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CYCLIC STATIONARY PROBABILITY DISTRIBUTION OF SECOND ORDER MARKOV CHAINS AND ITS APPLICATIONS*

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Abstract: In this paper, we define a system of cyclic stationary probability distribution equations for a second order Markov chain process in case that all states are independent each other, which improves the system of equations in [W. Li, and M.K. Ng, On the limiting probability distribution of a transition probability tensor, Linear and Multilinear Algebra. 62(2014): 362-385]. There are two applications for the new model. First, the proposed model can be seen as a rank-3 approximation of a second order Markov chain with non-independent states. Second, unlike the previous tensor model, if the fixed point algorithm for solving the new model is convergent, the second order Markov chain process in the independent state cyclic-converges. Furthermore, we investigate properties of the solutions for the proposed stationary equation.

Key words: *Markov chain, stochastic tensor, stationary probability distribution*

Mathematics Subject Classification: *15A48, 15A69, 65F10, 65H10*

1 Introduction

Higher-order Markov chains have crucial applications in data analysis and optimization such as data clustering, community detection, and network classification (e.g.,see [1, 11, 16]). Recently, theoretical analysis and applications of tensor model for Markov chains have attracted many scholars' attention [9, 11, 16, 19, 28, 29, 30]. Let a space have n states $\{1, 2, ..., n\}$ and S_t denote the state at time t. Assume that the transition probability of the next state just depends on the last two states. Consider a transition probability of the state being i at time $t + 1$ from the last two states j and k given by

$$\Pr\{S_{t+1} = i | S_t = j, S_{t-1} = k\} = p_{ijk}.$$

Let $\langle n\rangle = \{1, 2, ..., n\}$. For all $j, k \in \langle n\rangle$, it is clear that

$$p_{ijk} \geq 0\ ,\ \sum_{i=1}^{n} p_{ijk} = 1. \tag{1.1}$$

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We say $\mathcal{P} = (p_{ijk})$ to be a *stochastic tensor* or a *transition probability tensor*(In [8], $\mathcal{P}$ satisfying (1.1) was also called as a *three-dimensional line-stochastic matrix*). If $t \longrightarrow \infty$, the process runs to infinity, and we can get the *system of stationary distribution equations* as follows:

$$x_{ij} = \sum_{k=1}^{n} p_{ijk} x_{jk}, \ x_{ij} \geq 0, \ \sum_{i,j=1}^{n} x_{ij} = 1, \tag{1.2}$$

where the unknown solution $X = (x_{ij})$ is the *stationary distribution matrix* which is denoted the stationary probability related to the pairs of states (i, j). Let $\mathbf{x}^{(t)} = (x_i^{(t)})$ denote the probability distribution of all states at time t. Assume that all states are independent each other, we can get a second order Markov chain process:

$$\mathbf{x}^{(t)} = \mathcal{P}\mathbf{x}^{(t-1)}\mathbf{x}^{(t-2)}. \tag{1.3}$$

If $\lim_{t\to\infty} \mathbf{x}^{(t)} = \mathbf{x}$, by (1.2), we have

$$\mathcal{P}\mathbf{x}^2 = \mathbf{x}, x_i \geq 0, \sum_{i=1}^{n} x_i = 1 \tag{1.4}$$

with $x_{ij} = x_i x_j$, where $\mathbf{x} = (x_i)$ is called a stochastic vector, i.e., $x_i \geq 0, \sum_{i=1}^{n} x_i = 1$. For other cases, the system of equations (1.4) can be seen as a rank-1 approximation model of the system of equations (1.2) (e.g., see [1, 11]). This approximation can overcome the difficulty of storage for computing the stationary probability X in (1.2) with large datasets, i.e., the requirement $O(n^2)$ storage of a matrix is replaced by $O(n)$ for a vector.

It's worth noting that the approximation model (1.4) was first presented by Li and Ng in [19] and can be employed to establish multilinear PageRank problem [11]. Furthermore, many applications for the system of equations (1.4) in the spacey random work such as population genetic, transportation, ranking and clustering and so on (see [1, 11]) were given. Theory and numerical analysis for the system of equations (1.4) were presented such as uniqueness of solutions (e.g., see [4, 5, 10, 11, 14, 19, 20, 22]), algorithms (e.g., see [6, 11, 12, 18, 23, 24]) and error and perturbation analysis (e.g., see [13, 21, 22]).

Let $\mathbb{S}^n$ denote the set of all stochastic vectors. In this paper, we main concentrate on a *system of cyclic stationary probability distribution equations* of the second order Markov chain as follows:

$$\begin{cases} \mathbf{x} = \mathcal{P}\mathbf{y}\mathbf{z} \\ \mathbf{y} = \mathcal{P}\mathbf{z}\mathbf{x} \\ \mathbf{z} = \mathcal{P}\mathbf{x}\mathbf{y} \end{cases}, \ \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{S}^n. \tag{1.5}$$

We call the solution $\boldsymbol{\omega} \equiv (\mathbf{x}^T, \mathbf{y}^T, \mathbf{z}^T)^T \in \mathbb{S}^n \times \mathbb{S}^n \times \mathbb{S}^n$ of the system of equations (1.5) as a *cyclic stationary probability distribution triple*. If furthermore there are at least two different vectors in $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, we call $\boldsymbol{\omega}$ a *non-degenerate cyclic stationary probability distribution triple*. If $\mathbf{x} = \mathbf{y} = \mathbf{z}$, then we call $\boldsymbol{\omega}$ a *degenerate cyclic stationary probability distribution triple*. In this case the system of equations (1.5) reduces to (1.4).

In this paper, the main contributions are given as follows:

(1) We establish the system of cyclic stationary probability distribution equations (1.5) and give the theoretical analysis.

(2) Two applications of the system of equations (1.5) are considered. First, the proposed model can be seen as an approximation of the stationary distribution equation (1.2). Second, the new model can deal with the problem encountered by the existing model (1.4): the convergence of the fixed-point algorithm (i.e., $\mathbf{x}_k = \mathcal{P}\mathbf{x}_{k-1}^2$) to solve the system of equations (1.4) does not imply the convergence of the second order Markov chain process (1.3) itself. For the proposed model, if the fixed-point algorithm to solve the system of equations (1.5) is convergent, the second order Markov chain process (1.3) is cyclic-convergent.

The rest of this paper is organized as follows. In Section 2, we introduce some definitions and notations. Furthermore, in Section 3, some applications for the cyclic stationary probability distribution are given. In Section 4, we investigate the properties for the system of equations (1.5). The final section is a concluding remark.

2 Preliminaries

Let $\mathbb{R}$ be the real field. An order m dimension n tensor $\mathcal{A}$ with n^m entries is defined as follows:

$$\mathcal{A} = (a_{i_1\cdots i_m}),\ a_{i_1\cdots i_m} \in \mathbb{R},\ i_j \in \langle n\rangle,\ j \in \langle m\rangle.$$

Conventionally, we use curlicue letters such as $\mathcal{A}$ to denote tensors. Besides, a matrix is denoted by capital roman font in italics such as A, and lowercase italics with black Roman letter such as $\mathbf{x}$ is used to denote a vector. If every entry of a vector (matrix or tensor) is nonnegative (positive), we call the vector (matrix or tensor) being nonnegative (positive). Let $\mathbb{R}^n$ and $\mathbb{R}^{[m,n]}$ denote the sets of all dimension-n real vectors and dimension-n order-m real tensors respectively. We denote the set of all stochastic order-3 dimension-n stochastic tensors by $\mathbb{S}^{[3,n]}$ respectively. The 1-norm and ∞-norm of a matrix or a vector are denoted by $||\cdot||_1$ and $||\cdot||_\infty$.

Next we give some definitions and lemmas which will be used in the sequel. We first recall some products between a tensor and vectors.

Definition 2.1 ([25])**.** Let $\mathcal{A} \in \mathbb{R}^{[3,n]}$, $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in \mathbb{R}^n$ and $x_i^{(l)}$ denote the i-th entry of $\mathbf{x}^{(l)}$, $l = 1, 2$. We define $\mathcal{A}\mathbf{x}^{(1)}\mathbf{x}^{(2)}$ to be an n dimensional vector whose the i-th entry is given by

$$(\mathcal{A}\mathbf{x}^{(1)}\mathbf{x}^{(2)})_i = \sum_{j,k\in\langle n\rangle} a_{ijk} x_j^{(1)} x_k^{(2)}.$$

Remark 2.2. If $\mathbf{x} \equiv \mathbf{x}^{(1)} = \mathbf{x}^{(2)}$, then $\mathcal{A}\mathbf{x}^{(1)}\mathbf{x}^{(2)}$ reduces to $\mathcal{A}\mathbf{x}^2$ which was defined in [32].

In [25], Lim gave the definition of an irreducible tensor.

Definition 2.3 ([25])**.** An $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called reducible if there exists a nonempty proper index subset $\mathbb{I} \subseteq \{1, 2, ..., n\}$ such that

$$a_{i_1 i_2\cdots i_m} = 0,\ \forall i_1 \in \mathbb{I},\ \forall i_2, ..., i_m \notin \mathbb{I}.$$

If $\mathcal{A}$ is not reducible, we call $\mathcal{A}$ irreducible.

Li and Ng [19] proposed (1.4) and gave the related existence conditions of the solutions.

Lemma 2.4 ([19])**.** *If $\mathcal{P}$ is a stochastic tensor, the system of equations* (1.4) *has a solution. In particular, $\mathcal{P}$ is an irreducible stochastic tensor, all solutions of the system of equations* (1.4) *are positive.*

Remark 2.5. It is known that the positive solution is unique for the first Markov chain $\mathbf{x} = P\mathbf{x}$ if P is an irreducible transition probability matrix [27]. However, the system of equations (1.4) may have multiple positive solutions even $\mathcal{P}$ is a positive tensor. For details, please refer to the reference [33].

Lemma 2.6 ([23])**.** *Let* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{S}^n$ *and* $\mathcal{J}^{(2)}, \mathcal{J}^{(3)} \in \mathbb{R}^{[3,n]}$ *with*

$$\begin{cases} (\mathcal{J}^{(2)})_{ijk} = J_{ik}^{(2)} \in \mathbb{R} \\ (\mathcal{J}^{(3)})_{ijk} = J_{ij}^{(3)} \in \mathbb{R} \end{cases}, i, j, k \in \langle n \rangle .$$

Then

$$\mathcal{J}^{(2)}\Delta\mathbf{x}\mathbf{z} = 0,\ \mathcal{J}^{(3)}\mathbf{z}\Delta\mathbf{x} = 0,$$

where $\Delta\mathbf{x} = \mathbf{x} - \mathbf{y}$.

Let $\mathcal{J}^{(2)}$ and $\mathcal{J}^{(3)}$ be defined by Lemma 2.6. In [22], Li et al. defined the following formula.

$$\mu(\mathcal{J}^{(2)}, \mathcal{J}^{(3)}) = \max_{t\in\langle n\rangle} \left(\max_{i_3\in\langle n\rangle} \sum_{i=1}^{n} |p_{iti_3} - J_{ii_3}^{(2)}| + \max_{i_2\in\langle n\rangle} \sum_{i=1}^{n} |p_{ii_2t} - J_{ii_2}^{(3)}| \right), \tag{2.1}$$

$$\nu(\mathcal{J}^{(2)}, \mathcal{J}^{(3)}) = \max_{i\in\langle n\rangle} \left(\max_{i_3\in\langle n\rangle} \sum_{t=1}^{n} |p_{iti_3} - J_{ii_3}^{(2)}| + \max_{i_2\in\langle n\rangle} \sum_{t=1}^{n} |p_{ii_2t} - J_{ii_2}^{(3)}| \right). \tag{2.2}$$

Let $\mathbb{S}$ be a proper subset of $\langle n\rangle$ and $\mathbb{S}^{'}$ be its complementary set in $\langle n\rangle$, i.e., $\mathbb{S}^{'} = \langle n\rangle\backslash\mathbb{S}$. Let

$$\gamma_{\mathbb{S}}^{(1)} = \min_{i_3\in\langle n\rangle} \Big(\min_{i_2\in\mathbb{S}} \sum_{i\in\mathbb{S}'} p_{ii_2i_3} + \min_{i_2\in\mathbb{S}'} \sum_{i\in\mathbb{S}} p_{ii_2i_3} \Big), \tag{2.3}$$

$$\gamma_{\mathbb{S}}^{(2)} = \min_{i_2\in\langle n\rangle} \Big(\min_{i_3\in\mathbb{S}} \sum_{i\in\mathbb{S}'} p_{ii_2i_3} + \min_{i_3\in\mathbb{S}'} \sum_{i\in\mathbb{S}} p_{ii_2i_3} \Big) \tag{2.4}$$

and

$$\gamma = \min_{\mathbb{S}\subset\langle n\rangle} \Big(\gamma_{\mathbb{S}}^{(1)} + \gamma_{\mathbb{S}}^{(2)} \Big).$$

Furthermore, Li et al. [22] gave the following uniqueness conditions of a solution for the system of equations (1.4).

Lemma 2.7 ([22])**.** *If there exist* $\mathcal{J}^{(2)}$ *and* $\mathcal{J}^{(3)}$ *such that one of the following conditions holds:*

(1) $\mu(\mathcal{J}^{(2)}, \mathcal{J}^{(3)}) < 1$,

(2) $\nu(\mathcal{J}^{(2)}, \mathcal{J}^{(3)}) < 1$,

(3) $\gamma > 1$,

then the system of equations (1.4) *has a unique solution.*

Fasino et al. [10] defined ergodicity coefficients for order-3 tensors as follows:

$$\tau_L = \max_{\mathbf{x}\in\mathbb{S}^n} \max_{\mathbf{z}\in\mathbb{Z}^n} ||\mathcal{P}\mathbf{x}\mathbf{z}||_1,$$

$$\tau_R = \max_{\mathbf{x}\in\mathbb{S}^n}\max_{\mathbf{z}\in\mathbb{Z}^n} ||\mathcal{P}\mathbf{z}\mathbf{x}||_1,$$
$$\tau = \max_{\mathbf{x}\in\mathbb{S}^n}\max_{\mathbf{z}\in\mathbb{Z}^n} ||\mathcal{P}\mathbf{x}\mathbf{z} + \mathcal{P}\mathbf{z}\mathbf{x}||_1,$$

where $\mathbb{Z}^n = \left\{\mathbf{z} = (z_i) \in \mathbb{R}^n, \sum_{i=1}^{n} z_i = 0, ||\mathbf{z}||_1 = 1\right\}$. Furthermore, they showed that

$$\tau_L = \frac{1}{2}\max_{j,k_1,k_2\in\langle n\rangle}\sum_{i=1}^{n}\left|p_{ijk_1} - p_{ijk_2}\right|,$$
$$\tau_R = \frac{1}{2}\max_{j_1,j_2,k\in\langle n\rangle}\sum_{i=1}^{n}\left|p_{ij_1k} - p_{ij_2k}\right|,$$
$$\tau = \frac{1}{2}\max_{j,k_1,k_2\in\langle n\rangle}\sum_{i=1}^{n}\left|p_{ijk_1} - p_{ijk_2} + p_{ik_1j} - p_{ik_2j}\right|,$$

and then a uniqueness condition of the solutions for the system of equations (1.4) was given.

Lemma 2.8 ([10]). *If $\tau < 1$, the system of equations* (1.4) *has a unique solution.*

Remark 2.9. Li et al. [22] showed that it is difficult to compare these uniqueness conditions given in Lemmas 2.7 and 2.8. In this paper, we employ these coefficients to investigate the properties of the solutions for the system of equations (1.5) in Subsection 4.3.

3 Applications for Cyclic Stationary Probability Distributions of Second Order Markov Chains

3.1 Low-rank approximation of X in the system of equations (1.2)

In this section, we consider a new low-rank approximation of X in the system of equations (1.2) by a system of cyclic stationary probability distribution equations (1.5).

Let $\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v},\mathbf{a},\mathbf{b} \in \mathbb{S}^n$. We consider the following approximation of X in (1.2):

$$X \approx \frac{1}{3}(\mathbf{x}\circ\mathbf{y} + \mathbf{u}\circ\mathbf{v} + \mathbf{a}\circ\mathbf{b}), \tag{3.1}$$

where $\circ$ denotes the vector outer product, i.e.,

$$x_{ij} \approx \frac{1}{3}(x_iy_j + u_iv_j + a_ib_j).$$

Thus, substituting (3.1) into (1.2) yields

$$x_iy_j + u_iv_j + a_ib_j \approx \sum_{k=1}^{n} p_{ijk}(x_jy_k + u_jv_k + a_jb_k),$$

which gives

$$x_i + u_i + a_i \approx \sum_{j,k\in\langle n\rangle} p_{ijk}(x_jy_k + u_jv_k + a_jb_k).$$

Thus, we have

$$\mathbf{x} + \mathbf{u} + \mathbf{a} \approx \mathcal{P}\mathbf{x}\mathbf{y} + \mathcal{P}\mathbf{u}\mathbf{v} + \mathcal{P}\mathbf{a}\mathbf{b}. \tag{3.2}$$

It is not easy to compute the unknown vectors $\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v},\mathbf{a}$ and $\mathbf{b}$ by the system of equations (3.2). Therefore, we further consider the approximation of model (3.2) by $\mathbf{a} = \mathcal{P}\mathbf{u}\mathbf{v}, \mathbf{v} = \mathcal{P}\mathbf{a}\mathbf{b}, \mathbf{u} = \mathcal{P}\mathbf{x}\mathbf{y}$ with $\mathbf{z} \approx \mathbf{u} \approx \mathbf{b}$, $\mathbf{y} \approx \mathbf{a}$, $\mathbf{x} \approx \mathbf{v}$. Thus, we can get an approximation solution by solving the system of equations (1.5).

3.2 Convergence of the Markov chain process (1.3)

In the current works (e.g., [1, 2, 3, 7, 15, 19, 26]), the convergence of second order Markov chains with all states being independent each other is defined as follows:

Definition 3.1. For every initial states $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in \mathbb{S}^n$, if the sequence $\{\mathbf{x}^{(t)}\}$ generated by (1.3) is convergent as $t \longrightarrow \infty$, we call the second order Markov chain process being convergent.

In other words, the convergence of the second order Markov chains is based on the assumption:

$$\lim_{t\to\infty} \mathbf{x}^{(t)} = \mathbf{x}. \tag{3.3}$$

In this assumption, it leads to two problem: **(i)** If the fixed-point method $\mathbf{x}_k = \mathcal{P}\mathbf{x}_{k-1}^2$ ([11, 19]) is convergent for any $\mathbf{x}_0 \in \mathbb{S}^n$, the second order Markov chain process (1.3) may be divergent, which is different from first order Markov chain (see [15]). **(ii)** If a non-degenerate cyclic probability distribution triple $\boldsymbol{\omega}$ exists in (1.5), the second order Markov chain (1.3) itself can not converge from any starting states in $\mathbb{S}^n$. Actually, if we let $\mathbf{x}^{(1)} = \mathbf{x}$ and $\mathbf{x}^{(2)} = \mathbf{y}$, then we get a sequence $\{\mathbf{x}^{(k)}\}$ by the second order Markov chain (1.3) with

$$\lim_{t\to\infty} \mathbf{x}^{(3t+1)} = \mathbf{x}, \lim_{t\to\infty} \mathbf{x}^{(3t+2)} = \mathbf{y}, \lim_{t\to\infty} \mathbf{x}^{(3t+3)} = \mathbf{z}. \tag{3.4}$$

In this case, it is weird that we can not call it being convergent though the second order Markov chain (1.3) is stable as $t \longrightarrow \infty$. Therefore, we reconsider the new definition of the convergence for the second order Markov chain process under the new hypothesis (3.4).

Definition 3.2. For every initial states $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in \mathbb{S}^n$, if the sequence $\{\mathbf{x}^{(t)}\}$ generated by (1.3) satisfy (3.4), we call the second order Markov chain process being cyclic-convergent.

Remark 3.3. It is easy to check that if the second order Markov chain process (1.3) is convergent, thus it is cyclic-convergent.

It is note that if the fixed-point method for solving the system of equations (1.5) is convergent, the second order Markov chain process (1.3) itself is cyclic-convergent. By the above analysis, it is necessary to study such cyclic stationary probability distributions.

4 Properties of A Solution for the System of Equations (1.5)

In this section, we will consider the existence of the solutions for the system of equations (1.5) and its related properties.

4.1 Existence of solutions for the system of equations (1.5)

First, for given $\mathcal{P} \in \mathbb{S}^{[3,n]}$, we define a nonlinear mapping: $\mathbb{S}^n \times \mathbb{S}^n \times \mathbb{S}^n \longrightarrow \mathbb{S}^n \times \mathbb{S}^n \times \mathbb{S}^n$ with

$$G(\boldsymbol{\omega}) := \begin{pmatrix} \mathcal{P}\mathbf{yz} \\ \mathcal{P}\mathbf{zx} \\ \mathcal{P}\mathbf{xy} \end{pmatrix}, \tag{4.1}$$

where $\boldsymbol{\omega} \equiv (\mathbf{x}^T, \mathbf{y}^T, \mathbf{z}^T)^T \in \mathbb{S}^n \times \mathbb{S}^n \times \mathbb{S}^n$. Since $G(\boldsymbol{\omega})$ is a continuous function and $\mathbb{S}^n \times \mathbb{S}^n \times \mathbb{S}^n$ is a compact set, by the Brouwer fixed-point theorem (see [17, 34]), there exists a fixed-point $\boldsymbol{\omega}$ such that $\boldsymbol{\omega} = G(\boldsymbol{\omega})$. Thus, we have the following assertion.

Theorem 4.1. *For given $\mathcal{P} \in \mathbb{S}^{[3,n]}$, the solution of the system of equations* (1.5) *always exists.*

Remark 4.2. (a) It is noted that if $\overline{\mathbf{x}}$ is a solution of the system of equations (1.4), the system of equations (1.5) always has a solution $\boldsymbol{\omega} = (\overline{\mathbf{x}}^T, \overline{\mathbf{x}}^T, \overline{\mathbf{x}}^T)^T$. This implies that a degenerate cyclic stationary probability distribution always exists.

(b) A fixed-point formula is proposed for solving the system of equations (1.5) as follows:

$$\boldsymbol{\omega}_{k+1} = G(\boldsymbol{\omega}_k). \tag{4.2}$$

It is clear that if the fixed-point method (4.2) is convergent, the second order Markov chain process is cyclic-convergent.

Next we give the following example to illustrate the existence of a non-degenerate cyclic stationary probability distribution triple for the system of equations (1.5).

Example 4.3. Let $\mathcal{P} \in \mathbb{R}^{[3,2]}$ be a stochastic tensor with $\mathcal{P}_{(1)} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Let $\mathbf{e}_1 = (1,0)^T, \mathbf{e}_2 = (0,1)^T$ and $\mathbf{u} = (\frac{\sqrt{5}-1}{2}, \frac{3-\sqrt{5}}{2})^T$. All solutions of the system of equations (1.5) are

$$\boldsymbol{\omega}_1 = (\mathbf{e}_1^T, \mathbf{e}_1^T, \mathbf{e}_2^T)^T, \boldsymbol{\omega}_2 = (\mathbf{e}_1^T, \mathbf{e}_2^T, \mathbf{e}_1^T)^T, \boldsymbol{\omega}_3 = (\mathbf{e}_2^T, \mathbf{e}_1^T, \mathbf{e}_1^T)^T, \boldsymbol{\omega}_4 = (\mathbf{u}^T, \mathbf{u}^T, \mathbf{u}^T)^T.$$

It is seen that $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$ are non-degenerate cyclic stationary probability distribution triple and $\boldsymbol{\omega}_4$ is a degenerate cyclic stationary probability distribution triple.

It is known that if $\mathcal{P} \in \mathbb{S}^{[3,n]}$ is an irreducible tensor, all solutions of the system of equations (1.4) are positive by Lemma 2.4. However, the system of equations (1.5) has non-positive solutions even if $\mathcal{P} \in \mathbb{S}^{[3,n]}$ is an irreducible tensor. Actually, for Example 4.3, since the majorization matrix $M(\mathcal{P}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is an irreducible matrix, $\mathcal{P}$ is an irreducible tensor (see Theorem 2.3 in [31]). Therefore, it is necessary to derive the conditions of $\mathcal{P}$ such that all solutions of the system of equations (1.5) are positive. Then we introduce the concept of a cyclic-irreducible tensor.

Definition 4.4. Let $\mathcal{P} \in \mathbb{R}^{[3,n]}$. $\mathcal{P}$ is called as a cyclic-reducible tensor if there exist proper subsets $\mathbb{I}_1$, $\mathbb{I}_2$ and $\mathbb{I}_3$ of $\langle n \rangle$ (at least one nonempty set for $\mathbb{I}_1$, $\mathbb{I}_2$ and $\mathbb{I}_3$) which satisfy the following conditions:

(1) $i \in \mathbb{I}_1,\ j \notin \mathbb{I}_2^{'}, k \notin \mathbb{I}_3^{'}, p_{ijk} = 0$;

(2) $i \in \mathbb{I}_2,\ j \notin \mathbb{I}_3^{'}, k \notin \mathbb{I}_1^{'}, p_{ijk} = 0$;

(3) $i \in \mathbb{I}_3,\ j \notin \mathbb{I}_1^{'}, k \notin \mathbb{I}_2^{'}, p_{ijk} = 0$.

If $\mathcal{P}$ is not a cyclic-reducible tensor, then $\mathcal{P}$ is called as cyclic-irreducible.

Remark 4.5. If $\mathcal{P}$ is a reducible tensor, $\mathcal{P}$ must be a cyclic-reducible tensor. Thus, if $\mathcal{P}$ is a cyclic-irreducible tensor, $\mathcal{P}$ is an irreducible tensor.

Theorem 4.6. *If $\mathcal{P} \in \mathbb{S}^{[3,n]}$ is a cyclic-irreducible tensor, all solutions of the system of equations* (1.5) *are positive.*

Proof. Let $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ are solutions of the system of equations (1.5) and $\mathbb{I}_{\mathbf{x}} = \{i|x_i = 0\}$, $\mathbb{I}_{\mathbf{y}} = \{i|y_i = 0\}$ and $\mathbb{I}_{\mathbf{z}} = \{i|z_i = 0\}$. Obviously, $\mathbb{I}_{\mathbf{x}}$, $\mathbb{I}_{\mathbf{y}}$ and $\mathbb{I}_{\mathbf{z}}$ are all proper sets of $\langle n\rangle$. Assume that the system of equations (1.5) have a solution including zero entries if $\mathcal{P}$ is cyclic-irreducible, i.e., there is at least one of $\mathbb{I}_{\mathbf{x}}$, $\mathbb{I}_{\mathbf{y}}$ and $\mathbb{I}_{\mathbf{z}}$ is nonempty. Let $\delta_{\mathbf{x}} = \min_{i\in\mathbb{I}'_{\mathbf{x}}}\{x_i\}$, $\delta_{\mathbf{y}} = \min_{i\in\mathbb{I}'_{\mathbf{y}}}\{y_i\}$, $\delta_{\mathbf{z}} = \min_{i\in\mathbb{I}'_{\mathbf{z}}}\{z_i\}$ and $\delta = \min\{\delta_{\mathbf{x}}, \delta_{\mathbf{y}}, \delta_{\mathbf{z}}\}$. Then for $i \in \mathbb{I}_{\mathbf{x}}$

$$0 = x_i = \sum_{j\in\mathbb{I}'_{\mathbf{y}},k\in\mathbb{I}'_{\mathbf{z}}} p_{ijk}y_jz_k \geq \sum_{j\in\mathbb{I}'_{\mathbf{y}},k\in\mathbb{I}'_{\mathbf{z}}} p_{ijk}\delta^2,$$

which gives $p_{ijk} = 0$ for $j \notin \mathbb{I}'_{\mathbf{z}}, k \notin \mathbb{I}'_{\mathbf{x}}$. Similarly, we have

$$\text{for } i \in \mathbb{I}_{\mathbf{y}}, j \notin \mathbb{I}'_{\mathbf{z}}, k \notin \mathbb{I}'_{\mathbf{x}}, p_{ijk} = 0,$$

and

$$\text{for } i \in \mathbb{I}_{\mathbf{z}}, j \notin \mathbb{I}'_{\mathbf{x}}, k \notin \mathbb{I}'_{\mathbf{y}}, p_{ijk} = 0,$$

which contradicts to the cyclic-irreducibility of $\mathcal{P}$. □

Remark 4.7. Since $\mathcal{P}$ is an irreducible tensor if $\mathcal{P}$ is a cyclic-irreducible tensor, it is easy to check that the positive solution of the system of equations (1.5) are not unique by Remark 2.5.

4.2 The existence of non-degenerate cyclic stationary probability distribution triple

It is easy to check that there exists a cyclic-reducible tensor but an irreducible stochastic tensor by Example 4.3. Thus, inspired by this, we get the existence of non-degenerate cyclic stationary probability distribution triple for the system of equations (1.5).

Theorem 4.8. *If $\mathcal{P} \in \mathbb{S}^{[3,n]}$ is an irreducible and cyclic-reducible tensor, there exists an non-degenerate cyclic stationary probability distribution triple for the system of equations* (1.5).

Proof. Since $\mathcal{P}$ is an irreducible stochastic tensor, degenerate cyclic stationary probability distribution triples of the system of equations (1.5) are positive. Next, we prove that there exists an nonnegative but no-positive triple for the system of equations (1.5).

Because $\mathcal{P}$ is a cyclic-reducible tensor, there exist $\mathbb{I}_1$, $\mathbb{I}_2$ and $\mathbb{I}_3$ such that **(1)-(3)** hold in Definition 4.4. Let $\mathbb{S}_{\mathbf{x}} = \{\mathbf{x} \in \mathbb{S}^n | x_i = 0, \forall i \in \mathbb{I}_1\}$, $\mathbb{S}_{\mathbf{y}} = \{\mathbf{y} \in \mathbb{S}^n | y_i = 0, \forall i \in \mathbb{I}_2\}$ and $\mathbb{S}_{\mathbf{z}} = \{\mathbf{z} \in \mathbb{S}^n | z_i = 0, \forall i \in \mathbb{I}_3\}$. Thus, we can get a bounded closed and convex subset $\mathbb{W} \equiv \{\boldsymbol{\omega} = (\mathbf{x}^T, \mathbf{y}^T, \mathbf{z}^T)^T | \mathbf{x} \in \mathbb{S}_{\mathbf{x}}, \mathbf{y} \in \mathbb{S}_{\mathbf{y}}, \mathbf{z} \in \mathbb{S}_{\mathbf{z}}\}$. By Definition 4.4, for any $\boldsymbol{\omega} \in \mathbb{W}$, we have

$$\sum_{j,k\in\langle n\rangle} p_{ijk}y_jz_k = \sum_{j\in\mathbb{S}'_{\mathbf{y}},k\in\mathbb{S}'_{\mathbf{z}}} p_{ijk}y_jz_k = 0, \quad \forall i \in \mathbb{I}_{\mathbf{x}},$$

$$\sum_{j,k\in\langle n\rangle} p_{ijk}z_jx_k = \sum_{j\in\mathbb{S}'_{\mathbf{z}},k\in\mathbb{S}'_{\mathbf{x}}} p_{ijk}z_jx_k = 0, \quad \forall i \in \mathbb{I}_{\mathbf{y}}$$

and

$$\sum_{j,k\in\langle n\rangle} p_{ijk}x_j y_k = \sum_{j\in\mathbb{S}'_{\mathbf{z}},k\in\mathbb{S}'_{\mathbf{x}}} p_{ijk}x_j y_k = 0, \quad \forall i \in \mathbb{I}_{\mathbf{z}},$$

which imply $G(\mathbb{W}) \subseteq \mathbb{W}$. Due to the continuity of the function $G(\omega)$, there exists a fixed-point in the set $\mathbb{W}$ by Browder's fixed-point theorem. Therefore, the desired assertion is followed. □

4.3 Equivalent conditions for the systems (1.4) and (1.5)

In this section, based on the conditions given by Lemma 2.7, we obtain the following uniqueness theorem.

Theorem 4.9. *If one of the following conditions holds:*

(1) $\mu(\mathcal{J}^{(2)}, \mathcal{J}^{(3)}) < 1$;

(2) $\nu(\mathcal{J}^{(2)}, \mathcal{J}^{(3)}) < 1$;

then the system of equations (1.5) *has no non-degenerate cyclic probability distribution triple.*

Proof. By Lemma 2.4, the system of equations (1.4) has a unique solution $\overline{\mathbf{x}}$. Next we prove that the system of equations (1.5) has only the solution $\mathbf{x} = \mathbf{y} = \mathbf{z} = \overline{\mathbf{x}}$. Assume that there are two different solutions for the system of equations (1.5) $\boldsymbol{\omega}_1 \equiv (\mathbf{x}_1^T, \mathbf{y}_1^T, \mathbf{z}_1^T)^T$ and $\boldsymbol{\omega}_2 \equiv (\mathbf{x}_2^T, \mathbf{y}_2^T, \mathbf{z}_2^T)^T$. Let $\Delta\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$, $\Delta\mathbf{y} = \mathbf{y}_1 - \mathbf{y}_2$ and $\Delta\mathbf{z} = \mathbf{z}_1 - \mathbf{z}_2$.

Assertion **(1)**: the system of equations (1.5) has a unique degenerate cyclic probability distribution triple.

It is easy to check that

$$\begin{aligned} ||\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2||_1 &= ||\mathbf{x}_1 - \mathbf{x}_2||_1 + ||\mathbf{y}_1 - \mathbf{y}_2||_1 + ||\mathbf{z}_1 - \mathbf{z}_2||_1 &(4.3)\\ &= ||\mathcal{P}\mathbf{y}_1\mathbf{z}_1 - \mathcal{P}\mathbf{y}_2\mathbf{z}_2||_1 + ||\mathcal{P}\mathbf{z}_1\mathbf{x}_1 - \mathcal{P}\mathbf{z}_2\mathbf{x}_2||_1 + ||\mathcal{P}\mathbf{x}_1\mathbf{y}_1 - \mathcal{P}\mathbf{x}_2\mathbf{y}_2||_1, &(4.4)\end{aligned}$$

Note that

$$||\mathcal{P}\mathbf{y}_1\mathbf{z}_1 - \mathcal{P}\mathbf{y}_2\mathbf{z}_2||_1 \le ||\mathcal{P}\Delta\mathbf{y}\mathbf{z}_1||_1 + ||\mathcal{P}\mathbf{y}_2\Delta\mathbf{z}||_1, \tag{4.5}$$

$$||\mathcal{P}\mathbf{z}_1\mathbf{x}_1 - \mathcal{P}\mathbf{z}_2\mathbf{x}_2||_1 \le ||\mathcal{P}\Delta\mathbf{z}\mathbf{x}_1||_1 + ||\mathcal{P}\mathbf{z}_2\Delta\mathbf{x}||_1 \tag{4.6}$$

and

$$||\mathcal{P}\mathbf{x}_1\mathbf{y}_1 - \mathcal{P}\mathbf{x}_2\mathbf{y}_2||_1 \le ||\mathcal{P}\Delta\mathbf{x}\mathbf{y}_1||_1 + ||\mathcal{P}\mathbf{x}_2\Delta\mathbf{y}||_1. \tag{4.7}$$

Let $x_{s,i}$, $y_{s,i}$ and $z_{s,i}$ denote i-th entry of $\mathbf{x}_s$, $\mathbf{y}_s$ and $\mathbf{z}_s$ $(s = 1, 2)$, respectively and Δx_i, Δy_i and Δz_i denote the i-th entry of $\Delta\mathbf{x}$, $\Delta\mathbf{y}$ and $\Delta\mathbf{z}$, respectively. By Lemma 2.6, it is easy to obtain

$$\begin{aligned} ||\mathcal{P}\Delta\mathbf{y}\mathbf{z}_1||_1 + ||\mathcal{P}\mathbf{y}_2\Delta\mathbf{z}||_1 &= ||(\mathcal{P} - \mathcal{J}^{(2)})\Delta\mathbf{y}\mathbf{z}_1||_1 + ||(\mathcal{P} - \mathcal{J}^{(3)})\mathbf{y}_2\Delta\mathbf{z}||_1\\ &\le \sum_{j,k\in\langle n\rangle}\sum_{i=1}^n |p_{ijk} - J_{ik}^{(2)}||\Delta y_j|z_{1,k} + \sum_{j,k\in\langle n\rangle}\sum_{i=1}^n |p_{ijk} - J_{ij}^{(3)}|y_{2,j}|\Delta z_k|.\end{aligned}$$

Similarly, we have

$$||\mathcal{P}\Delta\mathbf{z}\mathbf{x}_1||_1 + ||\mathcal{P}\mathbf{z}_2\Delta\mathbf{x}||_1 \le \sum_{j,k\in\langle n\rangle}\sum_{i=1}^{n}|p_{ijk} - J_{ik}^{(2)}||\Delta z_j|x_{1,k} + \sum_{j,k\in\langle n\rangle}\sum_{i=1}^{n}|p_{ijk} - J_{ij}^{(3)}|z_{2,j}|\Delta x_k|$$

and

$$||\mathcal{P}\Delta\mathbf{x}\mathbf{y}_1||_1 + ||\mathcal{P}\mathbf{x}_2\Delta\mathbf{y}||_1 \le \sum_{j,k\in\langle n\rangle}\sum_{i=1}^{n}|p_{ijk} - J_{ik}^{(2)}||\Delta x_j|y_{1,k} + \sum_{j,k\in\langle n\rangle}\sum_{i=1}^{n}|p_{ijk} - J_{ij}^{(3)}|x_{2,j}|\Delta y_k|,$$

which together with (4.3)-(4.4) give

$$\begin{aligned}||\boldsymbol{\omega_1} - \boldsymbol{\omega_2}||_1 \le&\Big(\sum_{j,k\in\langle n\rangle}\sum_{i=1}^{n}|p_{ijk} - J_{ik}^{(2)}||\Delta x_j|y_{1,k} + \sum_{j,k\in\langle n\rangle}\sum_{i=1}^{n}|p_{ijk} - J_{ij}^{(3)}|z_{2,j}|\Delta x_k|\Big)\\ &+\Big(\sum_{j,k\in\langle n\rangle}\sum_{i=1}^{n}|p_{ijk} - J_{ik}^{(2)}||\Delta y_j|z_{1,k} + \sum_{j,k\in\langle n\rangle}\sum_{i=1}^{n}|p_{ijk} - J_{ij}^{(3)}|x_{2,j}|\Delta y_k|\Big)\\ &+\Big(\sum_{j,k\in\langle n\rangle}\sum_{i=1}^{n}|p_{ijk} - J_{ik}^{(2)}||\Delta z_j|x_{1,k} + \sum_{j,k\in\langle n\rangle}\sum_{i=1}^{n}|p_{ijk} - J_{ij}^{(3)}|y_{2,j}|\Delta z_k|\Big).\end{aligned} \tag{4.8}$$

Consider the first term of the right hand side of (4.8), we obtain

$$\begin{aligned}&\sum_{j,k\in\langle n\rangle}\sum_{i=1}^{n}|p_{ijk} - J_{ik}^{(2)}||\Delta x_j|y_{1,k} + \sum_{j,k\in\langle n\rangle}\sum_{i=1}^{n}|p_{ijk} - J_{ij}^{(3)}|z_{2,j}|\Delta x_k|\\ \le&\max_{k\in\langle n\rangle}\sum_{j=1}^{n}\sum_{i=1}^{n}|p_{ijk} - J_{ik}^{(2)}||\Delta x_j| + \max_{j\in\langle n\rangle}\sum_{k=1}^{n}\sum_{i=1}^{n}|p_{ijk} - J_{ij}^{(3)}||\Delta x_k|\\ =&\Big(\max_{k\in\langle n\rangle}\sum_{i=1}^{n}|p_{i1k} - J_{ik}^{(2)}| + \max_{j\in\langle n\rangle}\sum_{i=1}^{n}|p_{ij1} - J_{ij}^{(3)}|\Big)|\Delta x_1|\\ &+\Big(\max_{k\in\langle n\rangle}\sum_{i=1}^{n}|p_{i2k} - J_{ik}^{(2)}| + \max_{j\in\langle n\rangle}\sum_{i=1}^{n}|p_{ij2} - J_{ij}^{(3)}|\Big)|\Delta x_2|\\ &+\cdots+\Big(\max_{k\in\langle n\rangle}\sum_{i=1}^{n}|p_{ink} - J_{ik}^{(2)}| + \max_{j\in\langle n\rangle}\sum_{i=1}^{n}|p_{ijn} - J_{ij}^{(3)}|\Big)|\Delta x_n|\\ \le&\max_{t\in\langle n\rangle}\left(\max_{k\in\langle n\rangle}\sum_{i=1}^{n}|p_{itk} - J_{ik}^{(2)}| + \max_{j\in\langle n\rangle}\sum_{i=1}^{n}|p_{ijt} - J_{ij}^{(3)}|\right)\sum_{t=1}^{n}|\Delta x_t|\\ =&\mu(\mathcal{J}^{(2)},\mathcal{J}^{(3)})||\Delta\mathbf{x}||_1.\end{aligned} \tag{4.9}$$

By the analogous technique to (4.9), we get

$$\sum_{j,k\in\langle n\rangle}\sum_{i=1}^{n}|p_{ijk}-J^{(2)}_{ik}||\Delta y_j|z_{1,k}+\sum_{j,k\in\langle n\rangle}\sum_{i=1}^{n}|p_{ijk}-J^{(3)}_{ij}|x_{2,j}|\Delta y_k|\leq\mu(\mathcal{J}^{(2)},\mathcal{J}^{(3)})||\Delta\mathbf{y}||_1 \tag{4.10}$$

and

$$\sum_{j,k\in\langle n\rangle}\sum_{i=1}^{n}|p_{ijk}-J^{(2)}_{ik}||\Delta z_j|x_{1,k}+\sum_{j,k\in\langle n\rangle}\sum_{i=1}^{n}|p_{ijk}-J^{(3)}_{ij}|y_{2,j}|\Delta z_k|\leq\mu(\mathcal{J}^{(2)},\mathcal{J}^{(3)})||\Delta\mathbf{z}||_1. \tag{4.11}$$

By (4.8)-(4.11), we have

$$||\boldsymbol{\omega_1}-\boldsymbol{\omega_2}||_1\leq\mu(\mathcal{J}^{(2)},\mathcal{J}^{(3)})||\boldsymbol{\omega_1}-\boldsymbol{\omega_2}||_1,$$

which contradicts to the condition **(1)** of this theorem. Thus, $\boldsymbol{\omega_1}=\boldsymbol{\omega_2}$.

Assertion **(2)**: the system of equations (1.5) has a unique degenerate cyclic probability distribution triple if $\nu(\mathcal{J}^{(2)},\mathcal{J}^{(3)})<1$.

Without loss of generality, we assume that $s=\arg||\boldsymbol{\omega}_1-\boldsymbol{\omega}_2||_\infty$ with $1\leq s\leq n$, i.e.,

$$||\boldsymbol{\omega}_1-\boldsymbol{\omega}_2||_\infty=||\mathbf{x}_1-\mathbf{x}_2||_\infty=|\Delta x_s|. \tag{4.12}$$

Note that

$$\begin{aligned}|\Delta x_s|&=\Big|\sum_{i_2,i_3\in\langle n\rangle}p_{si_2i_3}(y_{1,i_2}z_{1,i_3}-y_{2,i_2}z_{2,i_3})\Big|\\&=\Big|\sum_{i_2,i_3\in\langle n\rangle}(p_{si_2i_3}-J^{(2)}_{si_3})\Delta y_{i_2}z_{1,i_3}+\sum_{i_2,i_3\in\langle n\rangle}(p_{si_2i_3}-J^{(3)}_{si_2})y_{2,i_2}\Delta z_{i_3}\Big|\\&\leq\sum_{i_2,i_3\in\langle n\rangle}|p_{si_2i_3}-J^{(2)}_{si_3}||\Delta y_{i_2}|z_{1,i_3}+\sum_{i_2,i_3\in\langle n\rangle}|p_{si_2i_3}-J^{(3)}_{si_2}|y_{2,i_2}|\Delta z_{i_3}|\\&\leq\max_{j\in\langle n\rangle}\Big(\sum_{i_2\in\langle n\rangle}|p_{si_2j}-J^{(2)}_{sj}|+\sum_{i_3\in\langle n\rangle}|p_{sji_3}-J^{(3)}_{sj}|\Big)\max\Big\{||\Delta\mathbf{x}||_\infty,||\Delta\mathbf{z}||_\infty\Big\}\\&\leq\nu(\mathcal{J}^{(2)},\mathcal{J}^{(3)})\max\Big\{||\Delta\mathbf{x}||_\infty,||\Delta\mathbf{z}||_\infty\Big\}\\&=\nu(\mathcal{J}^{(2)},\mathcal{J}^{(3)})|\Delta x_s|,\end{aligned}$$

which implies that $\nu(\mathcal{J}^{(2)},\mathcal{J}^{(3)})\geq1$. This contradicts the condition **(2)** of this theorem. By (4.12), we have $\boldsymbol{\omega_1}=\boldsymbol{\omega_2}$.

From Assertions **(1)** and **(2)**, the desired conclude is followed. □

Let $\mathbb{S}_1$, $\mathbb{S}_2$ and $\mathbb{S}_3$ be a proper subset of $\langle n\rangle$, and $\mathbb{S}_1^{'}$, $\mathbb{S}_2^{'}$ and $\mathbb{S}_3^{'}$ be their complementary set in $\langle n\rangle$, i.e., $\mathbb{S}_1^{'}=\langle n\rangle\backslash\mathbb{S}_1$, $\mathbb{S}_2^{'}=\langle n\rangle\backslash\mathbb{S}_2$ and $\mathbb{S}_3^{'}=\langle n\rangle\backslash\mathbb{S}_3$. Let

$$\gamma^{(1)}_{\mathbb{S}_1,\mathbb{S}_2}=\min_{i_3\in\langle n\rangle}\Big(\min_{i_2\in\mathbb{S}_2}\sum_{i\in\mathbb{S}_1^{'}}p_{ii_2i_3}+\min_{i_2\in\mathbb{S}_2^{'}}\sum_{i\in\mathbb{S}_1}p_{ii_2i_3}\Big),$$

$$\gamma^{(2)}_{\mathbb{S}_2,\mathbb{S}_3}=\min_{i_2\in\langle n\rangle}\Big(\min_{i_3\in\mathbb{S}_2}\sum_{i\in\mathbb{S}_3^{'}}p_{ii_2i_3}+\min_{i_2\in\mathbb{S}_2^{'}}\sum_{i\in\mathbb{S}_3}p_{ii_2i_3}\Big)$$

and

$$\overline{\gamma}=\min_{\mathbb{S}_1\subset\langle n\rangle,\mathbb{S}_2\subset\langle n\rangle,\mathbb{S}_3\subset\langle n\rangle}\Big(\gamma^{(1)}_{\mathbb{S}_1,\mathbb{S}_2}+\gamma^{(2)}_{\mathbb{S}_2,\mathbb{S}_3}\Big).$$

Theorem 4.10. *If $\overline{\gamma} > 1$, the system of equations* (1.5) *has a unique degenerate cyclic probability distribution triple.*

Proof. Since $1 < \overline{\gamma} \leq \gamma$, by Lemma 2.4, the system of equations (1.4) has a unique solution $\overline{\mathbf{x}}$. Next we prove that the system of equations (1.5) has the unique solution $\mathbf{x} = \mathbf{y} = \mathbf{z} = \overline{\mathbf{x}}$. Assume that there are two different solutions for the system of equations (1.5) $\boldsymbol{\omega}_1 \equiv (\mathbf{x}_1^T, \mathbf{y}_1^T, \mathbf{z}_1^T)^T$ and $\boldsymbol{\omega}_2 \equiv (\mathbf{x}_2^T, \mathbf{y}_2^T, \mathbf{z}_2^T)^T$. Let $\Delta\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$, $\Delta\mathbf{y} = \mathbf{y}_1 - \mathbf{y}_2$ and $\Delta\mathbf{z} = \mathbf{z}_1 - \mathbf{z}_2$.

Let $\mathbb{V}_1 = \{i|\Delta x_i \geq 0\}$, $\mathbb{V}_2 = \{i|\Delta y_i \geq 0\}$ and $\mathbb{V}_3 = \{i|\Delta z_i \geq 0\}$. Since $\boldsymbol{\omega}_1 \neq \boldsymbol{\omega}_2$, then there is at least a nonempty set among $\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3$ with $\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3 \subset \langle n \rangle$. Taking $\mathbb{V}_1' = \langle n \rangle / \mathbb{V}_1$, $\mathbb{V}_2' = \langle n \rangle / \mathbb{V}_2$ and $\mathbb{V}_3' = \langle n \rangle / \mathbb{V}_3$, thus, there is at least a nonempty set among $\mathbb{V}_1', \mathbb{V}_2', \mathbb{V}_3'$. Note that

$$\sum_{i \in \mathbb{V}_1} \Delta x_i = \sum_{i_2, i_3 \in \langle n \rangle} \sum_{i \in \mathbb{V}_1} p_{ii_2i_3} \Delta y_{i_2} z_{1,i_3} + \sum_{i_2, i_3 \in \langle n \rangle} \sum_{i \in \mathbb{V}_1} p_{ii_2i_3} y_{2,i_2} \Delta z_{i_3}. \tag{4.13}$$

For the first term of the right part in the inequality (4.13),

$$\begin{aligned}
&\sum_{i_2, i_3 \in \langle n \rangle} \sum_{i \in \mathbb{V}_1} p_{ii_2i_3} \Delta y_{i_2} z_{1,i_3} \\
=& \sum_{i_3 \in \langle n \rangle} \Big(\sum_{i_2 \in \mathbb{V}_2} \sum_{i \in \mathbb{V}_1} p_{ii_2i_3} + \sum_{i_2 \in \mathbb{V}_2'} \sum_{i \in \mathbb{V}_1} p_{ii_2i_3} \Big) \Delta y_{i_2} z_{1,i_3} \\
\leq& \max_{i_3 \in \langle n \rangle} \Big(\sum_{i_2 \in \mathbb{V}_2} \sum_{i \in \mathbb{V}_1} p_{ii_2i_3} + \sum_{i_2 \in \mathbb{V}_2'} \sum_{i \in \mathbb{V}_1} p_{ii_2i_3} \Big) \Delta y_{i_2} \\
\leq& \max_{i_3 \in \langle n \rangle} \Big(\max_{i_2 \in \mathbb{V}_2} \sum_{i \in \mathbb{V}_1} p_{ii_2i_3} - \max_{i_2 \in \mathbb{V}_2'} \sum_{i \in \mathbb{V}_1} p_{ii_2i_3} \Big) \sum_{i_2 \in \mathbb{V}_2} \Delta y_{i_2} \\
=& \Big[1 - \min_{i_3 \in \langle n \rangle} \Big(\min_{i_2 \in \mathbb{V}_2} \sum_{i \in \mathbb{V}_1'} p_{ii_2i_3} + \min_{i_2 \in \mathbb{V}_2'} \sum_{i \in \mathbb{V}_1} p_{ii_2i_3} \Big) \Big] \sum_{i_2 \in \mathbb{V}_2} \Delta y_{i_2}.
\end{aligned} \tag{4.14}$$

With the same technique, we have

$$\sum_{i_2, i_3 \in \langle n \rangle} \sum_{i \in \mathbb{V}_1} p_{ii_2i_3} y_{i_2} \Delta z_{1,i_3} = \Big[1 - \min_{i_2 \in \langle n \rangle} \Big(\min_{i_3 \in \mathbb{V}_3} \sum_{i \in \mathbb{V}_1'} p_{ii_2i_3} + \min_{i_3 \in \mathbb{V}_3'} \sum_{i \in \mathbb{V}_1} p_{ii_2i_3} \Big) \Big] \sum_{i_3 \in \mathbb{V}_3} \Delta z_{i_3}. \tag{4.15}$$

Combining (4.15) with (4.13) together gives

$$\begin{aligned}
\sum_{i \in \mathbb{V}_1} \Delta x_i \leq& \Big[1 - \min_{i_3 \in \langle n \rangle} \Big(\min_{i_2 \in \mathbb{V}_2} \sum_{i \in \mathbb{V}_1'} p_{ii_2i_3} + \min_{i_2 \in \mathbb{V}_2'} \sum_{i \in \mathbb{V}_1} p_{ii_2i_3} \Big) \Big] \sum_{i_2 \in \mathbb{V}_2} \Delta y_{i_2} \\
&+ \Big[1 - \min_{i_2 \in \langle n \rangle} \Big(\min_{i_3 \in \mathbb{V}_3} \sum_{i \in \mathbb{V}_1'} p_{ii_2i_3} + \min_{i_3 \in \mathbb{V}_3'} \sum_{i \in \mathbb{V}_1} p_{ii_2i_3} \Big) \Big] \sum_{i_3 \in \mathbb{V}_3} \Delta z_{i_3}.
\end{aligned} \tag{4.16}$$

Similarly, we get

$$\begin{aligned}
\sum_{i \in \mathbb{V}_2} \Delta y_i \leq& \Big[1 - \min_{i_3 \in \langle n \rangle} \Big(\min_{i_2 \in \mathbb{V}_3} \sum_{i \in \mathbb{V}_2'} p_{ii_2i_3} + \min_{i_2 \in \mathbb{V}_3'} \sum_{i \in \mathbb{V}_2} p_{ii_2i_3} \Big) \Big] \sum_{i_2 \in \mathbb{V}_3} \Delta z_{i_2} \\
&+ \Big[1 - \min_{i_2 \in \langle n \rangle} \Big(\min_{i_3 \in \mathbb{V}_1} \sum_{i \in \mathbb{V}_2'} p_{ii_2i_3} + \min_{i_3 \in \mathbb{V}_1'} \sum_{i \in \mathbb{V}_2} p_{ii_2i_3} \Big) \Big] \sum_{i_3 \in \mathbb{V}_1} \Delta x_{i_3}
\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}\sum_{i\in\mathbb{V}_3}\Delta x_i \leq & \Big[1-\min_{i_3\in\langle n\rangle}\Big(\min_{i_2\in\mathbb{V}_1}\sum_{i\in\mathbb{V}_3'}p_{ii_2i_3}+\min_{i_2\in\mathbb{V}_1'}\sum_{i\in\mathbb{V}_3}p_{ii_2i_3}\Big)\Big]\sum_{i_2\in\mathbb{V}_1}\Delta x_{i_2}\\ & +\Big[1-\min_{i_2\in\langle n\rangle}\Big(\min_{i_3\in\mathbb{V}_2}\sum_{i\in\mathbb{V}_3'}p_{ii_2i_3}+\min_{i_3\in\mathbb{V}_2'}\sum_{i\in\mathbb{V}_3}p_{ii_2i_3}\Big)\Big]\sum_{i_3\in\mathbb{V}_2}\Delta y_{i_3},\end{aligned} \tag{4.18}$$

which together with (4.16)-(4.17) gives

$$\sum_{i\in\mathbb{V}_1}\Delta x_i+\sum_{i\in\mathbb{V}_2}\Delta x_i+\sum_{i\in\mathbb{V}_3}\Delta x_i\leq(2-\overline{\gamma})(\sum_{i\in\mathbb{V}_1}\Delta x_i+\sum_{i\in\mathbb{V}_2}\Delta x_i+\sum_{i\in\mathbb{V}_3}\Delta x_i).$$

This is a contradiction with $\overline{\gamma}>1$.

□

Now based on the technique in [10], it follows the following statements.

Theorem 4.11. *If $\overline{\tau}<1$, the system of equations* (1.5) *has a unique degenerate cyclic probability distribution triple, where $\overline{\tau}=\tau_L+\tau_R$.*

Proof. Since $\tau\leq\overline{\tau}<1$, by Lemma 2.8, the system of equations (1.4) has a unique solution $\overline{\mathbf{x}}$. Next we prove that the system of equations (1.5) has the unique solution $\mathbf{x}=\mathbf{y}=\mathbf{z}=\overline{\mathbf{x}}$. By the definition of τ_R and τ_L, it is seen that for any $\mathbf{x},\mathbf{y}\in\mathbb{S}^n$, we have

$$||\mathcal{P}\mathbf{x}(\mathbf{x}-\mathbf{y})||_1\leq\tau_L||\mathbf{x}-\mathbf{y}||_1, \tag{4.19}$$

and

$$||\mathcal{P}(\mathbf{x}-\mathbf{y})\mathbf{x}||_1\leq\tau_R||\mathbf{x}-\mathbf{y}||_1, \tag{4.20}$$

combining with (4.3)-(4.7), it can be shown that

$$||\boldsymbol{\omega_1}-\boldsymbol{\omega_2}||_1\leq\overline{\tau}||\boldsymbol{\omega_1}-\boldsymbol{\omega_2}||_1,$$

which is a contradiction with the known condition. □

By a similar technique with Theorem 4.9 and Theorem 4.10, we give the following proposition.

Proposition 4.12. *If $\mu(\mathcal{J}^{(2)},\mathcal{J}^{(3)})<1$, for any $\boldsymbol{\omega}_1,\boldsymbol{\omega}_2\in\mathbb{S}^n\times\mathbb{S}^n\times\mathbb{S}^n$, we have*

$$||G(\boldsymbol{\omega}_1)-G(\boldsymbol{\omega}_2)||_1\leq\mu(\mathcal{J}^{(2)},\mathcal{J}^{(3)})||\boldsymbol{\omega}_1-\boldsymbol{\omega}_2||_1,$$

$$||G(\boldsymbol{\tau}_1)-G(\boldsymbol{\tau}_2)||_1\leq(2-\overline{\gamma})||\boldsymbol{\tau}_1-\boldsymbol{\tau}_2||_1,$$

and

$$||G(\boldsymbol{\tau}_1)-G(\boldsymbol{\tau}_2)||_\infty\leq\nu(\mathcal{J}^{(2)},\mathcal{J}^{(3)})||\boldsymbol{\tau}_1-\boldsymbol{\tau}_2||_\infty.$$

Proof. The proof is similar to those in Theorems 4.9 and 4.10. So we omit it. □

Remark 4.13. If one of the conditions (1)-(2) in Theorem 4.9 or one in Theorem 4.10(or one in Theorem 4.11) holds, it is easy to give an error analysis for every iterative step of the fixed-point method (4.2) by Proposition 4.12, i.e., for any initial vector $\boldsymbol{\omega}_0$,

$$||\boldsymbol{\omega}_{k+1}-\boldsymbol{\omega}_k||_1\leq\epsilon^k||\boldsymbol{\omega}_1-\boldsymbol{\omega}_0||_1,$$

where $\epsilon=\min\Big\{\mu(\mathcal{J}^{(2)},\mathcal{J}^{(3)}),2-\overline{\gamma},\overline{\tau}\Big\}$.

Let $\boldsymbol{\omega} \equiv (\mathbf{x}^T, \mathbf{y}^T, \mathbf{z}^T)^T \in \mathbb{S}^n \times \mathbb{S}^n \times \mathbb{S}^n$ be a solution of the system of equations (1.5). Next we consider the property of solution for the system of equations (1.5) if arbitrary two of the vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are same. Firstly, we give the following lemma which will be used in the sequel.

Lemma 4.14. *Let $\mathcal{P} \in \mathbb{S}^{[3,n]}$ and $\mathbf{x} \in \mathbb{S}^n$ be a positive vector.*

(1) *For any $i \in \langle n \rangle$, if there exists $k \in \langle n \rangle$ such that $p_{iik} > 0$, the matrix $\mathcal{P} \times_3 \mathbf{x}$ does not have eigenvalue -1.*

(2) *For any $i \in \langle n \rangle$, if there exists $k \in \langle n \rangle$ such that $p_{iki} > 0$, the matrix $\mathcal{P} \times_2 \mathbf{x}$ does not have eigenvalue -1.*

Where

$$(\mathcal{P} \times_2 \mathbf{x})_{ik} = \sum_{j \in \langle n \rangle} p_{ijk} x_j, \ (\mathcal{P} \times_3 \mathbf{x})_{ij} = \sum_{k \in \langle n \rangle} p_{ijk} x_k.$$

Proof. We only prove **(1)**. For any positive vectors $\mathbf{x} \in \mathbb{S}^n$, a matrix $A = (a_{ij})$ is defined by $a_{ij} = \sum_{k=1}^{n} p_{ijk} x_k$. Note that $\sum_{i=1}^{n} a_{ij} = 1$. Thus, A is a column stochastic matrix. Since there exists $k \in \langle n \rangle$ such that $p_{iik} > 0$ for any $i \in \langle n \rangle$, we get $a_{ii} = \sum_{k=1}^{n} p_{iik} x_k > 0$. i.e., all diagonal entries of the matrix A are positive. Thus for any $i \in \langle n \rangle$,

$$|-1 - a_{ii}| = 1 + a_{ii} > 1 - a_{ii} = \sum_{i \neq j} a_{ij}.$$

By using Gerschgorin theorem, it is seen that there is no eigenvalue -1 for the matrix A^T. The desired conclusion is obtained. Furthermore, the assertion **(2)** is also give by an analogous technique. □

Next we give the following theorem.

Theorem 4.15. *Let $\boldsymbol{\omega} \equiv (\mathbf{x}^T, \mathbf{y}^T, \mathbf{z}^T)^T \in \mathbb{S}^n \times \mathbb{S}^n \times \mathbb{S}^n$ be a solution of the system of equations* (1.5) *with $\mathcal{P} \in \mathbb{S}^{[3,n]}$ being a cyclic-irreducible tensor satisfying one of the following two statements.*

(1) *For any $i \in \langle n \rangle$, there exists $k \in \langle n \rangle$ such that $p_{iik} > 0$;*

(2) *For any $i \in \langle n \rangle$, there exists $k \in \langle n \rangle$ such that $p_{iki} > 0$.*

If both of the vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are same, we have $\mathbf{x} = \mathbf{y} = \mathbf{z}$.

Proof. Since $\mathcal{P}$ is a cyclic-irreducible tensor, $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are positive by Theorem 4.6. Without loss of generality, we assume that $\mathbf{x} = \mathbf{y}$. The system of equations (1.5) is rewritten as follows:

$$\begin{cases} \mathbf{x} = \mathcal{P}\mathbf{x}\mathbf{z}, \\ \mathbf{x} = \mathcal{P}\mathbf{z}\mathbf{x}, \\ \mathbf{z} = \mathcal{P}\mathbf{x}\mathbf{x}. \end{cases} \tag{4.21}$$

Let $\Delta\mathbf{u} = \mathbf{x} - \mathbf{z}$. Then

$$\Delta\mathbf{u} = -\mathcal{P}\Delta\mathbf{u}\mathbf{x} \tag{4.22}$$

and

$$\Delta\mathbf{u} = -\mathcal{P}\mathbf{x}\Delta\mathbf{u}, \tag{4.23}$$

which are equivalent to

$$(I + P_1)\Delta\mathbf{u} = \mathbf{0} \tag{4.24}$$

and

$$(I + P_2)\Delta\mathbf{u} = \mathbf{0}, \tag{4.25}$$

respectively, where $(P_1)_{ij} = \sum_{k=1}^{n} p_{ijk}x_k$ and $(P_2)_{ij} = \sum_{k=1}^{n} p_{ikj}x_k$. By Lemma 4.14, $I + P_1$ or $I + P_2$ is nonsingular. Therefore, $\Delta\mathbf{u} = \mathbf{0}$. □

Immediately, by Theorem 4.15, it is easy to check that the following result holds.

Corollary 4.16. *Let $\boldsymbol{\omega} \equiv (\mathbf{x}^T, \mathbf{y}^T, \mathbf{z}^T)^T \in \mathbb{S}^n \times \mathbb{S}^n \times \mathbb{S}^n$ be a solution of the system of equations* (1.5) *with $\mathcal{P} \in \mathbb{S}^{[3,n]}$ being positive. If both of the vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are same, we have $\mathbf{x} = \mathbf{y} = \mathbf{z}$.*

Based on the coefficients τ_L and τ_R, we can imply the following result by (4.19)-(4.20) and (4.22)-(4.23).

Proposition 4.17. *Let $\boldsymbol{\omega} \equiv (\mathbf{x}^T, \mathbf{y}^T, \mathbf{z}^T)^T \in \mathbb{S}^n \times \mathbb{S}^n \times \mathbb{S}^n$ be a solution of the system of equations* (1.5) *with $\tau_L < 1$ or $\tau_R < 1$. If both of the vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are same, we have $\mathbf{x} = \mathbf{y} = \mathbf{z}$.*

5 Concluding Remarks

In this paper, we propose a system of cyclic stationary probability distribution equations for a second order Markov chain process when all states are independent each other. The new model is seen as a rank-3 approximation of the system of equations (1.2). Comparing with results given in previous works, the convergence of the fixed-point method to solve the proposed model leads to the convergence of the second order Markov chain process with the independent state converges. Besides, we investigate the properties of the solutions for the proposed equation such as the conditions of positive solutions and the existence of a non-degenerate cyclic stationary probability distribution triple and so on. We also give some equivalent statements between the system of equations (1.5) and (1.4).

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