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CHARACTERIZATION OF S-DIAGONAL TENSORS FROM T-SVD FACTORIZATION OF THIRD ORDER TENSORS*

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Abstract: Recently, the tensor singular value decomposition (t-SVD) based on t-product operation of third order tensors has draw intensive scholarly interest. Lately, a set of sufficient and necessary conditions were proposed to characterize the resulting f-diagonal tensor of the t-SVD procedure. Those special f-diagonal tensors are called s-diagonal tensors. In this paper, we eliminate the complex numbers involved in the previous conditions and present a new characterization in real matrix multiplication form. In addition, we find a way to construct s-diagonal tensors with some nonnegative vectors. Finally, some specific checking matrices are presented for reference.

Key words: *T-SVD factorization, s-diagonal tensor, f-diagonal tensor, sufficient and necessary conditions*

Mathematics Subject Classification: *15A69, 15A18*

1 Introduction

The t-product operation and t-SVD (tensor singular value decomposition) factorization were first introduced by Kilmer and Martin [2] in 2011. Since then, they are widely applied in various fields, such as image processing, signal processing, low rank tensor recovery and so on [5, 9, 10, 11, 12, 13, 14, 15, 17]. Meanwhile, theoretical analysis on t-product and t-SVD have been conducted with many meaningful results being obtained in [4, 6, 7, 8, 16].

Through the t-product operation, the t-SVD factorization can be demonstrated as following: An $m \times n \times p$ real tensor $\mathcal{T}$ is decomposed into the t-product of an $m \times m \times p$ orthogonal tensor $\mathcal{U}$, an $m \times n \times p$ f-diagonal tensor $\mathcal{D}$ and an $n \times n \times p$ orthogonal tensor $\mathcal{V}$, i.e.,

$$\mathcal{T} = \mathcal{U} * \mathcal{D} * \mathcal{V}^\top,$$

where $*$ is the t-product operation. According to [8], two $m \times n \times p$ real tensors $\mathcal{A}$ and $\mathcal{B}$ are called orthogonally equivalent, if there exist an $m \times m \times p$ tensor $\mathcal{U}$ and an $n \times n \times p$ tensor $\mathcal{V}$ such that $\mathcal{A} = \mathcal{U} * \mathcal{B} * \mathcal{V}^\top$. Then the t-SVD implies that every third order tensor is orthogonally equivalent with a special f-diagonal tensor. It was further affirmed by Qi et al [8] that tensors which are orthogonally equivalent to a common f-diagonal tensor are mutually orthogonal equivalent and vise versa.

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In [2] , the t-SVD factorization of a third order tensor is implemented by going through three main steps, namely the Discrete Fourier Transform (DFT), matrix singular value decomposition (SVD), and the inverse DFT. This procedure was regarded as a mapping from third order tensors to f-diagonal tensors in [4], which is denoted as $\mathcal{K}(\cdot)$ throughout this paper. In order to depict the property of the resulting tensors of t-SVD through the Kilmer-Martin mapping $\mathcal{K}(\cdot)$, Ling et al. proposed the definition of s-diagonal tensors in [4] to separate them from the general f-diagonal tensors. It was found that an f-diagonal tensor $\mathcal{T}$ is s-diagonal if and only if $\mathcal{K}(\mathcal{T}) = \mathcal{T}$. Based on this property, four meaningful necessary conditions for s-diagonal tensors were obtained. Right after that, a set of sufficient and necessary conditions for s-diagonal tensors were proposed. However, some of these conditions are expressed with complex numbers, which make them seemingly inconvenient and not straightforward for practical use. Hence in this paper, we attempt to provide a different version of sufficient and necessary conditions without complex numbers and perhaps facilitates the computation.

The remaining of this article is distributed as follows. In Section 2, we provide some preliminary knowledge on t-product, t-SVD factorization and s-diagonal tensors. In Section 3, the characterization and construction of s-diagonal tensors through real matrix multiplication are proposed. Finally, we present some specific tool matrices for tensors with frontal slice number $p \leq 20$.

2 Preliminaries

In the following scenario, capital letters $A, B, \ldots$ are used to denote matrices and Euler script letters $\mathcal{A}, \mathcal{B}, \ldots$ are used to denote tensors. The real number field is denoted by $\mathbb{R}$, and the complex number field is denoted by $\mathbb{C}$. Let $\mathcal{T}$ be a third order tensor in $\mathbb{R}^{m\times n\times p}$. The (i,j,k)-th entry of $\mathcal{T}$ is denoted as t_{ijk}, and the i-th horizontal, lateral and frontal slice of $\mathcal{T}$ are represented by $\mathcal{T}_{i..}$, $\mathcal{T}_{.i.}$ and $\mathcal{T}_{..i}$ respectively. Specifically, we use $\mathcal{T}^{(i)}$ to denote the i-th frontal slice $\mathcal{T}_{..i}$. The (i,j)-th tube of $\mathcal{T}$, denoted as $\mathcal{T}_{ij}$, refers to the vector $(t_{ij1}, t_{ij2}, \ldots, t_{ijp})^\top$. Furthermore, for convenience, we denote $[n] := \{1, 2, \ldots, n\}$ for a positive integer n.

2.1 Basic definitions

We now introduce some basic definitions and notations from [1, 2] in the beginning. A third order tensor $\mathcal{T}$ in $\mathbb{R}^{m\times n\times p}$ is called f-diagonal if all of its frontal slices $\mathcal{T}^{(1)}, \ldots, \mathcal{T}^{(p)}$ are diagonal. The diagonal entries of its frontal slices are called the diagonal entries of $\mathcal{T}$.

The block circulant matrix of a third order tensor $\mathcal{T} \in \mathbb{R}^{m\times n\times p}$ is defined as

$$\operatorname{bcirc}(\mathcal{T}) := \begin{pmatrix} \mathcal{T}^{(1)} & \mathcal{T}^{(p)} & \mathcal{T}^{(p-1)} & \ldots & \mathcal{T}^{(2)} \\ \mathcal{T}^{(2)} & \mathcal{T}^{(1)} & \mathcal{T}^{(p)} & \ldots & \mathcal{T}^{(3)} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \mathcal{T}^{(p)} & \mathcal{T}^{(p-1)} & \mathcal{T}^{(p-2)} & \ldots & \mathcal{T}^{(1)} \end{pmatrix},$$

and $\operatorname{bcirc}^{-1}(\operatorname{bcirc}(\mathcal{T})) := \mathcal{T}$.

The identity tensor $\mathcal{I}_{nnp}$ is defined as

$$\mathcal{I}_{nnp} = \operatorname{bcirc}^{-1}(I_{np}),$$

where I_{np} is the identity matrix of order np.

The transpose of a tensor $\mathcal{T} \in \mathbb{R}^{m\times n\times p}$, denoted as $\mathcal{T}^\top$, is a tensor in $\mathbb{R}^{n\times m\times p}$ satisfying

$$\mathrm{bcirc}(\mathcal{T}^\top) = (\mathrm{bcirc}(\mathcal{T}))^\top.$$

In [2], the unfold and fold operations for a third order tensor $\mathcal{T} \in \mathbb{R}^{m\times n\times p}$ are defined as

$$\mathrm{unfold}(\mathcal{T}) := \begin{pmatrix} \mathcal{T}^{(1)} \\ \mathcal{T}^{(2)} \\ \vdots \\ \mathcal{T}^{(p)} \end{pmatrix} \in \mathbb{R}^{mp\times n},$$

and $\mathrm{fold}(\mathrm{unfold}(\mathcal{T})) := \mathcal{T}$.

The t-product of two third order tensors $\mathcal{U} \in \mathbb{R}^{m\times s\times p}$ and $\mathcal{V} \in \mathbb{R}^{s\times n\times p}$ is defined as

$$\mathcal{U} * \mathcal{V} := \mathrm{fold}[\mathrm{bcirc}(\mathcal{U})\mathrm{unfold}(\mathcal{V})] \in \mathbb{R}^{m\times n\times p}.$$

It can be seen that

$$\mathrm{bcirc}(\mathcal{U} * \mathcal{V}) = \mathrm{bcirc}(\mathcal{U})\mathrm{bcirc}(\mathcal{V}). \tag{2.1}$$

According to [2], by applying the fast Fourier transform (FFT), the t-product operation (2.1) can be done with computational cost of $O(mnsp)$ flops.

A tensor $\mathcal{P} \in \mathbb{R}^{n\times n\times p}$ is called an orthogonal tensor if

$$\mathcal{P} * \mathcal{P}^\top = \mathcal{P}^\top * \mathcal{P} = \mathcal{I}_{nnp}.$$

$\boxed{2.2}$ The Kilmer-Martin mapping and s-diagonal tensors

Based on the t-product operation, Kilmer and Martin [2] proposed the t-SVD factorization for third order tensors, which can be regarded as a procedure mapping an arbitrary third order tensor to a special f-diagonal third order tensor.

In the following, this procedure will be introduced in details by three steps. It is called the Kilmer-Martin mapping in [8].

Let $\mathcal{T} \in \mathbb{R}^{m\times n\times p}$. At first, we conduct the block-diagonalization of $\mathrm{bcirc}(\mathcal{T})$ through a Discrete Fourier Transform (DFT) as displayed below:

$$M(\mathcal{T}) := (F_p \otimes I_m)\mathrm{bcirc}(\mathcal{T})(F_p^* \otimes I_n) = \begin{pmatrix} M^{(1)} & & & \\ & M^{(2)} & & \\ & & \ddots & \\ & & & M^{(p)} \end{pmatrix}, \tag{2.2}$$

where F_p is the Fourier matrix of order p, whose (i,j)-th entry is

$$\omega^{(i-1)(j-1)} = \left[\cos\frac{2\pi(i-1)(j-1)}{p} - \sqrt{-1}\sin\frac{2\pi(i-1)(j-1)}{p}\right],$$

F_p^* is the conjugate of F_p, $\otimes$ denotes the Kronecker product, and $M^{(k)} \in \mathbb{C}^{m\times n}$ for $k = 1,\ldots,p$. Here, the notation $\sqrt{-1}$ is the imaginary unit and $\omega, \omega^2, \ldots, \omega^p$ are the p complex number roots of $x^p = 1$.

Secondly, we apply the standard SVD procedure to each $M^{(k)}$ as

$$M^{(k)} = U^{(k)} N^{(k)} {V^{(k)}}^*,$$

where $U^{(k)} \in \mathbb{C}^{m\times m}$ and $V^{(k)} \in \mathbb{C}^{n\times n}$ are unitary matrices, $N^{(k)} \in \mathbb{R}^{m\times n}$ is diagonal with the diagonal entries (i.e. the singular values of $M^{(k)}$) following a non-increasing order. Denote

$$N(\mathcal{T}) := \begin{pmatrix} N^{(1)} & & & \\ & N^{(2)} & & \\ & & \ddots & \\ & & & N^{(p)} \end{pmatrix} \tag{2.3}$$

and

$$U(\mathcal{T}) := \begin{pmatrix} U^{(1)} & & & \\ & U^{(2)} & & \\ & & \ddots & \\ & & & U^{(p)} \end{pmatrix}, V(\mathcal{T}) := \begin{pmatrix} V^{(1)} & & & \\ & V^{(2)} & & \\ & & \ddots & \\ & & & V^{(p)} \end{pmatrix}.$$

Then we have $M = UNV^*$. Finally, an inverse DFT is used to produce a block circulant matrix D as below:

$$D = (F_p^* \otimes I_m) N(\mathcal{T}) (F_p \otimes I_n) = \begin{pmatrix} D^{(1)} & D^{(p)} & \dots & D^{(2)} \\ D^{(2)} & D^{(1)} & \dots & D^{(3)} \\ \vdots & \vdots & \ddots & \vdots \\ D^{(p)} & D^{(p-1)} & \dots & D^{(1)} \end{pmatrix}, \tag{2.4}$$

and let $\mathcal{D} := \mathrm{bcirc}^{-1}(D)$. Then the k-th frontal slice of $\mathcal{D}$ is $\mathcal{D}^{(k)} = D^{(k)}$ in (2.4).

The above Kilmer-Martin mapping is denoted as $\mathcal{K}(\cdot)$, and the previous discussion states $\mathcal{K}(\mathcal{T}) = \mathcal{D}$. Now we construct two tensors $\mathcal{U}$ and $\mathcal{V}$ from the two block diagonal matrices $U(\mathcal{T})$ and $V(\mathcal{T})$ such that

$$\mathrm{bcirc}(\mathcal{U}) = (F_p^* \otimes I_m) U(\mathcal{T}) (F_p \otimes I_m),$$

$$\mathrm{bcirc}(\mathcal{V}) = (F_p^* \otimes I_n) V(\mathcal{T}) (F_p \otimes I_n).$$

Since F_p and each diagonal block of U, V are unitary matrices, the tensors $\mathcal{U} \in \mathbb{R}^{m\times m\times p}$ and $\mathcal{V} \in \mathbb{R}^{n\times n\times p}$ are orthogonal tensors. Then $\mathcal{T}$ has its t-SVD

$$\mathcal{T} = \mathcal{U} * \mathcal{D} * \mathcal{V}^\top.$$

It should be aware that if the matrix SVD in the second step is not taking the standard order, i.e. the diagonal entries of each $N^{(k)}$ do not follow a non-increasing order, then the proceeding step may yield an f-diagonal tensor having distinct diagonal entry set with $\mathcal{D}$.

In order to facilitate the discussions in the subsequent sections, we now present the specific transformation formulas of DFT and inverse DFT among $\mathcal{T}^{(k)}$, $M^{(k)}$ and $N^{(k)}$, $D^{(k)}$. Denote M_{ij} as the (i, j)-th entry of a matrix M.

By (2.2) and (2.4), with an explicit form of F_p and F_p^*, we have for $i = 1, \dots, m$, $j = 1, \dots, n$ and $k = 1, \dots, p$ that

$$M_{ij}^{(k)} = \sum_{l=1}^{p} \omega^{(k-1)(l-1)} \mathcal{T}_{ij}^{(l)}, \tag{2.5}$$

$$D_{ij}^{(k)} = \frac{1}{p} \sum_{l=1}^{p} \bar{\omega}^{(k-1)(l-1)} N_{ij}^{(l)}, \tag{2.6}$$

and thus

$$\mathcal{T}_{ij}^{(k)} = \frac{1}{p} \sum_{l=1}^{p} \bar{\omega}^{(k-1)(l-1)} M_{ij}^{(l)}, \tag{2.7}$$

$$N_{ij}^{(k)} = \sum_{l=1}^{p} \omega^{(k-1)(l-1)} D_{ij}^{(l)}, \tag{2.8}$$

where $\bar{\omega} = \omega^{-1}$ is the conjugate number of ω.

It can be seen from (2.6) that each $D^{(k)}$ is diagonal as $N(\mathcal{T})$ is a block diagonal matrix. Therefore, the resulting tensor of the Kilmer-Martin mapping $\mathcal{K}(\mathcal{T}) = \mathcal{D} = \mathrm{bcirc}^{-1}(D)$ is f-diagonal.

However, an arbitrary f-diagonal tensor may not be the mapping result of a third order tensor, i.e. the set of all f-diagonal tensors in $\mathbb{R}^{m\times n\times p}$ differs from the set of all resulting tensors $\mathcal{K}(\mathcal{T})$ for $\mathcal{T} \in \mathbb{R}^{m\times n\times p}$. We denote the former set as $\mathbb{F}^{m\times n\times p}$ and the later set as $\mathbb{S}^{m\times n\times p}$, then $\mathbb{S}^{m\times n\times p} \subsetneq \mathbb{F}^{m\times n\times p}$. To distinguish these two types of tensors, a tensor in $\mathbb{S}^{m\times n\times p}$ is called an s-diagonal tensor [4].

In [4], the invariance of an s-diagonal tensor under the Kilmer-Martin mapping was found, which may be considered as an intrinsic property of s-diagonal tensors.

Theorem 2.1 ([4]). *Let $\mathcal{T} \in \mathbb{F}^{m\times n\times p}$. Then $\mathcal{T} \in \mathbb{S}^{m\times n\times p}$ if and only if*

$$\mathcal{K}(\mathcal{T}) = \mathcal{T}. \tag{2.9}$$

The equation (2.9) actually means the equivalence of the two matrices $M(\mathcal{T})$ and $N(\mathcal{T})$ stated in the second step of the Kilmer-Martin mapping. Base on that, sufficient and necessary conditions of s-diagonal tensors have been deduced. They are now displayed below in a slightly different way.

Theorem 2.2 ([4]). *Let $\mathcal{T} \in \mathbb{F}^{m\times n\times p}$ such that*

$$\mathcal{T}^{(k)} = \mathcal{T}^{(p-k+2)} \quad \text{for all } k = 2, \dots, p.$$

Let $q = \min\{m, n\}$. Then $\mathcal{T} \in \mathbb{S}^{m\times n\times p}$ if and only if for all $k \in [p]$,

(1) *the vector $\delta_i \in \mathbb{C}^p$ with the k-th element being $\sum_{l=1}^{p} \omega^{(k-1)(l-1)} \mathcal{T}_{ii}^{(l)}$ is real and nonnegative, where $i \in [q]$;*

(2) *the vector $\delta_i - \delta_{i+1} \in \mathbb{C}^p$ is real and nonnegative, where $i \in [q-1]$.*

The above conditions involve the p-th roots of unity ω, which is a complex number and may cause some inconvenience in computation. In practice, it can be replaced by real number in expression of cosine value. In the next section, we provide a sufficient and necessary condition of s-diagonal tensors without any complex number.

$\boxed{3}$ Characterization and Construction of s-Diagonal Tensors

For a tensor $\mathcal{T} \in \mathbb{F}^{m\times n\times p}$, let $M = M(\mathcal{T})$ and $N = N(\mathcal{T})$ be the block diagonal matrices in (2.2) and (2.3).

We use $\mathrm{Re}(\alpha)$ to denote the real part of a complex number $\alpha \in \mathbb{C}$. For a complex matrix A, the notation $\mathrm{Re}(A)$ represents the matrix obtained from A by removing the imaginary part of each entry in A.

Theorem 3.1. *Let $\mathcal{T} \in \mathbb{F}^{m\times n\times p}$ and $q = \min\{m, n\}$. Then $\mathcal{T} \in \mathbb{S}^{m\times n\times p}$ if and only if*

(1) *$\mathcal{T}^{(k)} = \mathcal{T}^{(p-k+2)}$, for each $k = 2, \ldots, p$,*

(2) *the matrix R_pT is nonnegative,*

where $R_p = \mathrm{Re}(F_p) \in \mathbb{R}^{p\times p}$, F_p is the Fourier matrix of order p and

$$T = (\mathcal{T}_{11}, \ldots, \mathcal{T}_{qq}, \mathcal{T}_{11} - \mathcal{T}_{22}, \ldots, \mathcal{T}_{q-1,q-1} - \mathcal{T}_{qq}) \in \mathbb{R}^{p\times(2q-1)}, \tag{3.1}$$

while $\mathcal{T}_{ii}$ refers to the (i, i)-th tube of $\mathcal{T}$ for $i \in [q]$.

Proof. Suppose $\mathcal{T} \in \mathbb{S}^{m\times n\times p}$. It has been shown in Theorem 5.1 of [4] that the frontal slices of $\mathcal{T}$ satisfy the condition (1), which is called the third mode symmetry property there.

Moreover, as stated in the last section, the equivalent condition $\mathcal{K}(\mathcal{T}) = \mathcal{T}$ of a tensor $\mathcal{T}$ being s-diagonal in Theorem 2.1 actually means $M = N$. Then $M^{(k)} = N^{(k)}$ for each $k \in [p]$, which implies $M_{ii}^{(k)} \in \mathbb{R}^+$ and $M_{ii}^{(k)} - M_{i+1,i+1}^{(k)} \in \mathbb{R}^+$. Here $\mathbb{R}^+$ is the set of all nonnegative real numbers. Therefore, according to (2.5) we have

$$\begin{aligned}
&\mathrm{Re}\left(1, \omega^{(k-1)}, \ldots, \omega^{(k-1)(p-1)}\right) \mathcal{T}_{ii} \\
= \;& \mathrm{Re}\left(1, \omega^{(k-1)}, \ldots, \omega^{(k-1)(p-1)}\right) \begin{pmatrix} \mathcal{T}_{ii}^{(1)} \\ \mathcal{T}_{ii}^{(2)} \\ \vdots \\ \mathcal{T}_{ii}^{(p)} \end{pmatrix} \\
= \;& \sum_{l=1}^{p} \mathrm{Re}(\omega^{(k-1)(l-1)}) \mathcal{T}_{ii}^{(l)} \\
= \;& \mathrm{Re}(M_{ii}^{(k)}) = M_{ii}^{(k)} \in \mathbb{R}^+
\end{aligned}$$

for $i \in [q]$, and thus for each $i \in [q-1]$,

$$\mathrm{Re}\left(1, \omega^{(k-1)}, \ldots, \omega^{(k-1)(p-1)}\right) (\mathcal{T}_{ii} - \mathcal{T}_{i+1,i+1}) = M_{ii}^{(k)} - M_{i+1,i+1}^{(k)} \in \mathbb{R}^+.$$

Note that the p-dimensional row vector $\mathrm{Re}\left(1, \omega^{(k-1)}, \ldots, \omega^{(k-1)(p-1)}\right)$ is the k-th row of the matrix $\mathrm{Re}(F_p) = R_p$. Hence the entries in each row of R_pT are real and nonnegative, which directly approves the condition (2).

On the other hand, suppose $\mathcal{T} \in \mathbb{F}^{m\times n\times p}$ satisfies the conditions (1) and (2). Theorem 6.1 in [4] shows that the third mode symmetry property of $\mathcal{T}$ will result in a real $M = M(\mathcal{T})$, i.e. $\mathrm{Re}(M) = M$. Together with the condition $R_pT \geq 0$, we have

$$M_{ii}^{(k)} = \mathrm{Re}(M_{ii}^{(k)}) = \mathrm{Re}\left(1, \omega^{(k-1)}, \ldots, \omega^{(k-1)(p-1)}\right) \mathcal{T}_{ii} \geq 0,$$

for $i \in [q]$, and for every $i \in [q-1]$,

$$M_{ii}^{(k)} - M_{i+1,i+1}^{(k)} = \mathrm{Re}\left(1, \omega^{(k-1)}, \ldots, \omega^{(k-1)(p-1)}\right)(\mathcal{T}_{ii} - \mathcal{T}_{i+1,i+1}) \geq 0.$$

In conclusion, each diagonal block of M is a real nonnegative diagonal matrix with diagonal entries following a non-increasing order, which yields $M^{(k)} = N^{(k)}$ for each $k \in [p]$. Thus $M(\mathcal{T}) = N(\mathcal{T})$ and then $\mathcal{K}(\mathcal{T}) = \mathcal{T}$. By Theorem 2.1, the sufficiency of (1) and (2) is also affirmed. □

The above theorem characterizes s-diagonal tensors in a more concise form through real matrix multiplication condition instead of equations using complex numbers. Furthermore, utilizing the symmetry properties of $\mathcal{T}^{(k)}$ and $M^{(k)}$, the matrices R_p and T in Theorem 3.1 can be reduced to matrices in nearly half of their original sizes.

First we should illustrate the conjugate symmetry on diagonal blocks $M^{(2)}, \ldots, M^{(p)}$ of $M(\mathcal{T})$ for $\mathcal{T} \in \mathbb{R}^{m\times n\times p}$.

Note that $\omega^{[(p-k+2)-1](l-1)} = \omega^{(1-k)(l-1)} = \bar{\omega}^{(k-1)(l-1)}$. By the equation (2.5) we have

$$M^{(p-k+2)} = \sum_{l=1}^{p} \bar{\omega}^{(k-1)(l-1)}\mathcal{T}^{(l)} = \overline{\sum_{l=1}^{p} \omega^{(k-1)(l-1)}\mathcal{T}^{(l)}} = \overline{M^{(k)}},$$

i.e. each entry of $M^{(p-k+2)}$ is the conjugate number of the corresponding entry of $M^{(k)}$. Furthermore, if the tensor $\mathcal{T}$ satisfies the third mode symmetry property (the condition (1) in Theorem 3.1), then $M(\mathcal{T})$ is real, and thus

$$M^{(p-k+2)} = M^{(k)},\ k = 2, \ldots, p.$$

Considering the symmetry of $\mathcal{T}^{(2)}, \ldots, \mathcal{T}^{(p)}$ as well as $M^{(2)}, \ldots, M^{(p)}$, if $\mathcal{T}$ satisfies the condition (1), then by equation (2.5), for every $i \in [q]$ we have

$$\begin{aligned} M_{ii}^{(k)} &= \sum_{l=1}^{p} \mathrm{Re}\{\omega^{(k-1)(l-1)}\}\mathcal{T}_{ii}^{(l)} \\ &= \mathcal{T}_{ii}^{(1)} + \begin{cases} \sum_{l=2}^{\frac{p+1}{2}} 2\mathrm{Re}\{\omega^{(k-1)(l-1)}\}\mathcal{T}_{ii}^{(l)}, & p \text{ is odd}, \\ \sum_{l=2}^{\frac{p}{2}} 2\mathrm{Re}\{\omega^{(k-1)(l-1)}\}\mathcal{T}_{ii}^{(l)} + (-1)^{(k-1)}\mathcal{T}_{ii}^{(\frac{p}{2}+1)}, & p \text{ is even}. \end{cases} \end{aligned}$$

Denote $\theta = \frac{2\pi}{p}$. Now $\mathrm{Re}(\omega) = \cos\theta$ and the above equation is transformed to

$$M_{ii}^{(k)} = \mathcal{T}_{ii}^{(1)} + \begin{cases} \sum_{l=2}^{\frac{p+1}{2}} 2\cos[(k-1)(l-1)\theta]\mathcal{T}_{ii}^{(l)}, & p \text{ is odd}, \\ \sum_{l=2}^{\frac{p}{2}} 2\cos[(k-1)(l-1)\theta]\mathcal{T}_{ii}^{(l)} + (-1)^{(k-1)}\mathcal{T}_{ii}^{(\frac{p}{2}+1)}, & p \text{ is even}. \end{cases} \tag{3.2}$$

Therefore, to judge whether an f-diagonal tensor $\mathcal{T}$ is s-diagonal, we only need $\mathcal{T}^{(1)}, \ldots, \mathcal{T}^{(\lceil \frac{p+1}{2} \rceil)}$ to determine the nonnegativity and diagonal decaying property of $M^{(1)}, \ldots, M^{(\lceil \frac{p+1}{2} \rceil)}$. The notation "$\lceil \frac{p+1}{2} \rceil$" represents the smallest integer greater than or equal to $\frac{p+1}{2}$.

Define $\hat{R}_p \in \mathbb{R}^{\lceil \frac{p+1}{2} \rceil \times \lceil \frac{p+1}{2} \rceil}$ as

$$\hat{R}_p = \begin{pmatrix} 1 & 2 & \dots & 2 \\ 1 & 2\cos\theta & \dots & 2\cos\frac{(p-1)\theta}{2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2\cos\frac{(p-1)\theta}{2} & \dots & 2\cos\frac{(p-1)^2\theta}{4} \end{pmatrix} \text{ for odd } p, \tag{3.3}$$

and

$$\hat{R}_p = \begin{pmatrix} 1 & 2 & \dots & 2 & 1 \\ 1 & 2\cos\theta & \dots & 2\cos(\frac{p}{2}-1)\theta & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2\cos\frac{p}{2}\theta & \dots & 2\cos\frac{p}{2}(\frac{p}{2}-1)\theta & (-1)^{k-1} \end{pmatrix} \text{ for even } p. \tag{3.4}$$

Define the matrix

$$S = (S_{11}, \dots, S_{qq}, S_{11} - S_{22}, \dots, S_{q-1,q-1} - S_{qq}) \in \mathbb{R}^{\lceil \frac{p+1}{2} \rceil \times (2q-1)}, \tag{3.5}$$

where each S_{ii} is a column vector formed by the first $\lceil \frac{p+1}{2} \rceil$ entries of the (i,i)-th tube $\mathcal{T}_{ii}$ of $\mathcal{T} \in \mathbb{R}^{m\times n\times p}$. Now by equation (3.2), we have

$$\begin{pmatrix} M_{ii}^{(1)} \\ M_{ii}^{(2)} \\ \vdots \\ M_{ii}^{(\lceil \frac{p+1}{2} \rceil)} \end{pmatrix} = \hat{R}_p S_{ii} \quad \text{and} \quad \begin{pmatrix} M_{ii}^{(1)} - M_{i+1,i+1}^{(1)} \\ M_{ii}^{(2)} - M_{i+1,i+1}^{(2)} \\ \vdots \\ M_{ii}^{(\lceil \frac{p+1}{2} \rceil)} - M_{i+1,i+1}^{(\lceil \frac{p+1}{2} \rceil)} \end{pmatrix} = \hat{R}_p(S_{ii} - S_{i+1,i+1}),$$

whose nonnegativity coincides the nonnegativity of $R_p T$ under the third mode symmetry property of $\mathcal{T}$.

Then according to Theorem 3.1, the following corollary is obtained directly.

Corollary 3.2. *Let $\mathcal{T} \in \mathbb{F}^{m\times n\times p}$ and $q = \min\{m,n\}$. Then $\mathcal{T} \in \mathbb{S}^{m\times n\times p}$ if and only if*

(1) *$\mathcal{T}^{(k)} = \mathcal{T}^{(p-k+2)}$, for each $k = 2, \dots, p$,*

(2) *the matrix $\hat{R}_p S$ is nonnegative,*

where $\hat{R}_p \in \mathbb{R}^{\lceil \frac{p+1}{2} \rceil \times \lceil \frac{p+1}{2} \rceil}$ is defined by (3.3) *and* (3.4)*, and $S \in \mathbb{R}^{\lceil \frac{p+1}{2} \rceil \times (2q-1)}$ is defined in* (3.5)*.*

We may call $\hat{R}_p$ the checking matrix of s-diagonal tensors with p frontal slices. Denote

$$\Delta = (\delta_1, \dots, \delta_q, \delta_1 - \delta_2, \dots, \delta_{q-1} - \delta_q) \in \mathbb{C}^{p\times(2q-1)},$$

where δ_i is the vector defined in Theorem 2.2 (1) for $i \in [q]$. Let $\hat{\Delta} \in \mathbb{C}^{\lceil \frac{p+1}{2} \rceil \times (2q-1)}$ be the matrix formed by the first $\lceil \frac{p+1}{2} \rceil$ rows of Δ. Observe that the k-th element of δ_i is $M_{ii}^{(k)}$. If $\mathcal{T}$ possesses the third mode property, then Δ and $\hat{\Delta}$ are real and

$$R_p T = \Delta, \quad \hat{R}_p S = \hat{\Delta}.$$

By (2.7), with a similar process as we derive Theorem 3.1 and Corollary 3.2, we have

$$T = \mathrm{Re}(\frac{1}{p}F_p^*)\Delta = \frac{1}{p}R_p\Delta,$$

and thus by the third mode symmetry property,

$$S = \frac{1}{p}\hat{R}_p\hat{\Delta}. \tag{3.6}$$

The equation (3.6) provides a way to construct an s-diagonal tensor by matrix multiplication using a series of nonnegative vectors. We present it in the following table.

Steps	
1	Pick q nonnegative vectors $\Delta_1, \ldots, \Delta_{q-1}, d_q \in \mathbb{R}^{\lceil \frac{p+1}{2} \rceil}$
2	Set $d_{i-1} = \Delta_{i-1} + d_i$, $i = 2, \ldots, q$
3	Compute $\tilde{S} = \frac{1}{p}\hat{R}_p(d_1, \ldots, d_q) \in \mathbb{R}^{\lceil \frac{p+1}{2} \rceil \times q}$
4	Construct an s-diagonal tensor $\mathcal{D} \in \mathbb{R}^{m\times n\times p}$ with $\min\{m, n\} = q$: 1° for $k = 1, \ldots, \lceil \frac{p+1}{2} \rceil$, set $\mathcal{D}_{ii}^{(k)} = \tilde{S}_{ki}$, $i \in [q]$ and other non-diagonal entries being zero; 2° for $k = \lceil \frac{p+1}{2} \rceil + 1, \ldots, p$, set $\mathcal{D}^{(k)} = \mathcal{D}^{(p-k+2)}$.

In fact, the vectors and matrices generated by those steps are linked with the ones in the previous analysis:

The vectors $d_1, \ldots, d_q$ created by steps 1 and 2 actually correspond to the first q column vectors of $\hat{\Delta}$, which are nonnegative vectors

The q column vectors of $\tilde{S}$ correspond to the first q column vectors of S in (3.6)

The f-diagonal tensor $\mathcal{D}$ is constructed by filling the first $\lceil \frac{p+1}{2} \rceil$ entries of its (i, i)-th tube with the i-th column vector of $\tilde{S}$, and then generating the remaining entries by the third mode symmetry property along each tube.

According to the above corresponding relations, it is clear from Corollary 3.2 that the tensor $\mathcal{D}$ is s-diagonal.

Observe that the matrices $\hat{R}_p$ and $\frac{1}{p}\hat{R}_p$ represent the linear transformation from S to $\hat{\Delta}$ and the inverse transformation from $\hat{\Delta}$ to S, respectively. The following proposition show that they have a further relation.

Proposition 3.3. *Let $\hat{R}_p \in \mathbb{R}^{\lceil \frac{p+1}{2} \rceil \times \lceil \frac{p+1}{2} \rceil}$ be the matrix defined in* (3.3) *and* (3.4)*, where p is an positive integer no less than 2. Then $\frac{1}{p}\hat{R}_p$ is the inverse matrix of $\hat{R}_p$.*

Proof. Denote the i-th row vector and the j-th column vector of $\hat{R}_p$ as $\mathbf{a}_i^\top$ and $\mathbf{b}_j$ respectively for $i, j = 1, \ldots, \lceil \frac{p+1}{2} \rceil$. It suffices to show that $\hat{R}_p^2 = pI_p$, which means

$$\mathbf{a}_i^\top \mathbf{b}_j = \begin{cases} p, & i = j, \\ 0, & i \neq j, \end{cases} \qquad \text{for } i, j = 1, \ldots, \lceil \frac{p+1}{2} \rceil.$$

Note from (3.3) and (3.4) that the i-th row sum of $\hat{R}_p$ equals the i-th row sum of $R_p = \mathrm{Re}(F_p)$. Thus for $i = 1, \dots, \lceil \frac{p+1}{2} \rceil$,

$$\sum_{l=1}^{\lceil \frac{p+1}{2} \rceil} \mathbf{a}_i^\top(l) = \sum_{l=1}^{p} \mathrm{Re}\left(\omega^{(l-1)(i-1)}\right) = \mathrm{Re}\left(\sum_{l=1}^{p} \omega^{(l-1)(i-1)}\right) = \begin{cases} p, & i = 1, \\ 0, & i \neq 1. \end{cases}$$

The last equation follows from the property of ω that

$$\sum_{l=1}^{p} \omega^{(l-1)m} = \begin{cases} p, & m \equiv 0 \pmod{p}, \\ 0, & \text{otherwise.} \end{cases}$$

If p is even, then there are three cases to discuss. When $j = 1$, we have

$$\mathbf{a}_i^\top \mathbf{b}_1 = \sum_{l=1}^{\lceil \frac{p+1}{2} \rceil} \mathbf{a}_i^\top(l) = \begin{cases} p, & i = 1, \\ 0, & i \neq 1. \end{cases}$$

When $j = 2, \dots, \frac{p}{2}$, we have

$$\begin{aligned} \mathbf{a}_i^\top \mathbf{b}_j &= \sum_{l=1}^{p} 2\cos[(l-1)(i-1)\theta]\cos[(l-1)(j-1)\theta] \\ &= \sum_{l=1}^{p} \{\cos[(l-1)(i+j-2)\theta] + \cos[(l-1)(i-j)\theta]\} \\ &= \mathrm{Re}\left(\sum_{l=1}^{p} \omega^{(l-1)(i+j-2)}\right) + \mathrm{Re}\left(\sum_{l=1}^{p} \omega^{(l-1)(i-j)}\right) \\ &= \begin{cases} 0 + p = p, & i = j, \\ 0 + 0 = 0, & i \neq j. \end{cases} \end{aligned}$$

The first equation follows from $\cos(l-1)(i-1)\theta = \cos[(p-l+2)-1](i-1)\theta$, while the second one is derived from the product-to-sum formula $2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$.

When $j = \frac{p}{2} + 1$, we have $\mathbf{b}_{\frac{p}{2}+1}(l) = (-1)^{(l-1)} = \cos(l-1)\frac{p}{2}\theta$. Thus

$$\begin{aligned} \mathbf{a}_i^\top \mathbf{b}_{\frac{p}{2}+1} &= \sum_{l=1}^{p} \cos[(l-1)(i-1)\theta]\cos[(l-1)\frac{p}{2}\theta] \\ &= \frac{1}{2}\mathrm{Re}\left(\sum_{l=1}^{p} \omega^{(l-1)(i+\frac{p}{2}-1)}\right) + \frac{1}{2}\mathrm{Re}\left(\sum_{l=1}^{p} \omega^{(l-1)(i-\frac{p}{2}-1)}\right) \\ &= \begin{cases} \frac{p}{2} + \frac{p}{2} = p, & i = \frac{p}{2} + 1, \\ 0 + 0 = 0, & i \neq \frac{p}{2} + 1. \end{cases} \end{aligned}$$

Therefore, the conclusion holds when p is even. While p is odd, the two cases $j = 1$ and $j \neq 1$ can be verified similarly as we deal with the above first two cases. Now the proof is completed. $\square$

In the final section, we present some specific $\hat{R}_p$ with special values of p.

$\boxed{4}$ Checking Matrices for Some $p \leq 20$

According to the equations (3.3) and (3.4), we can write out the specific matrix $\hat{R}_p$ when p is taking some values with $\cos\theta = \cos\frac{2\pi}{p}$ easy to compute.

For $p = 2$, $\theta = \pi$ and $\cos\theta = -1$,

$$\hat{R}_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

For $p = 3$, $\theta = \frac{2}{3}\pi$ and $\cos\theta = -\frac{1}{2}$,

$$\hat{R}_3 = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}.$$

For $p = 4$, $\theta = \frac{\pi}{2}$ and $\cos\theta = 0, \cos 2\theta = -1$,

$$\hat{R}_4 = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -2 \\ 1 & -2 & 2 \end{pmatrix}.$$

For $p = 6$, $\theta = \frac{\pi}{3}$ and $\cos\theta = \frac{1}{2} = \cos 5\theta, \cos 2\theta = -1 = \cos 4\theta, \cos 3\theta = -1$,

$$\hat{R}_6 = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -2 & 2 & -1 \end{pmatrix},$$

For $p = 8$, $\theta = \frac{\pi}{4}$ and $\cos\theta = \frac{\sqrt{2}}{2} = \cos 7\theta = -\cos 3\theta = -\cos 5\theta, \cos 2\theta = 0 = \cos 6\theta, \cos 4\theta = -1$,

$$\hat{R}_8 = \begin{pmatrix} 1 & 2 & 2 & 2 & 1 \\ 1 & \sqrt{2} & 0 & -\sqrt{2} & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & -\sqrt{2} & 0 & \sqrt{2} & -1 \\ 1 & -2 & 2 & -2 & 1 \end{pmatrix}.$$

For $p = 12$, $\theta = \frac{\pi}{6}$ and

$$\begin{aligned} & (1, \cos\theta, \cos 2\theta, \ldots, \cos 11\theta) \\ = \; & (1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1, -\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{\sqrt{3}}{2}), \end{aligned}$$

$$\hat{R}_{12} = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & \sqrt{3} & 1 & 0 & -1 & -\sqrt{3} & -1 \\ 1 & 1 & -1 & -2 & -1 & 1 & 1 \\ 1 & 0 & -2 & 0 & 2 & 0 & -1 \\ 1 & -1 & -1 & 2 & -1 & -1 & 1 \\ 1 & -\sqrt{3} & 1 & 0 & -1 & \sqrt{3} & -1 \\ 1 & -2 & 2 & -2 & 2 & -2 & 1 \end{pmatrix}.$$

It can be observed that once we have the value $\cos\theta$ and the vector

$$(1, \cos\theta, \cos 2\theta, \ldots, \cos(p-1)\theta),$$

the entries of $\hat{R}_p$ can be determined by them. In fact, since $\cos k\theta = \cos(p-k)\theta$ for $k \in [p]$, the computation of $\cos\theta, \ldots, \cos\lceil \frac{p+1}{2} \rceil\theta$ is already enough. In addition to the common values of $\cos\theta$ as presented above for $p = 2, 3, 4, 6, 8, 12$, we can derive the matrix $\hat{R}_{24}$ by employing the formula $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$ to obtain $\cos\frac{\pi}{12} = \frac{1}{2}\sqrt{2+\sqrt{3}}$. Furthermore, we have for $p = 5$ that $\cos\theta = \cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$, thus $\cos\frac{\theta}{2} = \sqrt{\frac{\sqrt{5}+3}{8}}$, $\cos\frac{\theta}{4} = \frac{\sqrt{\sqrt{5}+12}}{4}$, then the matrices $\hat{R}_5, \hat{R}_{10}, \hat{R}_{20}$ can be derived.

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