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TENSOR QUADRATIC EIGENVALUE COMPLEMENTARITY PROBLEM*

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Abstract: In this paper, we introduce a class of tensor quadratic eigenvalue complementarity problems, which is an interesting generalization of matrix quadratic eigenvalue complementarity problems on higher order tensors. We give a sufficient condition to guarantee the existence of solutions, and propose a semismooth Newton-type method to solve the tensor quadratic eigenvalue complementarity problem. Finally, some numerical results are reported to show the efficiency of the proposed method.

Key words: *tensor quadratic eigenvalue complementarity problem, strictly copositive tensor, semismooth Newton method*

Mathematics Subject Classification: *15A69, 15A18, 90C33*

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1 Introduction

Given matrices $A, B \in \mathbb{R}^{n\times n}$, the eigenvalue complementarity problem (EiCP) (denoted by EiCP(A,B); see e.g.[25, 37]) is to find a scalar $\lambda \in \mathbb{R}$ and a vector $x \in \mathbb{R}^n \setminus \{0\}$ such that

$$\begin{cases} \omega = \lambda Ax - Bx, \\ \omega \geq 0, x \geq 0, \\ x^T\omega = 0. \end{cases} \tag{1.1}$$

We call (x, λ) a Pareto eigenvector-eigenvalue pair of (A, B). EiCP is a special kind of complementarity problems, which first appeared in the study of static equilibrium states of mechanical system with unilateral friction [9] and has been widely studied due to its important applications in the engineering, economics and sciences [1, 24, 25, 26, 27, 37].

Recently, an extension of EiCP has been introduced in [38], which is called quadratic eigenvalue complementarity problem (QEiCP). This problem differs from EiCP through adding a quadratic term in λ. Its formal definition is as follows. Given matrices $A, B, C \in \mathbb{R}^{n\times n}$, QEiCP$(A, B, C)$ is to find a scalar $\lambda \in \mathbb{R}$ and a vector $x \in \mathbb{R}^n \setminus \{0\}$ such that

$$\begin{cases} \omega = \lambda^2 Ax + \lambda Bx + Cx, \\ \omega \geq 0, x \geq 0, \\ x^T\omega = 0. \end{cases} \tag{1.2}$$

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Clearly, QEiCP(A, B, C) reduces to EiCP$(B, -C)$ when $A = O$. The wide applications of QEiCP attract many scholars to study [3, 4, 15, 23]. When A, B, C are symmetric, QEiCP (1.2) is symmetric, which has been well studied in [15].

It is well-known that tensor eigenvalue complementarity problem (TEiCP), as a generalization of EiCP (1.1) on higher order matrices [29], has been studied extensively [5, 7, 14, 22, 28, 47, 48]. TEiCP is an emerging subject from the tensor community and has closed relation with the tensor complementarity problem. Tensor complementarity problem (TCP) is a class of nonlinear complementarity problems with the involved function being defined by a tensor [34]. Complementarity problems have been developed well due to the wide applications in economics, engineering and related fields. In the last few years, TCP has attracted a lot of attention, and has been studied extensively, from theory [2, 11, 19, 40, 43, 44, 45] to solution methods [12, 17, 46] and applications [20, 21, 36]. In the last decades, the research on tensor eigenvalue problems has also received extensive attention [6, 8, 34, 35, 39]. From the above analysis, we can see that EiCP, TEiCP and TCP are well studied and used widely in engineering problems and related optimization problems. Naturally, we will think about whether we can extend QEiCP from matrices to tensors. If yes, how to express the mathematical model of the QEiCP with tensor? Motivated by these questions, we introduce the tensor quadratic eigenvalue complementarity problem (TQEiCP) in this paper and study its solvability.

Now, we collect some basic notations and definitions, which will be used in this paper. A real mth order n-dimensional tensor $\mathcal{A} = (a_{i_1 \ldots i_m})$ is a multi-array of real entries $a_{i_1 \ldots i_m}$, where $i_j \in J_n$ for $j \in J_m$, $J_m := \{1, \ldots, m\}$. The set of all real mth order n-dimensional tensors is denoted as $\mathbb{R}^{[m,n]}$. $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called a symmetric tensor if its entries $a_{i_1 \ldots i_m}$ are invariant under any permutation of its indices. For a vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ and a tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$, $\mathcal{A}x^{m-1}$ and $x^{[m-1]}$ are two vectors in $\mathbb{R}^n$ with the ith component defined by

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \ldots, i_m = 1}^{n} a_{i i_2 \ldots i_m} x_{i_2} \ldots x_{i_m} \quad \text{and} \quad (x^{[m-1]})_i = x_i^{m-1},$$

respectively. $\mathcal{A}x^m$ is a value at x of a homogeneous polynomial defined by

$$\mathcal{A}x^m = \sum_{i_1, \ldots, i_m = 1}^{n} a_{i_1 i_2 \ldots i_m} x_{i_1} \ldots x_{i_m}.$$

$\mathcal{A}x^{m-2}$ is a matrix in $\mathbb{R}^{n \times n}$ with its (i, j)-th component defined by

$$(\mathcal{A}x^{m-2})_{i,j} = \sum_{i_3, \ldots, i_m = 1}^{n} a_{i j i_3 \ldots i_m} x_{i_3} \ldots x_{i_m}.$$

With the above basic notations and definitions, we propose the formal definition of TQEiCP as follows. Given $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{R}^{[m,n]}$, TQEiCP$(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is to find $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n$ such that

$$\begin{cases} \omega = (\lambda^2 \mathcal{A} + \lambda \mathcal{B} + \mathcal{C}) x^{m-1}, \\ \omega \geq 0, x \geq 0, \\ x^T \omega = 0, \\ \mathbf{1}_n^T x = 1, \end{cases} \tag{1.3}$$

where $\mathbf{1}_n = (1, \ldots, 1)^T \in \mathbb{R}^n$. From the last equality, we know that $x \neq 0$. Certainly, the last equality can be replaced with $x^T x = 1$, which also ensures $x \neq 0$. The solution λ of

(1.3) is called a tensor quadratic complementarity eigenvalue of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. TQEiCP$(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is called symmetric when $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are symmetric tensors.

We know that QEiCP has rich theoretical results and numerical methods. Here, can we obtain more interesting and important properties and simple, direct and efficient methods for TQEiCP (1.3)? Given tensors $\mathcal{A}, \mathcal{B}, \mathcal{C}$, when does TQEiCP (1.3) have a solution? In the following of this paper, we will study the above interesting questions about TQEiCP (1.3). We will give a sufficient condition to guarantee that TQEiCP (1.3) has solutions based on structured tensors, and we will present a semismooth Newton-type method to solve TQEiCP (1.3).

The rest of this paper is organized as follows. In Section 2, we recall some definitions and existing results on tensors and TEiCP. In Section 3, we establish a sufficient condition for the existence of solutions to TQEiCP (1.3). In Section 4, we propose a Newton-type method for solving TQEiCP (1.3), and some preliminary numerical results are reported. Finally, we give some conclusions in Section 5.

2 Preliminaries

In this section, we recall some definitions on tensors and some existing results on TEiCP.

Given $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{[m,n]}$, TEiCP$(\mathcal{A}, \mathcal{B})$ is to find $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n \setminus \{0\}$ such that

$$\begin{cases} \omega = (\lambda\mathcal{A} - \mathcal{B})x^{m-1}, \\ \omega \geq 0, x \geq 0, \\ x^T\omega = 0. \end{cases} \tag{2.1}$$

The solution (λ, x) is called a Pareto eigenvalue-eigenvector pair of $(\mathcal{A}, \mathcal{B})$, and λ is called a complementarity eigenvalue. TEiCP has been also extensively studied [5, 7, 14, 22, 28, 47, 48]. Particularly, Chang [5] extended QEiCP (1.2) from matrix to tensor and introduced a class of tensor generalized higher-degree eigenvalue complementarity problem (TGHDEiCP). TGHDEiCP$(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is to find $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n \setminus \{0\}$ such that

$$\begin{cases} (\lambda^k\mathcal{A} + \lambda^l\mathcal{B} + \mathcal{C})x^{m-1} \geq 0, \\ x \geq 0, \\ x^T(\lambda^k\mathcal{A} + \lambda^l\mathcal{B} + \mathcal{C})x^{m-1} = 0, \end{cases} \tag{2.2}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{R}^{[m,n]}$, m is even and k, l are natural numbers satisfying $m \geq k > l \geq 1$. The (λ, x) satisfying (2.2) is called a (k, l) degree eigenpair of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. Note that TGHDEiCP (2.2) reduces to tensor high-degree eigenvalue complementarity problem (THDEiCP) [28] and QEiCP when $\mathcal{A} = \mathcal{O}$ and $m = 2$, respectively. Chang [5] proved that TGHDEiCP can be transformed into THDEiCP under some mild conditions and established the relationship between the solutions of TGHDEiCP and THDEiCP when $k = 2l$, where l is odd and m is even. It follows from (1.3) and (2.2) that TQEiCP is a special class of TGHDEiCP. But, some nice properties of TQEiCP (1.3) can be shown in this paper which differ from those given in [5]. Moreover, Yan and Ling [47] introduced QEiCP of tensor on second-order cone, which is to find $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n \setminus \{0\}$ such that

$$x \in K, \quad \omega = (\lambda^2\mathcal{A} + \lambda\mathcal{B} + \mathcal{C})x^{m-1} \in K, \quad x^T\omega = 0,$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{R}^{[m,n]}$, $K = K^{n_1} \times K^{n_2} \times \cdots \times K^{n_r}$ is the second-order cone in $\mathbb{R}^n$ with $\Sigma_{i=1}^r n_i = n$, $K^{n_i} = \{x^i = (x^i_\bullet, (x^i_\circ)^T)^T \in \mathbb{R} \times \mathbb{R}^{n_i-1} | x^i_\bullet \geq \|x^i_\circ\|\}$, and $\|\cdot\|$ is Euclidean norm. The authors proposed the the nonlinear programming models related with QEiCPs of tensor. Under mild conditions, the authors also proved the relations between the solutions of QEiCPs of tensor and the optimal solutions or stable points of the corresponding nonlinear programming problems. In this paper, we will choose an approach with different form to discuss the solutions of TQEiCP (1.3) without the symmetry assumptions on the tensors involved in the problem.

In order to establish a sufficient condition for existence of solutions of TQEiCP (1.3) in the next section, we recall some definitions on tensors. For more details, please refer to [29, 33, 41, 42] and the references therein.

Definition 2.1. A tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called a strictly copositive tensor, if $\mathcal{A}x^m > 0$ for any $x \in \mathbb{R}^n \setminus \{0\}$ with $x \geq 0$.

Definition 2.2. A tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called an S_0-tensor, if there exists $x \in \mathbb{R}^n \setminus \{0\}$ with $x \geq 0$ such that $\mathcal{A}x^{m-1} \geq 0$.

The following proposition was given in [29].

Proposition 2.3. *Given $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{[m,n]}$ and assume that $\mathcal{A}$ is a strictly copositive tensor, then TEiCP($\mathcal{A}, \mathcal{B}$) is solvable for any $\mathcal{B}$.*

In the next section, we will establish a connection between TQEiCP and TEiCP. Based on the connection, we will give a condition to guarantee the solvability of TQEiCP.

3 The Solvability of TQEiCP

In this section, we present a sufficient condition for the existence of solutions of TQEiCP (1.3).

We consider TQEiCP (1.3) with $\mathcal{A} = (a_{i_1 \ldots i_m}),\ \mathcal{B} = (b_{i_1 \ldots i_m}),\ \mathcal{C} = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[m,n]}$. Define $\mathcal{D}, \mathcal{G}, \mathcal{H} \in \mathbb{R}^{[m,2n]}$ as

$$\mathcal{D} = (d_{i_1 i_2 \ldots i_m}) = \begin{cases} a_{i_1 \ldots i_m}, & i_1, i_2, \ldots, i_m \in J_n, \\ 1, & i_1 = i_2 = \cdots = i_m \in J_{2n} \setminus J_n, \\ 0, & otherwise, \end{cases} \tag{3.1}$$

$$\mathcal{G} = (g_{i_1 i_2 \ldots i_m}) = \begin{cases} -b_{i_1 \ldots i_m}, & i_1, i_2, \ldots, i_m \in J_n, \\ -c_{i_1 (i_2 - n) \ldots (i_m - n)}, & i_1 \in J_n,\ i_2, \ldots, i_m \in J_{2n} \setminus J_n, \\ 1, & i_1 \in J_{2n} \setminus J_n,\ i_1 - n = i_2 = \cdots = i_m \in J_n, \\ 0, & otherwise, \end{cases} \tag{3.2}$$

$$\mathcal{H} = (h_{i_1 i_2 \ldots i_m}) = \begin{cases} b_{i_1 \ldots i_m}, & i_1, i_2, \ldots, i_m \in J_n, \\ -c_{i_1 (i_2 - n) \ldots (i_m - n)}, & i_1 \in J_n,\ i_2, \ldots, i_m \in J_{2n} \setminus J_n, \\ 1, & i_1 \in J_{2n} \setminus J_n,\ i_1 - n = i_2 = \cdots = i_m \in J_n, \\ 0, & otherwise. \end{cases} \tag{3.3}$$

The following simple example illustrates the structures of $\mathcal{D}, \mathcal{G}, \mathcal{H}$ defined by (3.1), (3.2) and (3.3), respectively.

Example 3.1. Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{R}^{[3,2]}$ be defined as

$$\mathcal{A}(1,:,:) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix},\ \mathcal{A}(2,:,:) = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix},\ \mathcal{B}(1,:,:) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathcal{B}(2,:,:) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ \mathcal{C}(1,:,:) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix},\ \mathcal{C}(2,:,:) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then, it follows from (3.1)-(3.3) that $\mathcal{D}, \mathcal{G}, \mathcal{H}$ take their components as

$$\mathcal{D}(1,:,:) = \begin{pmatrix} \mathcal{A}(1,:,:) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},\ \mathcal{D}(2,:,:) = \begin{pmatrix} \mathcal{A}(2,:,:) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$
$$\mathcal{D}(3,:,:) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & E_{11} \end{pmatrix},\ \mathcal{D}(4,:,:) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & E_{22} \end{pmatrix},$$
$$\mathcal{G}(1,:,:) = \begin{pmatrix} -\mathcal{B}(1,:,:) & \mathbf{0} \\ \mathbf{0} & -\mathcal{C}(1,:,:) \end{pmatrix},\ \mathcal{G}(2,:,:) = \begin{pmatrix} -\mathcal{B}(2,:,:) & \mathbf{0} \\ \mathbf{0} & -\mathcal{C}(2,:,:) \end{pmatrix},$$
$$\mathcal{G}(3,:,:) = \begin{pmatrix} E_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},\ \mathcal{G}(4,:,:) = \begin{pmatrix} E_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$
$$\mathcal{H}(1,:,:) = \begin{pmatrix} \mathcal{B}(1,:,:) & \mathbf{0} \\ \mathbf{0} & -\mathcal{C}(1,:,:) \end{pmatrix},\ \mathcal{H}(2,:,:) = \begin{pmatrix} \mathcal{B}(2,:,:) & \mathbf{0} \\ \mathbf{0} & -\mathcal{C}(2,:,:) \end{pmatrix},$$
$$\mathcal{H}(3,:,:) = \begin{pmatrix} E_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},\ \mathcal{H}(4,:,:) = \begin{pmatrix} E_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where $\mathbf{0} \in \mathbb{R}^{2\times 2}$ denotes zero matrix and $E_{ii} \in \mathbb{R}^{2\times 2}$ denotes the matrix whose entries are 0 except (i,i)-element being 1 for $i = 1, 2$.

Consider TEiCP$(\mathcal{D}, \mathcal{G})$, which is to find $(\lambda, z) \in \mathbb{R} \times \mathbb{R}^{2n}$ such that

$$\begin{cases} \omega = (\lambda\mathcal{D} - \mathcal{G})z^{m-1}, \\ \omega \geq 0, z \geq 0, \\ z^T\omega = 0, \\ \mathbf{1}_{2n}^T z = 1. \end{cases} \tag{3.4}$$

Let $z = (u_1, u_2, \ldots, u_n, v_1, \ldots, v_n)^T$. With the structures of $\mathcal{D}$ and $\mathcal{G}$, the entries of $\omega \in \mathbb{R}^{2n}$ are given by (3.4) as follows

$$\omega_i = \begin{cases} ((\lambda\mathcal{A} + \mathcal{B})u^{m-1} + \mathcal{C}v^{m-1})_i, & i \in J_n, \\ \lambda v_{i-n}^{m-1} - u_{i-n}^{m-1}, & i \in J_{2n} \setminus J_n. \end{cases} \tag{3.5}$$

Then we have

$$z^T\omega = (\lambda\mathcal{A} + \mathcal{B})u^m + u^T\mathcal{C}v^{m-1} + v^T(\lambda v^{[m-1]} - u^{[m-1]}). \tag{3.6}$$

Similarly, TEiCP$(\mathcal{D}, \mathcal{H})$ is to find $(\lambda, z) \in \mathbb{R} \times \mathbb{R}^{2n}$ such that

$$\begin{cases} \omega = (\lambda\mathcal{D} - \mathcal{H})z^{m-1}, \\ \omega \geq 0, z \geq 0, \\ z^T\omega = 0, \\ \mathbf{1}_{2n}^T z = 1. \end{cases} \tag{3.7}$$

Let $z = (u_1, u_2, \ldots, u_n, v_1, \ldots, v_n)^T$. With the structures of $\mathcal{D}$ and $\mathcal{H}$, the entries of $\omega \in \mathbb{R}^{2n}$ in (3.7) are given as follows

$$\omega_i = \begin{cases} ((\lambda\mathcal{A} - \mathcal{B})u^{m-1} + \mathcal{C}v^{m-1})_i, & i \in J_n, \\ \lambda v_{i-n}^{m-1} - u_{i-n}^{m-1}, & i \in J_{2n} \setminus J_n. \end{cases} \tag{3.8}$$

Then we have

$$z^T\omega = (\lambda\mathcal{A} - \mathcal{B})u^m + u^T\mathcal{C}v^{m-1} + v^T(\lambda v^{[m-1]} - u^{[m-1]}). \tag{3.9}$$

Next we establish a relationship between the solution sets of TQEiCP (1.3) and TEiCP($\mathcal{D}, \mathcal{G}$), TEiCP($\mathcal{D}, \mathcal{H}$), respectively.

Proposition 3.2. *Suppose that $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n$ is a solution of TQEiCP($\mathcal{A}, \mathcal{B}, \mathcal{C}$) given in* (1.3) *and $\mathcal{D}, \mathcal{G}, \mathcal{H} \in \mathbb{R}^{[m,2n]}$ are defined by* (3.1)*,* (3.2) *and* (3.3)*, respectively. Then we have the following statements:*

(i) *If $\lambda = 0$, then (λ, z), with $z = (0, x) \in \mathbb{R}^{2n}$, solves both TEiCP($\mathcal{D}, \mathcal{G}$) and TEiCP($\mathcal{D}, \mathcal{H}$).*

(ii) *If $\lambda > 0$, then (λ, z), with $z = (1 + \lambda^{\frac{1}{m-1}})^{-1}(\lambda^{\frac{1}{m-1}}x, x) \in \mathbb{R}^{2n}$, solves TEiCP($\mathcal{D}, \mathcal{G}$).*

(iii) *If $\lambda < 0$, then $(-\lambda, z)$, with $z = (1 + (-\lambda)^{\frac{1}{m-1}})^{-1}((-\lambda)^{\frac{1}{m-1}}x, x) \in \mathbb{R}^{2n}$, solves TEiCP($\mathcal{D}, \mathcal{H}$).*

Proof. For (i), we need to verify whether $(0, (0, x))$ satisfies both (3.4) and (3.7) or not. Combining (3.5), (3.6), (3.8) and (3.9), we only need to check whether the following formulation holds:

$$\mathcal{C}x^{m-1} \geq 0, \quad x \geq 0, \quad \mathcal{C}x^m = 0, \quad \mathbf{1}_n^T x = 1. \tag{3.10}$$

Since $(0, x)$ satisfies (1.3), it is obvious that (3.10) holds.

For (ii), we need to verify whether $(\lambda, z) \in \mathbb{R} \times \mathbb{R}^{2n}$ with $z = (1 + \lambda^{\frac{1}{m-1}})^{-1}(\lambda^{\frac{1}{m-1}}x, x)$ satisfies (3.4). Combining (3.5) and (3.6), it is equivalent to verifying whether the following system of inequalities holds:

$$\begin{cases} (1 + \lambda^{\frac{1}{m-1}})^{1-m}(\lambda^2\mathcal{A} + \lambda\mathcal{B} + \mathcal{C})x^{m-1} \geq 0, \\ (1 + \lambda^{\frac{1}{m-1}})^{1-m}(\lambda x_j^{m-1} - \lambda x_j^{m-1}) \geq 0, \\ \frac{\lambda^{\frac{1}{m-1}}}{(1+\lambda^{\frac{1}{m-1}})^m}(\lambda^2\mathcal{A} + \lambda\mathcal{B} + \mathcal{C})x^m = 0, \\ \mathbf{1}_n^T x = 1, \quad x \geq 0. \end{cases} \tag{3.11}$$

Since (λ, x) with $\lambda > 0$ satisfies (1.3), we immediately obtain (3.11).

For (iii), since (λ, x) with $\lambda < 0$ satisfies (1.3), we have $(-\lambda, x)$ being a solution of TQEiCP($\mathcal{A}, -\mathcal{B}, \mathcal{C}$). By (ii) and $-\lambda > 0$, we can replace λ with $-\lambda$ and B with $-B$ in (3.5) and (3.6), and then we immediately obtain (3.8) and (3.9). Hence, we get the desired result in (iii) by taking into account the definitions of z and $\mathcal{H}$. □

Proposition 3.3. *Let TQEiCP($\mathcal{A}, \mathcal{B}, \mathcal{C}$) be given in* (1.3) *and $\mathcal{D}, \mathcal{G}, \mathcal{H} \in \mathbb{R}^{[m,2n]}$ be defined by* (3.1)*,* (3.2) *and* (3.3)*, respectively. Then the following statements holds:*

(i) *If (λ, z) is a solution of TEiCP($\mathcal{D}, \mathcal{G}$) with $z = (y, x) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \neq 0$, then $\lambda > 0$ and $(\lambda, (1 + \lambda^{\frac{1}{m-1}})x)$ is a solution of TQEiCP($\mathcal{A}, \mathcal{B}, \mathcal{C}$).*

(ii) *If (λ, z) is a solution of TEiCP($\mathcal{D}, \mathcal{H}$) with $z = (y, x) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \neq 0$, then $\lambda > 0$ and $(-\lambda, (1 + \lambda^{\frac{1}{m-1}})x)$ is a solution of TQEiCP($\mathcal{A}, \mathcal{B}, \mathcal{C}$).*

Proof. Let $x_* = (1+\lambda^{\frac{1}{m-1}})x$. To prove (i), we need to confirm whether we can get

$$\begin{cases} x_* \geq 0, \quad (\lambda^2\mathcal{A}+\lambda\mathcal{B}+\mathcal{C})x_*^{m-1} \geq 0, \\ (\lambda^2\mathcal{A}+\lambda\mathcal{B}+\mathcal{C})x_*^m = 0, \\ \mathbf{1}_n^T x_* = 1. \end{cases} \tag{3.12}$$

Since (λ, z) satisfies (3.4), it follows from (3.5) that

$$\begin{cases} (\lambda\mathcal{A}+\mathcal{B})y^{m-1} + \mathcal{C}x^{m-1} \geq 0, \ \lambda x^{[m-1]} - y^{[m-1]} \geq 0, \ y \geq 0, \ x \geq 0, \\ y^T((\lambda\mathcal{A}+\mathcal{B})y^{m-1} + \mathcal{C}x^{m-1}) + x^T(\lambda x^{[m-1]} - y^{[m-1]}) = 0, \\ \mathbf{1}_n^T y + \mathbf{1}_n^T x = 1. \end{cases} \tag{3.13}$$

Since $\lambda x^{[m-1]} - y^{[m-1]} \geq 0$, $y \geq 0$, $x \geq 0$ and $\lambda \neq 0$, we have $\lambda > 0$. From the first two expressions in (3.13), we get

$$x^T(\lambda x^{[m-1]} - y^{[m-1]}) = 0.$$

Hence, $x_i(\lambda x_i^{m-1} - y_i^{m-1}) = 0$ for all $i \in J_n$. If $x_i > 0$, then $\lambda x_i^{m-1} - y_i^{m-1} = 0$ which implies that $y_i = \lambda^{\frac{1}{m-1}} x_i$. If $x_i = 0$, we have $-y_i \geq 0$ from $\lambda x_i^{m-1} - y_i^{m-1} \geq 0$. It follows from the fact $y \geq 0$ that $y_i = 0$. Hence, $y_i = \lambda^{\frac{1}{m-1}} x_i$ also holds in this case. Thus, we have $\lambda > 0$ and $y = \lambda^{\frac{1}{m-1}} x$. By (3.13) and the definition of x_*, (3.12) holds.

We next prove (ii). Since the difference between $\mathcal{G}$ and $\mathcal{H}$ is the sign of $\mathcal{B}$, it follows from (i) that $\lambda > 0$ and $(\lambda, (1+\lambda^{\frac{1}{m-1}})x)$ solves TQEiCP$(\mathcal{A}, -\mathcal{B}, \mathcal{C})$. Therefore, $(-\lambda, (1+\lambda^{\frac{1}{m-1}})x)$ is a solution of TQEiCP$(\mathcal{A}, \mathcal{B}, \mathcal{C})$. □

We have some remarks on Propositions 3.2 and 3.3.

Remark 3.4. Propositions 3.2 and 3.3 show the following results on complementarity eigenvalues:

(i) Each tensor quadratic complementarity eigenvalue for $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is either a complementarity eigenvalue for $(\mathcal{D}, \mathcal{G})$ or a complementarity eigenvalue for $(\mathcal{D}, \mathcal{H})$.

(ii) All nonzero complementarity eigenvalues for $(\mathcal{D}, \mathcal{G})$ are positive, and are tensor quadratic complementarity eigenvalues for $(\mathcal{A}, \mathcal{B}, \mathcal{C})$.

(iii) All nonzero complementarity eigenvalues for $(\mathcal{D}, \mathcal{H})$ are positive, and their additive inverse are tensor quadratic complementarity eigenvalues for $(\mathcal{A}, \mathcal{B}, \mathcal{C})$.

Remark 3.5. The conclusions of Propositions 3.2 and 3.3 can also be extended to TGHDEiCP (2.2). Consequently, we have the following conclusions for TGHDEiCP (2.2):

(i) Every (k, l) degree eigenvalue for $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is either an l degree eigenvalue for $(\mathcal{D}, \mathcal{G})$ or a complementarity eigenvalue for $(\mathcal{D}, \mathcal{H})$.

(ii) All nonzero l (l is odd and $k = 2l$) degree eigenvalues for $(\mathcal{D}, \mathcal{G})$ are positive, and are tensor quadratic complementarity eigenvalues for $(\mathcal{A}, \mathcal{B}, \mathcal{C})$.

(iii) All nonzero l (l is odd and $k = 2l$) degree eigenvalues for $(\mathcal{D}, \mathcal{H})$ are positive, and their additive inverse are tensor quadratic complementarity eigenvalues for $(\mathcal{A}, \mathcal{B}, \mathcal{C})$.

Clearly, Propositions 3.2 and 3.3 provide us a way to establish a sufficient condition for the existence of solutions of TQEiCP (1.3). It is equivalent to find a sufficient condition for the solvability of TEiCP($\mathcal{D},\mathcal{G}$) or TEiCP($\mathcal{D},\mathcal{H}$). We also need to impose some conditions to guarantee that 0 is a tensor quadratic complementarity eigenvalue of $(\mathcal{A},\mathcal{B},\mathcal{C})$ or that 0 is a complementarity eigenvalue of neither $(\mathcal{D},\mathcal{G})$ nor $(\mathcal{D},\mathcal{H})$. Thus, by Proposition 2.3 and Definition 2.2, we can obtain a sufficient condition for existence of solutions of TQEiCP (1.3). We first propose the following lemma.

Lemma 3.6. *Given $\mathcal{A},\mathcal{B},\mathcal{C}\in\mathbb{R}^{[m,n]}$ and let $\mathcal{D},\mathcal{G},\mathcal{H}\in\mathbb{R}^{[m,2n]}$ be defined as* (3.1), (3.2) *and* (3.3)*, respectively. Suppose that tensor $\mathcal{C}$ is not an S_0-tensor, then* 0 *is neither a tensor complementarity eigenvalue of $(\mathcal{D},\mathcal{G})$ nor that of $(\mathcal{D},\mathcal{H})$.*

Proof. Suppose by contradiction that 0 is a complementarity eigenvalue of TEiCP($\mathcal{D},\mathcal{G}$) and the corresponding complementarity eigenvector is $z=(u,v)\in R^n\times R^n$. Then, $(0,z)$ satisfies (3.4). Combining (3.4) and (3.5), we have

$$\begin{cases} \mathcal{B}u^{m-1}+\mathcal{C}v^{m-1}\geq 0, \\ u\geq 0, \quad v\geq 0, \quad -u^{[m-1]}\geq 0. \end{cases} \tag{3.14}$$

The second inequality in (3.14) yields $u=0$. Since $z=(u,v)\neq 0$, we have $v\neq 0$. Furthermore, the first inequality in (3.14) implies that there exists a vector $v\in\mathbb{R}^n\setminus\{0\}$ with $v\geq 0$ such that $\mathcal{C}v^{m-1}\geq 0$ holds. By Definition 2.2, $\mathcal{C}$ is an S_0-tensor, which contradicts with the hypothesis that $\mathcal{C}$ is not an S_0-tensor. Hence, 0 is not a complementarity eigenvalue of $(\mathcal{D},\mathcal{G})$. Similarly, we can easily prove that the conclusion also holds for $(\mathcal{D},\mathcal{H})$. □

Given TQEiCP($\mathcal{A},\mathcal{B},\mathcal{C}$), the following theorem shows that it is solvable under the assumptions that $\mathcal{A}$ is a strictly copositive tensor and $\mathcal{C}$ is not an S_0-tensor.

Theorem 3.7. *For any given tensors $\mathcal{A},\mathcal{B},\mathcal{C}\in\mathbb{R}^{[m,n]}$, assume that $\mathcal{A}$ is a strictly copositive tensor and $\mathcal{C}$ is not an S_0-tensor, then TQEiCP* (1.3) *has a solution. Moreover, $(\mathcal{A},\mathcal{B},\mathcal{C})$ admits at least one positive tensor quadratic complementarity eigenvalue and one negative tensor quadratic complementarity eigenvalue.*

Proof. By (3.1), we know that strict copositivity of $\mathcal{A}$ implies strict copositivity of $\mathcal{D}$. Hence, by Proposition 2.3, TEiCP($\mathcal{D},\mathcal{G}$) and TEiCP($\mathcal{D},\mathcal{H}$) are solvable. Since $\mathcal{C}$ is not an S_0-tensor, by Lemma 3.6, $(\mathcal{D},\mathcal{G})$ and $(\mathcal{D},\mathcal{H})$ have nonzero tensor complementarity eigenvalues. By Proposition 3.3, there exist at least one positive and one negative tensor quadratic complementarity eigenvalue for $(\mathcal{A},\mathcal{B},\mathcal{C})$. □

4 Method and numerical results

In this section, firstly, we also collect some definitions and results from [10, 13, 30, 32], which will be used in the sequel.

Let $G:\mathbb{R}^n\to\mathbb{R}^n$ be locally Lipschitzian. The B-subdifferential of G at x is defined as

$$\partial_B G(x)=\{H:\exists x_k, x_k\in D_G, lim_{x_k\to x}G'(x_k)=H\},$$

where D_G is the differentiable set of G. The Clarke subdifferential of G at x is defined as

$$\partial G(x)=co\partial_B G(x),$$

where co denotes the convex hull of a set. The related definitions of semismooth function and strongly semismooth function are also can be found in [30, 32]. We also know that a semismooth function G is BD-regular at x if all $H\in\partial_B G(x)$ are nonsingular.

From [10, 13, 30, 32], we can see that the semismooth Newton-type method is a class of effective methods for solving linear and nonlinear complementarity problems. Next, we will apply a semismooth Newton-type method to solve TQEiCP (1.3) and will give some preliminary numerical results. Now, we rewrite TQEiCP (1.3) as follows: Find $(\lambda, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ such that

$$y = (\lambda^2 \mathcal{A} + \lambda \mathcal{B} + \mathcal{C})x^{m-1} \geq 0, \quad x \geq 0, \quad x^T y = 0, \quad \mathbf{1}_n^T x = 1. \tag{4.1}$$

Define a function $\Phi : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ by

$$\Phi(z) = \Phi(x, y, \lambda) := \begin{pmatrix} \phi(x, y) \\ (\lambda^2 \mathcal{A} + \lambda \mathcal{B} + \mathcal{C})x^{m-1} - y \\ \mathbf{1}_n^T x - 1 \end{pmatrix}, \tag{4.2}$$

where $\phi(x, y) = (\varphi(x_1, y_1), \ldots, \varphi(x_n, y_n))^T$ with [16]

$$\varphi(x_i, y_i) = x_i + y_i - \sqrt{x_i^2 + y_i^2} \quad \forall i \in J_n.$$

It is obvious that $\Phi(z) = 0$ at a point $z = (x, y, \lambda)$ if and only if z is a solution of (4.1). Since the function Φ is semismooth for any $(x, y, \lambda) \in \mathbb{R}^{2n+1}$, we can apply semismooth Newton-type methods to solve the system of semismooth equations $\Phi(z) = 0$.

In order to introduce the framework of a semismooth Newton-type method, we also need some existing results on tensor function. For any tensor $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ and a vector $x \in \mathbb{R}^n$, by [31, Lemma 2.1], there is the unique semi-symmetric tensor $\hat{\mathcal{A}} \in \mathbb{R}^{[m,n]}$ such that $\hat{\mathcal{A}} x^{m-1} = \mathcal{A} x^{m-1}$ for all $x \in R^n$. Hence, we always assume that $\mathcal{A} \in T_{m,n}$ is semi-symmetric. By [31, Lemma 3.3], the Jacobian of $\mathcal{A}x^{m-1}$ at x is given by

$$(m-1)\mathcal{A}x^{m-2}. \tag{4.3}$$

Define a merit function for (4.2)

$$\Psi(z) = \frac{1}{2}\|\Phi(z)\|^2.$$

Clearly, Ψ is continuously differential for any $z \in \mathbb{R}^{2n+1}$. From (4.3), the gradient of Ψ at z is given by

$$\nabla \Psi(z) = H^T \Phi(z),$$

where

$$H \in \partial \Phi(z) = \begin{pmatrix} D_a & D_b & 0 \\ (m-1)(\lambda^2 \mathcal{A} + \lambda \mathcal{B} + \mathcal{C})x^{m-2} & -I & (2\lambda \mathcal{A} + \mathcal{B})x^{m-1} \\ \mathbf{1}_n^T & 0 & 0 \end{pmatrix},$$

and

$$D_a := diag\{a_1, \ldots, a_n\}, \quad D_b := diag\{b_1, \ldots, b_n\},$$

with

$$(a_i, b_i) := \begin{cases} (1 - \frac{x_i}{\sqrt{x_i^2 + y_i^2}}, 1 - \frac{y_i}{\sqrt{x_i^2 + y_i^2}}), \ if (x_i, y_i) \neq (0, 0), \\ (1 - \zeta, 1 - \varsigma), \ if (x_i, y_i) = (0, 0), \end{cases}$$

where (ζ, ς) satisfies $\|(\zeta, \varsigma)\| \leq 1$, for any $i \in J_n$.

Now, we give the following method.

Algorithm 4.1 (A semismooth Newton-type method)

Step 0 Choose $\rho > 0$, $\beta \in (0,1)$, $\sigma \in (0,1/2)$, $p > 2$ and $\epsilon \geq 0$. Let $z^0 \in \mathbb{R}^{2n+1}$ be an arbitrary vector. Set $k := 0$.

Step 1 If $\|\Phi(z^k)\| \leq \epsilon$, stop.

Step 2 Choose an element $H_k \in \partial\Phi(z^k)$ and compute $\Delta z^k = (\Delta x^k, \Delta y^k, \Delta\lambda^k) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ by

$$H_k \Delta z^k = -\Phi(z^k). \tag{4.4}$$

If the system (4.4) has no solution or the following condition

$$\nabla\Psi(z^k)^T \Delta z^k \leq -\rho\|\Delta z^k\|^p$$

is not satisfied, set $\Delta z^k := -\nabla\Psi(z^k)$.

Step 3 Let α_k be the maximum of the values $\{1, \beta, \beta^2, \dots\}$ such that

$$\Psi(z^k + \alpha_k \Delta z^k) \leq \Psi(z^k) + \sigma\alpha_k \nabla\Psi(z^k)^T \Delta z^k.$$

Step 4 Set $z^{k+1} := z^k + \alpha_k \Delta z^k$ and $k := k+1$. Go to Step 1.

The above algorithmic framework was proposed in [30]. We also have the following convergence theorem from [30, Theorem 11].

Theorem 4.1. *Let sequence $\{z^k\}$ be generated by Algorithm 4.1. If Φ defined as (4.2) is BD-regular, then any accumulation point of $\{z^k\}$ is a stationary point of Ψ and hence a solution of TQEiCP$(\mathcal{A}, \mathcal{B}, \mathcal{C})$.*

In the following, we give some preliminary numerical results of Algorithm 4.1. From these numerical results, we can see that Algorithm 4.1 is effective for solving TQEiCPs. By referring to some examples in [29], we construct some numerical examples for TQEiCP. Throughout our experiments, the parameters used in Algorithm 4.1 are chosen as $\epsilon = 10^{-6}$, $\rho = 0.1$, $\beta = 0.2$, $\sigma = 0.4$ and $p = 2.1$. We also set a maximum iteration step for the algorithm, i.e., Nmax= 500. In our numerical experiments, all codes are run in Matlab Version R2014a and Tensor Toolbox on a laptop with an Intel(R) Core(TM) i5-2520M CPU(2.50GHz) and RAM of 4.00GB.

Example 4.2. Consider TQEiCP (1.3) with symmetric tensors $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{R}^{[4,2]}$ and $\mathcal{C} = -\mathcal{A}$, where $\mathcal{A}, \mathcal{B}$ are given by

$$\mathcal{A}(:,:,1,1) = \begin{pmatrix} 1.6324 & 1.1880 \\ 1.1880 & 1.5469 \end{pmatrix}, \quad \mathcal{A}(:,:,1,2) = \begin{pmatrix} 1.1880 & 1.5469 \\ 1.5469 & 1.9340 \end{pmatrix},$$

$$\mathcal{A}(:,:,2,1) = \begin{pmatrix} 1.1880 & 1.5469 \\ 1.5469 & 1.9340 \end{pmatrix}, \quad \mathcal{A}(:,:,2,2) = \begin{pmatrix} 1.5469 & 1.9340 \\ 1.9340 & 1.0318 \end{pmatrix},$$

$$\mathcal{B}(:,:,1,1) = \begin{pmatrix} 0.8147 & 0.5164 \\ 0.5164 & 0.9134 \end{pmatrix}, \quad \mathcal{B}(:,:,1,2) = \begin{pmatrix} 0.5164 & 0.9134 \\ 0.9134 & 0.9595 \end{pmatrix},$$

$$\mathcal{B}(:,:,2,1) = \begin{pmatrix} 0.5164 & 0.9134 \\ 0.9134 & 0.9595 \end{pmatrix}, \quad \mathcal{B}(:,:,2,2) = \begin{pmatrix} 0.9134 & 0.9595 \\ 0.9595 & 0.3922 \end{pmatrix}.$$

We use Algorithm 4.1 to solve the corresponding TQEiCP (1.3) with random initial point (λ^0, x^0) uniformly distributed in $(0,1)$ and $y^0 = ((\lambda^0)^2\mathcal{A} + \lambda^0\mathcal{B} + \mathcal{C})(x^0)^{m-1}$. In order to get all possible solutions, 50 random initial points are used. The numerical results are reported in Table 1, where **No** denotes number of each solution detected by the method within 50

Table 1: The numerical results of Example 4.2

No	**Eigvalue**	**Eigvector**	**Niter**	**Time**(sec.)
28	0.8278	$(0.0000, 1.0000)^T$	12.3	0.3559
10	0.7851	$(0.9911, 0.1330)^T$	9.1	0.2229
7	0.7750	$(0.7045, 0.7097)^T$	6.8	0.1767
1	-1.2802	$(1.0000, 0.0000)^T$	42	1.3185
4	failure			

random initial points. **Eigvalue** denotes the tensor quadratic eigenvalue, **Eigvector** denotes the corresponding eigenvector, **Niter** denotes the average number of iterations, and **Time** denotes the average elapsed CPU time in seconds.

From Table 1, we can see that Algorithm 4.1 is able to detect three positive tensor quadratic complementarity eigenvalues, and one negative tensor quadratic complementarity eigenvalue. We also find that Algorithm 4.1 has 8% points which can not detect the tensor quadratic complementarity eigenvalues for $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in Example 4.2.

Example 4.3. Consider TQEiCP (1.3) with symmetric tensors $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{R}^{[4,2]}$, where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are given by

$$\mathcal{A}(:,:,1,1) = \begin{pmatrix} 0.0109 & 0.4967 \\ 0.4967 & 0.5465 \end{pmatrix}, \quad \mathcal{A}(:,:,1,2) = \begin{pmatrix} 0.4967 & 0.5465 \\ 0.5465 & 0.4407 \end{pmatrix},$$

$$\mathcal{A}(:,:,2,1) = \begin{pmatrix} 0.4967 & 0.5465 \\ 0.5465 & 0.4407 \end{pmatrix}, \quad \mathcal{A}(:,:,2,2) = \begin{pmatrix} 0.5465 & 0.4407 \\ 0.4407 & 0.4074 \end{pmatrix},$$

$$\mathcal{B}(:,:,1,1) = \begin{pmatrix} 0.4873 & 0.6825 \\ 0.6825 & 0.2822 \end{pmatrix}, \quad \mathcal{B}(:,:,1,2) = \begin{pmatrix} 0.6825 & 0.2822 \\ 0.2822 & 0.3345 \end{pmatrix},$$

$$\mathcal{B}(:,:,2,1) = \begin{pmatrix} 0.6825 & 0.2822 \\ 0.2822 & 0.3345 \end{pmatrix}, \quad \mathcal{B}(:,:,2,2) = \begin{pmatrix} 0.2822 & 0.3345 \\ 0.3345 & 0.1159 \end{pmatrix},$$

$$\mathcal{C}(:,:,1,1) = -\begin{pmatrix} 0.8147 & 0.6557 \\ 0.6557 & 0.5615 \end{pmatrix}, \quad \mathcal{C}(:,:,1,2) = -\begin{pmatrix} 0.6557 & 0.5615 \\ 0.5615 & 0.7008 \end{pmatrix},$$

$$\mathcal{C}(:,:,2,1) = -\begin{pmatrix} 0.6557 & 0.5615 \\ 0.5615 & 0.7008 \end{pmatrix}, \quad \mathcal{C}(:,:,2,2) = -\begin{pmatrix} 0.5615 & 0.7008 \\ 0.7008 & 0.1419 \end{pmatrix}.$$

We use Algorithm 4.1 to solve the corresponding TQEiCP (1.3) with 50 random initial points (λ^0, x^0) uniformly distributed in $(0,1)$ and $y^0 = ((\lambda^0)^2\mathcal{A} + \lambda^0\mathcal{B} + \mathcal{C})(x^0)^{m-1}$. The last equality $\mathbf{1}_n^T x = 1$ in (1.3) is used to ensure $x \neq 0$. We find that this condition is ill-condition for (4.4). Here, we use $x^T x = 1$ to replace $\mathbf{1}_n^T x = 1$ to ensure $x \neq 0$. The numerical results with different initial points are reported in Table 2.

From Table 2, we can see that Algorithm 4.1 is able to detect three positive tensor quadratic complementarity eigenvalues. We also find that Algorithm 4.1 has 24% points which can not detect the tensor quadratic complementarity eigenvalues for $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in Example 4.3.

Example 4.4. Consider TQEiCP (1.3) with symmetric tensors $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{R}^{[6,2]}$, where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are given by

$$\mathcal{A}(:,:,1,1,1,1) = \begin{pmatrix} 0.1518 & 0.4321 \\ 0.4321 & 0.4093 \end{pmatrix}, \quad \mathcal{A}(:,:,1,1,1,2) = \begin{pmatrix} 0.4321 & 0.4093 \\ 0.4093 & 0.3593 \end{pmatrix},$$

Table 2: The numerical results of Example 4.3

No	**Eigvalue**	**Eigvector**	**Niter**	**Time**(sec.)
19	0.7994	$(0.5039, 0.8637)^T$	10.7	0.3869
15	0.7933	$(0.7636, 0.6457)^T$	11.2	0.4341
4	1.6140	$(1.0000, 0.0000)^T$	10	0.3728
12	failure			

$$\mathcal{A}(:,:,1,1,2,1) = \begin{pmatrix} 0.4321 & 0.4093 \\ 0.4093 & 0.3593 \end{pmatrix}, \quad \mathcal{A}(:,:,1,1,2,2) = \begin{pmatrix} 0.4093 & 0.3593 \\ 0.3593 & 0.4671 \end{pmatrix},$$

$$\mathcal{A}(:,:,1,2,1,1) = \begin{pmatrix} 0.4321 & 0.4093 \\ 0.4093 & 0.3593 \end{pmatrix}, \quad \mathcal{A}(:,:,1,2,1,2) = \begin{pmatrix} 0.4093 & 0.3593 \\ 0.3593 & 0.4671 \end{pmatrix},$$

$$\mathcal{A}(:,:,1,2,2,1) = \begin{pmatrix} 0.4093 & 0.3593 \\ 0.3593 & 0.4671 \end{pmatrix}, \quad \mathcal{A}(:,:,1,2,2,2) = \begin{pmatrix} 0.3593 & 0.4671 \\ 0.4671 & 0.2735 \end{pmatrix},$$

$$\mathcal{A}(:,:,2,1,1,1) = \begin{pmatrix} 0.3593 & 0.4671 \\ 0.4671 & 0.2735 \end{pmatrix}, \quad \mathcal{A}(:,:,2,1,1,2) = \begin{pmatrix} 0.4093 & 0.3593 \\ 0.3593 & 0.4671 \end{pmatrix},$$

$$\mathcal{A}(:,:,2,1,2,1) = \begin{pmatrix} 0.4093 & 0.3593 \\ 0.3593 & 0.4671 \end{pmatrix}, \quad \mathcal{A}(:,:,2,1,2,2) = \begin{pmatrix} 0.3593 & 0.4671 \\ 0.4671 & 0.2735 \end{pmatrix},$$

$$\mathcal{A}(:,:,2,2,1,1) = \begin{pmatrix} 0.4093 & 0.3593 \\ 0.3593 & 0.4671 \end{pmatrix}, \quad \mathcal{A}(:,:,2,2,1,2) = \begin{pmatrix} 0.3593 & 0.4671 \\ 0.4671 & 0.2735 \end{pmatrix},$$

$$\mathcal{A}(:,:,2,2,2,1) = \begin{pmatrix} 0.3593 & 0.4671 \\ 0.4671 & 0.2735 \end{pmatrix}, \quad \mathcal{A}(:,:,2,2,2,2) = \begin{pmatrix} 0.4671 & 0.2735 \\ 0.2735 & 0.9138 \end{pmatrix},$$

$$\mathcal{B}(:,:,1,1,1,1) = \begin{pmatrix} 0.0366 & 0.7056 \\ 0.7056 & 0.5357 \end{pmatrix}, \quad \mathcal{B}(:,:,1,1,1,2) = \begin{pmatrix} 0.7056 & 0.5357 \\ 0.5357 & 0.5043 \end{pmatrix},$$

$$\mathcal{B}(:,:,1,1,2,1) = \begin{pmatrix} 0.4321 & 0.4093 \\ 0.4093 & 0.3593 \end{pmatrix}, \quad \mathcal{B}(:,:,1,1,2,2) = \begin{pmatrix} 0.5357 & 0.5043 \\ 0.5043 & 0.5231 \end{pmatrix},$$

$$\mathcal{B}(:,:,1,2,1,1) = \begin{pmatrix} 0.4321 & 0.4093 \\ 0.4093 & 0.3593 \end{pmatrix}, \quad \mathcal{B}(:,:,1,2,1,2) = \begin{pmatrix} 0.5357 & 0.5043 \\ 0.5043 & 0.5231 \end{pmatrix},$$

$$\mathcal{B}(:,:,1,2,2,1) = \begin{pmatrix} 0.5357 & 0.5043 \\ 0.5043 & 0.5231 \end{pmatrix}, \quad \mathcal{B}(:,:,1,2,2,2) = \begin{pmatrix} 0.5043 & 0.5231 \\ 0.5231 & 0.3836 \end{pmatrix},$$

$$\mathcal{B}(:,:,2,1,1,1) = \begin{pmatrix} 0.4321 & 0.4093 \\ 0.4093 & 0.3593 \end{pmatrix}, \quad \mathcal{B}(:,:,2,1,1,2) = \begin{pmatrix} 0.5357 & 0.5043 \\ 0.5043 & 0.5231 \end{pmatrix},$$

$$\mathcal{B}(:,:,2,1,2,1) = \begin{pmatrix} 0.5357 & 0.5043 \\ 0.5043 & 0.5231 \end{pmatrix}, \quad \mathcal{B}(:,:,2,1,2,2) = \begin{pmatrix} 0.5043 & 0.5231 \\ 0.5231 & 0.3836 \end{pmatrix},$$

$$\mathcal{B}(:,:,2,2,1,1) = \begin{pmatrix} 0.5357 & 0.5043 \\ 0.5043 & 0.5231 \end{pmatrix}, \quad \mathcal{B}(:,:,2,2,1,2) = \begin{pmatrix} 0.5043 & 0.5231 \\ 0.5231 & 0.3836 \end{pmatrix},$$

$$\mathcal{B}(:,:,2,2,2,1) = \begin{pmatrix} 0.5043 & 0.5231 \\ 0.5231 & 0.3836 \end{pmatrix}, \quad \mathcal{B}(:,:,2,2,2,2) = \begin{pmatrix} 0.5231 & 0.3836 \\ 0.3836 & 0.0875 \end{pmatrix},$$

$$\mathcal{C}(:,:,1,1,1,1) = -\begin{pmatrix} 0.6401 & 0.3181 \\ 0.3181 & 0.5562 \end{pmatrix}, \quad \mathcal{C}(:,:,1,1,1,2) = -\begin{pmatrix} 0.3181 & 0.5562 \\ 0.5562 & 0.5425 \end{pmatrix},$$

$$\mathcal{C}(:,:,1,1,2,1) = -\begin{pmatrix} 0.3181 & 0.5562 \\ 0.5562 & 0.5425 \end{pmatrix}, \quad \mathcal{C}(:,:,1,1,2,2) = -\begin{pmatrix} 0.5562 & 0.5425 \\ 0.5425 & 0.5029 \end{pmatrix},$$

$$\mathcal{C}(:,:,1,2,1,1) = -\begin{pmatrix} 0.3181 & 0.5562 \\ 0.5562 & 0.5425 \end{pmatrix}, \quad \mathcal{C}(:,:,1,2,1,2) = -\begin{pmatrix} 0.5562 & 0.5425 \\ 0.5425 & 0.5029 \end{pmatrix},$$

$$\mathcal{C}(:,:,1,2,2,1) = -\begin{pmatrix} 0.5562 & 0.5425 \\ 0.5425 & 0.5029 \end{pmatrix}, \quad \mathcal{C}(:,:,1,2,2,2) = -\begin{pmatrix} 0.5425 & 0.5029 \\ 0.5029 & 0.6992 \end{pmatrix},$$

$$\mathcal{C}(:,:,2,1,1,1) = -\begin{pmatrix} 0.3181 & 0.5562 \\ 0.5562 & 0.5425 \end{pmatrix}, \quad \mathcal{C}(:,:,2,1,1,2) = -\begin{pmatrix} 0.5562 & 0.5425 \\ 0.5425 & 0.5029 \end{pmatrix},$$

$$\mathcal{C}(:,:,2,1,2,1) = -\begin{pmatrix} 0.5562 & 0.5425 \\ 0.5425 & 0.5029 \end{pmatrix}, \quad \mathcal{C}(:,:,2,1,2,2) = -\begin{pmatrix} 0.5425 & 0.5029 \\ 0.5029 & 0.6992 \end{pmatrix},$$
$$\mathcal{C}(:,:,2,2,1,1) = -\begin{pmatrix} 0.5562 & 0.5425 \\ 0.5425 & 0.5029 \end{pmatrix}, \quad \mathcal{C}(:,:,2,2,1,2) = -\begin{pmatrix} 0.5425 & 0.5029 \\ 0.5029 & 0.6992 \end{pmatrix},$$
$$\mathcal{C}(:,:,2,2,2,1) = -\begin{pmatrix} 0.5425 & 0.5029 \\ 0.5029 & 0.6992 \end{pmatrix}, \quad \mathcal{C}(:,:,2,2,2,2) = -\begin{pmatrix} 0.5029 & 0.6992 \\ 0.6992 & 0.5118 \end{pmatrix}.$$

We use Algorithm 4.1 to solve the corresponding TQEiCP (1.3) with 50 random initial points (λ^0, x^0) uniformly distributed in $(0,1)$ and $y^0 = ((\lambda^0)^2\mathcal{A}+\lambda^0\mathcal{B}+\mathcal{C})(x^0)^{m-1}$. Here, we also use $x^Tx = 1$ to replace $\mathbf{1}_n^T x = 1$ to ensure $x \neq 0$. The numerical results with different initial points are reported in Table 3.

Table 3: The numerical results of Example 4.4

No	**Eigvalue**	**Eigvector**	**Niter**	**Time**(sec.)
19	0.7919	$(0.1795, 0.9838)^T$	15.7	0.7150
19	0.6412	$(0.9074, 0.4202)^T$	19.3	0.8152
12	failure			

From Table 3, we can see that Algorithm 4.1 is able to detect two positive tensor quadratic complementarity eigenvalues. We also find that Algorithm 4.1 has 24% points which can not detect the tensor quadratic complementarity eigenvalues for $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in Example 4.4.

Example 4.5. Consider TQEiCP (1.3) with symmetric tensors $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{R}^{[4,3]}$, where $\mathcal{C} = -\mathcal{A}$ and $\mathcal{A}, \mathcal{B}$ are given by

$$\mathcal{A}(:,:,1,1) = \begin{pmatrix} 0.6954 & 0.4018 & 0.1406 \\ 0.4018 & 0.9957 & 0.0483 \\ 0.1406 & 0.0483 & 0.0988 \end{pmatrix}, \quad \mathcal{A}(:,:,1,2) = \begin{pmatrix} 0.6730 & 0.5351 & 0.4473 \\ 0.5351 & 0.2853 & 0.3071 \\ 0.4473 & 0.3071 & 0.9665 \end{pmatrix},$$
$$\mathcal{A}(:,:,1,3) = \begin{pmatrix} 0.7585 & 0.6433 & 0.2306 \\ 0.6433 & 0.8986 & 0.3427 \\ 0.2306 & 0.3427 & 0.5390 \end{pmatrix}, \quad \mathcal{A}(:,:,2,2) = \begin{pmatrix} 0.3608 & 0.3941 & 0.5230 \\ 0.3941 & 0.6822 & 0.5516 \\ 0.5230 & 0.5516 & 0.7091 \end{pmatrix},$$
$$\mathcal{A}(:,:,2,3) = \begin{pmatrix} 0.4632 & 0.2043 & 0.2823 \\ 0.2043 & 0.7282 & 0.7400 \\ 0.2823 & 0.7400 & 0.9369 \end{pmatrix}, \quad \mathcal{A}(:,:,3,3) = \begin{pmatrix} 0.8200 & 0.5914 & 0.4983 \\ 0.5914 & 0.0762 & 0.2854 \\ 0.4983 & 0.2854 & 0.1266 \end{pmatrix},$$
$$\mathcal{B}(:,:,1,1) = \begin{pmatrix} 0.6229 & 0.2644 & 0.3567 \\ 0.2644 & 0.0475 & 0.7367 \\ 0.3567 & 0.7367 & 0.1259 \end{pmatrix}, \quad \mathcal{B}(:,:,1,2) = \begin{pmatrix} 0.7563 & 0.5878 & 0.5406 \\ 0.5878 & 0.1379 & 0.0715 \\ 0.5406 & 0.0715 & 0.3725 \end{pmatrix},$$
$$\mathcal{B}(:,:,1,3) = \begin{pmatrix} 0.0657 & 0.4918 & 0.9312 \\ 0.4918 & 0.7788 & 0.9045 \\ 0.9312 & 0.9045 & 0.8711 \end{pmatrix}, \quad \mathcal{B}(:,:,2,2) = \begin{pmatrix} 0.7689 & 0.3941 & 0.6034 \\ 0.3941 & 0.3577 & 0.3465 \\ 0.6034 & 0.3465 & 0.4516 \end{pmatrix},$$
$$\mathcal{B}(:,:,2,3) = \begin{pmatrix} 0.8077 & 0.4910 & 0.2953 \\ 0.4910 & 0.5054 & 0.5556 \\ 0.2953 & 0.5556 & 0.9608 \end{pmatrix}, \quad \mathcal{B}(:,:,3,3) = \begin{pmatrix} 0.7581 & 0.7205 & 0.9044 \\ 0.7205 & 0.0782 & 0.7240 \\ 0.9044 & 0.7240 & 0.3492 \end{pmatrix}$$

We use Algorithm 4.1 to solve the corresponding TQEiCP$(\mathcal{A}, \mathcal{B}, \mathcal{C})$ (1.3) with 50 random initial points (λ^0, x^0) uniformly distributed in $(0,1)$ and $y^0 = ((\lambda^0)^2\mathcal{A} + \lambda^0\mathcal{B} + \mathcal{C})(x^0)^{m-1}$. The numerical results with different initial points are reported in Table 4.

From Table 4, we can see that Algorithm 4.1 is able to detect three positive tensor quadratic complementarity eigenvalues and one negative tensor quadratic complementarity eigenvalue. We also find that Algorithm 4.1 has 56% points which can not detect the tensor quadratic complementarity eigenvalues for $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in Example 4.5.

Table 4: The numerical results of Example 4.5

No	**Eigvalue**	**Eigvector**	**Niter**	**Time**(sec.)
12	0.4898	$(0.1769, 0.1262, 0.6969)^T$	27.6	0.8023
4	0.7716	$(0.0000, 1.0000, 0.0000)^T$	6.8	0.2303
5	0.5267	$(0.0000, 0.1998, 0.8002)^T$	11.6	0.3481
1	-2.0418	$(0.1767, 0.1262, 0.6971)^T$	38	1.2279
28	failure			

Example 4.6. Consider TQEiCP (1.3) with symmetric tensors $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{R}^{[9,4]}$, where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are randomly generated by using the following commands

$$\text{X=tenrand}([4,4,4,4,4,4,4,4,4]);\text{A=symmetrize}(X)$$
$$\text{Y=tenrand}([4,4,4,4,4,4,4,4,4]);\text{B=symmetrize}(Y)$$
$$\text{Z=tenrand}([4,4,4,4,4,4,4,4,4]);\text{C=symmetrize}(Z)$$

We use Algorithm 4.1 to solve the corresponding TQEiCP (1.3) with 50 random initial points (λ^0, x^0) uniformly distributed in $(0,1)$ and $y^0 = ((\lambda^0)^2\mathcal{A} + \lambda^0\mathcal{B} + \mathcal{C})(x^0)^{m-1}$. Here, we use $x^Tx = 1$ to replace $\mathbf{1}_n^T x = 1$ to ensure $x \neq 0$. The numerical results with different initial points are reported in Table 5. From Table 5, we can see that Algorithm 4.1 is able to detect one positive tensor quadratic complementarity eigenvalue.

Table 5: The numerical results of Example 4.6

No	**Eigvalue**	**Eigvector**	**Niter**	**Time**(sec.)
5	0.6181	$(0.5296, 0.1064, 0.5627, 0.6257)^T$	35.6	18.0563
45	failure			

Example 4.7. Consider TQEiCP (1.3) with symmetric tensors $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{R}^{[4,10]}$, where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are randomly generated by using the following commands

$$\text{X=tenrand}([10,10,10,10]);\text{A=symmetrize}(X)$$
$$\text{Y=tenrand}([10,10,10,10]);\text{B=symmetrize}(Y)$$
$$\text{Z=tenrand}([10,10,10,10]);\text{C=symmetrize}(Z)$$

We use Algorithm 4.1 to solve the corresponding TQEiCP (1.3) with 50 random initial points (λ^0, x^0) uniformly distributed in $(0,1)$ and $y^0 = ((\lambda^0)^2\mathcal{A} + \lambda^0\mathcal{B} + \mathcal{C})(x^0)^{m-1}$. Here, we use $x^Tx = 1$ to replace $\mathbf{1}_n^T x = 1$ to ensure $x \neq 0$. The numerical results with different initial points are reported in Table 6.

From Table 6, we can see that Algorithm 4.1 is able to detect seven approximate positive tensor quadratic complementarity eigenvalue. We also find that Algorithm 4.1 has 86% points which can not detect the tensor quadratic complementarity eigenvalues for $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in Example 4.7.

Table 6: The numerical results of Example 4.7

No	Eigvalue	Eigvector	Niter	Time(sec.)
1	0.6516	$(0.1877, 0.0212, 0.0000, 0.1956, 0.0000, 0.7778, 0.0000, 0.1432, 0.0000, 0.5482)^T$	15	1.4591
1	0.6302	$(0.6527, 0.5674, 0.0000, 0.1135, 0.0000, 0.0869, 0.4812, 0.0000, 0.0000, 0.0078)^T$	18	1.7809
1	0.6334	$(0.4921, 0.0157, 0.0000, 0.0000, 0.0000, 0.5855, 0.0000, 0.1883, 0.4926, 0.3698)^T$	9	0.8143
1	0.6292	$(0.7503, 0.3593, 0.0000, 0.0837, 0.0000, 0.1520, 0.5271, 0.0000, 0.0000, 0.0000)^T$	22	2.2438
1	0.6153	$(0.5456, 0.2394, 0.0000, 0.0590, 0.5186, 0.0000, 0.6104, 0.0000, 0.0000, 0.0000)^T$	8	0.6169
1	0.6880	$(0.0000, 0.0000, 0.0569, 0.9756, 0.0000, 0.2048, 0.0000, 0.0550, 0.0000, 0.0000)^T$	183	20.0649
1	0.6227	$(0.0000, 0.0000, 0.0000, 0.7463, 0.2747, 0.1558, 0.0000, 0.2529, 0.5258, 0.0218)^T$	9	0.6709
43	failure			

$\boxed{5}$ Conclusions

In this paper, we proposed a class of TQEiCPs, which is an interesting extension of a class of QEiCPs on tensors. We showed that this problem can be reduced to two corresponding TEiCPs. Based on this result, we established a sufficient condition for the existence of solutions of this problem. For any given TQEiCP($\mathcal{A}, \mathcal{B}, \mathcal{C}$), if $\mathcal{A}$ is a strictly copositive tensor and $\mathcal{C}$ is not an S_0-tensor, then TQEiCP($\mathcal{A}, \mathcal{B}, \mathcal{C}$) is soluble. Finally, we reformulated TQEiCP($\mathcal{A}, \mathcal{B}, \mathcal{C}$) as an equivalent seimsmooth equation and applied a semismooth Newton-type method to solve it. Some preliminary numerical results showed the efficiency of the proposed method.

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