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THE INVERSE EIGENVALUE PROBLEMS OF TENSORS WITH AN APPLICATION IN TENSOR NEARNESS PROBLEMS

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Abstract: In this paper, we are concerned with the solution of the inverse eigenvalue problem of tensors. The problem under consideration can be regarded as an extension of the matrix case. Explicitly, for several given eigenpairs of a tensor, it is shown that this problem can be converted into solving the system of tensor equations with Einstein product. Using the Moore-Penrose inverses of tensors, we obtain the sufficient and necessary conditions for the solvability of the tensor inverse eigenvalue problem as well as the general solution. As an application of the obtained results, we address the tensor nearness problem corresponding to the tensor inverse eigenvalue problem, and derived its unique solution under some conditions. In order to solve those problems more easily, we also develop the associated iterative approaches originating from the classical conjugate gradient method. The performed numerical results illustrate the feasibility and effectiveness of the proposed methods.

Key words: *tensors, inverse eigenvalue problem, tensor equations, Moore-Penrose inverse, tensor nearness problem*

Mathematics Subject Classification: *15A69, 65F18, 65F20*

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1 Introduction

Since the notion of eigenvalues and eigenvectors for higher-order tenors was put forward independently by Qi [1] and Lim [2], there have been an increasing interest studies on the eigenvalue problems of tensors due to their widespread applications in various fields, including numerical multilinear algebra, image processing, and spectral hypergraph theory, and so on; See the pioneer work [3] for a relatively systematic treatment of the basic theory on this theme.

The motivation to research the inverse eigenvalue problem of tensors originates from the inverse eigenvalue problem of matrices, since the former can be regarded as formal generalization of the latter. An inverse eigenvalue problem concerns the reconstruction of a matrix or a tensor from prescribed spectral data, which has crucial applications in science and engineering areas [4, 5], including control theory [6], structural design [7, 8], electromagnetics [9] etc., and many researchers have contribute their attention to this kind of inverse problems with certain constraints; see, e.g., [10, 11, 12, 13, 14].

Nevertheless, there have been a few literature with respect to the inverse eigenvalue problem of tensos. In [15], the author considered the inverse eigenvalue problem for arbitrary,

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multiply connected, bounded domain in R^3. For the discretized case, Li and Zhang [16] constructed the second order Markov chains that admit every probability distribution vector, which can be viewed as an inverse eigenvalue problem for 3rd-order tensors. Very recently, Ye and Hu [17] investigated the inverse eigenvalue problem corresponding to higher order tensors: Given a multiset $S \in \mathbb{C}^{nm^{n-1}}/\mathbb{S}^{nm^{n-1}}$ of total multiplicity nm^{n-1}, where $\mathbb{S}^{nm^{n-1}}$ is the group of permutations on nm^{n-1} elements. Is there a tensor $\mathcal{A}$ of order $m+1$ and dimension n such that the multiset of eigenvalues of which is exactly S? The authors proved that the inverse problem mentioned above is solvable for the cases: a) $m = 1$; b) $n = 2$; c) $[n, m] = [3, 2], [4, 2], [3, 3]$.

As seen in the literature, the inverse eigenvalue problems of matrices with special structure or not have been studied extensively, especially for the cases with special structures due to the need of practical applications, but the research on inverse eigenvalue problem of tensors is just in its infancy, and the related theory is far from perfect. To the best of our knowledge, the research of inverse eigenvalue problems for general tensors has not been found in the literature. In view of the influence of errors to the structure of a tensor, in this paper we consider the inverse eigenvalue problem of a general tensor for given partial eigenvalues and eigenvectors. To be precise, the problem under consideration can be expressed as follows:

Let $\lambda_l \in \mathbb{C}$ and $\mathbf{x}_l \in \mathbb{C}^n$, $l = 1, 2, \ldots, L$, be given complex numbers and nonzero complex vectors, respectively. Find an mth-order and n-dimensional tensor $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ over the real field $\mathbb{R}$ such that

$$\mathcal{A}\mathbf{x}_l^{m-1} = \lambda_l \mathbf{x}_l^{[m-1]}, l = 1, 2, \ldots, L, \tag{1.1}$$

where the ith element of the vectors $\mathcal{A}\mathbf{x}_l^{m-1}$ and $\mathbf{x}_l^{[m-1]}$, is respectively defined by [1]

$$(\mathcal{A}\mathbf{x}_l^{m-1})(i) := \sum_{i_2, \ldots, i_m} a_{i i_2 \ldots i_m} \mathbf{x}_l(i_2) \ldots \mathbf{x}_l(i_m),$$

$$\mathbf{x}_l^{[m-1]}(i) := \mathbf{x}_l(i)^{m-1}.$$

In this paper, we shall establish the conditions guaranteeing the existence of the solution to the tensor inverse eigenvalue problem (1.1), and derive its solutions by transforming this problem as the tensor equations with Einstein product. Particularly, the general solution of which can be obtained by using the Moore-Penrose inverses of tensors [18].

In addition, we are also interested in the nearness problem associated with the inverse eigenvalue problem mentioned above, which can be viewed as an application of the tensor inverse eigenvalue problem, and can be mathematically expressed as follows:

Let $\mathcal{A}_0$ be a given tensor. Find the tensor $\widehat{\mathcal{A}} \in \Phi$ satisfying

$$\|\widehat{\mathcal{A}} - \mathcal{A}_0\| = \min_{\mathcal{A} \in \Phi} \|\mathcal{A} - \mathcal{A}_0\|, \tag{1.2}$$

where Φ is the solution set of the tensor inverse eigenvalue problem (1.1), hereafter, $\|\cdot\|$ denotes the Frobenius norm of a tensor, see Section 2 for details.

The aforementioned optimal approximation problem for a given tensor is a generalization of the matrix case that arises in many areas of applied matrix computations [19, 20], and has been widely studied; see, e.g., [21, 22, 23, 24]. Here, the tensor $\mathcal{A}_0$ in (1.2), may be obtained by experimental observation values and statistical distribution information, but it may not satisfy the needed form and the minimum residual requirement, while the optimal estimation $\widehat{\mathcal{A}}$ is the tensor that not only satisfies those constraints but also best approximates the tensor $\mathcal{A}_0$. Under appropriate assumptions, it will be shown that the solution to the tensor nearness problem (1.2) exists uniquely, and can be represented by means of the Moore-Penrose inverses of the known tensors as well.

The remainder of this paper is organized as follows. In Section 2, we review basic definitions and terminology related to tensors. In Section 3, we solve the tensor inverse eigenvalue problem (1.1) and the tensor nearness problem (1.2) turning to the properties of the Moore-Penrose inverse of tensors. In Section 4, we propose the corresponding iterative algorithms to solve the problems mentioned above. In Section 5, some numerical experiments will be given to illustrate the obtained results presented in this paper. Finally, we conclude this paper with some remarks.

2 Preliminary Knowledge

Throughout this paper, and unless otherwise specified, we denote scalars, vectors, matrices, and tensors, by lower-case letters, boldface lower-case letters, boldface capital letters, and calligraphic letters, respectively. An mth-order and $d_1 \times \cdots \times d_m$-dimensional tensor $\mathcal{A}$ consists of $d_1 \cdot \ldots \cdot d_m$ entries $a_{i_1 \ldots i_m}$ with $1 \leq i_j \leq d_j$ and $j = 1, 2, \ldots, m$. The set of m-order and $d_1 \times \cdots \times d_m$-dimensional tensors over the real (complex) field is denoted by $\mathbb{R}^{d_1 \times \cdots \times d_m}$ ($\mathbb{C}^{d_1 \times \cdots \times d_m}$), and is particularly denoted by $\mathbb{R}^{[m,n]}$ ($\mathbb{C}^{[m,n]}$) when $d_1 = d_2 = \cdots = d_m = n$. A tensor $\mathcal{I}_k = (e_{i_1 \ldots i_k j_1 \ldots j_k}) \in \mathbb{R}^{d_1 \times \cdots \times d_k \times d_1 \times \cdots \times d_k}$ is used to represent the identity tensor of order $2k$, whose entries are defined by $e_{i_1 \ldots i_k j_1 \ldots j_k} = \prod_{l=1}^{k} \delta_{i_l j_l}$, in which $\delta_{i_l j_l} = 1$ if $i_l = j_l$, and otherwise $\delta_{i_l j_l} = 0$. The symbol $\mathcal{O}$ denotes the null tensor whose entries are zeros.

We need the following definitions; see, e.g., [25, 18, 26] for details.

Definition 2.1. Let $\mathcal{A} = (a_{i_1 \ldots i_k j_1 \ldots j_l}) \in \mathbb{C}^{d_1 \times \cdots \times d_k \times f_1 \times \cdots \times f_l}$ be a given tensor, then its conjugate transpose, denoted by $\mathcal{A}^H = (\hat{a}_{i_1 \ldots i_k j_1 \ldots j_l})$, is defined as

$$\hat{a}_{j_1 \ldots j_l i_1 \ldots i_k} = \overline{a}_{i_1 \ldots i_k j_1 \ldots j_l},$$

where the over line denotes the conjugate of the entry $a_{i_1 \ldots i_k j_1 \ldots j_l}$. Particularly, if the tensor $\mathcal{A}$ is real, then it reduces to its transpose, denoted by $\mathcal{A}^T$.

Definition 2.2. For tensors $\mathcal{A} = (a_{i_1 \ldots i_s j_1 \ldots j_k}) \in \mathbb{C}^{d_1 \times \cdots \times d_s \times f_1 \times \cdots \times f_k}$ and $\mathcal{B} = (b_{j_1 \ldots j_k l_1 \ldots l_t}) \in \mathbb{C}^{f_1 \times \cdots \times f_k \times g_1 \times \cdots \times g_t}$, the Einstein product $\mathcal{A} *_k \mathcal{B} = (c_{i_1 \cdots i_s l_1 \ldots l_t}) \in \mathbb{C}^{d_1 \times \cdots \times d_s \times g_1 \times \cdots \times g_t}$ is defined by

$$c_{i_1 \ldots i_s l_1 \ldots l_t} = \sum_{j_1, \ldots, j_k}^{f_1, \ldots, f_k} a_{i_1 \ldots i_s j_1 \ldots j_k} b_{j_1 \ldots j_k l_1 \ldots l_t}.$$

This tensor product reduces to the standard matrix multiplication in the sense that $s = k = t = 1$, which also contains the tensor-vector product and the tensor-matrix product [27] as special cases. Moreover, for $\mathcal{A} = (a_{i_1 \ldots i_k}), \mathcal{B} = (b_{j_1 \ldots j_k}) \in \mathbb{C}^{f_1 \times \cdots \times f_k}$, the inner product of which can be defined as [27]

$$< \mathcal{A}, \mathcal{B} > = \sum_{i_1, \ldots, i_k}^{f_1, \ldots, f_k} a_{i_1 i_2 \ldots i_k} b_{i_1 i_2 \ldots i_k}.$$

This definition leads to the Frobenius norm of a tensor, i.e., $\|\mathcal{A}\| = \sqrt{< \mathcal{A}, \mathcal{A} >}$. Particularly, we say that the tensors $\mathcal{A}$ and $\mathcal{B}$ are orthogonal each other, if they satisfy that $< \mathcal{A}, \mathcal{B} > = 0$. Additionally, the outer product of the tensors $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \circ \mathcal{B} = (c_{i_1 \ldots i_k j_1 \ldots j_k})$, is defined by $c_{i_1 \ldots i_k j_1 \ldots j_k} = a_{i_1 i_2 \ldots i_k} b_{j_1 \ldots j_k}$.

Furthermore, the concept of the Moore-Penrose inverse of a tensor can be defined as follows [18, 28]:

Definition 2.3. Let $\mathcal{A} \in \mathbb{C}^{d_1\times\cdots\times d_m\times f_1\times\cdots\times f_n}$. If the tensor $\mathcal{X} \in \mathbb{C}^{f_1\times\cdots\times f_n\times d_1\times\cdots\times d_m}$ satisfies the following tensor equations

$$\begin{array}{ll} (1)\ \mathcal{A} *_n \mathcal{X} *_m \mathcal{A} = \mathcal{A}, & (2)\ \mathcal{X} *_m \mathcal{A} *_n \mathcal{X} = \mathcal{X}, \\ (3)\ (\mathcal{A} *_n \mathcal{X})^H = \mathcal{A} *_n \mathcal{X}, & (4)\ (\mathcal{X} *_m \mathcal{A})^H = \mathcal{X} *_m \mathcal{A}, \end{array} \tag{2.1}$$

we say that it is the Moore-Penrose inverse of the tensor $\mathcal{A}$, denoted by $\mathcal{A}^\dagger$.

Obviously, in the case of an invertible tensor $\mathcal{A}$ [29], it holds that $\mathcal{A}^\dagger = \mathcal{A}^{-1}$, the inverse of the tensor $\mathcal{A}$ [25]. Notably, if $m = n$, it reduces to the case given in [18].

3 Main Results

3.1 Solving the tensor inverse eigenvalue problem (1.1)

In this subsection, we investigate the solvability concerning the tensor inverse eigenvalue problem (1.1). Actually, one can know from the definition of the Einstein product that, for the eigenpair $(\lambda_l, \mathbf{x}_l)$ with $l \in \{1, 2, \ldots, L\}$, the tensor-vector product $\mathcal{A}\mathbf{x}_l^{m-1}$ can be rewritten as the Einstein product which contracts the last $m-1$ modes of $\mathcal{A}$ and all the modes of the CP-rank 1 tensor $\mathcal{X}_l := \mathbf{x}_l \circ \mathbf{x}_l \circ \cdots \circ \mathbf{x}_l \in \mathbb{C}^{[m-1,n]}$, the outer product of vectors $\mathbf{x}_l$, namely,

$$\mathcal{A}\mathbf{x}_l^{m-1} = \mathcal{A} *_{m-1} \mathcal{X}_l.$$

This means that the tensor inverse eigenvalue problem (1.1) can be transformed equivalently into the solution of the tensor equations

$$\mathcal{A} *_{m-1} \mathcal{X}_l = \mathcal{B}_l, l = 1, 2, \ldots, L, \tag{3.1}$$

where $\mathcal{B}_l := \lambda_l \mathbf{x}_l^{[m-1]}$ are vectors of size n.

Furthermore, we can prove the following conclusion.

Lemma 3.1. *The system of tensor equations* (3.1) *is equivalent to the tensor equation*

$$\mathcal{A} *_{m-1} \mathcal{X} = \mathcal{B}, \tag{3.2}$$

in which $\mathcal{X}$ is an mth-order and $n \times \cdots \times n \times L$-dimensional tensor defined by

$$\mathcal{X}(:, \ldots, :, l) = \mathcal{X}_l, \tag{3.3}$$

and $\mathcal{B} \in \mathbb{C}^{n\times L}$ defined by $\mathcal{B}(:, l) = \mathcal{B}_l$, $l = 1, 2, \ldots, L$.

Proof. Unfolding the tensors $\mathcal{A}$, $\mathcal{X}_l$, and $\mathcal{B}_l$ as $\mathbf{A} \in \mathbb{C}^{n\times n^{m-1}}$, $\mathbf{x}_l \in \mathbb{C}^{n^{m-1}}$, and $\mathbf{b}_l \in \mathbb{C}^n$, $l = 1, 2, \ldots, L$, then the tensor equations (3.1) can be rewritten as

$$\mathbf{A}[\mathbf{x}_1, \ \cdots, \ \mathbf{x}_L] = [\mathbf{b}_1, \ \cdots, \ \mathbf{b}_L]. \tag{3.4}$$

Making using of the definition of the tensors $\mathcal{X}$ and $\mathcal{B}$, together with the equality (3.4), it is known that the conclusion holds true. □

We should point out that the definition of the block tensor $\mathcal{X}$ in Lemma 3.1 is different from the one given by Sun *et al.* in [18].

The following lemma is derived from Lemma 4.2 of the Reference [28].

Lemma 3.2. *For given tensors $\mathcal{B} \in \mathbb{C}^{k_1\times\cdots\times k_p\times d_1\times\cdots\times d_m}$, $\mathcal{C} \in \mathbb{C}^{g_1\times\cdots\times g_n\times l_1\times\cdots\times l_q}$ and $\mathcal{D} \in \mathbb{C}^{k_1\times\cdots\times k_p\times l_1\times\cdots\times l_q}$, then tensor equation*

$$\mathcal{B} *_m \mathcal{Z} *_n \mathcal{C} = \mathcal{D}$$

is solvable if and only if $\mathcal{B} *_m \mathcal{B}^\dagger *_p \mathcal{D} *_q \mathcal{C}^\dagger *_n \mathcal{C} = \mathcal{D}$. In this case, its general solution can be expressed as

$$\mathcal{Z} = \mathcal{B}^\dagger *_p \mathcal{D} *_q \mathcal{C}^\dagger + \mathcal{U} - \mathcal{B}^\dagger *_p \mathcal{B} *_m \mathcal{U} *_n \mathcal{C} *_q \mathcal{C}^\dagger$$

with arbitrary tensor $\mathcal{U} \in \mathbb{C}^{d_1\times\cdots\times d_m\times g_1\times\cdots\times g_n}$.

By Lemmas 3.1 and 3.2, we can obtain the following theorem which provides a necessary and sufficient condition for the solvability of the tensor inverse eigenvalue problem (1.1).

Theorem 3.3. *For given $(\lambda_l, \boldsymbol{x}_l) \in \mathbb{C} \times \mathbb{C}^n \setminus \{0\}$, $l = 1, 2, \ldots, L$, there exists a tensor $\mathcal{A}$ of order m and dimension n such that the equations given in* (1.1) *hold simultaneously, if and only if*

$$\mathcal{B} *_1 \mathcal{X}^\dagger *_{m-1} \mathcal{X} = \mathcal{B}. \tag{3.5}$$

At this time, the general solution of the tensor inverse eigenvalue problem (1.1) *can be represented as*

$$\mathcal{A} = \mathcal{B} *_1 \mathcal{X}^\dagger + \mathcal{W} *_{m-1} (\mathcal{I}_{m-1} - \mathcal{X} *_1 \mathcal{X}^\dagger). \tag{3.6}$$

where $\mathcal{I}_{m-1} \in \mathbb{R}^{[2(m-1),n]}$ is the identity tensor, and $\mathcal{W} \in \mathbb{C}^{[m,n]}$ is arbitrary.

Proof. Based on the previous analysis, it is known that the inverse eigenvalue problem is equivalent to the solution of the tensor equation (3.2). Using the properties of Moore-Penrose inverse, it follows from (3.2) that

$$\mathcal{A} *_{m-1} \mathcal{X} *_1 \mathcal{X}^\dagger *_{m-1} \mathcal{X} = \mathcal{B},$$

which, together with Lemma 3.2, deduces that the tensor inverse eigenvalue problem (1.1) is solvable if and only if the equality (3.5) holds, and the solution of which can be expressed as the form of (3.6) for arbitrary tensor $\mathcal{W}$. □

Remark 3.4. Using the properties of Moore-Penrose inverses of tensors [18, 28], the tensors $\mathcal{B} *_1 \mathcal{X}^\dagger *_{m-1} (\mathcal{W} *_{m-1} (\mathcal{I}_{m-1} - \mathcal{X} *_1 \mathcal{X}^\dagger))^H = \mathcal{O}$, which implies that $\mathcal{B} *_1 \mathcal{X}^\dagger$ is the least norm solution to the tensor inverse eigenvalue problem (1.1).

As a special case of Theorem 3.3, we have the following corollary.

Corollary 3.5. *Let $L = 1$ in Theorem 3.3. Then the general solution to the inverse eigenvalue problem* (1.1) *is*

$$\mathcal{A} = \lambda_1 \cdot \boldsymbol{x}_1^{[m-1]} \circ \mathcal{X}_1^\dagger + \mathcal{W} *_{m-1} (\mathcal{I}_{m-1} - \mathcal{X}_1 \circ {\mathcal{X}_1}^\dagger) \tag{3.7}$$

where $\mathcal{X}_1^\dagger = \boldsymbol{x}_1^H \circ \cdots \circ \boldsymbol{x}_1^H \in \mathbb{C}^{[m-1,n]}$, and $\mathcal{W} \in \mathbb{C}^{[m,n]}$ is arbitrary.

$\boxed{3.2}$ Solving the tensor nearness problem (1.2)

In this section, we are going to address the nearness problem (1.2) for the given tensor $\mathcal{A}_0$. We need the following lemma that derives from the reference [28], which is crucial for solving this problem.

Lemma 3.6. *Let $\mathcal{H} \in \mathbb{C}^{d_1\times\cdots\times d_m\times f_1\times\cdots\times f_n}$, and suppose $\mathcal{F} \in \mathbb{C}^{d_1\times\cdots\times d_m\times d_1\times\cdots\times d_m}$, and $\mathcal{G} \in \mathbb{C}^{f_1\times\cdots\times f_n\times f_1\times\cdots\times f_n}$ satisfying the conditions $\mathcal{F} *_m \mathcal{F} = \mathcal{F} = \mathcal{F}^H$ and $\mathcal{G} *_n \mathcal{G} = \mathcal{G} = \mathcal{G}^H$, then*

$$\|\mathcal{H} - \mathcal{F} *_m \mathcal{H} *_n \mathcal{G}\| = \min_{\mathcal{Z}\in\mathbb{C}^{d_1\times\cdots\times d_m\times f_1\times\cdots\times f_n}} \| \mathcal{H} - \mathcal{F} *_m \mathcal{Z} *_n \mathcal{G} \|,$$

if and only if $\mathcal{F} *_m (\mathcal{H} - \mathcal{Z}) *_n \mathcal{G} = \mathcal{O}$.

Then we obtain the following result.

Theorem 3.7. *Let $\mathcal{A}_0 \in \mathbb{C}^{[m,n]}$ be a given tensor, and assume that the solution set Φ of the tensor inverse eigenvalue problem (1.2) is nonempty, then there exists unique tensor $\widehat{\mathcal{A}}$ such that*

$$\|\widehat{\mathcal{A}} - \mathcal{A}_0\| = \min_{\mathcal{A}\in\Phi} \|\mathcal{A} - \mathcal{A}_0\|, \tag{3.8}$$

namely,

$$\widehat{\mathcal{A}} = \mathcal{B} *_1 \mathcal{X}^\dagger + \mathcal{A}_0 *_{m-1} (\mathcal{I}_{m-1} - \mathcal{X} *_1 \mathcal{X}^\dagger). \tag{3.9}$$

Proof. According to the assumption that the tensor inverse eigenvalue problem (1.2) is solvable, it follows from Theorem 3.3 that the solution set Φ can be explicitly represented as

$$\Phi = \{\mathcal{A} \mid \mathcal{A} = \mathcal{B} *_1 \mathcal{X}^\dagger + \mathcal{W} *_{m-1} (\mathcal{I}_{m-1} - \mathcal{X} *_1 \mathcal{X}^\dagger) \text{ for any } \mathcal{W} \in \mathbb{C}^{[m,n]}\}.$$

It is not difficult to verify that Φ is a closed and convex set, and thus the solution to the problem (3.8) exists uniquely. On the other hand,

$$\|\mathcal{A} - \mathcal{A}_0\|^2 = \|\mathcal{A}_0 - \mathcal{A}\|^2 = \|\mathcal{A}_0 - \mathcal{B} *_1 \mathcal{X}^\dagger - \mathcal{W} *_{m-1} (\mathcal{I}_{m-1} - \mathcal{X} *_1 \mathcal{X}^\dagger)\|^2. \tag{3.10}$$

By Definition 2.3 and simple algebra, we know that the tensor $\mathcal{I}_{m-1} - \mathcal{X} *_1 \mathcal{X}^\dagger$ satisfies the conditions required in Lemma 3.6, namely,

$$(\mathcal{I}_{m-1} - \mathcal{X} *_1 \mathcal{X}^\dagger) *_{m-1} (\mathcal{I}_{m-1} - \mathcal{X} *_1 \mathcal{X}^\dagger) = \mathcal{I}_{m-1} - \mathcal{X} *_1 \mathcal{X}^\dagger = (\mathcal{I}_{m-1} - \mathcal{X} *_1 \mathcal{X}^\dagger)^H.$$

Together with (3.10), it follows that

$$\begin{aligned}
&\min_{\mathcal{W}\in\mathbb{C}^{[m,n]}} \|\mathcal{A}_0 - \mathcal{B} *_1 \mathcal{X}^\dagger - \mathcal{W} *_{m-1} (\mathcal{I}_{m-1} - \mathcal{X} *_1 \mathcal{X}^\dagger)\| \\
&= \|\mathcal{A}_0 - \mathcal{B} *_1 \mathcal{X}^\dagger - (\mathcal{A}_0 - \mathcal{B} *_1 \mathcal{X}^\dagger) *_{m-1} (\mathcal{I}_{m-1} - \mathcal{X} *_1 \mathcal{X}^\dagger)\| \\
&= \|\mathcal{A}_0 - \mathcal{B} *_1 \mathcal{X}^\dagger - \mathcal{A}_0 *_{m-1} (\mathcal{I}_{m-1} - \mathcal{X} *_1 \mathcal{X}^\dagger)\|,
\end{aligned}$$

if and only if $(\mathcal{A}_0 - \mathcal{B} *_1 \mathcal{X}^\dagger - \mathcal{W}) *_{m-1} (\mathcal{I}_{m-1} - \mathcal{X} *_1 \mathcal{X}^\dagger) = \mathcal{O}$. In this case,

$$\begin{aligned}
\widehat{\mathcal{A}} &= \mathcal{B} *_1 \mathcal{X}^\dagger + (\mathcal{A}_0 - \mathcal{B} *_1 \mathcal{X}^\dagger) *_{m-1} (\mathcal{I}_{m-1} - \mathcal{X} *_1 \mathcal{X}^\dagger) \\
&= \mathcal{B} *_1 \mathcal{X}^\dagger + \mathcal{A}_0 *_{m-1} (\mathcal{I}_{m-1} - \mathcal{X} *_1 \mathcal{X}^\dagger).
\end{aligned}$$

The proof is completed. □

4 Algorithms

In previous section, we establish the solvability conditions of the inverse eigenvalue problem for a general tensor, and derive its general solution under certain hypotheses. Besides, we also study the tensor nearness problem corresponding to the tensor inverse eigenvalue problem, and represent its unique solution by using the Moore-Penrose inverses of the known tensors.

However, the problems related to tensors are often large-scale, which means that the direct methods developed in previous section may be untractable to deal with those problems in practice. Next, we establish iterative methods to solve the aforementioned problems.

Just as shown in Subsection 3.1, the tensor inverse eigenvalue problem (1.1) is equivalent to solving the tensor equation (3.2). Motivated by the gradient-type methods given in [30], we can design a similar approach for this problem, i.e., Algorithm 4.1.

Algorithm 4.1

Step 1: Input $\lambda_l \in \mathbb{C}$, $\mathbf{x}_l \in \mathbb{C}^n$, $l = 1, 2, \ldots, L$, and initial iteration tensor $\mathcal{A}^{(0)} \in \mathbb{R}^{[m,n]}$.
Step 2: Formulate the tensors $\mathcal{X}$ and $\mathcal{B}$ by (3.2) and (3.3), respectively.
Step 3: Compute $\mathcal{R}^{(0)} = \mathcal{B} - \mathcal{A}^{(0)} *_{m-1} \mathcal{X}$, and $\mathcal{P}^{(0)} = \mathcal{R}^{(0)} *_1 \mathcal{X}^H$.
Step 4: For $k = 1, 2, \ldots$, compute
$\mathcal{A}^{(k)} = \mathcal{A}^{(k-1)} + \alpha_{k-1}\, \mathcal{P}^{(k-1)}$ with $\alpha_{k-1} := \frac{\|\mathcal{R}^{(k-1)}\|^2}{\|\mathcal{P}^{(k-1)}\|^2}$.
Step 5: Compute $\mathcal{R}^{(k)} = \mathcal{B} - \mathcal{A}^{(k)} *_{m-1} \mathcal{X}$ with $\beta_k := \frac{\|\mathcal{R}^{(k)}\|^2}{\|\mathcal{R}^{(k-1)}\|^2}$, and then calculate
$\mathcal{P}^{(k)} = \mathcal{R}^{(k)} *_1 \mathcal{X}^H + \beta_k\, \mathcal{P}^{(k-1)}$.
If $\mathcal{R}^{(k)} = \mathcal{O}$, or $\mathcal{R}^{(k)} \neq \mathcal{O}$, $\mathcal{P}^{(k-1)} = \mathcal{O}$, stop; Otherwise, goto Step 4.

For any initial tensor $\mathcal{A}^{(0)}$, as shown in [30], we can prove that Algorithm 4.1 converges to a solution of (3.2) within finite iteration steps in the absence of roundoff errors, and particularly, the solvability of the tensor inverse eigenvalue problem (1.1) can be determined automatically. Those results are concluded in the following two theorems, and the proofs of which are omitted.

Theorem 4.1. *Let $(\lambda_l, \boldsymbol{x}_l) \in \mathbb{C} \times \mathbb{C}^n \backslash \{0\}$, $l = 1, 2, \ldots, L$, be given eigenpairs, and suppose that they satisfy the condition (3.5), then for any initial tensor $\mathcal{A}^{(0)}$, the solution to the tensor inverse eigenvalue problem (1.1) can be obtained by Algorithm 4.1 within finite iteration steps.*

On the other hand, the tensor inverse eigenvalue problem (1.1) is not solvable, if and only if there exists a positive integer $\widehat{k}$ such that $\mathcal{R}^{(\widehat{k})} \neq \mathcal{O}$ and $\mathcal{P}^{(\widehat{k})} = \mathcal{O}$.

The following theorem claims that the least norm solution to the tensor inverse eigenvalue problem (1.1) can be derived by choosing special initial iteration tensors.

Theorem 4.2. *If the tensor inverse eigenvalue problem (1.1) is solvable, and let the initial tensor $\mathcal{A}^{(0)} = \mathcal{Z} *_1 \mathcal{X}^T$ with arbitrary $\mathcal{Z} \in \mathbb{R}^{n \times L}$, then the solution generated by Algorithm 4.1 is the unique least norm solution.*

It should be mentioned that the proposed algorithm in this paper can also be used to solve the tensor nearness problem (1.2). In fact, for the given tensor $\mathcal{A}_0$, since the solutions of the tensor nearness problem (1.2) is equivalent to finding the least norm solution (denoted by $\widetilde{\mathcal{H}}$) of the following tensor equation

$$\mathcal{H} *_{m-1} \mathcal{X} = \widetilde{\mathcal{B}}, \tag{4.1}$$

where $\mathcal{H} = \mathcal{A} - \mathcal{A}_0$ and $\widetilde{\mathcal{B}} = \mathcal{B} - \mathcal{A}_0 *_{m-1} \mathcal{X}$. Then, from Theorem 4.2 it follows that the least norm solution $\widetilde{\mathcal{H}}$ can be derived by applying Algorithm 4.1 to (4.1) with the initial tensor $\mathcal{A}^{(0)} = \mathcal{W} *_1 \mathcal{X}^T$ for some $\mathcal{W} \in \mathbb{R}^{n \times L}$. In this case, the optimal approximation solution to the problem (1.2) can be represented by $\widehat{\mathcal{A}} = \widetilde{\mathcal{H}} + \mathcal{A}_0$. For simplicity, we always let $\mathcal{W} = \mathcal{O}$, i.e., $\mathcal{A}^{(0)} = \mathcal{O}$ be the initial iteration tensor in practice.

5 Numerical Experiments

In this section, we shall demonstrate the feasibility of the theoretical analysis, as well as the proposed algorithm, upon the tensor inverse eigenvalue problem and the associated optimal approximation problem. All codes were implemented in MATLAB (version R2016a), and run by using the tensor toolbox [31] on a personal computer with Inter(R) Core(TM) i5-4200M and 4.00G memory.

Example 5.1. Let $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{[4,3]}$ be the symmetric tensor presented in Example 1 of [32]. The tensor $\mathcal{A}$ has 11 real H-eigenpairs (see Table 1) [33].

Table 1: H-eigenvalue λ and its eigenvector $\mathbf{x}$ of the tensor $\mathcal{A}$ in Example 5.1

λ	$\mathbf{x}$		
2.3129	[0.7875	0.6483	-0.8138]
1.9316	[0.8749	-0.6536	0.6936]
0.9780	[0.1474	-0.9540	0.6432]
0.8944	[0.5223	0.8048	0.8434]
0.7228	[0.8526	0.4939	0.8012]
0.4108	[0.2035	-0.5145	-0.9816]
0.2528	[1.0000	0.1020	-0.0868]
0.2499	[0.4178	0.9917	0.2184]
-0.0887	[0.9158	0.7376	0.1559]
-0.6665	[0.2291	-0.2579	0.9982]
-2.6841	[0.7793	-0.8675	-0.5044]

We first address the inverse H-eigenvalue problem for the given 11 real eigenpairs in Table 1. It is easy to very verify that the unfolding rank [29] of the tensor $\mathcal{X}$ defined in (3.3) equals to 10, and that the solvability conditions required in Theorem 3.3 are satisfied. Using the given eigenvectors formulates the known tensor $\mathcal{X}$ in tensor equation (3.2). Particularly, depending on the Moore-Penrose inverse theory of tensors [28], we can obtain the Moore-Penrose inverse $\mathcal{X}^\dagger$ of the tensor $\mathcal{X}$ (see, Table 2). Moreover, by Theorem 3.7, we obtain the least norm solution to the inverse eigenvalue problem corresponding to the tensor $\mathcal{A}$ (see, Table 3). Obviously, the obtained least norm solution $\mathcal{A}$ is symmetric, whose entries are the same as those given in Example 1 of [32].

Next, we consider the tensor nearness problem associated the tensor inverse eigenvalue problem in Example 5.1. Assume that the tensor $\mathcal{A}_0 \in \mathbb{R}^{[4,3]}$ is as follows. By Theorem 3.3, we obtain the optimal approximation solution $\widehat{\mathcal{A}}$ (see, Table 5). At this time, $\|\widehat{\mathcal{A}} - \mathcal{A}_0\| = 773.2102$.

Additionally, in order to illustrate the efficiency of the proposed algorithm, i.e., Algorithm 4.1, take the given eigenpairs in Example 5.1 as an example. The iterations will be terminated if the norm of the residual, i.e., RES$= \|\mathcal{B} - \mathcal{A}^{(k)} *_{m-1} \mathcal{X}\| < \varepsilon = 1.0e-5$, or the number of iteration steps exceeds the maximal positive integer number $k_{\max} = 1000$. Choosing zero tensor as the initial iteration value, after 11 iterations, we obtain the same solution as in Table 3 with the residual RES$=5.2022e-05$. Furthermore, for the given tensor $\mathcal{A}_0$ in Table 4, letting zero tensor be the initial iteration value, we can also obtain the unique optimal approximation solution with the residual RES$=5.1995e-05$ after 11 iteration steps. In Figure 1, we plot the curve of the residual versus the iteration steps.

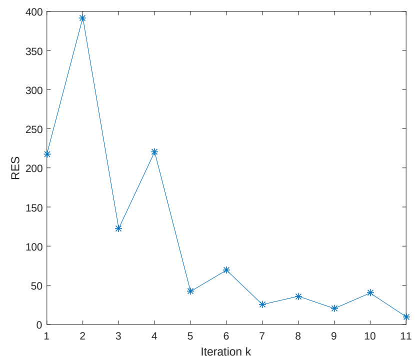

Figure 1: Convergence of Algorithm 4.1 for the tensor nearness problem in Example 5.1.

In the following example, we demonstrate the effectiveness of Algorithm 4.1 presented in this paper when it is used to solve the tensor nearness problem (1.2), in which the given eigenpairs are chosen randomly.

Example 5.2. For given positive integer numbers m, n and L, Let $\lambda = \texttt{randn}(1, L)$ be given eigenvalues and each column of the matrix $\mathbf{X} = \texttt{randn}(n, L)$ represents the corresponding eigenvector, and assume $\mathcal{A}_0 = \texttt{tensor}(1 : n^m, v)$ with $v = \texttt{repmat}(n, 1, m)$ is the given tensors, where the function `tensor` reshapes numbers $1 : n^m$ as an mth-order and n-dimensional tensor.

Under the same initial iteration conditions and accuracy requirements, we respectively ran Algorithm 4.1 for different choices of the order m and the dimension n, and display the convergence behavior in Figure 2, those curves reveal that our algorithm is efficient.

6 Conclusions

In this paper, we addressed the tensor inverse eigenvalue problem (1.1) and the associated tensor nearness problem (1.2), obtained their solvability conditions and general solutions by using the Moore-Penrose inverses of the known tensors. Besides, we also proposed a conjugate gradient type algorithm for the underling problems in this paper. The performed numerical experiments reflect that the obtained theoretical results and the corresponding iterative algorithm are feasible and efficient. We should mention that there have been several kinds of eigenvalues for tensors due to different applications in practice [3], so one can also consider the inverse problems for those tensor eigenvalues. Moreover, the size of the testing problems in Section 5 is small, which is mainly limited to our experimental platform. Alternatively, how to design some other more efficient iterative methods to solve the inverse eigenvalue problems of tensors with or without special structures is necessary.

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Table 2: The Moore-Penrose inverse of the tensor $\mathcal{X}$ in Example 5.1.

$\mathcal{X}^\dagger(:,:,1,1)$	$\mathcal{X}^\dagger(:,:,2,1)$	$\mathcal{X}^\dagger(:,:,3,1)$
-0.0741 0.0184 0.9840	-0.0676 -0.1673 0.1211	-0.0968 -0.1141 -0.1507
0.1011 0.0049 -0.0214	0.0111 0.1252 0.0173	0.2872 -0.1410 -0.1576
-0.0872 -0.0060 -0.0309	0.0781 -0.1253 -0.1444	0.0795 0.2092 0.2200
$\mathcal{X}^\dagger(:,:,1,2)$	$\mathcal{X}^\dagger(:,:,2,2)$	$\mathcal{X}^\dagger(:,:,3,2)$
-0.0522 0.1951 0.0781	-0.1253 -0.1444 0.0795	0.1013 -0.0557 0.0810
-0.2880 -0.1839 -0.1673	0.1211 -0.0968 -0.1141	0.0461 -0.1938 0.3291
-0.1183 -0.0747 0.1252	0.0173 0.2872 0.0465	-0.0669 -0.4050 -0.1106
$\mathcal{X}^\dagger(:,:,1,3)$	$\mathcal{X}^\dagger(:,:,2,3)$	$\mathcal{X}^\dagger(:,:,3,3)$
0.2170 -0.0053 0.0952	-0.0465 -0.1410 -0.1576	-0.2880 -0.1839 -0.1166
-0.0941 0.0867 0.0746	0.0083 0.2092 0.2200	-0.1183 -0.0747 -0.0053
-0.1166 0.0565 0.0179	0.0753 -0.1507 -0.0522	0.1951 -0.0941 0.0867
$\mathcal{X}^\dagger(:,:,1,4)$	$\mathcal{X}^\dagger(:,:,2,4)$	$\mathcal{X}^\dagger(:,:,3,4)$
0.0565 0.0179 0.0753	0.0311 0.3173 -0.2940	0.0781 -0.1253 -0.1444
0.0952 -0.0465 0.1845	0.1654 0.0404 0.2146	-0.1673 0.1211 -0.0968
0.0746 0.0083 0.0004	0.1426 0.0070 -0.0481	0.1252 0.0173 0.2872
$\mathcal{X}^\dagger(:,:,1,5)$	$\mathcal{X}^\dagger(:,:,2,5)$	$\mathcal{X}^\dagger(:,:,3,5)$
0.0795 0.1013 -0.0557	0.0810 0.2170 -0.0053	0.0952 -0.0465 0.0465
-0.1141 0.0461 -0.1938	0.3291 -0.0941 0.0867	0.0746 0.0083 0.1013
0.0465 -0.0669 -0.4050	-0.1106 -0.1166 0.0565	0.0179 0.0753 0.0461
$\mathcal{X}^\dagger(:,:,1,6)$	$\mathcal{X}^\dagger(:,:,2,6)$	$\mathcal{X}^\dagger(:,:,3,6)$
-0.0669 -0.4050 -0.1106	0.0591 -0.2250 0.6412	-0.2067 0.2911 0.0894
-0.0557 0.0810 0.2170	-0.4048 0.2827 -0.2512	-0.0284 0.2649 0.2297
-0.1938 0.3291 -0.0690	0.1815 0.3471 0.2946	0.0066 -0.1279 0.1391
$\mathcal{X}^\dagger(:,:,1,7)$	$\mathcal{X}^\dagger(:,:,2,7)$	$\mathcal{X}^\dagger(:,:,3,7)$
-0.1691 -0.0941 0.0867	0.0746 0.0083 0.0066	-0.1279 0.1391 -0.1251
-0.1209 -0.1166 0.0565	0.0179 0.0753 0.2911	0.0894 -0.1691 0.0895
-0.1251 -0.0053 0.0952	-0.0465 -0.0284 0.2649	0.2297 -0.1209 0.0233
$\mathcal{X}^\dagger(:,:,1,8)$	$\mathcal{X}^\dagger(:,:,2,8)$	$\mathcal{X}^\dagger(:,:,3,8)$
-0.0770 -0.2835 -0.0311	-0.1410 -0.1576 -0.2880	-0.1839 -0.1166 0.0565
0.1016 -0.0845 -0.2887	0.2092 0.2200 -0.1183	-0.0747 -0.0053 0.0952
0.0509 -0.1507 0.0427	-0.1507 -0.0522 0.1951	-0.0941 0.0867 0.0746
$\mathcal{X}^\dagger(:,:,1,9)$	$\mathcal{X}^\dagger(:,:,2,9)$	$\mathcal{X}^\dagger(:,:,3,9)$
0.0179 0.0753 0.0311	0.3173 -0.2940 -0.0941	0.0867 0.0746 0.0083
-0.0465 0.1845 0.1654	0.0404 0.2146 -0.1166	0.0565 0.0179 0.0753
0.0083 0.0004 0.1426	0.0070 -0.0481 -0.0053	0.0952 -0.0465 -0.0284
$\mathcal{X}^\dagger(:,:,1,10)$	$\mathcal{X}^\dagger(:,:,2,10)$	$\mathcal{X}^\dagger(:,:,3,10)$
0.0066 -0.1279 0.1391	-0.1251 -0.0770 -0.2835	-0.0311 0.1845 0.1654
0.2911 0.0894 -0.1691	0.0895 0.1016 -0.0845	-0.2887 0.0004 0.1426
0.2649 0.2297 -0.1209	0.0233 0.0509 -0.1507	0.0427 0.0311 0.3173
$\mathcal{X}^\dagger(:,:,1,11)$	$\mathcal{X}^\dagger(:,:,2,11)$	$\mathcal{X}^\dagger(:,:,3,11)$
0.0404 0.2146 0.0233	0.0509 -0.1507 0.0427	-0.0869 -0.4333 0.1093
0.0070 -0.0481 -0.0770	-0.2835 -0.0311 -0.0522	-0.1015 -0.0554 0.6845
0.2940 0.0895 0.1016	-0.0845 -0.2887 -0.0891	0.0206 -0.0486 0.1215

Table 3: The least norm solution to the inverse eigenvalue problem in Example 5.1.

$\mathcal{A}(:,:,1,1)$	$\mathcal{A}(:,:,2,1)$	$\mathcal{A}(:,:,3,1)$
0.2883 -0.0031 0.1973	-0.0031 -0.2485 -0.2939	0.1973 -0.2939 0.3847
-0.0031 -0.2485 -0.2939	-0.2485 0.2971 0.1862	-0.2939 0.1862 0.0919
0.1973 -0.2939 0.3847	-0.2939 0.1862 0.0919	0.3847 0.0919 -0.3619
$\mathcal{A}(:,:,1,2)$	$\mathcal{A}(:,:,2,2)$	$\mathcal{A}(:,:,3,2)$
-0.0031 -0.2485 -0.2939	-0.2485 0.2973 0.1862	-0.2939 0.1862 0.0919
-0.2485 0.2971 0.1862	0.2971 0.1242 -0.3420	0.1862 -0.3420 0.2127
-0.2939 0.1862 0.0919	0.1862 -0.3420 0.2127	0.0919 0.2127 0.2727
$\mathcal{A}(:,:,1,3)$	$\mathcal{A}(:,:,2,3)$	$\mathcal{A}(:,:,3,3)$
0.1973 -0.2939 0.3847	-0.2939 0.1862 0.0919	0.3847 0.0919 -0.3619
-0.2939 0.1862 0.0919	0.1862 -0.3420 0.2127	0.0919 0.2127 0.2726
0.3847 0.0919 -0.3619	0.0919 0.2127 0.2727	-0.3619 0.2727 -0.3055

Table 4: The least norm solution to the inverse eigenvalue problem in Example 5.1.

$\mathcal{A}_0(:,:,1,1)$	$\mathcal{A}_0(:,:,2,1)$	$\mathcal{A}_0(:,:,3,1)$	$\mathcal{A}_0(:,:,1,2)$	$\mathcal{A}_0(:,:,2,2)$
1 4 7	10 13 16	19 22 25	28 31 34	37 40 43
2 5 8	11 14 17	20 23 26	29 32 35	38 41 44
3 6 9	12 15 18	21 24 27	30 33 36	39 42 45
$\mathcal{A}_0(:,:,3,2)$	$\mathcal{A}_0(:,:,1,3)$	$\mathcal{A}_0(:,:,2,3)$	$\mathcal{A}_0(:,:,3,3)$	
46 49 52	55 58 61	64 67 70	73 76 79	
47 50 53	56 59 62	65 68 71	74 77 80	
48 51 54	57 60 63	66 69 72	75 78 81	

Table 5: The approximate tensor $\widehat{\mathcal{A}}$ of the tensor nearness problem in Example 5.1.

$\widehat{\mathcal{A}}(:,:,1,1)$	$\widehat{\mathcal{A}}(:,:,2,1)$	$\widehat{\mathcal{A}}(:,:,3,1)$
0.2883 -14.0031 -26.8027	-14.0031 -27.2485 -40.2939	-26.8027 -40.2939 -52.6153
-0.0031 -15.2485 -28.2939	-15.2485 -27.7029 -40.8138	-28.2939 -40.8138 -53.9081
0.1973 -16.2939 -28.6153	-16.2939 -28.8138 -41.9081	-28.6153 -41.9081 -55.3619
$\widehat{\mathcal{A}}(:,:,1,2)$	$\widehat{\mathcal{A}}(:,:,2,2)$	$\widehat{\mathcal{A}}(:,:,3,2)$
-14.0031 -27.2485 -40.2939	-27.2485 0.2973 -52.8138	-40.2939 -52.8138 -65.9081
-15.2485 -27.7029 -40.8138	-27.7029 0.1242 -54.3420	-40.8138 -54.3420 -66.7873
-16.2939 -28.8138 -41.9081	-28.8138 -0.3420 -54.7873	-41.9081 -54.7873 -67.7273
$\widehat{\mathcal{A}}(:,:,1,3)$	$\widehat{\mathcal{A}}(:,:,2,3)$	$\widehat{\mathcal{A}}(:,:,3,3)$
-26.8027 -40.2939 -52.6153	-40.2939 -52.8138 -65.9081	-52.6153 -65.9081 -0.3619
-28.2939 -40.8138 -53.9081	-40.8138 -54.3420 -66.7873	-53.9081 -66.7873 0.2726
-28.6153 -41.9081 -55.3619	-41.9081 -54.7873 -67.7273	-55.3619 -67.7273 -0.3055

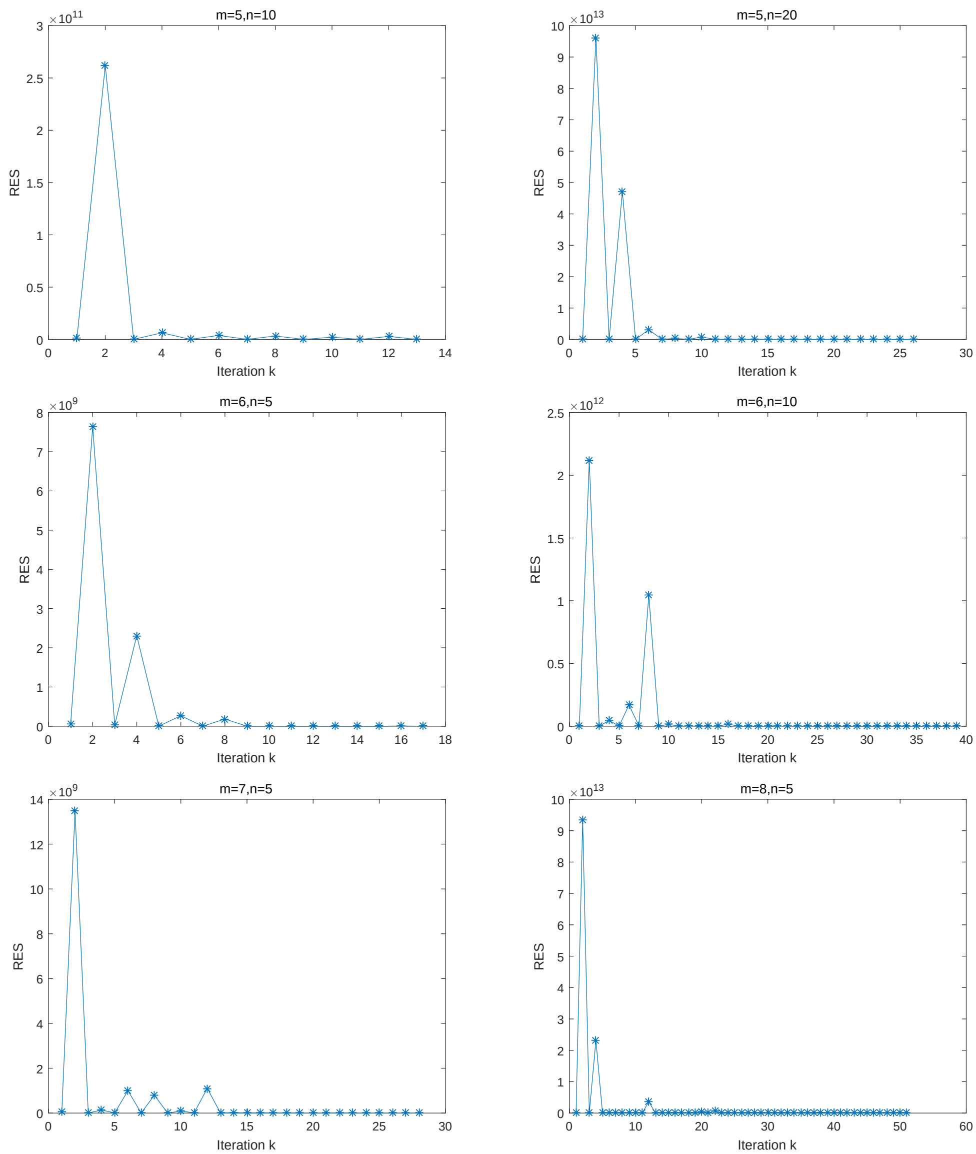

Figure 2: Convergence of Algorithm 4.1 for the tensor nearness problem in Example 5.2.