Pacific Journal of Optimization
Vol. 17, No. 3, pp. 507–519, 2021

THREE CLASSES OF COPOSITIVE-TYPE TENSORS AND TENSOR COMPLEMENTARITY PROBLEMS*

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Abstract: In the field of complementary problems, an important issue is to investigate under what conditions feasibility of the problem can lead to its solvability. For the linear complementarity problem, such an issue has been studied when the matrix involved is a copositive star matrix, a pseudomonotone matrix, or a copositive plus matrix. In this paper, we first introduce the concepts of copositive star tensors, pseudomonotone tensors, and copositive plus tensors, which are natural extensions of copositive star matrices, pseudomonotone matrices, and copositive plus matrices, respectively. We discuss the relationships among these three classes of tensors and give a complete characterization. Then we establish an existence result of solutions to the tensor complementarity problem under the assumption that the tensor involved is one of these three classes of tensors and an addition condition. Finally we show the equivalence of solvability and feasibility for the tensor complementarity problem with the tensor involved being one of these three classes of tensors.

Key words: *tensor complementarity problem, copositive star tensor, pseudomonotone tensor, copositive plus tensor*

Mathematics Subject Classification: *15A69, 90C33, 65K10*

1 Introduction

The theory and algorithm of the linear complementarity problem (LCP) have been studied a lot due to its large number of practical applications [6]. It is well known that some special types of matrices play an important role in the study of LCPs, and at least there are more than 50 matrix classes discussed in the literature of the LCP before 2010 [7]. These matrix classes describe some characteristics of the LCP and provide some good features from the perspective of algorithms.

In the era of big data, many practical problems need to be described by tensors (hypermatrices), so tensors and their related issues have become one of the research hotspots in recent years. As a generalization of the LCP, the tensor complementarity problem (TCP) has been developed rapidly since 2015, and has achieved fruitful results in both theory [1, 2, 20, 29, 30, 31, 34, 33, 37, 39] and algorithm [10, 13, 21, 22, 32, 35, 36, 38]. An application on multiplayer non-cooperative games was given in [14]. More research can be found in the survey papers [15, 26, 16]. It can be seen that various types of tensors play an important role in the study of the TCP.

*This work was supported by the National Natural Science Foundation of China (Nos. 11871051 and 11901600) and the Science and Technology Program of Guangzhou (No. 202002030280).

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It is well known that in the study of the LCP, an important issue is to investigate its solvability under the feasibility and some additional conditions. The study of this issue is closely related to three kinds of matrices: copositive star matrices, pseudomonotone matrices and copositive plus matrices, which was introduced by Cottle, Habetler and Lemke [8]. The related studies can be found in [9, 12, 18, 24]. For the TCP, although many results for the existence of solutions have been obtained, we have not seen the relevant results for the existence of solutions which are achieved by using feasibility and other conditions. In this paper, we consider such an issue. After the necessary notation, concepts, and results are introduced in the next section, we begin our investigation.

In Section 3, we extend the concepts of pseudomonotone matrices, copositive plus matrices and copositive star matrices to the cases of tensors, named as pseudomonotone tensors, copositive plus tensors and copositive star tensors, respectively. In particular, we give a complete characterization of the relationships among these three classes of tensors.

In Section 4, we first establish an existence result of solutions to the TCP with a copositive star tensor, and use an example to show that the obtained result is different from the existing related result; and then, we obtain the existence result of solutions to the TCP with a copositive plus tensor (or pseudomonotone tensor) by using the relationships among these three classes of tensors.

In Section 5, we first give an equivalence result between solvability and feasibility for a TCP with a copositive star tensor; and then, obtain the same result for a TCP with a copositive plus tensor (or pseudomonotone tensor) by using the relationships among these three classes of tensors.

Some conclusions are given in the last section.

2 Preliminaries

Throughout this paper, we assume that m and n are two positive integers with $m, n \geq 2$, and denote $[n] := \{1, 2, \ldots, n\}$. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space with Euclidean inner product denoted by $\langle \cdot, \cdot \rangle$, and denote it by $\mathbb{R}$ when $n = 1$. We denote $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i \in [n]\}$. A set $K \subseteq \mathbb{R}^n$ is called a cone if $\lambda x \in K$ holds for all $\lambda \geq 0$ and $x \in K$; and furthermore, it is called a closed convex cone if it is both closed set and convex set. For any nonempty set $C \subseteq \mathbb{R}^n$, we use $\mathrm{int}(C)$ to denote its interior and $C^* := \{x \in \mathbb{R}^n : \langle x, y \rangle \geq 0 \text{ for all } y \in C\}$ to denote its dual cone. Obviously, C^* is a closed convex cone with $0 \in C^*$.

An m-th order n-dimensional real tensor $\mathscr{A} = (a_{i_1 i_2 \cdots i_m})$ consists of n^m real entries: $a_{i_1 i_2 \cdots i_m} \in \mathbb{R}$ for any $i_j \in [n]$ with $j \in [m]$. We use $\mathbb{R}^{[m,n]}$ to denote the set of all m-th order n-dimensional real tensors. Suppose that $\mathcal{P}_m$ denotes the permutation group of m indices $\{1, 2, \ldots, m\}$, then a tensor $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is said to be symmetric if $a_{i_1 i_2 \cdots i_m} = a_{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(m)}}$ for all $\sigma \in \mathcal{P}_m$. For any $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ and $x \in \mathbb{R}^n$, we define

$$\mathscr{A} x^{m-1} \in \mathbb{R}^n \text{ with } (\mathscr{A} x^{m-1})_i = \sum_{i_2, i_3, \ldots, i_m = 1}^{n} a_{i i_2 i_3 \cdots i_m} x_{i_2} x_{i_3} \cdots x_{i_m} \ \forall i \in [n];$$

and

$$\mathscr{A} x^m = \sum_{i_1, i_2, \ldots, i_m = 1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}.$$

Let $K \subseteq \mathbb{R}^n$ be a closed convex cone, $\mathscr{A} \in \mathbb{R}^{[m,n]}$ and $p \in \mathbb{R}^n$. The tensor complementarity problem over K, denoted by TCP$(\mathscr{A}, K, p)$, is to find an $x \in K$ such that

$$x \in K, \quad \mathscr{A}x^{m-1} + p \in K^*, \quad \text{and} \quad \langle x, \mathscr{A}x^{m-1} + p\rangle = 0. \tag{2.1}$$

We denote its solution set by SOL$(\mathscr{A}, K, p)$ and its feasible set by FEA$(\mathscr{A}, K, p)$. We will use the following notation:

$$\mathcal{S} := \mathrm{SOL}(\mathscr{A}, K, 0). \tag{2.2}$$

When $K = \mathbb{R}^n_+$, this problem reduces to the TCP, and in this case, we denote it, its solution set and its feasible set by TCP$(\mathscr{A}, p)$, SOL$(\mathscr{A}, p)$ and FEA$(\mathscr{A}, p)$, respectively. Moreover, when $m = 2$, the TCP reduces to the LCP.

A symmetric tensor $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is called to be copositive if $\mathscr{A}x^m \geq 0$ for all $x \in \mathbb{R}^n_+$, which was introduced by Qi [25]. Since then, the copositive tensor has been studied extensively (see, for example, [28, 5, 3, 4, 19]). If $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is not symmetric, by defining $\tilde{\mathscr{A}} = (\tilde{a}_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ with

$$\tilde{a}_{i_1 i_2 \cdots i_m} = \frac{1}{m!} \sum_{\sigma \in \mathcal{P}_m} a_{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(m)}}, \ \forall i_j \in [n], \forall j \in [m],$$

we can see that $\mathscr{A}x^m = \tilde{\mathscr{A}}x^m$ and $\tilde{\mathscr{A}}$ is symmetric [27]; and hence, the copositivity of $\mathscr{A}$ can be judged by the copositivity of symmetric tensor $\tilde{\mathscr{A}}$. In this paper, we use the following definition.

Definition 2.1. A tensor $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is said to be copositive on cone K if

$$\mathscr{A}x^m \geq 0 \text{ for all } x \in K.$$

When $K = \mathbb{R}^n_+$, $\mathscr{A}$ is called to be copositive.

In Definition 2.1, the symmetry of $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is not required. In fact, the copositive mapping has been studied in the literature (see, for example, [11, 23]), and when the mapping is the form of $\mathscr{A}x^m$, the symmetry of $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is also not required. The following result is easy to be obtained from [11, Theorem 1] (or [23, Theorem 3.1]).

Lemma 2.2. *If $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is copositive on closed convex cone K and $p \in \mathrm{int}(\mathcal{S}^*)$, then the solution set of* TCP$(\mathscr{A}, K, p)$ *is nonempty and compact.*

3 Copositive-Type Tensors

In this section, we introduce three classes of copositive-type tensors, which are the natural extensions of the corresponding matrices; and then, we discuss the relationships among them.

Definition 3.1. Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$ and $K \subseteq \mathbb{R}^n$ be a closed convex cone.

(i) $\mathscr{A}$ is called to be a pseudomonotone tensor on K if

$$\langle \mathscr{A}x^{m-1}, y - x\rangle \geq 0, \ \forall x, y \in K \quad \Longrightarrow \quad \langle \mathscr{A}y^{m-1}, y - x\rangle \geq 0. \tag{3.1}$$

Especially, $\mathscr{A}$ is said to be a pseudomonotone tensor if it is a pseudomonotone tensor on K with $K = \mathbb{R}^n_+$.

(ii) $\mathscr{A}$ is called to be a copositive plus tensor on K if $\mathscr{A}$ is copositive on K and

$$\langle \mathscr{A}x^{m-1}, x\rangle = 0,\ \forall x \in K \implies \langle \mathscr{A}y^{m-1}, x\rangle + \langle \mathscr{A}x^{m-1}, y\rangle = 0,\ \forall y \in K. \tag{3.2}$$

Especially, $\mathscr{A}$ is said to be a copositive plus tensor if it is a copositive plus tensor on K with $K = \mathbb{R}^n_+$.

(iii) $\mathscr{A}$ is called to be a copositive star tensor on K if $\mathscr{A}$ is copositive on K and

$$\forall x \in \mathcal{S} \quad \implies \quad \langle \mathscr{A}y^{m-1}, x\rangle \le 0,\ \ \forall y \in K. \tag{3.3}$$

Especially, $\mathscr{A}$ is said to be a copositive star tensor if it is copositive star tensor on K with $K = \mathbb{R}^n_+$.

In the following, we discuss relationships among pseudomonotone tensors, copositive plus tensors, and copositive star tensors.

Proposition 3.2. *If $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is a copositive plus tensor on K, then it is a copositive star tensor on K.*

Proof. Since $\mathscr{A}$ is a copositive plus tensor on K, it follows from Definition 3.1(ii) that (3.2) holds. Thus, by Definition 3.1(iii), we need to show that (3.3) holds. For this purpose, for any $x \in \mathcal{S}$, by (3.2) we have

$$\langle \mathscr{A}y^{m-1}, x\rangle + \langle \mathscr{A}x^{m-1}, y\rangle = 0,\ \forall y \in K.$$

Since $y \in K$ and $\mathscr{A}x^{m-1} \in K^*$, it follows that $\langle \mathscr{A}x^{m-1}, y\rangle \ge 0$. Thus, by the above equality we obtain that

$$\langle \mathscr{A}y^{m-1}, x\rangle \le 0,\ \ \forall y \in K.$$

This means that (3.3) holds; and hence, the desired result holds.

Proposition 3.3. *If $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is a pseudomonotone tensor on K, then it is a copositive star tensor on K.*

Proof. Take $\hat{x} \in \mathcal{S}$, then for any $\alpha > 0$ and $y \in K$, we have

$$\langle \mathscr{A}\hat{x}^{m-1}, \alpha y - \hat{x}\rangle = \alpha\langle \mathscr{A}\hat{x}^{m-1}, y\rangle \ge 0.$$

Since $\mathscr{A}$ is a pseudomonotone tensor on K, it follows from Definition 3.1(i) that

$$\langle \mathscr{A}(\alpha y)^{m-1}, \alpha y - \hat{x}\rangle \ge 0,\ \ \forall y \in K, \forall \alpha > 0,$$

which implies that

$$\alpha\langle \mathscr{A}y^{m-1}, y\rangle \ge \langle \mathscr{A}y^{m-1}, \hat{x}\rangle,\ \ \forall y \in K, \forall \alpha > 0.$$

Let $\alpha \to 0$, we can further obtain that $\langle \mathscr{A}y^{m-1}, \hat{x}\rangle \le 0$ for all $y \in K$. This, together with the arbitrariness of $\hat{x}$, implies that (3.3) holds; and hence, $\mathscr{A}$ is a copositive star tensor on K.

Let $K = \mathbb{R}^n_+$, from Propositions 3.2 and 3.3 we have the following results.

Corollary 3.4. (i) *A copositive plus tensor must be a copositive star tensor.* (ii) *A pseudomonotone tensor must be a copositive star tensor.*

In the following, we construct several examples to further discuss the relationships among pseudomonotone tensors, copositive plus tensors and copositive star tensors.

Example 3.5. Let $K := \mathbb{R}^2_+$. Consider $\mathscr{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{[4,2]}$, where $a_{1111} = a_{2222} = 1$, $a_{1122} = -4$, $a_{2112} = 2$ and all other $a_{i_1i_2i_3i_4} = 0$.

For any $x \in \mathbb{R}^2$, it is obvious that

$$\mathscr{A}x^3 = \begin{pmatrix} x_1^3 - 4x_1x_2^2 \\ 2x_1^2x_2 + x_2^3 \end{pmatrix}.$$

First, we show that $\mathscr{A}$ is a copositive star tensor. On one hand, since

$$\langle \mathscr{A}x^3, x\rangle = (x_1^2 - x_2^2)^2 \geq 0, \ \ \forall x \in \mathbb{R}^2,$$

it follows that $\mathscr{A}$ is copositive. On the other hand, by noting that

$$\{x \in K : \mathscr{A}x^3 \in K^*, \langle \mathscr{A}x^3, x\rangle = 0\} = \{0\},$$

we always have that (3.3) holds. Thus, $\mathscr{A}$ is a copositive star tensor.

Second, we show that $\mathscr{A}$ is not a pseudomonotone tensor. Take $\bar{x} = (0,1)^\top$ and $\bar{y} = (\frac{1}{2}, 1)^\top$, then $\bar{y} - \bar{x} = (\frac{1}{2}, 0)^\top$, $\mathscr{A}\bar{x}^3 = (0,1)^\top$, and $\mathscr{A}\bar{y}^3 = (-\frac{15}{8}, \frac{3}{2})^\top$; and hence,

$$\langle \mathscr{A}\bar{x}^3, \bar{y} - \bar{x}\rangle = 0, \quad \langle \mathscr{A}\bar{y}^3, \bar{y} - \bar{x}\rangle = -\frac{15}{16} < 0.$$

Thus, $\mathscr{A}$ is not a pseudomonotone tensor.

Third, we show that $\mathscr{A}$ is not a copositive plus tensor. Obviously,

$$\{x \in K : \langle \mathscr{A}x^3, x\rangle = 0\} = \{x \in \mathbb{R}^2 : x_1 = x_2 \geq 0\}.$$

Take $\bar{x} = (1,1)^\top$, then $\mathscr{A}\bar{x}^3 = (-3,3)^\top$; and take $\bar{y} = (\frac{1}{2}, 1)^\top$, then $\mathscr{A}\bar{y}^3 = (-\frac{15}{8}, \frac{3}{2})^\top$. Then,

$$\langle \mathscr{A}\bar{y}^3, \bar{x}\rangle + \langle \mathscr{A}\bar{x}^3, \bar{y}\rangle = \frac{9}{8} \neq 0.$$

Thus, $\mathscr{A}$ is not a copositive plus tensor.

Example 3.5 indicates that the inverses of two results in Corollary 3.4 are not true. That is, a copositive star tensor might not be a pseudomonotone tensor; and a copositive star tensor might not be a copositive plus tensor. In the following, we further discuss the relationship between pseudomonotone tensors and copositive plus tensors.

Example 3.6. Let $K := \mathbb{R}^2_+$. Consider $\mathscr{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{[4,2]}$, where $a_{1111} = a_{2222} = 1$, $a_{1122} = a_{2112} = 4$ and all other $a_{i_1i_2i_3i_4} = 0$.

For any $x \in \mathbb{R}^2$, it is obvious that

$$\mathscr{A}x^3 = \begin{pmatrix} x_1^3 + 4x_1x_2^2 \\ 4x_1^2x_2 + x_2^3 \end{pmatrix}.$$

On one hand, since all the entries of $\mathscr{A}$ are nonnegative, it is obvious that $\langle \mathscr{A}x^3, x\rangle \geq 0$ for all $x \in \mathbb{R}^2_+$. Thus, $\mathscr{A}$ is copositive. Furthermore, it is easy to see that $\{x \in K : \langle \mathscr{A}x^3, x\rangle = 0\} = \{0\}$, which implies that (3.2) holds, and hence, $\mathscr{A}$ is a copositive plus tensor. On the other hand, take $\bar{y} := (2,1)^\top$ and $\bar{x} := (1,2)^\top$, then

$$\langle \mathscr{A}\bar{x}^3, \bar{y} - \bar{x}\rangle = 1 > 0, \quad \langle \mathscr{A}\bar{y}^3, \bar{y} - \bar{x}\rangle = -1 < 0.$$

Thus, $\mathscr{A}$ is not a pseudomonotone tensor.

Example 3.6 indicates that a copositive plus tensor might not be a pseudomonotone tensor.

Example 3.7. Let $K := \mathbb{R}^2_+$. Consider $\mathscr{A} = (a_{i_1 i_2 i_3}) \in \mathbb{R}^{[3,2]}$, where $a_{112} = a_{122} = -1$, $a_{211} = a_{212} = 2$ and all other $a_{i_1 i_2 i_3} = 0$.

For any $x \in \mathbb{R}^2$, it is obvious that

$$\mathscr{A} x^2 = \begin{pmatrix} -x_1 x_2 - x_2^2 \\ 2x_1^2 + 2x_1 x_2 \end{pmatrix}.$$

Thus,

$$\langle \mathscr{A} x^2, x \rangle = x_1 x_2 (x_1 + x_2), \quad \forall x \in \mathbb{R}^2,$$

which implies that $\mathscr{A}$ is copositive.

First, we show that $\mathscr{A}$ is a pseudomonotone tensor. For any $x, y \in \mathbb{R}^2$, we have

$$\langle \mathscr{A} x^2, y - x \rangle = (x_1 + x_2)[2x_1 y_2 - x_2(x_1 + y_1)], \tag{3.4}$$
$$\langle \mathscr{A} y^2, y - x \rangle = (y_1 + y_2)[y_2(x_1 + y_1) - 2x_2 y_1]. \tag{3.5}$$

Let

$$\Omega := \{x, y \in \mathbb{R}^2_+ : \langle \mathscr{A} x^2, y - x \rangle \geq 0\}. \tag{3.6}$$

We need to show that $\langle \mathscr{A} y^2, y - x \rangle \geq 0$ holds for all $x, y \in \Omega$. For any $x, y \in \Omega$, if $x_1 + x_2 = 0$, then $x = 0$. In this case, it follows from $x = 0$ and (3.5) that $\langle \mathscr{A} y^2, y - x \rangle \geq 0$. In the following, we assume that $x, y \in \Omega$ and $x_1 + x_2 \neq 0$. By (3.4) and (3.6), we have

$$2x_1 y_2 - x_2(x_1 + y_1) \geq 0. \tag{3.7}$$

If $x_1 = 0$, then it follows from (3.7) that $-x_2 y_1 \geq 0$. Thus, by (3.5), we can obtain that $\langle \mathscr{A} y^2, y - x \rangle \geq 0$. Next, we assume that $x_1 \neq 0$. Then, by (3.7) we have that

$$y_2 \geq (2x_1)^{-1} x_2 (x_1 + y_1).$$

Thus, by using (3.5) we can obtain that

$$\begin{aligned} \langle \mathscr{A} y^2, y - x \rangle &= (y_1 + y_2)[y_2(x_1 + y_1) - 2x_2 y_1] \\ &\geq (y_1 + y_2)\left[(2x_1)^{-1} x_2 (x_1 + y_1)^2 - 2x_2 y_1\right] \\ &= (y_1 + y_2)\left\{(2x_1)^{-1} x_2 \left[(x_1 + y_1)^2 - 4x_1 y_1\right]\right\} \\ &= (y_1 + y_2)(2x_1)^{-1} x_2 (x_1 - y_1)^2 \\ &\geq 0. \end{aligned}$$

So, $\mathscr{A}$ is a pseudomonotone tensor.

Second, we show that $\mathscr{A}$ is not a copositive plus tensor. We need to show that there exists some $x \in K$ satisfying $\langle \mathscr{A} x^2, x \rangle = 0$ and some $y \geq 0$ such that

$$\langle \mathscr{A} y^2, x \rangle + \langle \mathscr{A} x^2, y \rangle \neq 0.$$

Take $\bar{x} = (0, 3)^\top$, then $\bar{x} \geq 0$ and $\langle \mathscr{A} \bar{x}^2, \bar{x} \rangle = 0$; and take $\bar{y} = (1, 1)^\top$. Then, we have

$$\langle \mathscr{A} \bar{y}^2, \bar{x} \rangle + \langle \mathscr{A} \bar{x}^2, \bar{y} \rangle = 3 \neq 0.$$

So, $\mathscr{A}$ is not a copositive plus tensor.

Example 3.7 indicates that a pseudomonotone tensor might not be a copositive plus tensor.

Example 3.8. Let $K := \mathbb{R}^2_+$. Consider $\mathscr{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{[4,2]}$, where $a_{1111} = a_{2222} = 1$ and all other $a_{i_1i_2i_3i_4} = 0$.

For any $x \in \mathbb{R}^2$, it is obvious that

$$\mathscr{A}x^3 = \begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix}.$$

First, it is easy to show that $\langle \mathscr{A}x^3 - \mathscr{A}y^3, x - y\rangle \geq 0$ for all $x, y \in K$, which implies that $\mathscr{A}$ is a pseudomonotone tensor.

Second, it is easy to see that $\langle \mathscr{A}x^3, x\rangle = x_1^4 + x_2^4 \geq 0$ for all $x \in \mathbb{R}^2$, which implies that $\mathscr{A}$ is copositive. Moreover, it is also obvious that $\{x \in K : \langle \mathscr{A}x^3, x\rangle = 0\} = \{0\}$, which implies that (3.2) holds. Thus, $\mathscr{A}$ is a copositive plus tensor.

Example 3.8 indicates that the classes of copositive plus tensors and pseudomonotone tensors have a nonempty intersection.

Up to now, we can see that Corollary 3.4 and Examples 3.5-3.8 describe full relationships among pseudomonotone tensors, copositive plus tensors, and copositive star tensors. To see them more intuitively, we depict the full relationships among these tensors in Figure 1.

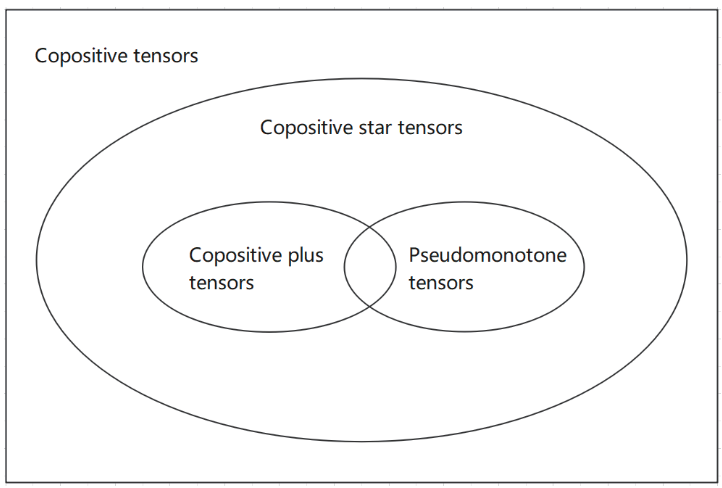

Figure 1: Relationships among several classes of copositive-type tensors

4 Solvability Results

In this section, we discuss the existence of solutions to the TCP with the tensor involved being one of the concerned copositive-type tensors.

Theorem 4.1. *Denote* $\Omega := \{x \in \mathcal{S} \setminus \{0\} : \langle x, p\rangle \leq 0\}$. *Suppose that* $\mathscr{A} \in \mathbb{R}^{[m,n]}$ *is copositive star on closed convex cone* K *and* $\mathrm{FEA}(\mathscr{A}, K, p) \neq \emptyset$. *If* $\Omega \bigcap \mathrm{FEA}(\mathscr{A}, K, p) \neq \emptyset$ *when* $\Omega \neq \emptyset$, *then* $\mathrm{TCP}(\mathscr{A}, K, p)$ *is solvable.*

Proof. On one hand, for any $x \in K$, it is obvious that $\langle x, \mathscr{A}y^{m-1} + p\rangle \geq 0$ for all $y \in \mathrm{FEA}(\mathscr{A}, K, p)$; and hence, for any $x \in \mathcal{S}$, we have that

$$\langle x, \mathscr{A}y^{m-1}\rangle + \langle x, p\rangle \geq 0, \quad \forall y \in \mathrm{FEA}(\mathscr{A}, K, p). \tag{4.1}$$

On the other hand, since $\mathscr{A}$ is copositive star on K, it follows from (3.3) that for any $x \in \mathcal{S}$, $\langle x, \mathscr{A} y^{m-1} \rangle \leq 0$ for all $y \in K$, and hence,

$$\langle x, \mathscr{A} y^{m-1} \rangle \leq 0, \quad \forall y \in \mathrm{FEA}(\mathscr{A}, K, p). \tag{4.2}$$

Combining (4.1) with (4.2), we may assert that

$$\langle x, p \rangle \geq 0, \ \ \forall x \in \mathcal{S}. \tag{4.3}$$

Obviously, $0 \in \mathcal{S}$. In the following, we divide the proof into two cases:

Case 1. Suppose that $\mathcal{S} = \{0\}$. In this case, we have $\mathcal{S}^* = \mathbb{R}^n$, which implies that $p \in \mathrm{int}(\mathcal{S}^*)$. Thus, by Lemma 2.2, we obtain that the solution set of $\mathrm{TCP}(\mathscr{A}, K, p)$ is nonempty and compact.

Case 2. Suppose that $\mathcal{S} \neq \{0\}$. If $\Omega = \emptyset$, then, by (4.3) we have that $\langle x, p \rangle > 0$ for all $x \in \mathcal{S} \setminus \{0\}$, which means that $p \in \mathrm{int}(\mathcal{S}^*)$. Furthermore, by Lemma 2.2, we obtain that the solution set of $\mathrm{TCP}(\mathscr{A}, K, p)$ is nonempty and compact. Moreover, if $\Omega \neq \emptyset$, then it follows from the assumed condition that there exists a point $\bar{x} \in \Omega \bigcap \mathrm{FEA}(\mathscr{A}, K, p)$. In this case, we have

$$0 \leq \langle \bar{x}, \mathscr{A} \bar{x}^{m-1} + p \rangle = \langle \bar{x}, \mathscr{A} \bar{x}^{m-1} \rangle + \langle \bar{x}, p \rangle = \langle \bar{x}, p \rangle \leq 0, \tag{4.4}$$

where the first inequality holds from $\bar{x} \in K$ and $\mathscr{A} \bar{x}^{m-1} + p \in K^*$, the second equality and the second inequality hold from $\bar{x} \in \mathcal{S} \setminus \{0\}$ and $\langle \bar{x}, p \rangle \leq 0$, respectively. From (4.4), we obtain that $\langle \bar{x}, \mathscr{A} \bar{x}^{m-1} + p \rangle = 0$. This, together with $\bar{x} \in \mathrm{FEA}(\mathscr{A}, K, p)$, implies that $\bar{x}$ is a solution to $\mathrm{TCP}(\mathscr{A}, K, p)$.

Combining *Case 1* with *Case 2*, we can obtain the desired result.

In the following, we construct an example, in which all the conditions of Theorem 4.1 are satisfied, but at least one condition of Lemma 2.2 is not satisfied.

Example 4.2. Consider $\mathrm{TCP}(\mathscr{A}, K, p)$ where $\mathscr{A} = (a_{i_1 i_2 i_3}) \in \mathbb{R}^{[3,2]}$ with $a_{121} = a_{122} = 1$, $a_{211} = a_{212} = -1$ and all other $a_{i_1 i_2 i_3} = 0$, $K = \{x \in \mathbb{R}^2 : x_1 \geq x_2 \geq 0\}$ and $p = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

It is obvious that

$$\mathscr{A} x^2 = \begin{pmatrix} x_2(x_1 + x_2) \\ -x_1(x_1 + x_2) \end{pmatrix}, \ \ \forall x \in \mathbb{R}^2.$$

Since $K^* = \{x \in \mathbb{R}^2 : x_1 \geq 0 \ \text{ and } \ x_1 + x_2 \geq 0\}$, it is easy to see that the feasible set of $\mathrm{TCP}(\mathscr{A}, K, p)$ is

$$\begin{aligned} & \mathrm{FEA}(\mathscr{A}, K, p) \\ = \ & \{x \in \mathbb{R}^2 : x_1 \geq x_2 \geq 0, \ x_2(x_1 + x_2) \geq 1, \ (x_2 - x_1)(x_1 + x_2) \geq 0\} \\ = \ & \{x \in \mathbb{R}^2 : x_1 = x_2 \geq \tfrac{1}{\sqrt{2}}\}. \end{aligned}$$

For any $x \in K$, we have that

$$\langle x, \mathscr{A} x^2 \rangle = x_1 x_2 (x_1 + x_2) - x_1 x_2 (x_1 + x_2) = 0,$$

which means that $\mathscr{A}$ is copositive on K. Furthermore, it can be verified that the solution set of $\mathrm{TCP}(\mathscr{A}, K, 0)$ is

$$\mathcal{S} = \mathrm{SOL}(\mathscr{A}, K, 0) = \{x \in \mathbb{R}^2_+ : x_1 = x_2\}.$$

It is easy to check that for any $x \in \mathcal{S}$, it follows that

$$\langle x, \mathscr{A} y^2 \rangle = x_1(y_2 - y_1)(y_1 + y_2) \leq 0, \quad \forall y \in K.$$

Thus, $\mathscr{A}$ is copositive star on K.

It is easy to see that $\text{FEA}(\mathscr{A}, K, p) \subset \mathcal{S}$.

Therefore, from Theorem 4.1, it follows that $\text{TCP}(\mathscr{A}, K, p)$ is solvable. In fact, the solution set of $\text{TCP}(\mathscr{A}, K, p)$ is $\{x \in \mathbb{R}^2 : x_1 = x_2 \geq \frac{1}{\sqrt{2}}\}$. However, it is obvious that $p \notin \text{int}(\mathcal{S}^*)$, i.e., one condition of Lemma 2.2 is not satisfied.

When $K = \mathbb{R}^n_+$, from Theorem 4.1 we have the following results.

Theorem 4.3. *Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$ be a copositive star tensor and $\text{FEA}(\mathscr{A}, p) \neq \emptyset$. If*

$$\{x \in \text{SOL}(\mathscr{A}, 0) \setminus \{0\} : \langle x, p \rangle \leq 0\} \bigcap \text{FEA}(\mathscr{A}, p) \neq \emptyset$$

when $\{x \in \text{SOL}(\mathscr{A}, 0) \setminus \{0\} : \langle x, p \rangle \leq 0\} \neq \emptyset$, then $\text{TCP}(\mathscr{A}, p)$ is solvable.

Remark 4.4. Combining the relationships among copositive star tensors, pseudomonotone tensors and copositive plus tensors established in Secton 3 with Theorem 4.3, we have the following result: For Theorem 4.3, if condition "$\mathscr{A}$ is a copositive star tensor" is replaced by "$\mathscr{A}$ is a copositive plus tensor" (or "$\mathscr{A}$ is a pseudomonotone tensor"), then the same results still hold.

5 Equivalence of Solvability and Feasibility

Theorem 5.1. *Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$ be copositive star on closed convex cone K and $\mathcal{S}$ be defined by (2.2). Then, the following three statements are equivalent:*

(a) $\text{TCP}(\mathscr{A}, K, p)$ *is solvable for all $p \in \mathbb{R}^n$;*

(b) $\text{TCP}(\mathscr{A}, K, p)$ *is feasible for all $p \in \mathbb{R}^n$;*

(c) $\mathcal{S} = \{0\}$.

Proof. The proof is similar to one in [12, Theorem 5.2], and we give it here for completeness.

First, "(a) $\Rightarrow$ (b)" is obvious.

Second, we show "(b) $\Rightarrow$ (c)". For any fixed $x^* \in \mathcal{S}$, since $\mathscr{A}$ is a copositive star tensor on K, it follows that

$$\langle x^*, \mathscr{A} y^{m-1} \rangle \leq 0, \quad \forall y \in K. \tag{5.1}$$

Take $p := -x^*$. By (b) we have that $\text{TCP}(\mathscr{A}, K, p)$ is feasible, i.e., there exists $\bar{x} \in \text{FEA}(\mathscr{A}, K, p) \subseteq K$ such that $\mathscr{A} \bar{x}^{m-1} - x^* \in K^*$. Thus,

$$\langle x^*, \mathscr{A} \bar{x}^{m-1} \rangle - \langle x^*, x^* \rangle = \langle x^*, \mathscr{A} \bar{x}^{m-1} - x^* \rangle \geq 0.$$

This, together with (5.1), implies that $-\langle x^*, x^* \rangle \geq 0$, which yields $x^* = 0$. Thus, (c) holds.

Third, we show "(c) $\Rightarrow$ (a)". In this case, by $\mathcal{S} = \{0\}$, we have $\mathcal{S}^* = \mathbb{R}^n$. Thus, all the conditions in Lemma 2.2 are satisfied. By Lemma 2.2 we obtain that (a) holds.

Therefore, (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

When $K = \mathbb{R}^n_+$, from Theorem 5.1 we have the following results.

Theorem 5.2. *Suppose that $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is a copositive star tensor, then the following three statements are equivalent:*

(a) $\mathrm{TCP}(\mathscr{A}, p)$ *is solvable for all* $p \in \mathbb{R}^n$;

(b) $\mathrm{TCP}(\mathscr{A}, p)$ *is feasible for all* $p \in \mathbb{R}^n$;

(c) $\mathrm{SOL}(\mathscr{A}, 0) = \{0\}$.

Remark 5.3. (i) Combining the relationships among copositive star tensors, pseudomonotone tensors and copositive plus tensors established in Secton 3 with Theorem 5.2, we have the following result: For Theorem 5.2, if condition "$\mathscr{A}$ is a copositive star tensor" is replaced by "$\mathscr{A}$ is a copositive plus tensor" (or "$\mathscr{A}$ is a pseudomonotone tensor"), then the same results still hold. (ii) When $m = 2$, the results of Theorem 5.2 reduce to those in [12].

As a natural extension of Q_0-matrix, we introduce the following concept.

Definition 5.4. $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is called a Q_0-tensor if for all $p \in \mathbb{R}^n$, feasibility of $\mathrm{TCP}(\mathscr{A}, p)$ implies its solvability.

Recall that $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is called a Q-tensor [30, 17] if $\mathrm{TCP}(\mathscr{A}, p)$ is solvable for all $p \in \mathbb{R}^n$. Theorem 5.2 and Remark 5.3 indicate that if $\mathscr{A}$ is one of copositive star tensors, pseudomonotone tensors and copositive plus tensors, then it is a Q-tensor if and only if it is a Q_0-tensor.

6 Concluding Remarks

The main purpose of this paper is to investigate the conditions under which the feasibility of the TCP can lead to its solvability. For this purpose, we introduced three classes of tensors: copositive star tensors, pseudomonotone tensors and copositive plus tensors, and gave a complete characterization of the relationships among these three classes of tensors. For the TCP with the tensor involved being one of these three classes tensors, we achieved an existence result of solutions if an additional condition holds. We also obtain the equivalence result between solvability and feasibility for these three classes of TCPs.

In recent years, many special types of tensors have been studied. A further issue is to study the properties of these three tensors themselves such as the eigenvalue theory. In this paper, we show that copositive star tensors, pseudomonotone tensors and copositive plus tensors are Q_0-tensors. It's worth investigating which tensors are Q_0-tensors besides these three classes of tensors. In addition, it is possible that the results obtained in this paper can be extended to more general complementarity problems such as the complementarity problem with a polynomial mapping (or more generally, a weakly homogeneous mapping).

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Manuscript received 22 February 2021
revised 10 May 2021
accepted for publication 12 May 2021

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