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NEW SMOOTHING MERIT FUNCTION FOR SYMMETRIC CONE COMPLEMENTARITY PROBLEM*

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Abstract: This paper considers the symmetric cone complementarity problems (SCCP), which receive much attention in the recent literature. In this paper, we introduce a new symmetric cone complementarity function and a new associated merit function for SCCP. We show that the new functions have several interesting properties. We also present a sufficient condition under which a stationary point of the new merit function must be a solution of SCCP. Preliminary numerical experience indicates that the new merit function has good performance.

Key words: *symmetric cone complementarity problem, Euclidean Jordan algebra, complementarity function, merit function*

Mathematics Subject Classification: *90C33, 65K05, 90C15*

1 Introduction

Let $\mathbb{V}$ be a finite dimensional real vector space with an inner product denoted by $\langle \cdot, \cdot \rangle$ and $\mathcal{K}$ be a symmetric cone in $\mathbb{V}$. Given two continuous mappings $F : \mathbb{V} \to \mathbb{V}$ and $G : \mathbb{V} \to \mathbb{V}$, the symmetric cone complementarity problem (SCCP) is a problem of finding a vector $\zeta \in \mathbb{V}$ such that

$$F(\zeta) \in \mathcal{K}, \quad G(\zeta) \in \mathcal{K}, \quad \langle F(\zeta), G(\zeta) \rangle = 0. \tag{1.1}$$

This model provides a simple, natural and unified framework for various existing complementarity problems such as the classical nonlinear complementarity problems (NCP) [5, 8], the second-order cone complementarity problems (SOCCP) [1, 18, 9, 27], and the semidefinite complementarity problems (SDCP) [29, 2]. Complementarity problems have wide applications in engineering, economics, management science, and so on; see [5, 8, 19] and the references therein.

In order to solve the SCCP (1.1), the so-called complementarity function $\phi : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$ associated with $\mathcal{K}$, which satisfies

$$\phi(x, y) = 0 \Longleftrightarrow x \in \mathcal{K}, \ y \in \mathcal{K}, \ \langle x, y \rangle = 0, \tag{1.2}$$

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is often used. Two well-known complementarity functions associated with $\mathcal{K}$ include the natural residual function

$$\phi_{\rm NR}(x,y) := x - (x-y)_+$$

and the Fischer-Burmeister function

$$\phi_{\rm FB}(x,y) := (x+y) - (x^2+y^2)^{1/2}, \tag{1.3}$$

where $(\cdot)_+$ denotes the projection operator onto $\mathcal{K}$ under the norm induced by the inner product $\langle\cdot,\cdot\rangle$, x^2 denotes the Jordan product of x and x, and $x^{1/2}$ denotes the Jordan square root of x; see Section 2 for details about Jordan algebra.

By means of some complementarity function ϕ defined as (1.2), the SCCP (1.1) can be easily transformed as nonlinear equations

$$\phi(F(\zeta), G(\zeta)) = 0$$

or an optimization problem

$$\min_{\zeta\in\mathbb{V}} \ \frac{1}{2}\|\phi(F(\zeta), G(\zeta))\|^2.$$

Along these ways, some Newton-type methods and descent methods have been developed; see, e.g., [12, 10, 30]. In addition, more complementarity functions associated with $\mathcal{K}$ are presented. For example, Kum and Lim [14] extended the penalized natural residual and Fischer-Burmeister functions to SCCP. Li et al.[16] proposed a new class of complementarity functions for SCCP, which contained the penalized natural residual and penalized Fischer-Burmeister functions as special cases. Kong et al. [13] extended the implicit Lagrangian function to SCCP. Pan and Chen [20] extended the one-parametric class of merit functions proposed by Kanzow and Kleinmichel [11] to SCCP. More rencently, Tang et al. [26] suggested a new complementarity function defined as

$$\phi_{\rm TLM}(x,y) := x_+ \circ y_+ + (x_-)^2 + (y_-)^2,$$

which is shown to be strongly semismooth. In this paper, by making a slight change over the above function, we obtain a new complementarity function and a new merit function associated with the symmetric cone. We show that the new merit function is differentiable and, under suitable conditions, we establish some results related to coercivity and error bounds for SCCP. Furthermore, we present a sufficient condition under which a stationary point of the new merit function must be a solution of the SCCP (1.1).

This paper is organized as follows. In Section 2, we will introduce some useful mathematical results on Euclidean Jordan algebras associated with symmetric cone. The new complementarity function and the corresponding merit function will be discussed in Section 3 and further properties will be given in Section 4. In Section 5, we report preliminary numerical results with our new merit function. Throughout, we use $\|\cdot\|$ to represent the norm induced by the inner product $\langle\cdot,\cdot\rangle$ and $\mathrm{int}(\mathcal{K})$ to denote the interior of the symmetric cone $\mathcal{K}$. For any $x \in \mathbb{V}$, x_+ and x_- denote the projections of x onto $\mathcal{K}$ and $-\mathcal{K}$ under the induced norm respectively. For any differentiable function F, we denote by $\nabla F(x)$ its transposed Jacobian at x.

$\boxed{2}$ Preliminaries

Jordan algebra plays an important role in the analysis of symmetric cone problems. In this section, we briefly recall some background materials of Euclidean Jordan algebras that are needed in the subsequent sections. See, e.g., [6, 7] for more details.

A Euclidean Jordan algebra is a triple $(\mathbb{V}, \langle\cdot,\cdot\rangle, \circ)$, where $(\mathbb{V}, \langle\cdot,\cdot\rangle)$ is a finite-dimensional inner product space and $\circ$ is a bilinear mapping from $\mathbb{V}\times\mathbb{V}$ to $\mathbb{V}$ satisfying

$$x\circ y = y\circ x, \quad x\circ(x^2\circ y) = x^2\circ(x\circ y), \quad \langle x\circ y, z\rangle = \langle x, y\circ z\rangle$$

for all $x, y, z \in \mathbb{V}$. In general, the vector $x\circ y$ is called the *Jordan product* of x and y. We use $\mathbb{A}$ to denote a Euclidean Jordan algebra $(\mathbb{V}, \langle\cdot,\cdot\rangle, \circ)$ and e to denote the *identity element* in $\mathbb{A}$, i.e., $x\circ e = e\circ x = x$ for every $x\in\mathbb{V}$.

An element $c\in\mathbb{V}$ is said to be *idempotent* if $c^2 = c \neq 0$ and c is said to be *primitive* if it cannot be written as the sum of two idempotents. A *complete system of orthogonal idempotents* is a finite set $\{c_1, c_2, \cdots, c_k\}$ of idempotents with $c_i\circ c_j = 0$ $(i\neq j)$ and $\sum_{i=1}^k c_i = e$. A complete system of orthogonal primitive idempotents is called a *Jordan frame* of $\mathbb{V}$. We denote by $\mathcal{W}(x)$ the *degree* of the vector $x\in\mathbb{V}$, i.e., $\mathcal{W}(x) := \min\{k|\{e, x, x^2, \cdots, x^k\}$ are linearly dependent$\}$, and the *rank* of $\mathbb{A}$ is defined as $\max\{\mathcal{W}(x)|x\in\mathbb{V}\}$. In what follows, we use r to denote the rank of the underlying Eucildean Jordan algebra.

Let $\mathcal{K} := \{x^2 \,|\, x\in\mathbb{V}\}$ be the set of squares of $\mathbb{A}$. It is well known that $\mathcal{K}$ is a symmetric cone with nonempty interior and there exists an invertible linear transformation $\Gamma : \mathbb{V}\to\mathbb{V}$ such that $\Gamma(\mathcal{K}) = \mathcal{K}$ and $\Gamma(x) = y$ for any $x, y\in \mathrm{int}(\mathcal{K})$. A popular example of symmetric cone is the so-called *second-order cone* given by $\mathcal{K} := \{x = (x_1, x_2)\in \mathbb{R}\times\mathbb{R}^{n-1} \,|\, \|x_2\| \le x_1\}$. The following result for Euclidean Jordan algebra can be found in [6].

Theorem 2.1 (Spectral decomposition theorem)**.** *Let $\mathbb{A}$ be a Euclidean Jordan algebra of rank r. Then, for any $x\in\mathbb{A}$, there exists a Jordan frame $c_1(x), c_2(x), \cdots, c_r(x)$ and real numbers $\lambda_1(x), \lambda_2(x), \cdots, \lambda_r(x)$ such that*

$$x = \lambda_1(x)c_1(x) + \lambda_2(x)c_2(x) + \cdots + \lambda_r(x)c_r(x).$$

The numbers $\{\lambda_1(x), \lambda_2(x), \cdots, \lambda_r(x)\}$, which are uniquely determined by x, are called the eigenvalues of x and $\mathrm{Tr}(x) := \sum_{j=1}^r \lambda_j(x)$ is called the trace of x.

We now recall the definition of Löwner operator [25].

Definition 2.2. Let $g : \mathbb{R}\to\mathbb{R}$ be a real-valued function. The *Löwner operator* $\mathcal{G}(x) : \mathbb{A}\to\mathbb{A}$ associated with the Euclidean Jordan algebra $\mathbb{A}$ is defined as

$$\mathcal{G}(x) := g(\lambda_1(x))c_1(x) + g(\lambda_2(x))c_2(x) + \cdots + g(\lambda_r(x))c_r(x),$$

where $x\in\mathbb{A}$ has the spectral decomposition $x = \sum_{j=1}^r \lambda_r(x)c_r(x)$.

In particular, when $g(t)$ is chosen as $t_+ := \max\{0, t\}$ or $t_- := \min\{0, t\}$, $\mathcal{G}$ becomes the following metric projection operators respectively:

$$x_+ := \sum_{i=1}^r (\lambda_i(x))_+ c_i(x), \qquad x_- := \sum_{i=1}^r (\lambda_i(x))_- c_i(x). \tag{2.1}$$

Note that $x\in\mathcal{K}$ if and only if $\lambda_i(x)\ge 0$ $(i = 1, 2, \cdots, r)$. It is easy to verify that

$$x_+\in\mathcal{K}, \qquad x_- = -(-x)_+ \in -\mathcal{K}.$$

This means that x_+ is the projection of x onto $\mathcal{K}$ and x_- is the projection of x onto $-\mathcal{K}$. Moreover, it is not difficult to observe that

$$x = x_+ + x_-, \quad x_+\circ x_- = 0, \quad x\circ x_+ = (x_+)^2, \quad x\circ x_- = (x_-)^2. \tag{2.2}$$

For any $x \in \mathbb{V}$, we define the *Lyapunov transformation* $L(x) : \mathbb{V} \to \mathbb{V}$ by

$$L(x)y = x \circ y, \qquad y \in \mathbb{V}.$$

By Proposition III.1.5 of [6], $L(x)$ is symmetric with respect to $\langle \cdot, \cdot \rangle$ in the sense that

$$\langle L(x)y, z \rangle = \langle y, L(x)z \rangle, \qquad y, z \in \mathbb{V},$$

which means

$$\langle x \circ y, z \rangle = \langle y, z \circ x \rangle = \langle z, x \circ y \rangle, \qquad x, y, z \in \mathbb{V}.$$

The *norm* on $\mathbb{A}$ induced by the inner product can be represented as

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\mathrm{Tr}(x^2)} = \sqrt{\textstyle\sum_{i=1}^{k} (\lambda_i(x))^2}, \qquad x \in \mathbb{V}.$$

Thus, each element in a Jordan frame $\{c_1(x), c_2(x), \cdots, c_r(x)\}$ has a unit norm, i.e., $\|c_i(x)\| = 1$ for each i. It follows from the definitions of $\| \cdot \|$ and $\mathrm{Tr}(\cdot)$ that

$$\langle x, y \rangle = \mathrm{Tr}(x \circ y) \le r\lambda_{\max}(x \circ y) \le r\|x \circ y\|, \qquad x, y \in \mathbb{V}. \tag{2.3}$$

In addition, by the Schwartz's inequality, it is easy to verify that

$$\|x \circ y\| \le \|x\|\|y\|, \qquad x, y \in \mathbb{V}. \tag{2.4}$$

$\boxed{3}$ New Merit Function for SCCP

In this section, we consider the following complementarity function associated with the symmetric cone:

$$\phi(x, y) := (x \circ y)^2 + (x_-)^2 + (y_-)^2. \tag{3.1}$$

By means of the Jordan algebra techniques, we can obtain the new merit function

$$\psi(x, y) := \frac{1}{2}\langle e, \phi(x, y) \rangle = \frac{1}{2}\|x \circ y\|^2 + \frac{1}{2}\|x_-\|^2 + \frac{1}{2}\|y_-\|^2. \tag{3.2}$$

Note that the above function can also be considered as a natural extension of the following merit function proposed by Tseng in [28] for the classical complementarity problem:

$$\widetilde{\psi}(u, v) := u^2v^2 + \min{}^2\{0, u\} + \min{}^2\{0, v\}, \quad u, v \in \mathbb{R},$$

which is another reason why we study the functions (3.1) and (3.2).

We next show that (3.1) is a continuously differentiable complementarity function for SCCP.

Theorem 3.1. *For any $x, y \in \mathbb{V}$, the following statements are equivalent:*
(i) $x \in \mathcal{K},\ y \in \mathcal{K},\ x \circ y = 0$;
(ii) $\phi(x, y) = 0$.

Proof. (i) $\Rightarrow$ (ii). Since $x \in \mathcal{K}$ if and only if $\lambda_i(x) \ge 0$ $(i = 1, 2, \cdots, r)$, we have from (2.1) that $x_- = \sum_{i=1}^{r} (\lambda_i(x))_- c_i(x) = 0$ and, similarly, $y_- = 0$. Together with $x \circ y = 0$, we have $\phi(x, y) = 0$.

(ii) $\Rightarrow$ (i). Since

$$0 = \langle e, \phi(x, y) \rangle = \langle e, (x \circ y)^2 \rangle + \langle e, (x_-)^2 \rangle + \langle e, (y_-)^2 \rangle$$

$$
\begin{aligned}
&= \langle x \circ y, x \circ y\rangle + \langle x_-, x_-\rangle + \langle y_-, y_-\rangle \\
&= \|x \circ y\|^2 + \|x_-\|^2 + \|y_-\|^2,
\end{aligned}
$$

we get $x_- = y_- = x \circ y = 0$. It follows that $x = x_+ + x_- = x_+ \in \mathcal{K}$, and $y = y_+ + y_- = y_+ \in \mathcal{K}$. The desired conclusion is proved. $\square$

The following result can be found in Lemma 3.3 in [13].

Lemma 3.2. *The functions $\{x, x^2, x \circ y, x_+^2, x_-^2\}$ are all continuously differentiable. Moreover, we have*
(i) $\nabla_x(x) = L(e)$;
(ii) $\nabla_x(x \circ x) = 2L(x)$;
(iii) $\nabla_x(x \circ y) = L(y)$;
(iv) $\nabla_y(x \circ y) = L(x)$.

Definition 3.3. Let $f : \mathbb{W} \subseteq \mathbb{V} \to \mathbb{V}$ be locally Lipschitz continuous. Let D_f be the set of points at which f is differentiable. We say that f is *semismooth* at $x \in \mathbb{W}$ if f is directionally differentiable at x and, for any $d \in \mathbb{W}$ and $V \in \partial f(x + d)$,

$$f(x + d) - f(x) - Vd = o(\|d\|),$$

where $\partial f(x) := \mathrm{conv}\partial_B f(x)$.

From [25], we know that both x_- and x_+ are semismooth. Note that the Jordan product $x \circ y$ is also semismooth. Since the composition of semismooth functions is still semismooth, we get immediately that the function ϕ is semismooth. From Lemma 3.2, we can get the following result immediately.

Theorem 3.4. *The function ϕ is continuously differentiable and semismooth everywhere.*

We next consider the function ψ defined by (3.2).

Theorem 3.5. *For any $x, y \in \mathbb{V}$, the following statements are equivalent:*
(i) $x \in \mathcal{K},\ y \in \mathcal{K},\ x \circ y = 0$;
(ii) $\psi(x, y) = \frac{1}{2}\langle e, \phi(x, y)\rangle = 0$.
That is, ψ is a merit function for SCCP. Moreover, it is continuously differentiable function and its gradient is given by

$$\nabla\psi(x, y) = \begin{pmatrix} y \circ (x \circ y) + x_- \\ x \circ (x \circ y) + y_- \end{pmatrix}. \tag{3.3}$$

Proof. From Theorem 3.1 and its proof, we can get the equivalence between (i) and (ii) easily, whereas the continuous differentiability of ψ follows from Theorem 3.4 immediately. We next show (3.3).

In fact, from the chain rule and Lemma 3.2, we have

$$\nabla_x(\frac{1}{2}\|x\circ y\|^2) = \frac{1}{2}\nabla_x(\langle e, (x\circ y)^2\rangle) = \frac{1}{2}\nabla_x((x\circ y)^2)e = L(y)L(x\circ y)e = L(y)(x\circ y) = y\circ(x\circ y).$$

On the other hand, since $x = x_+ + x_-$ and $-x_- = (-x)_+$, we have

$$\|x_-\|^2 = \|(-x)_+\|^2 = \|x_+ - x\|^2 = \min_{v\in\mathcal{K}} \|v - x\|^2,$$

which is convex and differentiable in x (see Page 255 of [24]), and hence

$$\frac{1}{2}\nabla_x(\|x_-\|^2) = \frac{1}{2}\nabla_x(\|(-x)_+\|^2) = -(-x)_+ = x_-.$$

As a result, we have $\nabla_x\psi(x,y) = y\circ(x\circ y) + x_-$. In a similar way, we can show

$$\nabla_y\psi(x,y) = x\circ(x\circ y) + y_-.$$

This completes the proof. □

By Theorem 3.5, we have the following equivalent formulation of SCCP.

Theorem 3.6. *For any $x, y \in \mathbb{V}$, the following statements are equivalent:*
(i) $x \in \mathcal{K},\ y \in \mathcal{K},\ x\circ y = 0$;
(ii) $\nabla\psi(x,y) = 0$.

Proof. (i) ⇒ (ii) is obvious by (3.3). We only need to prove (ii) ⇒ (i). Suppose that $\nabla\psi(x,y) = 0$. By (3.3), we have

$$y\circ(x\circ y) + x_- = 0, \quad x\circ(x\circ y) + y_- = 0. \tag{3.4}$$

That is,

$$\langle y\circ(x\circ y), x\rangle = \langle -x_-, x\rangle, \quad \langle x\circ(x\circ y), y\rangle = \langle -y_-, y\rangle.$$

By (2.2) and the definition of $\mathrm{Tr}(\cdot)$, we have

$$\langle x_-, x\rangle = \mathrm{Tr}(x_-\circ x) = \mathrm{Tr}(x_-\circ x_-) = \langle x_-, x_-\rangle \geq 0,$$

which yields $\langle y\circ(x\circ y), x\rangle = \langle x\circ y, x\circ y\rangle \leq 0$. Together with $\langle x\circ y, x\circ y\rangle \geq 0$, we get $x\circ y = 0$ immediately. From (3.4), we have $x_- = y_- = 0$. Noting that $x = x_+ + x_-$ and $y = y_+ + y_-$, we have $x = x_+ \in \mathcal{K}$ and $y = y_+ \in \mathcal{K}$. This completes the proof. □

4 Results Related to Stationarity, Coerciveness, and Error Bounds

Suppose that F and G are the same as in Section 1. Let

$$\Psi(\zeta) := \psi(F(\zeta), G(\zeta)), \quad \zeta \in \mathbb{V}. \tag{4.1}$$

It is obvious that $\Psi(\zeta) \geq 0$ for any $\zeta \in \mathbb{V}$ and ζ^* solves problem (1.1) if and only if $\Psi(\zeta^*) = 0$. Therefore, the SCCP (1.1) is equivalent to the following unconstrained minimization problem:

$$\min_{\zeta\in\mathbb{V}} \Psi(\zeta). \tag{4.2}$$

Since problem (4.2) is nonconvex, we are only able to find its stationary points in general. An interesting question is when a stationary point of (4.2) is a solution of (1.1). We next discuss this question.

Assume that both F and G are continuously differentiable functions. This means that Ψ is also a continuously differentiable function and its gradient can be written as

$$\nabla\Psi(\zeta) = \nabla F(\zeta)\nabla_1\psi(F(\zeta), G(\zeta)) + \nabla G(\zeta)\nabla_2\psi(F(\zeta), G(\zeta)), \tag{4.3}$$

where $\nabla_1\psi(x,y)$ and $\nabla_2\psi(x,y)$ denote the gradients of ψ with respect to x and y respectively. We have the following theorem.

Theorem 4.1. *Let $\zeta^* \in \mathbb{V}$ be a stationary point of problem* (4.2) *and the matrix $\nabla G(\zeta^*)$ have full column rank. Suppose that $\nabla_1 \psi(F(\zeta^*), G(\zeta^*)) = 0$. Then ζ^* is a solution of the SCCP* (1.1).

Proof. Since ζ^* is stationary to problem (4.2), from (4.3), we get

$$\begin{aligned}\nabla\Psi(\zeta^*) &= \nabla F(\zeta^*)\nabla_1\psi(F(\zeta^*), G(\zeta^*)) + \nabla G(\zeta^*)\nabla_2\psi(F(\zeta^*), G(\zeta^*)) \\ &= \nabla G(\zeta^*)\nabla_2\psi(F(\zeta^*), G(\zeta^*)) \\ &= 0.\end{aligned}$$

Noting that $\nabla G(\zeta^*)$ has full column rank, we have $\nabla_2\psi(F(\zeta^*), G(\zeta^*)) = 0$ and hence

$$\nabla\psi(F(\zeta^*), G(\zeta^*)) = \begin{pmatrix} \nabla_1\psi(F(\zeta^*), G(\zeta^*)) \\ \nabla_2\psi(F(\zeta^*), G(\zeta^*)) \end{pmatrix} = 0.$$

By Theorem 3.6, we know that ζ^* is a solution of the SCCP (1.1). □

When solving the minimization problem (4.2), in order to guarantee the iterative sequence having a limit point, one often hopes that Ψ has a nonempty and bounded level set, which generally can be implied by the coerciveness of Ψ, namely, $\lim_{\|\zeta\|\to\infty} \Psi(\zeta) = +\infty$. Based on this observation, we next study the coerciveness of the function Ψ. To this end, we first introduce the following lemma and definition.

Lemma 4.2. *Let $\psi : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{R}_+$ be given by* (3.2)*. For any $x \in \mathbb{V}$, let $\lambda_{\min}(x)$ denote the minimal eigenvalue of x. Assume that the sequences $\{x^k\}$ and $\{y^k\}$ satisfy one of the following conditions:*

(i) $\lambda_{\min}(x^k) \to -\infty$ *or* $\lambda_{\min}(y^k) \to -\infty$;

(ii) $\{\lambda_{\min}(x^k)\}$ *and* $\{\lambda_{\min}(y^k)\}$ *are bounded below,* $\langle x^k, y^k\rangle \to +\infty$.

Then we have $\psi(x^k, y^k) \to +\infty$.

Proof. Suppose that condition (i) holds. Noting that $\|c_i(x^k)\| = 1$ $(i = 1, 2, \cdots, r)$, we have from (2.1) that

$$\begin{aligned}\|x^k_-\|^2 &= \Big\langle \sum_{i=1}^{r}(\lambda_i(x^k))_- c_i(x^k), \sum_{i=1}^{r}(\lambda_i(x^k))_- c_i(x^k)\Big\rangle \\ &= \sum_{i=1}^{r}(\lambda_i(x^k))_-^2 \|c_i(x^k)\|^2 \\ &\geq (\lambda_{\min}(x^k))_-^2 \\ &= \max{}^2\{0, -\lambda_{\min}(x^k)\}.\end{aligned}$$

In a similar way, it can be shown that $\|y^k_-\|^2 \geq \max{}^2\{0, -\lambda_{\min}(y^k)\}$. By condition (i), either $\|x^k_-\|$ or $\|y^k_-\|$ tends to infinity. Thus, we have $\psi(x^k, y^k) \to +\infty$.

Suppose that condition (ii) holds. By (2.3), we have

$$\langle x^k, y^k\rangle = \mathrm{Tr}(x^k \circ y^k) \leq r\lambda_{\max}(x^k \circ y^k) \leq r\|x^k \circ y^k\|.$$

It follows from the assumption that $\|x^k \circ y^k\| \to +\infty$, which yields $\psi(x^k, y^k) \to +\infty$. □

Definition 4.3. The mappings F and G are called to be *joint R_0^s-functions* if, for any sequence $\{\zeta^k\} \subset \mathbb{V}$ with

$$\|\zeta^k\| \to \infty, \quad \frac{F(\zeta^k)_-}{\|\zeta^k\|} \to 0, \quad \frac{G(\zeta^k)_-}{\|\zeta^k\|} \to 0,$$

there holds

$$\liminf_{k\to\infty} \frac{\langle F(\zeta^k), G(\zeta^k)\rangle}{\|\zeta^k\|} > 0.$$

It is obvious that, when $F(\zeta) \equiv \zeta$, the joint R_0^s-functions reduce to the R_0^s-functions proposed by Pan et al. [21]. We now give the following result related to coerciveness of $\Psi(\zeta)$.

Theorem 4.4. *Let $\Psi : \mathbb{V} \longrightarrow \mathbb{R}_+$ be the function defined by* (4.1)*. If F and G are joint R_0^s-functions, then Ψ is coercive.*

Proof. We prove this result by contradiction. Without loss of generality, we may suppose that there exist a constant $\gamma > 0$ and a sequence $\{\zeta^k\}$ with $\|\zeta^k\| \to \infty$ such that $\Psi(\zeta^k) \leq \gamma$ for all k. It is not difficult to see that the sequence of the smallest eigenvalues of $\{F(\zeta^k)\}$ and $\{G(\zeta^k)\}$ are bounded below (in fact, if not, it follows from Lemma 4.2 that $\Psi(\zeta^k) = \psi(F(\zeta^k), G(\zeta^k)) \to +\infty$, which contradicts $\Psi(\zeta^k) \leq \gamma$). Therefore, we have $\lim_{k\to\infty} \|F(\zeta^k)_-\| < +\infty$ and $\lim_{k\to\infty} \|G(\zeta^k)_-\| < +\infty$, i.e., there exists a positive constant C such that

$$\max\{\|F(\zeta^k)_-\|, \|G(\zeta^k)_-\|\} < C.$$

Since the mappings F and G are joint R_0^s-functions, we immediately obtain

$$\lim_{k\to\infty} \langle F(\zeta^k), G(\zeta^k)\rangle \to +\infty.$$

By Lemma 4.2, we get

$$\Psi(\zeta^k) = \psi(F(\zeta^k), G(\zeta^k)) \to +\infty,$$

which is a contradiction to $\Psi(\zeta^k) \leq \gamma$. Thus, the desired result is proved. □

In the rest of this section, we devote to study error bound conditions, which play a key role in establishing convergence rate of numerical algorithms in general. Let SOL denote the solution set of the SCCP (1.1).

Definition 4.5. We say that $\Psi(\zeta)$ provides a *global* (*local*) *error bound* for SCCP if there exists some constant $c > 0$ (and $\delta > 0$) such that, for each $\zeta \in \mathbb{V}$ (with $\Psi(\zeta) \leq \delta$),

$$\operatorname{dist}(\zeta, \mathrm{SOL}) \leq c\Psi(\zeta),$$

where $\operatorname{dist}(\zeta, \mathrm{SOL})$ denotes the distance from a point ζ to the solution set SOL. In addition, a function f is said to be *BD-regular* at x if all elements in $\partial_B f(x)$ are nonsingular, where $\partial_B f(x)$ denotes the B-subdifferential of f at x.

Theorem 4.6. *Let ϕ and Ψ be defined by* (3.1) *and* (4.1) *respectively. Suppose that* SOL *is nonempty and $\phi(F(\zeta), G(\zeta))$ is BD-regular at all solutions of the SCCP* (1.1)*. Then $\Psi(\zeta)$ provides a local error bound for the SCCP if F and G are joint R_0^s-functions.*

Proof. Since F and G are joint R_0^s-functions, from Theorem 4.4, we know that the level set

$$\mathcal{L}(\gamma) := \{\zeta : \Psi(\zeta) \leq \gamma\}$$

is compact for any $\delta > 0$. Suppose that $\Psi(\zeta)$ does not provide a local error bound, which means that, for any k, there exists $\zeta^k \in \mathcal{L}(1/k) \subseteq \mathcal{L}(1)$ such that $\mathrm{dist}(\zeta^k, \mathrm{SOL}) > k\Psi(\zeta^k)$. It then follows that

$$\frac{\Psi(\zeta^k)}{\mathrm{dist}(\zeta^k, \mathrm{SOL})} \to 0, \quad \Psi(\zeta^k) \to 0$$

as $k \to +\infty$. By the compactness of $\mathcal{L}(1)$, we may assume without loss of generality that $\{\zeta^k\}$ is convergent to a vector, say ζ^*. Then we have

$$\Psi(\zeta^*) = \psi(F(\zeta^*), G(\zeta^*)) = 0, \tag{4.4}$$

which implies $\zeta^* \in \mathrm{SOL}$. It turns out that

$$\frac{\Psi(\zeta^k)}{\|\zeta^k - \zeta^*\|} \to 0 \ \text{ as } \ k \to +\infty. \tag{4.5}$$

By (4.4) and Theorems 3.1–3.5, we get $\phi(F(\zeta^*), G(\zeta^*)) = 0$. Furthermore, since $\phi(F(\zeta), G(\zeta))$ is BD-regular and semismooth, according to Proposition 3 in [22], there are positive constants c and δ such that, for any ζ satisfying $\|\zeta - \zeta^*\| \leq \delta$, we have $\|\phi(F(\zeta), G(\zeta))\| \geq c\|\zeta - \zeta^*\|$. Noting that

$$\begin{aligned}
\|\phi(F(\zeta), G(\zeta))\| &= \|(F(\zeta) \circ G(\zeta))^2 + (F(\zeta)_-)^2 + (G(\zeta)_-)^2\| \\
&\leq \|(F(\zeta) \circ G(\zeta))^2\| + \|(F(\zeta)_-)^2\| + \|(G(\zeta)_-)^2\| \\
&\leq \|F(\zeta) \circ G(\zeta)\|^2 + \|F(\zeta)_-\|^2 + \|G(\zeta)_-\|^2 \\
&= 2\psi(F(\zeta), G(\zeta)) \\
&= 2\Psi(\zeta),
\end{aligned}$$

where the second inequality comes from (2.4), we obtain $2\Psi(\zeta) \geq c\|\zeta - \zeta^*\|$, which contradicts (4.5) and hence $\Psi(\zeta)$ provides a local error bound for SCCP. $\square$

Definition 4.7 ([4])**.** The mappings F and G are called to be *jointly strongly monotone* if there exists a constant $\rho > 0$ such that

$$\langle F(\zeta) - F(\xi), G(\zeta) - G(\xi)\rangle \geq \rho\|\zeta - \xi\|^2, \qquad \zeta, \xi \in \mathbb{V}.$$

The following theorem indicates that the new merit function can provide a global error bound for SCCP.

Theorem 4.8. *Let $\mathbb{A}$ be a Euclidean Jordan algebra of rank r. Suppose that F and G are jointly strongly monotone and the SCCP* (1.1) *has a solution ζ^*. Then there holds*

$$\tau\|\zeta - \zeta^*\|^2 \leq (2 + \sqrt{2})\sqrt{\Psi(\zeta)}, \qquad \zeta \in \mathbb{V},$$

where $\tau := \frac{\rho}{r(1 + \|F(\zeta^*)\| + \|G(\zeta^*)\|)}$.

Proof. Noting that the symmetric cone $\mathcal{K}$ is a self dual cone, we have

$$\langle (-F(\zeta))_-, G(\zeta^*)\rangle \leq 0, \quad \langle F(\zeta^*), (-G(\zeta))_-\rangle \leq 0.$$

Then, by the jointly strong monotonicity assumption, there exists a scalar $\rho > 0$ such that, for any $\zeta \in \mathbb{V}$,

$$\begin{aligned}
\rho\|\zeta - \zeta^*\|^2 &\leq \langle F(\zeta) - F(\zeta^*), G(\zeta) - G(\zeta^*)\rangle \\
&= \langle F(\zeta), G(\zeta)\rangle + \langle -F(\zeta), G(\zeta^*)\rangle + \langle F(\zeta^*), -G(\zeta)\rangle \\
&= \langle F(\zeta), G(\zeta)\rangle + \langle (-F(\zeta))_+ + (-F(\zeta))_-, G(\zeta^*)\rangle \\
&\quad + \langle F(\zeta^*), (-G(\zeta))_+ + (-G(\zeta))_-\rangle \\
&\leq \langle F(\zeta), G(\zeta)\rangle + \langle (-F(\zeta))_+, G(\zeta^*)\rangle + \langle F(\zeta^*), (-G(\zeta))_+\rangle \\
&= \langle F(\zeta), G(\zeta)\rangle + \langle -F(\zeta)_-, G(\zeta^*)\rangle + \langle F(\zeta^*), -G(\zeta)_-\rangle.
\end{aligned}$$

From (2.3) and (2.4), we have

$$\begin{aligned}
\langle F(\zeta), G(\zeta)\rangle &\leq r\|F(\zeta) \circ G(\zeta)\|, \\
\langle -F(\zeta)_-, G(\zeta^*)\rangle &\leq r\|F(\zeta)_-\|\|G(\zeta^*)\|, \\
\langle F(\zeta^*), -G(\zeta)_-\rangle &\leq r\|F(\zeta^*)\|\|G(\zeta)_-\|.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\rho\|\zeta - \zeta^*\|^2 &\leq r\|F(\zeta) \circ G(\zeta)\| + r\|F(\zeta)_-\|\|G(\zeta^*)\| + r\|F(\zeta^*)\|\|G(\zeta)_-\| \\
&\leq r(1 + \|F(\zeta^*)\| + \|G(\zeta^*)\|)\{\|F(\zeta) \circ G(\zeta)\| + \|F(\zeta)_-\| + \|G(\zeta)_-\|\}. \quad (4.6)
\end{aligned}$$

Since $\Psi(\zeta) = \frac{1}{2}\|F(\zeta) \circ G(\zeta)\|^2 + \frac{1}{2}\|F(\zeta)_-\|^2 + \frac{1}{2}\|G(\zeta)_-\|^2$, we have

$$\|F(\zeta) \circ G(\zeta)\| \leq \sqrt{2\Psi(\zeta)}$$

and

$$\|F(\zeta)_-\| + \|G(\zeta)_-\| \leq \sqrt{2(\|F(\zeta)_-\|^2 + \|G(\zeta)_-\|^2)} \leq 2\sqrt{\Psi(\zeta)}.$$

Therefore, it follows from (4.6) that

$$\tau\|\zeta - \zeta^*\|^2 \leq (2 + \sqrt{2})\sqrt{\Psi(\zeta)}$$

with $\tau = \frac{\rho}{r(1+\|F(\zeta^*)\|+\|G(\zeta^*)\|)}$. This completes the proof. □

5 Preliminary Numerical Results

We report our preliminary numerical experience in this section. We first introduce some numerical experience for second-order cone complementarity problems. The tested examples are the same as in Section 4 of [3] for merit function approach (also from the DIMACS Implementation Challenge Library [23]). In our test, we applied the limited-memory Broyden-Fletcher-Goldfard-Shanno (L-BFGS) method applied by Chen and Pan [3], with different merit functions, for solving the linear second-order cone programming

$$\min \ c^T x \qquad \text{s.t.} \quad Ax = b, \ x \in \mathcal{K}. \tag{5.1}$$

Actually, as stated in [4], the KKT conditions of (5.1) can be reformulated as the SCCP (1.1) with

$$F(\zeta) := d + (I - A^T(AA^T)^{-1})\zeta, \quad G(\zeta) := c - A^T(AA^T)^{-1}A\zeta,$$

where $d \in \mathbb{R}^n$ is a fixed vector satisfying $Ad = b$. The merit functions employed here include the FB merit function

$$\psi_{\rm FB}(x,y) := \frac{1}{2}\|\phi_{\rm FB}(x,y)\|^2$$

with $\phi_{\rm FB}$ to be defined by (1.3), the MS merit function

$$\psi_{\rm MS}(x,y) := \langle x,y\rangle + \frac{1}{2\alpha}(\|(x-\alpha y)_+\|^2 - \|x\|^2 + \|(y-\alpha x)_+\|^2 - \|y\|^2)$$

proposed by Kong et al. [13], with $\alpha = 50$, the YF merit function

$$\psi_{\rm YF}(x,y) := \psi_{\rm FB}(x,y) + \psi_0(\langle x,y\rangle)$$

proposed by Yamashita and Fukushima [30], with $\psi_0(t) = \frac{1}{2}\max^2\{0,t\}$, and our new merit function given in (3.2) and denoted by $\psi_{\rm new}$ below.

We employed the limited Broyden-Fletcher-Goldfard-Shanno (L-BFGS) method with 5 limited-memory vector-updates proposed by Liu and Nocedal [17] to solve (5.1). To ease the readers, we briefly describe the method below. In what follows, we denote by ζ^k the current iterate and let $s_k := \zeta^{k+1} - \zeta^k$ and $y_k := \nabla\Psi(\zeta^{k+1}) - \nabla\Psi(\zeta^k)$. The method uses the inverse BFGS formula in the form

$$H_{k+1} = V_k^T H_k V_k + \gamma_k s_k^T s_k,$$

where $\gamma_k := 1/y_k^T s_k$ and $V_k := I - \gamma_k y_k s_k^T$. We revert to the steepest descent direction whenever $s_k^T y_k \le 10^5\|s_k\|\|y_k\|$ and adopt a nonmonotone line search to seek a suitable stepsize. The basic frame of the L-BFGS method is as follows:

Algorithm 5.1.

Step 0. Choose ζ^0, m, n, ρ, σ, $0 < \beta' < 1/2$, $\beta' < \beta < 1$, a symmetric and positive definite matrix H_0. Set $k := 0$.

Step 1. If termination criterion is satisfied, stop. Otherwise, compute

$$d_k := -H_k\nabla\Psi(\zeta^k), \quad \zeta^{k+1} := \zeta^k + \alpha_k d_k,$$

where α_k satisfies the Wolfe conditions

$$\begin{aligned}&\Psi(\zeta^k + \alpha_k d_k) \le \Psi(\zeta^k) + \beta'\alpha_k\nabla\Psi(\zeta^k)^T d_k,\\ &\nabla\Psi(\zeta^k + \alpha_k d_k)^T d_k \ge \beta\nabla\Psi(\zeta^k)^T d_k.\end{aligned}$$

Step 2. Let $\hat m := \{k, m-1\}$. Update H_0 for $\hat m + 1$ times by using the pairs $\{y_j, s_j\}_{j=k-\hat m}^k$, i.e., let

$$\begin{aligned}H_{k+1} \quad := \quad & (V_k^T \cdots V_{k-\hat m}^T)H_0(V_{k-\hat m}\cdots V_k)\\ & +\gamma_{k-\hat m}(V_k^T\cdots V_{k-\hat m+1}^T)s_{k-\hat m}s_{k-\hat m}^T(V_{k-\hat m+1}\cdots V_k)\\ & +\gamma_{k-\hat m+1}(V_k^T\cdots V_{k-\hat m+2}^T)s_{k-\hat m+1}s_{k-\hat m+1}^T(V_{k-\hat m+2}\cdots V_k)\\ & \vdots\\ & +\gamma_k s_k s_k^T.\end{aligned}$$

Step 3. If $s_k^T y_k \leq 10^{-5}\|s_k\|\|y_k\|$, then go to Step 4. Otherwise, set $k \leftarrow k+1$ and go to Step 1.

Step 4. Compute

$$d_k := -\nabla\Psi(\zeta^k), \quad \zeta^{k+1} := \zeta^k + \rho^{l_k} d_k,$$

where l_k is the smallest nonnegative integer l_k such that

$$\Psi(\zeta^k + \rho^{l_k} d_k) \leq \mathrm{W}_k + \sigma\rho^{l_k}\nabla\Psi(\zeta^k)^T d_k$$

with $\mathrm{W}_k := \max_{j=k-m_k,\cdots,k}\Psi(\zeta^j)$ and

$$m_k := \begin{cases} 0 & k \leq n, \\ \min\{m_{k-1}+1, m\} & otherwise. \end{cases}$$

Set $k \leftarrow k+1$ and go to Step 1.

In our test, we set the initial point and the parameters by $\zeta^0 = 0, m = 5, n = 5, \rho = 0.5, \sigma = 10^{-4}$, and terminated the iteration whenever one of the following conditions was satisfied: (1) $\Psi(\zeta^k) := \psi(F(\zeta^k), G(\zeta^k)) \leq 10^{-6}$ and $|\langle F(\zeta^k), G(\zeta^k)\rangle| \leq 10^{-4}$; (2) The step-length is less than 10^{-12}; (3) The number of iteration is over 10^5. The numerical results are summarized in Table 1, in which $\Psi(\zeta^k)$ denotes the merit function values at the final iterations, `Iter` indicates the number of iteration required for each problem, and the notation $*$ means that the iterations were terminated due to too small stepsizes. The numerical results show that our new merit function ψ_{new} is comparable with other merit functions. In particular, the values of our new merit function drops more quickly at the beginning of iteration for the tested problems, which can be observed from the iterative curves in Figure 1.

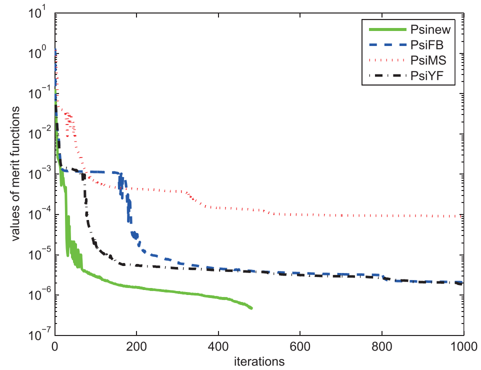
Table 1: Numerical results for linear SOCPs from DIMACS Library

Problem	$\Psi_{\text{FB}}(\zeta^k)$/Iter	$\Psi_{\text{MS}}(\zeta^k)$/Iter	$\Psi_{\text{YF}}(\zeta^k)$/Iter	$\Psi_{\text{new}}(\zeta^k)$/Iter
nb	9.91e-7/2775	4.98e-7/8619	9.99e-7/1514	5.03e-7/481
nb_L1	4.51e-3/10000	*/*	3.01e-3/10000	9.46e-3/8100
nb_L2	8.06e-7/382	9.58e-7/511	8.31e-7/245	*/*
nb_L2_bessel	9.07e-7/122	8.84e-7/162	9.86e-7/181	6.26e-7/308

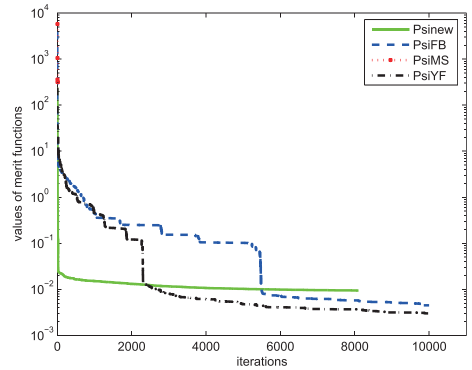
Next, we provide some numerical results for solving linear semidefinite cone complementarity problems (SDCP), which is defined as Find a matrix $X \in S^n$ such that

$$X \in S^n_+, \quad F(X) := M \circ X + q \in S^n_+, \quad \langle X, F(X)\rangle = 0, \tag{5.2}$$

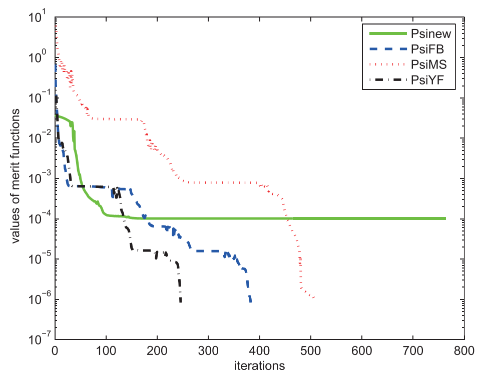
where S^n denotes the space of $n \times n$ real symmetric matrices, $S^n_+ \subseteq S^n$ denotes the closed convex cone comprising those elements of S^n that are positive semidefinite, M and q are symmetric matrices, $\circ$ is the Jordan product defined by $X \circ Y := \frac{X^TY+Y^TX}{2}$, and $\langle\cdot,\cdot\rangle$ is the inner product defined by $\langle X, Y\rangle := \mathrm{Tr}(X^TY)$.



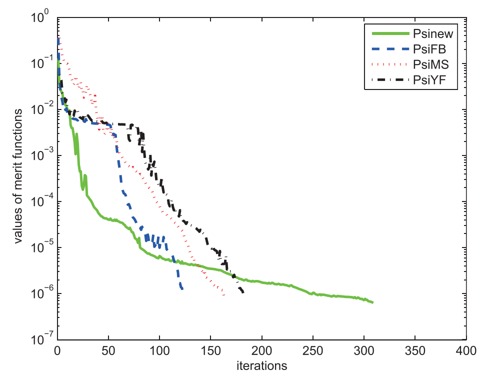
(a) Problem #nb from DIMACS Library



(b) Problem #nb_L1 from DIMACS Library



(c) Problem #nb_L2 from DIMACS Library



(d) Problem #nb_L2_bessel in DIMACS Library

Figure 1: Values of merit functions versus iterations

We employ a PRP-type conjugate gradient algorithm proposed in [15] for solving (5.2). Define

$$\Psi(X) := \psi(X, F(X))$$

and

$$\beta_{k+1}^{PRP} := \frac{\langle \nabla\Psi(X^{k+1}), \nabla\Psi(X^{k+1}) - \nabla\Psi(X^k)\rangle}{\|\nabla\Psi(X^k)\|^2}.$$

The basic frame of PRP-type conjugate gradient algorithm is as follows:

Algorithm 5.2.

Step 0. Choose $\delta,\ \alpha,\ \eta \in (0,1)$ and $\epsilon > 0$. Let $X^0 \in S^n$ and $D^0 := -\nabla\Psi(X^0)$. Set $k := 0$.

Step 1. If $\|\nabla\Psi(X^k)\| \le \epsilon$, stop. Otherwise, let m_k be the smallest nonnegative integer such that

$$\Psi(X^k + \delta^{m_k} D^k) \le \Psi(X^k) - \alpha\delta^{m_k}\|D^k\|^2$$

and set $X^{k+1} := X^k + \delta^{m_k} D^k$.

Step 2. Compute $M^{k+1} := -\nabla\Psi(X^{k+1}) + \beta_{k+1} D^k$ with $\beta_{k+1} := \max\{0, \beta_{k+1}^{PRP}\}$. If

$$\langle \nabla\Psi(X^{k+1}), M^{k+1} \rangle \ge 0$$

or

$$-\eta\|\nabla\Psi(X^{k+1})\|^2 < \langle \nabla\Psi(X^{k+1}), M^{k+1} \rangle < 0, \quad \langle \nabla\Psi(X^{k+1}), D^k \rangle > 0,$$

set $D^{k+1} := -\nabla\Psi(X^{k+1}) - \beta_{k+1} D^k$. Otherwise, set $D^{k+1} = M^{k+1}$. Let $k \leftarrow k+1$ and go to Step 1.

We started the PRP conjugate gradient algorithm with $\delta = 0.5, \alpha = 10^{-4}, \eta = 10^{-3}, \epsilon = 10^{-5}$ and chose $\{X^0, M, q\}$ randomly. Numerical results are reported in Table 2, in which $\Psi(X^0)$ and $\Psi(X^k)$ denote the initial function values and the function values at the final iterations respectively, $\langle X^0, F(X^0)\rangle$ and $\langle X^k, F(X^k)\rangle$ represent the initial inner products and the inner products at the final iterations respectively, $\Psi_{\rm FB}, \Psi_{\rm MS}$ and $\Psi_{\rm new}$ mean the FB merit function, the MS merit function, and our new merit function respectively, `Time` denotes the CPU time in second for solving each problem. The numerical results show again that the performance of our new merit function comparable with other merit functions.

Table 2: Numerical results for linear SDCPs

Dimension	$\Psi_{\rm FB}(X^0)/\Psi_{\rm FB}(X^k)/\langle X^0, F(X^0)\rangle/\langle X^k, F(X^k)\rangle$/Time	$\Psi_{\rm MS}(X^0)/\Psi_{\rm MS}(X^k)/\langle X^0, F(X^0)\rangle/\langle X^k, F(X^k)\rangle$/Time	$\Psi_{\rm new}(X^0)/\Psi_{\rm new}(X^k)/\langle X^0, F(X^0)\rangle/\langle X^k, F(X^k)\rangle$/Time
$n = 3$	5.61/9.80e-1/6.42e+1/7.28/4.95	6.42e+1/2.65/6.42e+1/2.65/3.36	1.99e+3/1.41e-8/6.42e+1/1.97e-4/3.01
$n = 9$	6.45e+1/6.03e+1/7.40e+2/6.58e+2/8.20	1.41e+4/2.17e+3/7.40e+2/1.23e+2/5.32	2.76e+5/1.16/7.40e+2/4.70e-1/8.44
$n = 15$	1.98e+2/1.79e+2/3.54e+3/3.42e+3/16.4	3.69e+4/1.65e+3/3.54e+3/8.78e+1/9.99	6.29e+6/1.17/3.54e+3/7.00e-1/13.6
$n = 30$	1.13e+3/1.01e+3/2.99e+4/3.69e+3/44.3	3.45e+5/3.54e+3/2.99e+4/3.95e+2/31.6	4.49e+8/6.88/2.99e+4/1.12/16.2
$n = 40$	2.25e+3/2.09e+3/6.57e+4/8.52e+4/67.6	8.09e+5/1.10e+4/6.57e+4/6.58e+1/54.0	2.16e+9/1.27e+1/6.57e+4/1.76/37.1
$n = 50$	3.78e+3/3.24e+3/1.28e+5/1.15e+5/91.0	1.51e+6/9.20e+3/1.28e+5/1.99e+2/85.4	8.21e+9/1.50e+1/1.28e+5/1.83/39.8
$n = 100$	4.34e+3/3.76e+3/1.25e+5/1.24e+5/99.0	1.77e+6/6.84e+3/1.25e+5/2.64e+2/85.7	7.78e+9/1.67e+1/1.25e+5/1.53/37.5
$n = 500$	2.26e+4/1.98e+4/1.01e+6/1.35e+5/315	1.10e+7/1.68e+4/1.01e+6/4.07e+2/219	5.07e+11/8.26e+1/1.01e+6/1.73/210

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