lagrange inclusion is simultaneously a dual relationship, which is satisfied by the pair of solutions of the primal and dual problems. Furthermore, solving numerical examples illustrates the application of these results.

Key words: *duality, Boundary value, polyhedral, Euler-Lagrange, transversality*

Mathematics Subject Classification: *90C46, 34A60,49N15*

1 Introduction

In terms of set-valued mappings, several extremal problems, significant from a practical point of view, such as classical problems of optimal control, differential games, economic dynamics models, are described. In the study of various dynamic processes represented by equations with discontinuous or set-valued mappings on the right-hand side, differential inclusions play a significant role as a tool, especially in the study of economic dynamics, variational analysis, and optimization theory [2, 3, 10, 14, 18, 20, 27]. Differential inclusions can also be studied for all problems usually studied in the theory of ordinary differential equations (existence and continuation of solutions, dependency on initial and boundary conditions and parameters, etc.).

In several cases, differential inclusions occur, including differential variational inequalities, projected dynamic systems, Moreau's sweeping process, dynamic linear and nonlinear complementarity systems, discontinuous ordinary differential equations, dynamic switching systems, and dynamic systems of Von Neumann-Gale, polyhedral problems [8, 9, 19, 23, 29]. Since there are typically several solutions to a differential inclusion starting at a given point, new types of problems arise, such as the investigation of the solution set's topological properties, the selection of solutions with properties given, and many others. They are also very convenient in proving the existence of theorems in the theory of control. Among the many articles devoted to differential inclusions, the work [13] in which results of various types of existence were obtained. In the theory of differential inclusions, the questions concerning the existence of the solutions included in a given set are significant; the field of research related to them bears the name of viability theory [26].

The importance of the consideration of differential inclusions follows naturally from the mathematical control theory. The development in the last decade of optimal control theory that culminated with the Pontryagin maximum principle has brought about a renewed interest in the calculus of variations, and particularly in such classical phases thereof as the theory of the first variation and first-order necessary conditions. It turns out that the classical Lagrange problem in the calculus of variations could be looked upon as a particular case of the optimal control problem. The necessary condition of optimality, the Pontryagin maximum principle, obtained for the latter problem contained all the classical first-order necessary conditions [7, 12].

This paper is written based on the analogy relationship between specific optimal control problem and the duality theory. Since the control theory constructed a completely new theory, duality theory can also develop by using concepts of variational analysis. There are as many alternative approaches to duality theory as individuals are working in the field of duality theory. These different approaches may be approximately classified, at the risk of gross simplification, into three groups. The first group of approaches is based on the conjugacy correspondence developed by Fenchel and extended by Rockafellar [30]. In its modern form, this theory states that given $f(x)$, a closed, proper, and convex function, its conjugate dual is also a closed proper convex function. Economically, this implies a one-to-one correspondence between the production function and the normalized profit function under the assumption of closure, properness, and convexity [31]. The second group of approaches is based on the symmetric duality between gauge functions, or distance functions, or polar cones of convex sets. The third group of approaches is based on the duality between the set of production possibilities and its support function.

Duality in the calculus of variations has existed in different forms in literature for a long time [6, 32, 33]. As is known, the principle of duality is one of the central directions in convex optimality problems due to the importance of its implementations, and it is interpreted differently for different particular cases. The paper [1] deals with optimality conditions for a convex bilevel programming problem via the Fenchel-Lagrange duality. The paper [5] presents a simple dual condition for the convex subdifferential sum formula. To obtain a generalized Clarke-Ekeland dual least action principle, the subdifferential sum formula is then used to derive necessary and sufficient optimality conditions for a general cone-constrained convex optimization problem under a much weaker dual constraint qualification.

In the paper [11], converse duality theorems for scalar and multiobjective second-order dual problems in nonlinear programming are established. The authors formulate a dual problem of approximate Mond-Weir type and develop the duality results in the paper [16]. The authors also study vector optimization problems with perturbed cone constraint by incorporating the concept of cone convex functions concerning mapping problems of vector optimization. Moreover, under that kind of cone convexity assumption, the necessary and sufficient optimality conditions and duality results for quasi-solutions to a mapping of vector optimization problems with perturbed cone constraint are formulated.

Mahmudov [21] gives a sufficient condition for optimality for the non-convex problem and proves duality theorems based on the apparatus of locally conjugate mappings. The paper [22] is devoted to investigating the optimization problem of partial Goursat-Darboux type differential inclusions and sufficient conditions are formulated for so-called Goursat-Darboux type convex and non-convex partial differential inclusions. Then the author constructs the dual problem to convex problem for differential inclusion of considered hyperbolic type. In the paper [25], applying the infimal convolution concept of convex functions, step by step, the author constructs the dual problems for third-order discrete, discrete-approximate, and differential inclusions and prove duality results. As a result, in the next investigations, the passage to the limit in the dual problem with discrete approximations plays a significant role, without which it is hardly ever possible to establish any duality to the continuous problem. In this way, relying on the described method for computation of the conjugate and support functions of discrete-approximate problems, a Pascal triangle with binomial coefficients can be successfully used for any higher-order calculations.

In this paper, we study the duality theory of optimal control problems given by second-order polyhedral differential inclusions. We formulate sufficient conditions of optimality for polyhedral differential inclusions based on the apparatus of Euler-Lagrange inclusions and transversality conditions. By the optimal control theory method and a careful analysis, we obtain some optimality results about the duality for boundary value problems and the duality theorems allow one to conclude that a sufficient condition for an extremum is an extremal relation for the primal and dual problems. Although the presented work is generally devoted to the dual optimization problem for polyhedral inclusions with special boundary conditions, in a sense it is also a logical continuation of the primal problem considered in [23].

There have been some applications of optimal control theory to boundary value problems for ordinary differential inclusions or\and equations [4]. The paper [15] deals with the existence-uniqueness problem in a class of Neumann boundary value problems for second-order ordinary differential equations probably across several points of resonance. Moreover, some global optimality results about the existence and uniqueness of solutions for boundary value problems are obtained. The necessary conditions of the Pontryagin Maximum Principle are discussed in the paper [31], by interpreting Boundary value problems for differential inclusions leading to a new indirect method for the computation of optimal trajectories with its focus on global convergence conditions for compact control domains.

The present paper is devoted to one of the difficult and interesting areas, i.e., the construction of duality for boundary value problems with second-order polyhedral differential inclusions. The problems posed and their dualities are novel. We organized this paper as follows.

Section 2 deals rather comprehensively with the set-valued mappings that are used on the right-hand side of differential inclusions. We propose a concise introduction to convex analysis, defining conjugate functions of convex functions, locally adjoint mappings, the cone of tangent directions, and applying their properties to the polyhedral set-valued mappings, and recalling the calculation of subdifferentials of convex functions. Then we introduce the boundary value problems for second-order polyhedral differential inclusions that are mainly concerned with dual problems.

Section 3 is concerned with the derivation of sufficient optimality conditions for second-order differential inclusions with boundary value conditions. Construction of the Euler-Lagrange inclusion and transversality condition is based on passing to the limit in the optimality conditions of discrete approximation problems associated with the differentiable problem. We formulate Theorem 3.1 on a sufficient optimality condition and do not use the passage to the limit in the corresponding discrete approximation problem for the following two reasons. First, the formation of optimality conditions of Theorem 3.1 requires more calculations, so we decided to omit them. Second, since we analyze optimality and duality in parallel in this paper, the sufficient conditions of optimality are enough for our needs. Consequently, a separate topic of consideration, in our view, is the substantiation of the laws of passage to the limit in the problem of discrete approximation and the establishment of the necessary conditions of optimality.

Section 4 is devoted to duality in the control problem with second-order polyhedral differential inclusions. We establish the dual problems for our main continuous problem. But the construction of the duality problem with the support of discrete approximation problems requires a lot of effort and attention to understand the computational aspects. We have omitted this so as not to deviate from the main question. A duality relationship between a pair of optimization problems with boundary value constraint is formulated. Therefore we prove that the Euler-Lagrange type adjoint inclusion simultaneously is a dual relation and the optimal values in the primal convex and the dual concave problems are equal.

In Section 5, the optimization of a second-order linear boundary value problem is considered to show the implementation of the used approach. Moreover, a numerical example is developed to solve a polyhedral boundary value problem with second-order differential inclusions. Then accordingly to Theorem 4.1, we show that under the conditions of Theorem 3.1, the optimal values of primal and dual problem coincide.

2 Needed facts and Problem Statements

We recall some necessary and recent results of set-valued mappings and refer the reader to the book [17, 28] for an introduction to the theory of differential inclusions and duality. Convex functions and convex set-valued mappings are studied in this paper in the setting of n-dimensional Euclidean space $\mathbb{R}^n$. Let $\langle x,y\rangle$ be an inner product of $x,y\in\mathbb{R}^n,(x,y)$ be a pair of x,y elements. Throughout this work we assume that $G:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ is a set-valued mapping from $\mathbb{R}^n$ into the set of subsets of $\mathbb{R}^n$. Then $G:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ is said to be convex if its graph $gphG=\{(x,y):y\in G(x)\}$ is a convex subset of $\mathbb{R}^{2n}=\mathbb{R}^n\times\mathbb{R}^n$. A set-valued mapping G is convex-closed if its $gphG$ is a convex-closed set in $\mathbb{R}^{2n}$. It is convex-valued if for each $x\in domG$ the set $G(x)$ is a convex set, where $domG=\{x:G(x)\neq\emptyset\}$.

The convex cone $K_A(z_0)$ is called the cone of tangent directions at the point $z_0=(x^0,y^0)\in A$ if from $\overline{z}=(\overline{x},\overline{y})\in K_A(z_0)$ it follows that $\overline{z}$ is a tangent vector to the set $A\subset\mathbb{R}^{2n}$ at the point z_0. In other words, there exists such function $\mu:\mathbb{R}\to\mathbb{R}^{2n}$ such that $z_0+\lambda\overline{z}+\mu(\lambda)\in A$ for sufficiently small $\lambda>0$ and $\lambda^{-1}\mu(\lambda)\to 0$ as $\lambda\downarrow 0$. Obviously, for a convex mapping G, the cone of tangent directions at the point $(x^0,y^0)\in gphG$ setting $\mu(\lambda)\equiv 0$, is defined as follows

$$\begin{aligned}K_{gphG}(x_0,y_0) &= cone[gphG-(x_0,y_0)]\\ &= \Big\{(\overline{x},\overline{y}):\overline{x}=\lambda(x-x^0),\ \ \overline{y}=\lambda(y-y^0)\ ,\lambda>0\Big\},\ \forall\ (x,y)\in gphG.\end{aligned}$$

A polyhedral convex set in $\mathbb{R}^n$ is a set that can be expressed as the intersection of some finite family of closed half-spaces, that is, as the set of solutions to some finite system of inequalities of the form $\langle x,x_k^*\rangle\le\beta_k,\ k=1,\dots,l$. In particular, if the finite system of inequalities is homogeneous, the set of solutions to this finite system of inequalities is called the polyhedral cone. One of the remarkable properties of a polyhedral set is that it can be interpreted as a sum of polyhedron and polyhedral cone. Conversely, the sum of any polyhedron and polyhedral cone is a polyhedral set.

A function $f(x)$ is called a proper function if it does not assume the value $-\infty$ and is not identically equal to $+\infty$. Clearly, f is proper if and only is $domf\neq\emptyset$ and $f(x)$ is finite for $x\in domf=\{(x):f(x)<+\infty\}$. The subgradient of a convex function $f(x_0)$ at x_0, denoted x^*, is defined by the system of inequalities $f(x)-f(x_0)\ge\langle x^*,x-x_0\rangle\ \forall\ x$. The set of all subgradients at x_0 denoted $\partial f(x_0)$ is referred to as the subdifferential of $f(x_0)$ at x_0. If the subdifferential at x_0 consists of only one element, it is equal to the gradient of f

at x_0, denoted $\nabla f(x_0)$. Given the subdifferential, a closed, proper, and convex function is determined up to an additive constant.

The definition of the conjugate of a function grows naturally out of the fact that the epigraph of a closed proper convex function on $\mathbb{R}^n$ is the intersection of the closed half-spaces in $\mathbb{R}^{n+1}$ that contain it. The function defined as $f^*(x^*) = \sup\limits_x\{\langle x, x^*\rangle - f(x)\}$ is called the conjugate of f which is one of the basic concepts both of convex analysis and duality theory. It is closed and convex. It is useful to remember, in particular, that Young's inequality $f(x) + f^*(x^*) \geq \langle x, x^*\rangle$ holds for any function. If here f is a proper convex function, then we shall refer to this relation as Fenchel's inequality.

For a convex set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, a set-valued mapping $G^*(\cdot;(x^0, y^0)) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$G^*\big(y^*;(x^0, y^0)\big) := \Big\{x^* : (x^*, -y^*) \in K^*_{gphG}(x^0, y^0)\Big\},$$

is called the LAM to G at a point $(x^0, y^0) \in gphG$, where $K^*_{gphG}(x^0, y^0)$ is the dual to the cone of tangent directions $K_{gphG}(x^0, y^0)$. Let us denote the following:

$$M_G(x^*, y^*) := \inf_{x,y}\Big\{\langle x, x^*\rangle - \langle y, y^*\rangle : \ (x, y) \in gphG\Big\}. \tag{2.1}$$

It is clear that for every $x \in \mathbb{R}^n$

$$M_G(x^*, y^*) \leq \langle x, x^*\rangle - H_G(x, y^*).$$

Here the Hamiltonian function $H_G(x, y^*) = \sup\limits_y\{\langle y, y^*\rangle : y \in G(x)\}, y^* \in \mathbb{R}^n$ and for convex set-valued mapping G, we put $H_G(x, y^*) = -\infty$, if $G(x) = \emptyset$. Moreover it is easy to see that the function

$$M_G(x^*, y^*) = \inf_x\{\langle x, x^*\rangle - H_G(x, y^*)\}$$

is a support function of the set $gphG$ taken with a minus sign. It follows that for a fixed y^* $M_G(x^*, y^*) = -\big(-H_G(\cdot, y^*)\big)^*(x^*)$, that is, M_G is the conjugate function for $-H_G(\cdot, y^*)$ taken with a minus sign.

From the applied point of view, the polyhedral mapping plays an important role. A polyhedral mapping is defined, the graph of which is the following polyhedral set in $\mathbb{R}^{2n}$:

$$gphG = \Big\{(x, y) : Ax - By \leq d\Big\}\ ,\ G(x) = \Big\{y : Ax - By \leq d\Big\},$$

where A and B, $m \times n$ dimensional matrices, d is a m-dimensional column-vector. The Hamiltonian function $H_G(\cdot, y^*)$ for a polyhedral mapping is closed and its LAM is the step function in the argument (x, y):

$$G^*\big(y^*;(x, y)\big) := \Big\{-A^*\lambda : y^* = B^*\lambda,\ \lambda \geq 0\ ,\ \langle Ax - By - d, \lambda\rangle = 0\Big\}.$$

In this paper, we consider the boundary value problem for second-order polyhedral differential inclusions, labeled by (PC):

$$\text{minimum } J(x(\cdot)) = \int_0^T f\big(x(t), t\big)dt, \tag{2.2}$$

$$(PC) \qquad x''(t) \in G(x(t), t), \quad \text{a.e.} \quad t \in [0, T], \tag{2.3}$$

$$x(0) - x(T) = \alpha_0\ ,\ x'(0) - x'(T) = \alpha_1\ , \tag{2.4}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is proper polyhedral function and $G(\cdot, t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a polyhedral set-valued mapping, $G(x) = \{y : Ax - By \leq d\}$, A and B, $m \times n$ dimensional matrices, d is a m-dimensional column-vector and α_0, α_1 are fixed vectors.

It is required to find a feasible trajectory $x(t)$, $t \in [0, T]$ minimizing Lagrange problem over a set of feasible trajectories. Here, a feasible trajectory $x(t); t \in [0, T]$ satisfies boundary value constraints almost everywhere (a.e.) in $[0, T]$, the second-order polyhedral differential inclusions whose second-order derivative in $[0, T]$ belongs to the standard Lebesgue space $L_1^n([0, T])$. In more detail, a feasible solution $x(\cdot)$ of the problem (PC) is a mapping $x(\cdot) : [0, T] \to \mathbb{R}^n$ satisfying $x''(t) \in G(x(t), t),$ a.e. $t \in [0, T]$, $x(0) - x(T) = \alpha_0$, $x'(0) - x'(T) = \alpha_1$ with $x(\cdot) \in AC([0, T]) \cap W_{1,2}^n([0, T])$ where $AC([0, T])$ is a space of absolutely continuous functions from $[0, T]$ into $\mathbb{R}^n$ and $W_{1,2}^n([0, T])$ is a Banach space of absolutely continuous functions from $[0, T]$ into $\mathbb{R}^n$ together with first order derivatives for which $x''(\cdot) \in L_1^n([0, T])$. Notice that a Banach space $W_{1,2}^n([0, T])$ can be equipped with the different equivalent norms.

The construction of the dual problems to the given primal polyhedral problem (PC) for ordinary differential inclusions is the main problem of this paper. Duality often makes it possible to simplify the computational procedure and construct a generalized solution to variational problems that do not have classical solutions. The duality theorems allow one to infer that a sufficient condition for an extremum is an extremal relation for the primal and dual problems. The latter means that if some pair (β, β^*) satisfies this extremal relation, then β and β^* are solutions to a primal and a dual problem, respectively. We remark that a significant part of the investigations of Ekeland and Temam [12] for simple variational problems are connected with such problems and that there are similar results for ordinary differential inclusions in Mahmudov [21]-[25].

In this paper, we purpose twofold goal. First, we prove sufficient optimality conditions of boundary value problem (PC). Second, we use this method to establish a duality relation problem to a Lagrange differential problem. Therefore, initially, we give the sufficient conditions of optimality for the problem (PC) which will play an important role in the next duality investigations.

3 Polyhedral Optimization of Boundary value problems for Differential Inclusions

Before giving the duality theorems and results, we formulate the sufficient conditions of optimality for the problem (PC) with second-order polyhedral differential inclusions.

Theorem 3.1. *Let f be continuous proper polyhedral function and G be a polyhedral set-valued mapping given in problem (PC). Then for the optimality of the trajectory $\tilde{x}(\cdot)$ in the problem (PC) with second-order differential inclusions, it is sufficient that there exist absolutely continuous function $x^*(\cdot)$, $t \in [0, T]$, together with the second-order derivatives and the function $\lambda(t)$ satisfying a.e. the following the second-order polyhedral Euler-Lagrange type inclusions (i), the equation (ii) and transversality conditions (iii):*

$$(i) \qquad x^{*''}(t) - A^*\lambda(t) \in \partial f(\widetilde{x}(t), t), \ \text{a.e. } t \in [0, T],$$

$$(ii) \quad \Big\langle A\tilde{x}(t) - B\tilde{x}''(t) - d \ , \ \lambda(t) \Big\rangle = 0 \ , \ \lambda(t) \geq 0, \ x^*(t) = B^*\lambda(t) \ \text{a.e. } t \in [0, T],$$

$$(iii) \ \ B^*\Big(\lambda(0) - \lambda(T)\Big) = 0 \ , \ B^*\Big(\lambda'(0) - \lambda'(T)\Big) = 0.$$

Proof. For all feasible solutions $x(\cdot)$, it follows from the definition of subdifferential that

$$f(x(t), t) - f(\widetilde{x}(t), t) \geq \Big\langle x^{*''}(t) - A^*\lambda(t) \ , x(t) - \widetilde{x}(t) \Big\rangle, \ \text{a.e. } t \in [0, T]. \tag{3.1}$$

It is easy to see from the condition (ii) that

$$\Big\langle Ax(t) - Bx''(t)\ ,\ \lambda(t)\Big\rangle \leq \Big\langle A\widetilde{x}(t) - B\widetilde{x}''(t)\ ,\ \lambda(t)\Big\rangle$$

for $\lambda(t) \geq 0, t \in [0,T]$, or more convenient form

$$-\Big\langle A^*\lambda(t)\ ,\ x(t) - \tilde{x}(t)\Big\rangle \geq -\Big\langle B^*\lambda(t)\ ,\ x''(t) - \widetilde{x}''(t)\Big\rangle. \tag{3.2}$$

Taking into account $x^*(t) = B^*\lambda(t)$ a.e. $t \in [0,T]$ and inequalities (3.1) and (3.2), we obviously have

$$f(x(t),t) - f(\widetilde{x}(t),t) \geq \Big\langle x^{*''}(t)\ ,x(t) - \widetilde{x}(t)\Big\rangle - \Big\langle x^*(t)\ ,\ x''(t) - \widetilde{x}''(t)\Big\rangle, \tag{3.3}$$

by integrating the inequality (3.3) over the interval $t \in [0,T]$, it follows that

$$\int_0^T \Big[f(x(t),t) - f(\widetilde{x}(t),t)\Big]dt \geq \int_0^T \Big[\Big\langle x^{*''}(t)\ ,x(t) - \widetilde{x}(t)\Big\rangle - \Big\langle x^*(t)\ ,\ x''(t) - \widetilde{x}''(t)\Big\rangle\Big]dt. \tag{3.4}$$

The formula in the square parentheses on the right-hand side of (3.4) can be expressed in the following equivalent way:

$$\Big\langle x^{*''}(t)\ ,x(t) - \widetilde{x}(t)\Big\rangle - \Big\langle x^*(t)\ ,\ x''(t) - \widetilde{x}''(t)\Big\rangle$$

$$= \frac{d}{dt}\Big\langle x^{*'}(t)\ ,x(t) - \widetilde{x}(t)\Big\rangle - \frac{d}{dt}\Big\langle x^*(t)\ ,\ x'(t) - \widetilde{x}'(t)\Big\rangle.$$

This allows us to conclude that

$$\int_0^T \Big[f(x(t),t) - f(\widetilde{x}(t),t)\Big]dt \geq \Big\langle x^{*'}(T)\ ,x(T) - \widetilde{x}(T)\Big\rangle - \Big\langle x^{*'}(0)\ ,x(0) - \widetilde{x}(0)\Big\rangle$$

$$-\Big\langle x^*(T)\ ,\ x'(T) - \widetilde{x}'(T)\Big\rangle + \Big\langle x^*(0)\ ,\ x'(0) - \widetilde{x}'(0)\Big\rangle. \tag{3.5}$$

Now denoting on the right hand-side of (3.5) by Λ and using the boundary condition of the problem (PC), we obtain that

$$\begin{aligned}
\Lambda &= \Big\langle x^{*'}(T)\ ,x(T) - \widetilde{x}(T)\Big\rangle - \Big\langle x^{*'}(0)\ ,x(0) - \widetilde{x}(0)\Big\rangle \\
&-\Big\langle x^*(T)\ ,\ x'(T) - \widetilde{x}'(T)\Big\rangle + \Big\langle x^*(0)\ ,\ x'(0) - \widetilde{x}'(0)\Big\rangle \\
&= \Big\langle x^{*'}(T)\ ,x(T) - \widetilde{x}(T)\Big\rangle - \Big\langle x^{*'}(0)\ ,x(T) - \widetilde{x}(T)\Big\rangle \\
&-\Big\langle x^*(T)\ ,\ x'(T) - \widetilde{x}'(T)\Big\rangle + \Big\langle x^*(0)\ ,\ x'(T) - \widetilde{x}'(T)\Big\rangle.
\end{aligned} \tag{3.6}$$

Then since $x^*(t) = B^*\lambda(t)$ a.e. $t \in [0,T]$ and taking into account the condition (iii) of the theorem, we have

$$\begin{aligned}
\Lambda &= \Big\langle x^{*'}(T) - x^{*'}(0)\ ,x(T) - \widetilde{x}(T)\Big\rangle - \Big\langle x^*(T) - x^*(0), x'(T) - \widetilde{x}'(T)\Big\rangle \\
&= \Big\langle B^*\lambda'(T) - B^*\lambda'(0)\ ,x(T) - \widetilde{x}(T)\Big\rangle - \Big\langle B^*\lambda(T) - B^*\lambda(0)\ ,x'(T) - \widetilde{x}'(T)\Big\rangle = 0.
\end{aligned} \tag{3.7}$$

Therefore this shows that

$$\int_0^T \Big[f(x(t),t) - f(\widetilde{x}(t),t)\Big]dt \geq 0$$

i.e. for all feasible solutions $x(t)$, we have $J\big(x(t)\big) \geq J\big(\tilde{x}(t)\big)$, so $\tilde{x}(t)$, $t \in [0,T]$ is optimal. The desired result is proved completely. □

Remark 3.2. It should be noted that this sufficient condition is also necessary conditions for the trajectory $\tilde{x}(t)$, $t \in [0,T]$, when properly reformulated. In this paper, we have used a specific peculiarity of the polyhedral differential inclusions. But in the proof of the necessary condition, it is hardly worthwhile. The argument is that certain conditions imposed on $f(\cdot,t)$ and LAM are required for a problem with differential inclusions in a general case. For a convex case, it is simple to realize the conditions put on the $f(\cdot,t)$ (not only for the polyhedral functions). It should be noted that the condition on boundness and upper semi-continuity of the LAM can be proven starting from the details of the polyhedral differential inclusions. In a more general case for first-order ordinary differential inclusions, all details related to the approximation problem are investigated using the familiar the Arzelá-Ascoli Theorem. Therefore, taking into account all the above difficulties, note that the transition to the continuous problem (PC), in any case, is a separate topic of discussion and omitted.

4 The Dual Problem and Main Results

The starting point of our investigations is a general approach for constructing a dual optimization problem to the primal problem (PC) based on the theory of conjugate functions. It is interesting to note that in the theory of mathematical programming problem the analogy of these results consists of the following. Suppose that we have a problem

$$\inf_{x\in M} f(x) \qquad (P)$$

where f is a closed, proper convex function and that M is a convex closed set. To establish the duality relations, we need the supplementary results of Theorem 3.15 [17] of the duality of operations of addition and infimal convolution of convex functions. By this result, if there exists a point $v \in M$, where f is continuous (f is continuous on $ridomf$, however, f may have a point of discontinuity in its boundary), the optimal value of problem (P) is

$$\begin{aligned}
\inf_{v\in M} f(v) = &\inf_{v\in M}\big\{f(v)+\delta_M(v)\big\} = -\sup_{v\in M}\big\{-f(v)-\delta_M(v)\big\}\\
= &-\sup_{v\in M}\big\{\langle v,0\rangle - \big(f(v)+\delta_M(v)\big)\big\} = -(f+\delta_M)^*(0) = -\big(f^*\oplus\delta_M^*\big)(0)\\
= &-\inf\big\{f^*(v^*)+\delta_M^*(-v^*)\big\} = \sup\big\{-f^*(v^*)-\delta_M^*(-v^*)\big\}.
\end{aligned}$$

Here $\delta_M(v) = \begin{cases} 0 & , v\in M \\ +\infty & , v\notin M\end{cases}$ is the indicator function of the set M and the operation of infimal convolution $\oplus$ of functions f^* and δ_M^* is defined as follows:

$$(f^*\oplus\delta_M^*)(v) = \inf\Big\{f^*(v_1)+\delta_M^*(v_2):\ v_1,v_2\in\mathbb{R}^n,\ v_1+v_2=v\Big\}.$$

In general, it can be noticed that $(f+\delta_M)^*(0) \leq \big(f^*\oplus\delta_M^*\big)(0)$ and so

$$\inf_{v\in M} f(v) \geq \sup\big\{-f^*(v^*)-\delta_M^*(-v^*)\big\}.$$

Then it is reasonable to announce that the dual problem to the primal problem (P) can be formulated as being

$$\sup\big\{-f^*(v^*)-\delta^*_M(-v^*)\big\}. \qquad (P^*)$$

The problem (P^*) is called the dual problem to the primal problem (P).

Further, we denote by β and β^* the optimal objective values of the problems (P) and (P^*), respectively. The next result shows that weak duality, namely the fact that the optimal objective value of the primal problem is always greater than or equal to the optimal objective value of the dual problem, is a consequence of the way in which the latter problem is defined. It holds $-\infty \le \beta^* \le \beta \le +\infty$. In addition, if the value of the problem (P) is finite, then the supremum in the problem (P^*) is attained for all v^*.

The construction of the duality problem would lead us too far astray from the main themes of this paper and is therefore omitted. We note that the dual problems are related to a large portion of Mahmudov's investigations for simple variational problems, and similar results occur for ordinary differential inclusions [21]-[25]. In general, in order to establish a dual problem to the convex problem (PC) (where f and G are convex functions and convex set-valued mapping, respectively), we have used a limiting process in dual problem for a discrete-approximate problem to continuous problem (PC); by passing to the formal limit as a discrete step tends to zero, the obtained maximization problem will be the dual problem to the continuous convex problem (PC):

$$\sup_{x^*,v^*}\Big\{-\int_0^T f^*(v^*(t),t)dt + \int_0^T M_G(x^{*''}(t)+v^*(t),x^*(t))dt+\langle\alpha_0,x^{*'}(0)\rangle-\langle\alpha_1,x^*(0)\rangle\Big\}. \quad (4.1)$$

Now we establish the dual pr oblem $(PC)^*$ to the main boundary value problem for second-order polyhedral differential inclusions. Then let's calculate the function M_G by using definition (2.1). It can be easily seen that, denoting $w=(x,y)\in\mathbb{R}^{2n}$, $w^*=(\xi^*,-\eta^*)\in\mathbb{R}^{2n}$ we have a linear programming problem

$$\inf\{\langle w,w^*\rangle:\ Cw\le d\} \quad (4.2)$$

where $C=[A:-B]$ is $m\times 2n$ block matrix. Then for a solution $\widetilde{w}=(\widetilde{x},\widetilde{y})$ of the problem (4.2) there exists m-dimensional vector $\lambda\ge 0$ such that $w^*=-C^*\lambda$, $\langle A\widetilde{x}-B\widetilde{y}-d,\lambda\rangle=0$. And vice versa, if these conditions are satisfied, then $\widetilde{w}=(\widetilde{x},\widetilde{y})$ is a solution of the problem (4.2). Therefore $w^*=-C^*\lambda$ implies that $\xi^*=-A^*\lambda$, $\eta^*=-B^*\lambda$, $\lambda\ge 0$. Thus we find that

$$M_G(\xi^*,\eta^*)=\langle\widetilde{x},-A^*\lambda\rangle-\langle\widetilde{y},-B^*\lambda\rangle=-\langle A\widetilde{x},\lambda\rangle+\langle B\widetilde{y},\lambda\rangle=-\langle A\widetilde{x}-B\widetilde{y},\lambda\rangle=-\langle d,\lambda\rangle. \quad (4.3)$$

Therefore by using the relation $x^{*''}(t)+v^*(t)=\xi^*(t)=-A^*\lambda(t)$ and $x^*(t)=\eta^*(t)=-B^*\lambda(t)$, we derive the following dual problem, labeled $(PC)^*$, to the primal continuous polyhedral problem (PC):

$$\sup_{\lambda(\cdot)}\Big\{-\int_0^T f^*\Big(B^*\lambda''(t)-A^*\lambda(t)\ ,\ t\Big)dt - \int_0^T\Big\langle d,\lambda(t)\Big\rangle dt-\Big\langle\alpha_0,B^*\lambda'(0)\Big\rangle+\Big\langle\alpha_1,B^*\lambda^*(0)\Big\rangle\Big\}. \quad (4.4)$$

Now we are in a position to prove the main result of this paper.

Theorem 4.1. *Let $f(\cdot,t)$ be a continuous proper polyhedral function and G be a polyhedral set-valued mapping. Moreover let $\widetilde{x}(t)$ be an optimal solution of the primal problem (PC) with polyhedral differential inclusions. Then for the optimality of dual variable $\lambda(t)$, $t \in [0,T]$ in the dual problem $(PC)^*$, it is necessary and sufficient that the conditions $(i)-(iii)$ of Theorem 3.1 are satisfied. Besides, the optimal values in the primal (PC) and dual $(PC)^*$ problems are equal.*

Proof. We prove that for all feasible solutions $x(\cdot)$ and dual variable $\lambda(\cdot)$ of the primal (PC) and dual $(PC)^*$ problems, respectively, the following inequality holds:

$$\int_0^T f(x(t),t)dt \geq -\int_0^T f^*\Big(B^*\lambda''(t) - A^*\lambda(t)\ ,\ t\Big)dt - \int_0^T \Big\langle d, \lambda(t)\Big\rangle dt \\ -\Big\langle \alpha_0, B^*\lambda'(0)\Big\rangle + \Big\langle \alpha_1, B^*\lambda(0)\Big\rangle. \tag{4.5}$$

From the Young's inequality, we write

$$-\int_0^T f^*\Big(B^*\lambda''(t) - A^*\lambda(t)\ ,\ t\Big)dt \leq \int_0^T f(x(t),t)dt \\ -\int_0^T \Big\langle B^*\lambda''(t) - A^*\lambda(t)\ ,\ x(t)\Big\rangle dt. \tag{4.6}$$

It is clear from the definition (2.1) that

$$-\int_0^T \Big\langle d, \lambda(t)\Big\rangle dt \leq -\int_0^T \Big\langle A^*\lambda(t), x(t)\Big\rangle dt + \int_0^T \Big\langle B^*\lambda(t)\ ,\ x''(t)\Big\rangle dt. \tag{4.7}$$

By summing up the inequalities (4.6)-(4.7), we conclude that

$$-\int_0^T f^*\Big(B^*\lambda''(t) - A^*\lambda(t)\ ,\ t\Big)dt - \int_0^T \Big\langle d, \lambda(t)\Big\rangle dt - \Big\langle \alpha_0, B^*\lambda'(0)\Big\rangle + \Big\langle \alpha_1, B^*\lambda(0)\Big\rangle \\ \leq \int_0^T f(x(t),t)dt - \int_0^T \Big\langle B^*\lambda''(t) - A^*\lambda(t)\ ,\ x(t)\Big\rangle dt - \int_0^T \Big\langle A^*\lambda(t), x(t)\Big\rangle dt \\ + \int_0^T \Big\langle B^*\lambda(t)\ ,\ x''(t)\Big\rangle dt - \Big\langle \alpha_0, B^*\lambda'(0)\Big\rangle + \Big\langle \alpha_1, B^*\lambda(0)\Big\rangle. \tag{4.8}$$

Now we rewrite inequality (4.8) in more convenience form

$$-\int_0^T f^*\Big(B^*\lambda''(t) - A^*\lambda(t)\ ,\ t\Big)dt - \int_0^T \Big\langle d, \lambda(t)\Big\rangle dt - \Big\langle \alpha_0, B^*\lambda'(0)\Big\rangle + \Big\langle \alpha_1, B^*\lambda(0)\Big\rangle \\ \leq \int_0^T f(x(t),t)dt + \int_0^T \Big[\langle B^*\lambda(t)\ ,\ x''(t)\rangle - \langle B^*\lambda''(t)\ ,\ x(t)\rangle\Big] dt \\ -\Big\langle \alpha_0, B^*\lambda'(0)\Big\rangle + \Big\langle \alpha_1, B^*\lambda(0)\Big\rangle. \tag{4.9}$$

Then it is easy to see that

$$\langle B^*\lambda(t)\ ,\ x''(t)\rangle - \langle B^*\lambda''(t)\ ,\ x(t)\rangle = \frac{d}{dt}\langle B^*\lambda(t)\ ,\ x'(t)\rangle - \frac{d}{dt}\langle B^*\lambda'(t)\ ,\ x(t)\rangle$$

and taking integral, we find that

$$-\int_0^T f^*\Big(B^*\lambda''(t)-A^*\lambda(t)\ ,\ t\Big)dt-\int_0^T\Big\langle d,\lambda(t)\Big\rangle dt-\Big\langle \alpha_0,B^*\lambda'(0)\Big\rangle+\Big\langle \alpha_1,B^*\lambda(0)\Big\rangle$$
$$\leq \int_0^T f(x(t),t)dt+\big\langle B^*\lambda(T)\ ,\ x'(T)\big\rangle-\big\langle B^*\lambda'(T)\ ,\ x(T)\big\rangle-\big\langle B^*\lambda(0)\ ,\ x'(0)\big\rangle$$
$$+\big\langle B^*\lambda'(0)\ ,\ x(0)\big\rangle-\Big\langle \alpha_0,B^*\lambda'(0)\Big\rangle+\Big\langle \alpha_1,B^*\lambda(0)\Big\rangle.$$

Then it follows from the boundary value constraint that

$$-\int_0^T f^*\Big(B^*\lambda''(t)-A^*\lambda(t)\ ,\ t\Big)dt-\int_0^T\Big\langle d,\lambda(t)\Big\rangle dt-\Big\langle \alpha_0,B^*\lambda'(0)\Big\rangle+\Big\langle \alpha_1,B^*\lambda(0)\Big\rangle$$
$$\leq \int_0^T f(x(t),t)dt+\big\langle B^*\lambda(T)\ ,\ x'(T)\big\rangle-\big\langle B^*\lambda'(T)\ ,\ x(T)\big\rangle-\big\langle B^*\lambda(0)\ ,\ x'(0)\big\rangle$$
$$+\big\langle B^*\lambda'(0)\ ,\ x(0)\big\rangle-\Big\langle x(0)-x(T),B^*\lambda'(0)\Big\rangle+\Big\langle x'(0)-x'(T),B^*\lambda(0)\Big\rangle.$$

or equivalently

$$-\int_0^T f^*\Big(B^*\lambda''(t)-A^*\lambda(t)\ ,\ t\Big)dt-\int_0^T\Big\langle d,\lambda(t)\Big\rangle dt-\Big\langle \alpha_0,B^*\lambda'(0)\Big\rangle+\Big\langle \alpha_1,B^*\lambda(0)\Big\rangle$$
$$\leq \int_0^T f(x(t),t)dt+\Big\langle B^*\lambda(T)-B^*\lambda(0)\ ,\ x'(T)\Big\rangle-\Big\langle B^*\lambda'(T)-B^*\lambda'(0)\ ,\ x(T)\Big\rangle.$$

By condition (iii) of Theorem 3.1, we derive that

$$-\int_0^T f^*\Big(B^*\lambda''(t)-A^*\lambda(t)\ ,\ t\Big)dt-\int_0^T\Big\langle d,\lambda(t)\Big\rangle dt-\Big\langle \alpha_0,B^*\lambda'(0)\Big\rangle+\Big\langle \alpha_1,B^*\lambda(0)\Big\rangle$$
$$\leq \int_0^T f(x(t),t)dt$$

which verifies the inequality (4.5) giving the desired result.

Furthermore, suppose that $\widetilde{\lambda}(t)$, $t\in[0,T]$ satisfies the conditions $(i)-(iii)$ of Theorem 3.1 by using the definition of function M_G in the interval $[0,T]$,

$$-\Big\langle d,\widetilde{\lambda}(t)\Big\rangle=-\Big\langle A^*\widetilde{\lambda}(t),\widetilde{x}(t)\Big\rangle+\Big\langle B^*\widetilde{\lambda}(t)\ ,\ \widetilde{x}''(t)\Big\rangle. \tag{4.10}$$

Moreover we have

$$-f^*\Big(B^*\widetilde{\lambda}''(t)-A^*\widetilde{\lambda}(t)\ ,\ t\Big)=f(\widetilde{x}(t),t)-\Big\langle B^*\widetilde{\lambda}''(t)-A^*\widetilde{\lambda}(t)\ ,\ \widetilde{x}(t)\Big\rangle. \tag{4.11}$$

Therefore taking into account the Eqs. (4.10)-(4.11) in relations (4.6)-(4.7), the inequality sign is replaced by equality, and hence for $\widetilde{x}(\cdot)$ and $\widetilde{\lambda}(t)$ the equality of values of the primal and dual problems is ensured. Moreover $\widetilde{x}(\cdot)$ and $\widetilde{\lambda}(t)$ are satisfy the conditions $(i)-(iii)$ of Theorem 3.1, the collection $(i)-(iii)$ is a dual relation for the primal (PC) and dual $(PC)^*$ problems. This completes the proof of theorem. □

$\boxed{5}$ Applications

In this section, we illustrate the main points of this paper through numerical examples.

Example 5.1. Let us consider the boundary value problems for second-order "linear" differential inclusions:

$$\begin{aligned} &\text{minimize } J(x(\cdot)) = \int_0^T f(x(t),t)dt \\ &x''(t) \in G(x(t)) \quad \text{a.e.} \quad t \in [0,T], \\ &G(x) = \{v :\ v = Ax + Bu,\ u \in U\}, \\ &x(0) - x(T) = \alpha_0,\ x'(0) - x'(T) = \alpha_1, \end{aligned} \tag{5.1}$$

where $f(\cdot,t)$ is continuously differentiable function, A and B are $n \times n$ and $n \times r$ matrices, respectively, U is a polyhedral subset of $\mathbb{R}^r$, α_0, α_1 are constant vectors. The problem is to find a controlling parameter $\widetilde{u}(t) \in U$ such that the arc $\widetilde{x}(t)$ corresponding to it minimizes $J(x(\cdot))$.

By elementary computations, we find that

$$G^*\big(v^*; (\widetilde{x}, \widetilde{v})\big) = \begin{cases} A^*v^*, & -B^*v^* \in (cone(U - \widetilde{u}))^*, \\ \emptyset, & -B^*v^* \notin (cone(U - \widetilde{u}))^*, \end{cases} \tag{5.2}$$

where $\widetilde{v} = A\widetilde{x} + B\widetilde{u}$, $u \in U$, A^* and B^* are transposed matrices. Besides $-B^*v^* \in (cone(U - \widetilde{u}))^*$ means that the Weierstrass-Pontryagin maximum condition

$$\Big\langle B\widetilde{u}(t),\ x^*(t)\Big\rangle = \sup_{u \in U} \Big\langle Bu,\ x^*(t)\Big\rangle$$

satisfied. Notice that the function f is continuous differentiable function, taking into account the definition of LAM and using Theorem 3.1, hence, we derive that the arc $\widetilde{x}(t)$ corresponding to the controlling parameter $\widetilde{u}(t)$ minimizes $J(x(\cdot))$ in the Example 5.1 if there exists an absolutely continuous function $x^*(t)$ satisfying second order adjoint differential inclusion(equation), the transversality and Weierstrass-Pontryagin conditions:

$$\begin{gathered} x^{*''}(t) = A^*\lambda(t) + f'_x(\tilde{x}(t),t)\ ,\ \text{a.e.}\ t \in [0,T], \\ B^*x^*(0) - f'_x(\tilde{x}(0),0) = B^*x^*(T) - f'_x(\tilde{x}(T),T), \\ B^*x^{*'}(0) - f'_x(\tilde{x}(0),0) = B^*x^{*'}(T) - f'_x(\tilde{x}(T),T), \\ \Big\langle B\widetilde{u}(t),\ x^*(t)\Big\rangle = \sup_{u \in U} \Big\langle Bu,\ x^*(t)\Big\rangle. \end{gathered}$$

Example 5.2. Consider the following second-order polyhedral differential problem with the boundary value constraints:

$$\min J(x(\cdot)) = \int_0^{10} x(t)dt, \tag{5.3}$$

$$2x(t) - x''(t) \le 0, \quad \text{a.e.} \quad t \in [0,10], \tag{5.4}$$

$$x(0) - x(10) = 1\ ,\ x'(0) - x'(10) = 4. \tag{5.5}$$

Here we assume that $f\big(x(t),t\big) = x(t)$ is continuously differentiable function, $T = 10$, and polyhedral set-valued mapping $G(x) = \{y :\ 2x - y \le 0\}$ where according to problem (PC), we choose that $A = [2], B = [1], d = [0]$ and $\alpha_0 = 1,\ \alpha_1 = 4$.

It can be easily seen that the subdifferential of f is the gradient vector, that is $\partial_x f(\widetilde{x}(t),t) = \{1\}$. Then according to Theorem 3.1, we have

$$\begin{gathered} {x^*}''(t) - 2\lambda(t) = 1, \text{ a.e. } t \in [0,10], \\ \big\langle 2\tilde{x}(t) - \tilde{x}''(t)\ ,\ \lambda(t)\big\rangle = 0\ ,\ \lambda(t) \geq 0,\ x^*(t) = \lambda(t) \text{ a.e. } t \in [0,10], \\ \lambda(0) = \lambda(10) \text{ and } \lambda'(0) = \lambda'(10). \end{gathered} \tag{5.6}$$

In view of first relations of (5.6) we have a boundary value problem given by second-order linear nonhomogeneous equation with constant coefficients

$$\begin{gathered} \lambda''(t) - 2\lambda(t) = 1, \\ \lambda(0) = \lambda(10) \text{ and } \lambda'(0) = \lambda'(10). \end{gathered} \tag{5.7}$$

We need to find the roots of the characteristic equation of the homogeneous differential equation. Obviously, characteristic equation $r^2 - 2 = 0$ has two real roots $r_1 = \sqrt{2}$ and $r_2 = -\sqrt{2}$ and the solution is $\lambda_h(t) = C_1 e^{\sqrt{2}t} + C_2 e^{-\sqrt{2}t}$ where C_1, C_2 are arbitrary constants. Moreover, the particular solution of (5.7) is $\lambda_p(t) = -\frac{1}{2}$. Thus the general solution of (5.7) is

$$\lambda(t) = \lambda_h(t) + \lambda_p(t) = C_1 e^{\sqrt{2}t} + C_2 e^{-\sqrt{2}t} - \frac{1}{2}$$

where C_1, C_2 are arbitrary constants. By using the boundary value condition, we find that $\lambda(t) = -\frac{1}{2}$.
On the other hand, since $\lambda(t) \neq 0$, $t \in [0,10]$ from the condition (5.6) of problem, we deduce that
$2\tilde{x}(t) - \tilde{x}''(t) = 0$, the general solution of which is $\widetilde{x}(t) = C_3 e^{\sqrt{2}t} + C_4 e^{-\sqrt{2}t}$, ($C_3, C_4$ are arbitrary constants). Using the boundary condition $x(0) - x(10) = 1,\ x'(0) - x'(10) = 4$, we find that $C_3 = \frac{1+2\sqrt{2}}{1-e^{10\sqrt{2}}}$ and $C_4 = \frac{1}{1-e^{-10\sqrt{2}}}$. Consequently, we have the solution of the stated problem in Example 5.2

$$\widetilde{x}(t) = \frac{1+2\sqrt{2}}{1-e^{10\sqrt{2}}} e^{\sqrt{2}t} + \frac{1}{1-e^{-10\sqrt{2}}} e^{-\sqrt{2}t}.$$

Then its value is

$$\begin{aligned} \int_0^{10} \widetilde{x}(t)dt &= \int_0^{10} \Big[\frac{1+2\sqrt{2}}{1-e^{10\sqrt{2}}} e^{\sqrt{2}t} + \frac{1}{1-e^{-10\sqrt{2}}} e^{-\sqrt{2}t}\Big] dt \\ &= \Big(\frac{(1+2\sqrt{2})e^{\sqrt{2}t}}{\sqrt{2}(1-e^{10\sqrt{2}})} - \frac{e^{-\sqrt{2}t}}{\sqrt{2}(1-e^{-10\sqrt{2}})}\Big)\Big|_0^{10} = -2. \end{aligned}$$

Example 5.3. Let us now construct the dual problem to the differential problem in Eqs. (5.3)-(5.5) and calculate its value. For this, we first compute the conjugate function f^* of the function f defined in the cost functional (5.3). By definition of the conjugate function, we find that

$$\begin{aligned} f^*(\eta^*) &= \sup_\eta \Big\{\langle \eta, \eta^* \rangle - f(\eta)\Big\} = \sup_\eta \Big\{\eta\eta^* - \eta\Big\} \\ &= \sup_\eta \Big\{\eta(\eta^* - 1)\Big\} = \begin{cases} 0\ , & \eta^* = 1, \\ +\infty, & \text{otherwise}. \end{cases} \end{aligned} \tag{5.8}$$

Then taking into account the dual problem (PC^*) in the formula (4.4), we write dual problem of the problem in Example 5.2 as follows:

$$\sup_{\lambda}\Big\{-\int_0^{10} f^*\big(\lambda''(t)-2\lambda(t)\big)dt-\lambda'(0)+4\lambda(0)\Big\}. \tag{5.9}$$

Then from (5.8), we have $f^*\big(\lambda''(t)-2\lambda(t)\big)=0$ under the condition $\lambda''(t)-2\lambda(t)=1$. Therefore, we get the dual problem (5.9) in more convenient form

$$\sup\Big\{-\lambda'(0)+4\lambda(0)\Big\}=4\lambda(0)-\lambda'(0). \tag{5.10}$$

Therefore we find the value of dual problem in Example 5.2 is $4\lambda(0)-\lambda'(0)=-2$. Thus accordingly to Theorem 4.1, we have showed that under the conditions of Theorem 3.1, the optimal values of primal (5.2) and dual problem (5.10) coincide. Consequently, we prove that if $\beta=\int_0^{10}\widetilde{x}(t)dt=-2$ and $\beta^*=4\lambda(0)-\lambda'(0)=-2$ are the values of second-order polyhedral differential problem in Example 5.2 and its dual problem, respectively, then $\beta=\beta^*=-2$ for an optimal pair primal and dual problems.

6 Conclusions

This paper can be divided conditionally into two parts. In the first part, a particular boundary value problem is considered for second-order polyhedral differential inclusions. To formulate sufficient conditions of optimality for polyhedral differential inclusions, we use constructions of convex analysis in terms of locally conjugate mappings for such problems. The arising conjugate polyhedral inclusions in the considered results are called Euler-Lagrange inclusions. The derivation of sufficient conditions is implemented by passing to the limit as the discrete steps tend to zero. It should be noted that the justification of sufficient conditions of optimality for the problem (PC) is complicated by the accompaniment of discrete and discrete-approximation problems therefore we omit and formulate only final results. In the second part, the duality theorem proved allows one to conclude that a sufficient condition for an extremum is an extremal relation for the primal and dual problems. Thus, we established a one-to-one relationship between the optimality conditions of polyhedral second order differential inclusions for the primal boundary value problems and the dual problem which are formulated by using the concepts of convex analysis and duality theory. Via numerical examples, we have shown how the conditions of optimality for the original polyhedral differential problem can be extended by comparing known duality relations to dual polyhedral differential inclusions. Besides, there can be no doubt that investigations of duality results in optimal control problems with second/higher-order polyhedral discrete/differential inclusions can have a great contribution to the modern development of the optimization theory. In addition to being of independent interest, such boundary value problems may also play a significant role in more complex approaches, including the theory of Sturm-Liouville and integral transformation techniques, as well as applying these to variational problems from different fields.

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