Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 17, No. 2, pp. 289–305, 2021

NONEMPTINESS AND BOUNDEDNESS OF SOLUTION SETS FOR PERTURBED GENERALIZED VECTOR VARIATIONAL INEQUALITIES*

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Abstract: In this paper we study the nonemptiness and boundedness of the solution set of a perturbed generalized vector variational inequality. We obtain an existence result when the mapping involved is perturbed by a nonlinear mapping. When the mapping is perturbed along a direction, we further establish the boundedness of the solution set. We also discuss the nonemptiness and boundedness of the solution set when the constraint set is perturbed by a unit ball. Finally, we investigate the nonemptiness and boundedness of the solution set when the mapping and the constraint set are perturbed simultaneously. Our results extend the existing results for the scalar perturbed generalized variational inequality to the vector case.

Key words: *perturbed generalized vector variational inequalities, nonemptiness and boundedness of the solution set; coercivity conditions*

Mathematics Subject Classification: *49J40; 47H10*

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1 Introduction

Let X be a real reflexive Banach space with the dual space X^* and Y be a real Banach space with the dual space Y^*. And let $\|x\|_X$ (resp. $\|y\|_Y$) be the norm of $x \in X$ (resp. $y \in Y$). The symbol $\to$ (resp. $\rightharpoonup$) stands for the strong convergence (resp. weak convergence). Suppose that $K \subset X$ is a nonempty closed convex set and $C \subset Y$ is a closed convex and pointed cone with $\text{int } C \neq \emptyset$, where $\text{int } C$ denotes the topological interior of C. Let $F : K \to 2^{L(X,Y)}$ be a set-valued mapping where $L(X, Y)$ is the space of all continuous linear mappings from X into Y. We consider the following generalized vector variational inequality associated with (F, K):

$GVVI(F, K)$ find $x \in K$, $\exists \xi \in F(x)$ such that

$$\langle \xi, y - x\rangle \notin -\text{ int } C, \quad \forall y \in K.$$

When $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, $GVVI(F, K)$ reduces to the following generalized variational inequality:

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*This work was partially supported by National Natural Science Foundation of China (No. 11471230) and Applied Basic Research Project of Sichuan Province (No.2018JY0169).

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$GVI(F,K)$ find $x \in K$, $\exists \xi \in F(x)$ such that

$$\langle \xi, y - x \rangle \geq 0, \quad \forall y \in K.$$

When F is a single-valued mapping, $GVVI(F,K)$ reduces to the following vector variational inequality introduced in [12]:

$VVI(F,K)$ find $x \in K$ such that

$$\langle F(x), y - x \rangle \notin -\text{ int } C, \quad \forall y \in K.$$

Denoted by $SOL_{GVVI}(F,K)$ the solution set of $GVVI(F,K)$. There are a large amount of papers investigating vector variational inequalities, its variants and applications in the literature. For details, we refer the reader to [3, 29, 14, 15, 10, 34, 35, 2, 16].

Perturbation analysis is one of important and interesting topics in variational inequalities and related problems. Fang, et al. [11] studied the well-posedness by perturbations of mixed variational inequalities in Banach spaces. Li and He [20] investigated the solvability of a perturbed generalized variational inequality in a finite dimensional space without assuming any kind of monotonicity. When the mapping is perturbed by a nonlinear mapping (resp. a direction in the interior of the barrier cone of the constraint set), they proved that the scalar perturbed generalized variational inequality with a coercivity assumption has a solution. Tang and Li [30] extended the results of Li and He [20] to the scalar perturbed mixed generalized variational inequality. Wang [32] improved the results of Li and He [20] by proving that the solution set of the scalar perturbed generalized variational inequality in a reflexive Banach space is nonempty and bounded. Recently, Luo [23] further investigated the nonemptiness and boundedness of the solution set of the scalar perturbed generalized variational inequality with the mapping and the constraint set being perturbed simultaneously in a reflexive Banach space. For more results on perturbation analysis of variational inequalities, we refer the reader to [28, 31, 13, 17, 21, 36]. Motivated by the works in [20, 23, 32, 30], in this paper, we attempt to discuss the perturbation analysis of the generalized vector variational problem in Banach spaces.

The rest of this paper is organized as follows: In Section 2, we present some notations and preliminary results. In Section 3, we investigate nonemptiness and boundedness of the solution set of $GVVI(F,K)$ when the mapping F is perturbed. In Section 4, we further investigate nonemptiness and boundedness of the solution set of $GVVI(F,K)$ when the mapping F and the constraint set K are perturbed simultaneously.

2 Preliminaries

In this section, we recall some concepts and results that are used in this paper. Denoted by $\bar{\mathbf{B}}_r$ the closed ball centered at zero with radius $r > 0$. Set $K_r := K \cap \bar{\mathbf{B}}_r$.

Definition 2.1 (See [5])**.** Let $P \subset X$ be a closed convex cone.

(i) The weak C-dual cone P_C^W of P is defined by

$$P_C^W = \{v \in L(X,Y) \mid \langle v, y \rangle \notin \text{ int } C, \quad \forall y \in P\}.$$

(ii) The strong C-dual cone P_C^S of P is defined by

$$P_C^S = \{v \in L(X,Y) \mid \langle v, y \rangle \in -C, \quad \forall y \in P\}.$$

Remark 2.2. When $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, both P_C^W and P_C^S reduce to the classical polar cone P^- of P which is defined by

$$P^- = \{v \in X^* \mid \langle v, y \rangle \leq 0, \quad \forall y \in P\}.$$

Set $P^+ = -P^-$. It is known that

$$x \in \text{ int } P \Leftrightarrow \langle v, x \rangle > 0, \quad \forall v \in P^+ \setminus \{0\}.$$

Definition 2.3 (See [13])**.** The recession cone K_∞ and the barrier cone $barr(K)$ of K in X are defined respectively as follows:

$$K_\infty = \{v \in X \mid \exists t_n \downarrow 0 \text{ and } x_n \in K \text{ such that } t_n x_n \rightharpoonup v\};$$

$$barr(K) = \{\xi \in X^* \mid \sup_{x \in K} \langle \xi, x \rangle < +\infty\}.$$

It is known that K_∞ is a closed, convex cone, $K_\infty = \{d \in X \mid x + \lambda d \in K, \quad \forall \lambda > 0\}$ with $x \in K$ and $K_\infty = (barr(K))^-$.

Definition 2.4 (See [4])**.** A mapping $T : K \subset X \to Y$ is said to be completely continuous, iff it maps weakly convergent sequence to strongly convergent sequence. Denote by $L_c(X, Y)$ the space of all completely continuous linear mappings from X into Y. Clearly, $L_c(X, Y)$ is a closed subspace of $L(X, Y)$ and so $L_c(X, Y)$ is also a Banach space.

Definition 2.5 (See [30])**.** A set-valued mapping $F : K \to 2^{L(X,Y)}$ is said to be upper semi-continuous at $x \in K$, iff for any neighborhood U of $F(x)$, there exists a neighborhood V of x, such that

$$F(x') \subseteq U, \quad \forall x' \in V \cap K.$$

F is upper semi-continuous on K, iff F is upper semi-continuous at every point of K. We say that F is completely upper semi-continuous iff it is upper semi-continuous with respect to the weak topology of X and the norm topology of $L(X, Y)$.

Lemma 2.6 (See [3])**.** *For any $x, y, z \in Y$, the following conclusions are true:*

(i) *If $x - y \notin -\text{int}\, C$ and $y - z \in C$, then $x - z \notin -\text{int}\, C$.*

(ii) *If $x - y \notin C$ and $y - z \in C$, then $x - z \notin -C$.*

(iii) *If $y - x \notin -\text{int}\, C$ and $x - z \in \text{int}\, C$, then $y - z \notin -C$.*

(iv) *If $y - x \in C$ and $x - z \in \text{int}\, C$, then $y - z \in \text{int}\, C$.*

Definition 2.7 (See [7])**.** Let E be a Hausdorff topological real vector space and $S \subseteq E$. A set-valued mapping $H : S \to 2^E$ is called a KKM mapping, iff for every finite set $\{x_1, x_2, ..., x_n\} \subseteq S$, one has

$$co\,\{x_1, x_2, ..., x_n\} \subseteq \bigcup_{i=1}^{n} H(x_i),$$

where co means the convex hull of a set.

Lemma 2.8 (See [7])**.** *Let E be a Hausdorff topological real vector space, $S \subseteq E$ be a nonempty subset and $H : S \to 2^E$ be a KKM mapping. If $H(x)$ is closed for every $x \in S$, and $\exists x_1 \in S$ such that $H(x_1)$ is compact, then $\bigcap_{x \in S} H(x) \neq \emptyset$.*

Lemma 2.9 (See [8])**.** *Let E be a compact Hausdorff space and D be an arbitrary set. Let f be a real-valued function on $E \times D$ such that, for any $x \in E, y \in D$, $f(\cdot, y)$ is lower semi-continuous and convex on E and $f(x, \cdot)$ is concave on D. Then the following conclusion holds:*

$$\min_{x\in E} \sup_{y\in D} f(x,y) = \sup_{y\in D} \min_{x\in E} f(x,y).$$

Lemma 2.10 (See [13])**.** *Let K be a nonempty, closed and convex set in a real reflexive Banach space X. If $\operatorname{int} barr(K) \neq \emptyset$, then there does not exist $\{x_n\} \subseteq K$ with $\|x_n\| \to +\infty$ such that $\frac{x_n}{\|x_n\|} \rightharpoonup 0$. If K is also a cone, then there does not exist $\{y_n\} \subseteq K$ with each $\|y_n\| = 1$ such that $\{y_n\} \rightharpoonup 0$.*

3 Results for GVVI(F, K) with F Being Perturbed

In this section, we investigate the nonemptiness and boundedness of the solution set of $GVVI(F, K)$ with F being perturbed. First, we prove an existence result when F is perturbed by a nonlinear mapping. Then, we prove that the solution set is nonempty and bounded when F is perturbed by an interior point of $C-$*completely dual cone* of K_∞ which is defined in Definition 3.6.

Lemma 3.1. *Let K be a nonempty, bounded and closed convex subset of a real reflexive Banach space X and Y be a real Banach space ordered by a closed convex and pointed cone C with $\operatorname{int} C \neq \emptyset$. Let $F : K \to 2^{L_c(X,Y)}$ be a completely upper semi-continuous mapping with nonempty and compact values. Then for each $y \in K$,*

$$M(y) = \{x \in K \mid \langle u, y - x\rangle \in -\operatorname{int} C, \quad \forall u \in F(x)\}$$

is open in K with respect to the weak topology of X.

Proof. Given $y \in K$, define $f_y : K \times L_c(X, Y) \to Y$ by

$$f_y(x, u) = \langle u, y - x\rangle, \qquad \forall x \in K, u \in L_c(X, Y).$$

We claim that the mapping $f_y(x, u)$ is a continuous mapping with respect to the weak topology of X, the norm topology of $L_c(X, Y)$ and the norm topology of Y. Indeed, for any given $x_0 \in K$, $u_0 \in L_c(X, Y)$ and $\varepsilon > 0$, since K is bounded, there exists a norm neighborhood U_0 of u_0 in $L_c(X, Y)$ such that

$$\|\langle u - u_0, y - z\rangle\|_Y < \frac{\varepsilon}{2}, \quad \forall u \in U_0, z \in K.$$

On the other hand, there exists a weak neighborhood V_0 of x_0 in X such that

$$\|\langle u_0, x_0 - x\rangle\|_Y < \frac{\varepsilon}{2}, \quad \forall x \in V_0 \cap K.$$

It follows that

$$\|\langle u, y - x\rangle - \langle u_0, y - x_0\rangle\|_Y \leq \|\langle u - u_0, y - x\rangle\|_Y + \|\langle u_0, x_0 - x\rangle\|_Y < \varepsilon$$

for all $u \in U_0$ and $x \in V_0 \cap K$. This proves that $f_y(\cdot, \cdot)$ is a continuous mapping with respect to the weak topology of X, the norm topology of $L_c(X, Y)$ and the norm topology of Y.

For any $y \in K$, let $w_0 \in M(y)$ and $u \in F(w_0)$. Then one has $f_y(u, w_0) \in -\operatorname{int} C$. Since $f_y(\cdot, \cdot)$ is continuous with respect to the weak topology of X, the norm topology of

$L_c(X, Y)$ and the norm topology of Y, there exist a norm neighborhood $U(u)$ of u and a weak neighborhood $V_u \cap K$ of w_0 in K such that

$$\langle u', y - x\rangle \in -\text{ int } C, \quad \forall u' \in U(u), \forall x \in V_u \cap K.$$

Because $F(w_0)$ is a compact set, there exist $u_1, u_2, \dots, u_k$ such that

$$\{U(u_1), U(u_2), \dots, U(u_k)\}$$

is an open cover of $F(w_0)$. Since F is completely upper semi-continuous, there exists a weak neighborhood $V_0 \cap K$ of w_0 in K such that

$$F(x) \subset \bigcup_{i=1}^{k} U(u_k), \quad \forall x \in V_0 \cap K.$$

Set $V = V_0 \cap (\bigcap_{i=1}^{k} V_{u_i}) \cap K$. Then, for any $x \in V$ and any $u \in F(x)$, there exists i such that $u \in U(u_i)$. This yields

$$\langle u, y - x\rangle \ \in -\text{ int } C.$$

As a consequence, one has $x \in M(y)$ for all $x \in V$. □

Remark 3.2. In [35], a set-valued mapping $F : K \to 2^{L_c(X,Y)}$ is said to be completely semi-continuous iff for each $y \in K$, $M(y) = \{x \in K \mid \langle u, y - x\rangle \in -\text{ int } C, \quad \forall u \in F(x)\}$ is open in K with respect to the weak topology X. Lemma 3.1 gives a sufficient condition for F being a completely semi-continuous mapping.

Theorem 3.3. *Let $K \subseteq X$ be a nonempty, bounded and closed convex subset of a real reflexive Banach space X and Y be a real Banach space ordered by a closed convex and pointed cone C with $\operatorname{int} C \neq \emptyset$. Let $F : K \to 2^{L_c(X,Y)}$ be a completely upper semi-continuous mapping with nonempty, compact and convex values. Then $SOL_{GVVI}(F, K) \neq \emptyset$.*

Proof. Let $x^* \in C^+ \backslash \{0\}$ and $y \in K$. By Lemma 3.1, $N(y) = \{x \in K \mid \langle x^* \circ u, y - x\rangle < 0, \quad \forall u \in F(x)\}$ is an open set in K with respect to the weak topology of X. Define $T : K \to 2^K$ by

$$T(y) = \{x \in K \mid \exists u \in F(x), \text{such that} \, \langle x^* \circ u, y - x\rangle \geq 0\}, \quad \forall y \in K.$$

Clearly, $T(y)$ is a nonempty closed set in K with respect to the weak topology of X. Since K is a weakly compact set, $T(y)$ is weakly compact. We claim that T is a KKM mapping. Indeed, if it is not true, then there exist $\mu_1, \mu_2, \dots, \mu_n$ with $\mu_n \in [0, 1], n = 1, 2, \dots, k$, $\sum_{n=1}^{k} \mu_n = 1$ and $y_0 = \sum_{n=1}^{k} \mu_n y_n$ with $y_1, y_2, \dots, y_n \in K$ such that $y_0 \notin \bigcup_{n=1}^{k} T(y_n)$. So $\langle x^* \circ u, y_n - y_0\rangle < 0$ for all $u \in T(y_0)$ and $n = 1, 2, \dots, k$. Let $u \in T(y_0)$. It follows that

$$0 > \sum_{n=1}^{k} \mu_n \langle x^* \circ u, y_n - y_0\rangle = \langle x^* \circ u, \sum_{n=1}^{k} \mu_n y_n - y_0\rangle = \langle x^* \circ u, y_0 - y_0\rangle = 0,$$

a contradiction. So, by Lemma 2.8, $\bigcap_{y \in K} T(y) \neq \emptyset$. Let $x_{x^*} \in \bigcap_{y \in K} T(y)$. Then, for any $y \in K$, there exists $u^* \in F(x_{x^*})$ such that $\langle x^* \circ u^*, y - x_{x^*}\rangle \geq 0$. Define $f : K \times F(x_{x^*}) \to R$ by

$$f(y, u) = \langle x^* \circ u, y - x_{x^*}\rangle, \quad \forall y \in K, u \in F(x_{x^*}).$$

It follows from Lemma 2.9 that

$$\max_{u\in F(x_{x^*})}\min_{y\in K} f(y,u) = \min_{y\in K}\max_{u\in F(x_{x^*})} f(y,u) \geq 0.$$

Since $F(x_{x^*})$ is compact, there exists $u_0 \in F(x_{x^*})$ such that

$$\langle x^* \circ u_0, y - x_{x^*}\rangle \geq 0, \quad \forall y \in K.$$

By Remark 2.2, one has $x_{x^*} \in SOL_{GVVI}(F, K)$. □

Let $m > 0$ and $\varepsilon > 0$. Suppose that $p_\varepsilon : K_m \to L_c(X, Y)$ is completely continuous such that $\|p_\varepsilon(x)\| \leq \varepsilon$ for $\forall x \in K_m$. The family of such mappings is denoted by $C(\varepsilon, K_m)$. Now we prove an existence result when F is perturbed by $p_\varepsilon \in C(\varepsilon, K_m)$.

Theorem 3.4. *Let K be a nonempty, closed convex subset of a finite dimensional space X and Y be a real Banach space ordered by a closed convex and pointed cone C with $\operatorname{int} C \neq \emptyset$ and $F : K \to 2^{L(X,Y)}$ be an upper semi-continuous mapping with nonempty, compact and convex values. Suppose that F satisfies the following coercivity condition (C_1): there exists $r > 0$ such that for every $x \in K \setminus K_r$, there is $y_x \in K$ with $\|y_x\| < \|x\|$ satisfying:*

$$\langle \xi, y_x - x\rangle \in -\text{int } C, \quad \forall \xi \in F(x).$$

Then for every $m > r$, there exists $\varepsilon > 0$ such that

$$SOL_{GVVI}(F + p_\varepsilon, K)\bigcap \bar{\mathbf{B}}_m \neq \emptyset.$$

Proof. If it is not true, then there exists $m > r$ such that for every $\varepsilon > 0$,

$$SOL_{GVVI}(F + p_\varepsilon, K)\bigcap \bar{\mathbf{B}}_m = \emptyset.$$

Let $x^* \in C^+\setminus\{0\}$. Consider the scalar problem $GVI(x^*\circ(F+p_\varepsilon), K_m)$ of $GVVI(F+p_\varepsilon, K_m)$. By Theorem 3.3, $SOL_{GVI}(x^* \circ (F + p_\varepsilon), K_m) \neq \emptyset$. Let $x_\varepsilon \in SOL_{GVI}(x^* \circ (F + p_\varepsilon), K_m)$. Then $\|x_\varepsilon\| \leq m$. Consider the following two cases.

(a) There exists some $\varepsilon > 0$ such that $\|x_\varepsilon\| < m$. In this case, we claim that $x_\varepsilon \in SOL_{GVI}(x^* \circ (F + p_\varepsilon), K)$. Indeed, for each $y \in K$, if $y \in K_m$, then, for $\forall t \in (0, 1)$, there is $z_t = x_\varepsilon + t(y - x_\varepsilon) \in K_m$ by the convexity of K_m. If $y \in K \setminus K_m$, it is obvious that is $\|y - x_\varepsilon\| \neq 0$. So $z_t = x_\varepsilon + t(y - x_\varepsilon) \in K_m$ for any $t \in (0, \frac{m-\|x_\varepsilon\|}{\|y-x_\varepsilon\|}) \subseteq (0, 1)$. Thus, for each $y \in K$, there exists $t \in (0, 1)$ such that $z_t = x_\varepsilon + t(y - x_\varepsilon) \in K_m$. Because $x_\varepsilon \in SOL_{GVI}(x^* \circ (F + p_\varepsilon), K_m)$, one has that there exists $\xi_\varepsilon \in F(x_\varepsilon)$ such that

$$\langle x^* \circ (\xi_\varepsilon + p_\varepsilon(x_\varepsilon)), z_t - x_\varepsilon\rangle = t\langle x^* \circ (\xi_\varepsilon + p_\varepsilon(x_\varepsilon)), y - x_\varepsilon\rangle \geq 0,$$

which implies

$$\langle x^* \circ (\xi_\varepsilon + p_\varepsilon(x_\varepsilon)), y - x_\varepsilon\rangle \geq 0, \quad \forall y \in K.$$

This yields

$$x_\varepsilon \in SOL_{GVVI}(F + p_\varepsilon, K)\bigcap \bar{\mathbf{B}}_m,$$

a contradiction.

(b) $\|x_\varepsilon\| = m$ for any $\varepsilon > 0$. In this case, without loss of generality, we assume that $x_\varepsilon \to d$ as $\varepsilon \to 0^+$. It is easy to see that $d \in K$ and $\|d\| = m > r$. By the coercivity condition (C_1), there exists $y_d \in K$ with $\|y_d\| < \|d\| = m$ such that

$$\langle \xi_d, y_d - d\rangle \in -\text{ int } C, \quad \forall \xi_d \in F(d).$$

By the compactness of $F(d)$ and Remark 2.2, one has

$$\sup_{\xi_d \in F(d)} \langle x^* \circ \xi_d, y_d - d \rangle < 0.$$

Since F is upper semi-continuous and compact-valued, the mapping $x \mapsto \sup_{\xi \in F(x)} \langle x^* \circ \xi, y_d - x \rangle$ is upper semi-continuous. It follows that

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} \{ \sup_{\xi_\varepsilon \in F(x_\varepsilon)} [\langle x^* \circ \xi_\varepsilon, y_d - x_\varepsilon \rangle] + \langle x^* \circ (p_\varepsilon(x_\varepsilon)), y_d - x_\varepsilon \rangle \} \\
&\quad \leq \limsup_{\varepsilon \to 0} \{ \sup_{\xi_\varepsilon \in F(x_\varepsilon)} [\langle x^* \circ \xi_\varepsilon, y_d - x_\varepsilon \rangle] \} + \lim_{\varepsilon \to 0} \langle x^* \circ (p_\varepsilon(x_\varepsilon)), y_d - x_\varepsilon \rangle \\
&\quad \leq \sup_{\xi_d \in F(d)} \langle x^* \circ \xi_d, y_d - d \rangle < 0.
\end{aligned}$$

Therefore, there exists $\delta > 0$ such that

$$\sup_{\xi_\varepsilon \in F(x_\varepsilon)} \langle x^* \circ \xi_\varepsilon, y_d - x_\varepsilon \rangle + \langle x^* \circ (p_\varepsilon(x_\varepsilon)), y_d - x_\varepsilon \rangle < 0, \quad \forall \varepsilon \in (0, \delta).$$

Since $\|y_d\| < m$, by same arguments as in case (a), for any $y \in K$, there exists $t \in (0,1)$ such that $a_t = y_d + t(y - y_d) \in K_m$. Since $x_\varepsilon \in SOL_{GVI}(x^* \circ (F + p_\varepsilon), K_m)$, it follows that for any $\varepsilon \in (0, \delta)$,

$$\begin{aligned}
& t[\sup_{\xi_\varepsilon \in F(x_\varepsilon)} \langle x^* \circ (\xi_\varepsilon + p_\varepsilon(x_\varepsilon)), y - x_\varepsilon \rangle] \\
& > t[\sup_{\xi_\varepsilon \in F(x_\varepsilon)} \langle x^* \circ (\xi_\varepsilon + p_\varepsilon(x_\varepsilon)), y - x_\varepsilon \rangle] \\
& +(1-t)[\sup_{\xi_\varepsilon \in F(x_\varepsilon)} \langle x^* \circ (\xi_\varepsilon + p_\varepsilon(x_\varepsilon)), y_d - x_\varepsilon \rangle] \\
& \geq \sup_{\xi_\varepsilon \in F(x_\varepsilon)} \langle x^* \circ (\xi_\varepsilon + p_\varepsilon(x_\varepsilon)), y_d + t(y - y_d) - x_\varepsilon \rangle \\
& = \sup_{\xi_\varepsilon \in F(x_\varepsilon)} \langle x^* \circ (\xi_\varepsilon + p_\varepsilon(x_\varepsilon)), a_t - x_\varepsilon \rangle \geq 0.
\end{aligned}$$

This implies that $x_\varepsilon \in SOL_{GVI}(x^* \circ (F + p_\varepsilon), K)$. By Remark 2.2, $x_\varepsilon \in SOL_{GVVI}(F + p_\varepsilon, K)$. Therefore,

$$x_\varepsilon \in SOL_{GVVI}(F + p_\varepsilon, K) \bigcap \bar{\mathbf{B}}_m \neq \emptyset,$$

a contradiction. □

Remark 3.5. Theorem 3.4 generalizes Theorem 3.1 of [20] to the vector case.

Next we need the concepts of weak and strong C-completely dual cones of K.

Definition 3.6. Given a set $K \subseteq X$, the $C-$*completely dual cones*(in short C_c-dual cones) of K are defined as follows:

(i) The weak C_c-dual cone $K^W_{C_c}$ of K is defined by:

$$K^W_{C_c} = \{ v \in L_c(X, Y) \mid \langle v, y \rangle \notin \text{ int } C, \quad \forall y \in K \};$$

(ii) The strong C_c-dual cone $K^S_{C_c}$ of set K is defined by:

$$K^S_{C_c} = \{ v \in L_c(X, Y) \mid \langle v, y \rangle \in -C, \quad \forall y \in K \}.$$

The following result gives characterizations of weak and strong C_c-dual cones of K, which is a slight modification of Lemma 3.3 of [10]. Here we omit its proof since it is similar as the one of Lemma 3.3 of [10].

Lemma 3.7. *Let K be a nonempty, closed and convex cone of a real reflexive Banach space X and Y be a real Banach space ordered by a proper closed convex and pointed cone C with $\operatorname{int} C \neq \emptyset$ and $\operatorname{int} barr(K) \neq \emptyset$. Then the following conclusions are true:*

(i) $\operatorname{int} K^S_{C_c} = \{v \in L_c(X,Y) \mid \langle v, y\rangle \in -\operatorname{int} C, \quad \forall y \in K \setminus \{0\}\}$;

(ii) $\operatorname{int} K^W_{C_c} = \{v \in L_c(X,Y) \mid \langle v, y\rangle \notin C, \quad \forall y \in K \setminus \{0\}\}$.

Remark 3.8. It is worth mentioning that a superfluous condition named Property C is assumed in Lemma 3.3 of [10]. Recall that in [10], a pair $\langle \cdot, \cdot \rangle$ between $L_c(X,Y)$ and X is said to satisfy Property (C) iff $\langle l, x\rangle \notin -\operatorname{int} C$ for all $l \in L_c(X,Y)$, implies that $x = 0$. In fact, Property (C) always holds in the setting of real Banach spaces. Indeed, suppose on the contrary that there exists $x_1 \neq 0$ such that $\langle l, x_1\rangle \notin -\operatorname{int} C$ for all $l \in L_c(X,Y)$. Choose $x^* \in X^*$ with $\langle x^*, x_1\rangle = 1$. Then for any fixed $y \in -\operatorname{int} C$, define $l(\cdot) = \langle x^*, \cdot\rangle y$. Let $x_0 \in X$. For any $\epsilon > 0$, there exist $\delta = \frac{\epsilon}{\|y\|_Y}$ and the weak neighborhood of x_0 denoted by $U(x_0) = \{x \in X : |\langle x^*, x\rangle - \langle x^*, x_0\rangle| < \delta\}$ such that $\|l(x) - l(x_0)\|_Y \leq |\langle x^*, x\rangle - \langle x^*, x_0\rangle| \cdot \|y\|_Y < \delta\|y\|_Y = \epsilon$ for any $x \in U(x_0)$. Thus, for any fixed $y \in -\operatorname{int} C$, $l(\cdot) = \langle x^*, \cdot\rangle y \in L_c(X,Y)$. Then $l(x_1) = y \in -\operatorname{int} C$, which is a contradiction.

Lemma 3.9. *Let K be a nonempty subset of a real reflexive Banach space X and $\operatorname{int} barr(K) \neq \emptyset$, Then $\operatorname{int} barr(K_\infty) \neq \emptyset$.*

Proof. By Proposition 2.1 of [1], $\operatorname{int}(K_\infty)^+ = -\operatorname{int} barr(K) \neq \emptyset$. Next, we assert $(K_\infty)^+ = -barr(K_\infty)$. Indeed, it is obvious that $(K_\infty)^+ \subseteq -barr(K_\infty)$. Assume that $\exists v \in -barr(K_\infty)$ such that $v \notin (K_\infty)^+$. Then there exists $x^* \in K_\infty$ such that $\langle v, x^*\rangle < 0$. Because K_∞ is a cone, one has $\varepsilon x^* \in K_\infty$ for $\forall \varepsilon > 0$. So $\langle v, \varepsilon x^*\rangle \to -\infty$ with $\varepsilon \to +\infty$. This implies $v \notin -barr(K_\infty)$, which is a contradiction. Therefore, $(K_\infty)^+ \supseteq -barr(K_\infty)$. Thus, $-barr(K_\infty) = (K_\infty)^+ \supseteq -barr(K) \supseteq -\operatorname{int} barr(K) \neq \emptyset$. □

Theorem 3.10. *Let K be a nonempty, closed and convex subset of a real reflexive Banach space X with $\operatorname{int} barr(K) \neq \emptyset$ and Y be a real Banach space ordered by a proper closed convex and pointed cone C with $\operatorname{int} C \neq \emptyset$. Suppose that $F : K \to 2^{L_c(X,Y)}$ is a completely upper semi-continuous mapping with nonempty, compact and convex values and satisfies the coercivity condition (C_2): there exists $r > 0$ such that for every $x \in K \setminus K_r$, there is $y_x \in K_r$ satisfying:*

$$\langle \xi, y_x - x\rangle \in -C, \quad \forall \xi \in F(x).$$

Then for any $p \in \operatorname{int}(K_\infty)^S_{C_c}$, there exists $m > r$ such that

(i) $SOL_{GVVI}(F - \varepsilon p, K) \neq \emptyset, \quad \forall \varepsilon \in (0, \frac{1}{m})$;

(ii) $SOL_{GVVI}(F - \varepsilon p, K) \subseteq \bar{\boldsymbol{B}}_{\frac{1}{\varepsilon}}, \quad \forall \varepsilon \in (0, \frac{1}{m})$.

Proof. To prove (i), suppose on the contrary that there exists $p \in \operatorname{int}(K_\infty)^S_{C_c}$ such that for any $m > r$, there exists $\varepsilon_m \in (0, \frac{1}{m})$ such that

$$SOL_{GVVI}(F - \varepsilon_m p, K) = \emptyset.$$

It is clear that $K_{\frac{1}{\varepsilon_m}} := \{x \in K \mid \|x\| \le \frac{1}{\varepsilon_m}\}$ is a nonempty, bounded and closed convex set for all sufficiently large m. By Theorem 3.3, $SOL_{GVVI}(F - \varepsilon_m p, K_{\frac{1}{\varepsilon_m}}) \neq \emptyset$. Let $x_m \in SOL_{GVVI}(F - \varepsilon_m p, K_{\frac{1}{\varepsilon_m}})$. Consider the following cases.

(a) $\|x_m\| < \frac{1}{\varepsilon_m}$ for some m. In this case we obtain $x_m \in SOL_{GVVI}(F - \varepsilon_m p, K)$ by same arguments as case (a) in the proof of Theorem 3.4. This arrives a contradiction.

(b) $\|x_m\| = \frac{1}{\varepsilon_m}$ for all $m > r$. Since $\frac{1}{\varepsilon_m} > m > r$, we have $x_m \in K \setminus K_r$. By the coercivity condition (C_2), there exists $y_m \in K_r$ such that

$$\langle \xi_m, y_m - x_m \rangle \in -C, \quad \forall \xi_m \in F(x_m). \tag{3.1}$$

Without loss of generality, we assume that $\frac{x_m}{\|x_m\|} \rightharpoonup d \in K_\infty$ as $m \to +\infty$. Since $\text{int}\, barr(K) \neq \emptyset$, we get $d \neq 0$ (by Lemma 2.10) and $\text{int}\, barr(K_\infty) \neq \emptyset$ (by Lemma 3.9). Since $p \in \text{int}\,(K_\infty)^S_{C_c}$, from Lemma 3.7 we get $\langle p, d\rangle \in -\text{int}\, C$. Let $y \in K$. Then $z_t = y_m + t(y - y_m) \in K_{\frac{1}{\varepsilon_m}}$ for all sufficiently small $t \in (0,1)$ since $\|y_m\| \le r < \frac{1}{\varepsilon_m}$. Since $x_m \in SOL_{GVVI}(F - \varepsilon_m p, K_{\frac{1}{\varepsilon_m}})$, there exists $\xi_m' \in F(x_m)$ such that

$$\begin{aligned} &\langle \xi_m' - \varepsilon_m p, z_t - x_m\rangle = \langle \xi_m' - \varepsilon_m p, y_m + t(y - y_m) - x_m\rangle \\ =&t\langle \xi_m' - \varepsilon_m p, y - x_m\rangle + (1-t)\langle \xi_m', y_m - x_m\rangle + \varepsilon_m(1-t)(\langle p, x_m\rangle - \langle p, y_m\rangle) \\ &\notin -\text{int}\, C. \end{aligned} \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$t\langle \xi_m' - \varepsilon_m p, y - x_m\rangle + \varepsilon_m(1-t)(\langle p, x_m\rangle - \langle p, y_m\rangle) \notin -\text{int}\, C. \tag{3.3}$$

Since $\varepsilon_m\langle p, x_m\rangle = \langle p, \frac{x_m}{\|x_m\|}\rangle \to \langle p, d\rangle \in -\text{int}\, C$ and $\varepsilon_m\langle p, y_m\rangle \to 0$ (as $m \to +\infty$), we get

$$\varepsilon_m(\langle p, x_m\rangle - \langle p, y_m\rangle) \to \langle p, d\rangle \in -\text{int}\, C \quad (\text{ as } m \to +\infty).$$

This implies that

$$\varepsilon_m(\langle p, x_m\rangle - \langle p, y_m\rangle) \in -\text{int}\, C, \tag{3.4}$$

for all sufficiently large m. It follows from Lemma 2.6, (3.3) and (3.4) that for all sufficiently large m,

$$\langle \xi_m' - \varepsilon_m p, y - x_m\rangle \notin -\text{int}\, C, \quad \forall y \in K,$$

which is a contradiction.

Next, we prove (ii). By (i), there exists $m > r$ such that for any $k \ge m$ and any $\varepsilon \in (0, \frac{1}{k})$, $SOL_{GVVI}(F - \varepsilon p, K) \neq \emptyset$. Suppose on the contrary that there exists $p \in \text{int}\,(K_\infty)^S_{C_c}$ such that for any $k \ge m$, there are $\varepsilon_k \in (0, \frac{1}{k})$ and $x_k \in SOL_{GVVI}(F - \varepsilon_k p, K)$ such that $x_k \notin \bar{\mathbf{B}}_{\frac{1}{\varepsilon_k}}$. Then $x_k \in K \setminus K_r$ since $\|x_k\| > \frac{1}{\varepsilon_k} \ge k \ge m > r$. By the coercivity condition, there exists $y_k \in K_r$ such that

$$\langle \xi_k, y_k - x_k\rangle \in -C, \quad \forall \xi_k \in F(x_k). \tag{3.5}$$

Since $x_k \in SOL_{GVVI}(F - \varepsilon_k p, K)$, there exists $\xi_k' \in F(x_k)$ such that

$$\langle \xi_k', y_k - x_k\rangle + \varepsilon_k(\langle p, x_k\rangle - \langle p, y_k\rangle) = \langle \xi_k' - \varepsilon_k p, y_k - x_k\rangle \notin -\text{int}\, C. \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\varepsilon_k(\langle p, x_k\rangle - \langle p, y_k\rangle) \notin -\text{int}\, C. \tag{3.7}$$

Without loss of generality, we assume that $\frac{x_k}{\|x_k\|} \rightharpoonup e \in K_\infty \backslash \{0\}$ as $k \to +\infty$. Since $\{y_k\} \in K_r$ is bounded and $\|x_k\| \to +\infty$ as $k \to +\infty$, it follows that

$$\langle p, \frac{x_k}{\|x_k\|}\rangle - \langle p, \frac{y_k}{\|x_k\|}\rangle \to \langle p, e\rangle \in - \text{ int } C,$$

as $k \to +\infty$. This implies that

$$\langle p, \frac{x_k}{\|x_k\|}\rangle - \langle p, \frac{y_k}{\|x_k\|}\rangle \in - \text{ int } C$$

for all sufficiently large k. This yields

$$\varepsilon_k(\langle p, x_k\rangle - \langle p, y_k\rangle) \in - \text{ int } C,$$

for all sufficiently large k, a contradiction to (3.7). □

Remark 3.11. When $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, Theorem 3.10 reduces to Theorem 3.1 of [32] and Theorem 3.1 of [21].

4 Results for GVVI(F, K) with F and K Being Perturbed Simultaneously

In this section we investigate nonemptiness and boundedness of the solution set of $GVVI(F, K)$ with F and K being perturbed simultaneously. Given $\delta > 0$, set $K^\delta = K + \bar{\mathbf{B}}_\delta$. Let $C_c(X, Y)$ be the space of all completely continuous mappings from X to Y. Given $\varepsilon > 0$, let $p : K^\delta \to C_c(X, Y)$ be a mapping such that $\|p(x)\| \leq \varepsilon$ for $\forall x \in K^\delta$. The set of all such mappings is denoted by $C(\varepsilon, K^\delta)$.

Theorem 4.1. *Let K be a nonempty, closed, convex and subset of a finite dimensional space X and Y be a real Banach space ordered by a pointed, closed and convex cone C with* $\text{int}\, C \neq \emptyset$. *Given $\delta > 0$, suppose that $F : K^\delta \to 2^{L_c(X,Y)}$ is a completely upper semi-continuous mapping with nonempty, compact and convex values and satisfies the coercivity condition (C_1) on K, i.e., there exists $r > 0$ such that for every $x \in K \setminus K_r$, there exists $y_x \in K$ with $\|y_x\| < \|x\|$ satisfying:*

$$\langle \xi, y_x - x\rangle \in -\text{int}\, C, \quad \forall \xi \in F(x).$$

Then for every $m > r$, there exist $\varepsilon > 0$ and $\delta_0 \in (0, \delta)$ such that

$$SOL_{GVVI}(F + p, K^\alpha) \bigcap \bar{\mathbf{B}}_m \neq \emptyset, \quad \forall p \in C(\varepsilon, K^{\delta_0}), \forall \alpha \in (0, \delta_0).$$

Proof. Suppose on the contrary that there exists $m > r$ such that for each $\varepsilon > 0$ and $\delta' \in (0, \delta)$, there exist $p_{\varepsilon,\delta'} \in C(\varepsilon, K^{\delta'})$ and $\alpha \in (0, \delta')$ such that

$$SOL_{GVVI}(F + p_{\varepsilon,\delta'}, K^\alpha) \bigcap \bar{\mathbf{B}}_m = \emptyset.$$

Let $x^* \in C^+ \setminus \{0\}$. Set $(K^\alpha)_m = \{x \in K^\alpha \mid \|x\| \leq m\}$. By Theorem 3.3, $SOL_{GVI}(x^* \circ (F + p_{\varepsilon,\delta'}), (K^\alpha)_m) \neq \emptyset$.

Let $x_{\varepsilon,\delta'} \in SOL_{GVI}(x^* \circ (F + p_{\varepsilon,\delta'}), (K^\alpha)_m)$. Clearly, $\|x_{\varepsilon,\delta'}\| \leq m$. Consider the following two cases.

(a) $\|x_{\varepsilon,\delta'}\| < m$ for some $\varepsilon > 0$ and $\delta' \in (0, \delta)$. By same arguments as case (a) in the proof of Theorem 3.4, we have $x_{\varepsilon,\delta'} \in SOL_{GVVI}(F + p_{\varepsilon,\delta'}, K^\alpha)$, a contradiction.

(b) $\|x_{\varepsilon,\delta'}\| = m$ for all $\varepsilon > 0$ and $\delta' \in (0, \delta)$. Without loss of generality, we assume that $x_{\varepsilon,\delta'} \to d$ with $\|d\| = m$ as $\varepsilon \to 0^+$ and $\delta' \to 0^+$. Since $x_{\varepsilon,\delta'} \in K + \bar{\mathbf{B}}_{\delta'}$, there exists $\widehat{x}_\varepsilon^{\delta'} \in K$ such that $\lim_{\delta \longrightarrow 0^+} \|x_{\varepsilon,\delta'} - \widehat{x}_\varepsilon^{\delta'}\| = 0$. This together with $x_{\varepsilon,\delta'} \to d$ yields $\{\widehat{x}_\varepsilon^{\delta'}\} \to d \in K \setminus K_r$ as $\varepsilon \to 0^+$ and $\delta' \to 0^+$. By the coercivity condition (C_1) on K, there exists $y_d \in K$ with $\|y_d\| < \|d\| = m$ such that

$$\langle \xi_d, y_d - d \rangle \in -\ \mathrm{int}\ C, \quad \forall \xi_d \in F(d).$$

Since $F(d)$ is compact and $x^* \in C^+ \setminus \{0\}$, we get

$$\sup_{\xi_d \in F(d)} \langle x^* \circ \xi_d, y_d - d \rangle < 0. \tag{4.1}$$

It follows from $\sup_{x \in (K^\alpha)_m} \|p_{\varepsilon,\delta'}(x)\| \leq \varepsilon$ that

$$\lim_{\varepsilon \to 0^+, \delta' \to 0^+} \langle x^* \circ (p_{\varepsilon,\delta'}(x_{\varepsilon,\delta'})), y_d - x_{\varepsilon,\delta'} \rangle = 0. \tag{4.2}$$

Since F is upper semi-continuous and compact-valued, the mapping $x \mapsto \sup_{\xi \in F(x)} \langle x^* \circ \xi, y_d - x \rangle$ is upper semi-continuous. It follows from (4.1) and (4.2) that

$$\begin{aligned}
&\limsup_{\varepsilon \to 0^+, \delta' \to 0^+} \{ \sup_{\xi_\varepsilon^{\delta'} \in F(x_{\varepsilon,\delta'})} [\langle x^* \circ \xi_\varepsilon^{\delta'}, y_d - x_{\varepsilon,\delta'} \rangle] + \langle x^* \circ (p_{\varepsilon,\delta'}(x_{\varepsilon,\delta'})), y_d - x_{\varepsilon,\delta'} \rangle \} \\
&\leq \limsup_{\varepsilon \to 0^+, \delta' \to 0^+} \{ \sup_{\xi_\varepsilon^{\delta'} \in F(x_{\varepsilon,\delta'})} [\langle x^* \circ \xi_\varepsilon^{\delta'}, y_d - x_{\varepsilon,\delta'} \rangle] \} \\
&+ \lim_{\varepsilon \to 0^+, \delta' \to 0^+} \langle x^* \circ (p_{\varepsilon,\delta'}(x_{\varepsilon,\delta'})), y_d - x_{\varepsilon,\delta'} \rangle \\
&\leq \sup_{\xi_d \in F(d)} \langle x^* \circ \xi_d, y_d - d \rangle < 0.
\end{aligned}$$

Therefore, there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that

$$\sup_{\xi_\varepsilon^{\delta'} \in F(x_{\varepsilon,\delta'})} \langle x^* \circ \xi_\varepsilon^{\delta'}, y_d - x_{\varepsilon,\delta'} \rangle + \langle x^* \circ (p_{\varepsilon,\delta'}(x_{\varepsilon,\delta'})), y_d - x_{\varepsilon,\delta'} \rangle < 0, \tag{4.3}$$

for $\forall \varepsilon \in (0, \varepsilon_0), \delta' \in (0, \delta_0)$. Let $y \in K^\alpha$. Then $a_t := y_d + t(y - y_d) \in (K^\alpha)_m$ for all sufficiently small $t \in (0, 1)$ since $y_d \in (K^\alpha)_m$ with $\|y_d\| < m$. Since $x_{\varepsilon,\delta'} \in SOL_{GVI}(x^* \circ (F + p_{\varepsilon,\delta'}), (K^\alpha)_m)$, it follows from (4.3) that

$$\begin{aligned}
&t[\sup_{\xi_\varepsilon^{\delta'} \in F(x_{\varepsilon,\delta'})} \langle x^* \circ (\xi_\varepsilon^{\delta'} + p_{\varepsilon,\delta'}(x_{\varepsilon,\delta'})), y - x_{\varepsilon,\delta'} \rangle] \\
&> t[\sup_{\xi_\varepsilon^{\delta'} \in F(x_{\varepsilon,\delta'})} \langle x^* \circ (\xi_\varepsilon^{\delta'} + p_{\varepsilon,\delta'}(x_{\varepsilon,\delta'})), y - x_{\varepsilon,\delta'} \rangle] \\
&+ (1-t)[\sup_{\xi_\varepsilon^{\delta'} \in F(x_{\varepsilon,\delta'})} \langle x^* \circ (\xi_\varepsilon^{\delta'} + p_{\varepsilon,\delta'}(x_{\varepsilon,\delta'})), y_d - x_{\varepsilon,\delta'} \rangle] \\
&\geq \sup_{\xi_\varepsilon^{\delta'} \in F(x_{\varepsilon,\delta'})} \langle x^* \circ (\xi_\varepsilon^{\delta'} + p_{\varepsilon,\delta'}(x_{\varepsilon,\delta'})), y_d + t(y - y_d) - x_{\varepsilon,\delta'} \rangle \\
&= \sup_{\xi_\varepsilon^{\delta'} \in F(x_{\varepsilon,\delta'})} \langle x^* \circ (\xi_\varepsilon^{\delta'} + p_{\varepsilon,\delta'}(x_{\varepsilon,\delta'})), a_t - x_{\varepsilon,\delta'} \rangle \geq 0.
\end{aligned}$$

This implies $x_{\varepsilon,\delta'} \in SOL_{GVI}(x^* \circ (F + p_{\varepsilon,\delta'}), K^\alpha)$. As a consequence, we get

$$x_{\varepsilon,\delta'} \in SOL_{GVVI}(F + p_{\varepsilon,\delta'}, K^\alpha) \bigcap \bar{\mathbf{B}}_m,$$

a contradiction. □

Remark 4.2. When $Y=\mathbb{R}$ and $C=\mathbb{R}_+$, Theorem 4.1 reduces to Theorem 6 of [23].

Theorem 4.3. *Let K be a nonempty, closed and convex set of a real reflexive Banach space X with $\operatorname{int} barr(K)\neq\emptyset$ and Y be a Banach space ordered by a pointed, closed and convex cone C with $\operatorname{int} C\neq\emptyset$. Given $\bar\delta>0$, suppose that $F: K^{\bar\delta}\to 2^{L_c(X,Y)}$ is a completely upper semi-continuous mapping with nonempty, compact and convex values and satisfies the following coercivity condition (C_3): there exist $r>0$ and $\delta_0\in(0,\bar\delta)$ such that for every $x\in K^{\delta_0}\setminus K_r$, there exists $y_x\in K_r$ such that*

$$\langle\xi, y_x-x\rangle\in -C,\quad \forall\xi\in F(x).$$

Let $p\in\operatorname{int}(K_\infty)^S_{C_c}$. Then there exists $m>\max\{r,\frac{1}{\delta}\}$ such that

(i) *$SOL_{GVVI}(F-\varepsilon p, K^{\delta'})\neq\emptyset$, for any $\varepsilon\in(0,\frac{1}{m})$ and any $\delta'\in(0,\frac{1}{m})$;*

(ii) *$SOL_{GVVI}(F-\varepsilon p, K^{\delta'})\subseteq \bar{\boldsymbol{B}}_{\frac{1}{\varepsilon}}$, for any $\varepsilon\in(0,\frac{1}{m})$ and any $\delta'\in(0,\frac{1}{m})$.*

Proof. To prove (i), suppose on the contrary that for any $m>\max\{r,\frac{1}{\delta}\}$, there exist $\varepsilon_m\in(0,\frac{1}{m})$ and $\delta_m\in(0,\frac{1}{m})$ such that $SOL_{GVVI}(F-\varepsilon_m p, K^{\delta_m})=\emptyset$. Set $T_m=\{x\in K^{\delta_m}\mid \|x\|\le\frac{1}{\varepsilon_m}\}$. We have $T_m\neq\emptyset$ for all sufficiently large $m>\max\{r,\frac{1}{\delta}\}$. Indeed, for any fixed $x\in K$, one has $x\in K\subseteq K+\bar{\mathbf{B}}_{\delta_1}$ for any $\delta_1\in(0,\bar\delta)$. And there exists sufficiently large $m'>\max\{r,\frac{1}{\delta}\}$ such that $\|x\|\le m'<\frac{1}{\varepsilon_{m'}}$. So $x\in K^{\delta_{m'}}\bigcap\{x\mid\|x\|\le\frac{1}{\varepsilon_{m'}}\}=T_{m'}$ for all sufficiently large $m'>\max\{r,\frac{1}{\delta}\}$. By Theorem 3.3, $SOL_{GVVI}(F-\varepsilon_m p, T_m)\neq\emptyset$. Let $x^{\delta_m}_m\in SOL_{GVVI}(F-\varepsilon_m p, T_m)$. Clearly, $\|x^{\delta_m}_m\|\le\frac{1}{\varepsilon_m}$. Consider the following two cases.

(a) $\|x^{\delta_m}_m\|<\frac{1}{\varepsilon_m}$ for some m. In this case we obtain $x^{\delta_m}_m\in SOL_{GVVI}(F-\varepsilon_m p, K^{\delta_m})$ by same arguments as case (a) in the proof of Theorem 3.4. This arrives a contradiction.

(b) $\|x^{\delta_m}_m\|=\frac{1}{\varepsilon_m}$ for all $m>\max\{r,\frac{1}{\delta}\}$. Since $\frac{1}{\varepsilon_m}>m>\max\{r,\frac{1}{\delta}\}\ge r$ and $x^{\delta_m}_m\in K^{\delta_m}$, we get $x^{\delta_m}_m\in K^{\delta_m}\backslash K_r$. By $\delta_m\in(0,\frac{1}{m})$, we have $\delta_m\le\delta_0$ for all sufficiently large m. Thus, $K^{\delta_m}\subseteq K^{\delta_0}$. So $x^{\delta_m}_m\in K^{\delta_m}\backslash K_r\subseteq K^{\delta_0}\backslash K_r$ for all sufficiently large m. By the coercivity condition (C_3), there exists $y_m\in K_r$ such that

$$\langle\eta_m, y_m-x^{\delta_m}_m\rangle\in -C,\quad\forall\eta_m\in F(x^{\delta_m}_m). \tag{4.4}$$

Without loss of generality, we assume that $\frac{x^{\delta_m}_m}{\|x^{\delta_m}_m\|}\rightharpoonup d$ as $m\to+\infty$. Since $x^{\delta_m}_m\in K^{\delta_m}$, there exist $\widehat{x}^{\delta_m}_m\in K$ and $k^{\delta_m}_m\in\bar{\mathbf{B}}$ (the closed unit ball) such that $x^{\delta_m}_m=\widehat{x}^{\delta_m}_m+\delta_m k^{\delta_m}_m$. Because $\bar{\mathbf{B}}$ is bounded and $\delta_m\in(0,\frac{1}{m})$, it is easy to verify that $\lim_{m\to+\infty}\|x^{\delta_m}_m-\widehat{x}^{\delta_m}_m\|=0$. It follows that

$$\begin{aligned}
\lim_{m\to+\infty}\left\|\frac{x^{\delta_m}_m}{\|x^{\delta_m}_m\|}-\frac{\widehat{x}^{\delta_m}_m}{\|\widehat{x}^{\delta_m}_m\|}\right\| &\le \lim_{m\to+\infty}\left[\left\|\frac{x^{\delta_m}_m}{\|x^{\delta_m}_m\|}-\frac{\widehat{x}^{\delta_m}_m}{\|x^{\delta_m}_m\|}\right\|+\left\|\frac{\widehat{x}^{\delta_m}_m}{\|x^{\delta_m}_m\|}-\frac{\widehat{x}^{\delta_m}_m}{\|\widehat{x}^{\delta_m}_m\|}\right\|\right]\\
&=\lim_{m\to+\infty}\frac{1}{\|x^{\delta_m}_m\|}\|x^{\delta_m}_m-\widehat{x}^{\delta_m}_m\|\\
&\quad+\lim_{m\to+\infty}\|\widehat{x}^{\delta_m}_m\|\cdot\left|\frac{1}{\|x^{\delta_m}_m\|}-\frac{1}{\|\widehat{x}^{\delta_m}_m\|}\right|\\
&=\lim_{m\to+\infty}\frac{1}{\|x^{\delta_m}_m\|}\|x^{\delta_m}_m-\widehat{x}^{\delta_m}_m\|\\
&\quad+\lim_{m\to+\infty}\frac{\big|\|x^{\delta_m}_m\|-\|\widehat{x}^{\delta_m}_m\|\big|}{\|x^{\delta_m}_m\|}
\end{aligned}$$

$$\leq \lim_{m\to+\infty} 2\varepsilon_m \|x_m^{\delta_m} - \widehat{x}_m^{\delta_m}\| = 0.$$

This together with $\frac{x_m^{\delta_m}}{\|x_m^{\delta_m}\|} \rightharpoonup d$ yields $\frac{\widehat{x}_m^{\delta_m}}{\|\widehat{x}_m^{\delta_m}\|} \rightharpoonup d \in K_\infty$ as $m \to +\infty$. Since int $barr(K) \neq \emptyset$, we get $d \neq 0$ (by Lemma 2.10) and int $barr(K_\infty) \neq \emptyset$ (by Lemma 3.9). Since $p \in$ int $(K_\infty)^S_{C_c}$, from Lemma 3.7 we get $\langle p, d\rangle \in -$ int C.

Let $y \in K^{\delta_m}$. Then $z_t = y_m + t(y - y_m) \in T_m$ for all sufficiently small $t \in (0,1)$ since $\|y_m\| \leq r < \frac{1}{\varepsilon_m}$ and $y_m \in K^{\delta_m}$. Since $x_m^{\delta_m} \in SOL_{GVVI}(F - \varepsilon_m p, T_m)$, there exists $\xi_m^{'} \in F(x_m^{\delta_m})$ such that

$$\begin{aligned}&\langle \xi_m^{'} - \varepsilon_m p, z_t - x_m^{\delta_m}\rangle = \langle \xi_m^{'} - \varepsilon_m p, y_m + t(y - y_m) - x_m^{\delta_m}\rangle\\ =&t\langle \xi_m^{'} - \varepsilon_m p, y - x_m^{\delta_m}\rangle + (1-t)\langle \xi_m^{'}, y_m - x_m^{\delta_m}\rangle + \varepsilon_m(1-t)(\langle p, x_m^{\delta_m}\rangle - \langle p, y_m\rangle)\\ &\notin -\text{ int } C.\end{aligned} \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$t\langle \xi_m^{'} - \varepsilon_m p, y - x_m^{\delta_m}\rangle + \varepsilon_m(1-t)(\langle p, x_m^{\delta_m}\rangle - \langle p, y_m\rangle) \notin -\text{ int } C. \tag{4.6}$$

Since $\varepsilon_m\langle p, x_m^{\delta_m}\rangle = \langle p, \frac{x_m^{\delta_m}}{\|x_m^{\delta_m}\|}\rangle \to \langle p, d\rangle \in -$ int C and $\varepsilon_m\langle p, y_m\rangle \to 0$ (as $m \to +\infty$), we have

$$\varepsilon_m(\langle p, x_m^{\delta_m}\rangle - \langle p, y_m\rangle) \to \langle p, d\rangle \in -\text{ int } C, \quad (m \to +\infty).$$

This implies that

$$\varepsilon_m(\langle p, x_m^{\delta_m}\rangle - \langle p, y_m\rangle) \in -C \tag{4.7}$$

for all sufficiently large m. It follows from Lemma 2.6, (4.6) and (4.7) that for all sufficiently large m,

$$\langle \xi_m^{'} - \varepsilon_m p, y - x_m^{\delta_m}\rangle \notin -\text{ int } C, \quad \forall y \in K^{\delta_m},$$

which is a contradiction.

Next, we prove (ii). By (i), there exists $m > \max\{r, \frac{1}{\delta}\}$ such that for any $k \geq m$, any $\varepsilon \in (0, \frac{1}{k})$ and any $\delta' \in (0, \frac{1}{k})$, $SOL_{GVVI}(F - \varepsilon p, K^{\delta'}) \neq \emptyset$. Suppose on the contrary that there exists $p \in$ int $(K_\infty)^S_{C_c}$ such that for any $k \geq m$, there are $\varepsilon_k \in (0, \frac{1}{k})$, $\delta_k \in (0, \frac{1}{k})$ and $x_k^{\delta_k} \in SOL_{GVVI}(F - \varepsilon_k p, K^{\delta_k})$ such that $x_k \notin \bar{\mathbf{B}}_{\frac{1}{\varepsilon_k}}$. Then $x_k^{\delta_k} \in K^{\delta_k} \setminus K_r \subseteq K^{\delta_0}\backslash K_r$ for all sufficiently large $k \geq m$ since $\|x_k^{\delta_k}\| > \frac{1}{\varepsilon_k} \geq k \geq m > \max\{r, \frac{1}{\delta}\} \geq r$ and $\delta_k \in (0, \frac{1}{k})$.

By the coercivity condition C_3, there exists $y_k \in K_r$ such that

$$\langle \eta_k, y_k - x_k^{\delta_k}\rangle \in -C, \quad \forall \eta_k \in F(x_k^{\delta_k}). \tag{4.8}$$

Because $y_k \in K \subseteq K^{\delta_k}$ and $x_k^{\delta_k} \in SOL_{GVVI}(F - \varepsilon_k p, K^{\delta_k})$, there exists $\xi_k^{'} \in F(x_k^{\delta_k})$ such that

$$\langle \xi_k^{'}, y_k - x_k^{\delta_k}\rangle + \varepsilon_k(\langle p, x_k^{\delta_k}\rangle - \langle p, y_k\rangle) = \langle \xi_k^{'} - \varepsilon_k p, y_k - x_k^{\delta_k}\rangle \notin -\text{ int } C. \tag{4.9}$$

It follows from (4.8) and (4.9) that

$$\varepsilon_k(\langle p, x_k^{\delta_k}\rangle - \langle p, y_k\rangle) \notin -\text{ int } C. \tag{4.10}$$

Without loss of generality, we assume that $\frac{x_k^{\delta_k}}{\|x_k^{\delta_k}\|} \rightharpoonup e$ as $(k \to +\infty)$. By similar arguments as in the proof of (i), there exists $\widehat{x}_k^{\delta_k} \in K$ such that $\frac{\widehat{x}_m^{\delta_m}}{\|\widehat{x}_m^{\delta_m}\|} \rightharpoonup e \in K_\infty$ as $(k \to +\infty)$.

Since int $barr(K) \neq \emptyset$, we get $e \neq 0$ (by Lemma 2.10) and int $barr(K_\infty) \neq \emptyset$ (by Lemma 3.9). Then, for $p \in$ int $(K_\infty)^S_{C_c}$, we get $\langle p, e\rangle \in -$ int C by the lemma 3.7. Since $\{y_k\} \in K_r$ is bounded and $\|x_k^{\delta_k}\| \to +\infty$ as $k \to +\infty$, it follows that

$$\langle p, \frac{x_k^{\delta_k}}{\|x_k^{\delta_k}\|}\rangle - \langle p, \frac{y_k}{\|x_k^{\delta_k}\|}\rangle \to \langle p, e\rangle \in -\text{ int } C.$$

This implies that

$$\langle p, \frac{x_k^{\delta_k}}{\|x_k^{\delta_k}\|}\rangle - \langle p, \frac{y_k}{\|x_k^{\delta_k}\|}\rangle \in -\text{ int } C$$

for all sufficiently large k. This yields

$$\varepsilon_k(\langle p, x_k^{\delta_k}\rangle - \langle p, y_k\rangle) = \varepsilon_k\|x_k^{\delta_k}\|(\langle p, \frac{x_k^{\delta_k}}{\|x_k^{\delta_k}\|}\rangle - \langle p, \frac{y_k}{\|x_k^{\delta_k}\|}\rangle) \in -\text{ int } C$$

for all sufficiently large k, a contradiction to (4.10). □

Remark 4.4. Recently, Luo [23] investigated nonemptiness and boundedness of the solution set of the scalar generalized variational inequality with the mapping being perturbed by $p \in \text{int}\,(barrK)$ and the constraint set K being perturbed by $K_\infty\backslash\{0\}$ under coercivity and stable quasimonotonicity conditions ([23, Theorem 9]). As a comparison, we established nonemptiness and boundedness of the solution set of the generalized vector variational inequality with the mapping being perturbed by $p \in \text{int}\,(K_\infty)^S_{C_c}$ and the constraint set K being perturbed by the closed unit ball without assuming any monotonicity.

5 Conclusion

In this paper we study the perturbed generalized vector variational inequality in the setting of Banach spaces. Under coercivity condition we prove nonemptiness and boundedness of the solution set of the perturbed vector variational inequality without assuming any monotonicity. Our results extend and improve the corresponding results of [20, 23, 21], where nonemptiness and boundedness of the solution set of the scalar perturbed variational inequality were established.

References

[1] S. Adly, T. Michel and E. Ernst, Stability of the solution set of non-coercive variational inequalities, *Commun. Contemp. Math.* 4 (2002) 145–160.

[2] L.C. Ceng and S. Huang, Existence theorems for generalized vector variational inequalities with a variable ordering relation, *J. Glob. Optim.* 46 (2010) 521–535.

[3] G.Y. Chen, Existence of solutions for a vector variational inequality: an extension of the Hartmann-Stampacchia theorem, *J. Optim. Theory Appl.* 74 (1992) 445–456.

[4] G.Y. Chen, X.X. Huang and X.Q. Yang, *Vector Optimization*, Springer-Verlag, Berlin, Heidelberg, 2005.

[5] G.Y. Chen and X.Q. Yang, The vector complementary problem and its equivalences with the weak minimal element in ordered spaces, *J. Math. Anal. Appl.* 153 (1990) 136–158.

[6] S. Dafermos, Sensitivity analysis in variational inequalities, *Math. Oper. Res.* 13 (1988) 421–434.

[7] K. Fan, A generalization of Tychonoff's fixed point theorem, *Math. Ann.* 142 (1961) 305–310.

[8] K. Fan, Minimax theorems, *Proc. Natl. Acad. Sci. USA* 39 (1953) 42–47.

[9] J. Fan and R. Zhong, Stability analysis for variational inequality in reflexive Banach spaces, *Nonlinear Anal.* 69 (2008) 2566–2574.

[10] Y.P. Fang and N.J. Huang, Feasibility and solvability for vector complementarity problems, *J. Optim. Theory Appl.* 129 (2006) 373–390.

[11] Y.P. Fang, N.J. Huang and J.C. Yao, Well-posedness by perturbations of mixed variational inequalities in Banach spaces, *Eur. J. Oper. Res.* 201 (2010) 682–692.

[12] F. Giannessi, Theorems of alternative, quadratic programs and complementarity problems in variational inequalities and complementarity Problems, in in: *Reformulation: Variational inequalities and complementarity problems*, R.W. Cottle, F. Giannessi and J.L. Lions (eds)., Wiley, 1980, pp. 151–186.

[13] Y.R. He, Stable pseudomonotone variational inequality in reflexive Banach spaces, *J. Math. Anal. Appl.* 330 (2007) 352–363.

[14] N. Hebestreit, Vector variational inequalities and related topics: A survey of theory and applications, *Appl. Set-Valued Anal. Optim.* 1 (2019) 231–305.

[15] N.J. Huang and Y.P. Fang, On vector variational inequalities in reflexive Banach spaces, *J. Glob. Optim.* 32 (2005) 495–505.

[16] I.V. Konnov and J.C. Yao, On the generalized vector variational inequality problem, *J. Math. Anal. Appl.* 206 (1997) 42–58.

[17] J. Kyparisis, Parametric variational inequalities with multivalued solution sets, *Math. Oper. Res.* 17 (1992) 341–364.

[18] J. Kyparisis, Sensitivity analysis framework for variational inequalities, *Math. Program.* 38 (1987) 203–213.

[19] J.L. Li, On the existence of solutions of variational inequalities in Banach spaces, *J. Math. Anal. Appl.* 295 (2004) 115–126.

[20] F.L. Li and Y.R. He, Solvability of a perturbed variational inequality, *Pacific J. Optim.* 10 (2014) 105–111.

[21] Z.J. Li and S.Q. Sun, Solvability of a perturbed variational inequality, *Acta Math. Sci. Chinese Series.* 36 (2016) 473–480 (in Chinese).

[22] D.T. Luc, Recession cones and the domination property in vector optimization, *Math. Program.* 49 (1990) 113–122.

[23] X.P. Luo, Solvability of some perturbed generalized variational inequalities in reflexive Banach spaces, *J. Func. Spaces*, 2018 (2018) Article number: 3897495.

[24] S. Makler-Scheimberg, V.H. Nguyen and J.J. Strodiot, Family of perturbation methods for variational inequalities, *J. Optim. Theory Appl.* 89 (1996) 42-3-452.

[25] R.N. Mukherjee and H.L. Verma, Sensitivity analysis of generalized variational inequalities, *J. Math. Anal. Appl.* 167 (1992) 299–304.

[26] M.A. Noor, Sensitivity analysis for quasi-variational inequalities, *J. Optim. Theory Appl.* 95 (1997) 399–407.

[27] M.A. Noor and K. Noor, T. Rassias, Some aspects of variational inequalities, *J. Comput. Appl. Math.* 47 (1993) 285–312.

[28] Y. Qiu and T.L. Magnanti, Sensitivity analysis for variational inequalities defined on polyhedral sets, *Math. Oper. Res.* 17 (1989) 61–76.

[29] Salahuddin, General set-valued vector variational inequality problems, *Commun. Optim. Theory* 2017 (2017) Article number: 13.

[30] G.J. Tang and Y.S. Li, Existence of solutions for mixed variational inequalities with perturbation in Banach spaces, *Optim. Lett.* 14 (2020) 637–651.

[31] R.T. Tobin, Sensitivity analysis for variational inequalities, *J. Optim. Theory Appl.* 48 (1986) 191–204.

[32] M. Wang, Existence theorems for perturbed variational inequalities in Banach spaces, *J. Fixed Point Theory Appl.* 20 (2018) Article number: 55.

[33] X.Q. Yang and J.C. Yao, Gap functions and existence of solutions to set-valued vector variational inequalities, *J. Optim. Theory Appl.* 115 (2002) 407–417.

[34] J.C. Yao and X.Y. Zheng, Existence of solutions and error bound for vector variational inequalities in Banach spaces, *Optimization* 67 (2018) 1333–1344.

[35] L.C. Zeng and J.C. Yao, Existence of solutions of generalized vector variational inequalities in reflexive Banach spaces, *J. Glob. Optim.* 36 (2006) 483–497.

[36] R.Y. Zhong and N.J. Huang, Stability analysis for Minty mixed variational inequality in reflexive Banach spaces, *J. Optim. Theory Appl.* 147 (2010) 454–472.

Manuscript received 20 September 2020
revised 21 February 2021
accepted for publication 28 February 2021

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