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# THE FIRST TWO HYPERTREES WITH LARGER SPECTRAL RADIUS AMONG ALL UNIFORM HYPERTREES WITH GIVEN SIZE AND STRONG STABILITY NUMBER*

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**Abstract:** In this paper, using matching polynomial method, we determine the first two hypertrees which uniquely attain the largest and second largest spectral radius among all hypertrees with given size and strong stability number.

**Key words:** *hypertree, adjacency tensor, spectral radius, matching polynomial*

**Mathematics Subject Classification:** *15A18, 05C65, 05C31*

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## 1 Introduction

A *hypergraph* $\mathcal{H}$ is a pair $(V, E)$, where $V$ is a finite set and $E \subseteq \mathcal{P}(V)$ and $\mathcal{P}(V)$ stands for the power set of $V$. The elements of $V$ are referred to as *vertices* and the elements of $E$ are called *edges*. Sometimes $V$ is denoted by $V(\mathcal{H})$ and $E$ by $E(\mathcal{H})$. We say $\mathcal{H}$ is *nontrivial* if $E \neq \emptyset$ and $\mathcal{H}$ is *r-uniform* if every edge $e \in E(\mathcal{H})$ contains precisely $r$ vertices. For a vertex $v \in V$, we denote by $E_{\mathcal{H}}(v)$ (or simply $E(v)$) the set of edges containing $v$. The cardinality $|E(v)|$ is the *degree* of $v$, denoted by $d_{\mathcal{H}}(v)$ (or simply $d(v)$). A vertex of $\mathcal{H}$ is called an *isolated vertex* if its degree is zero, and is a *core vertex* if its degree is equal to one and an *intersection vertex* otherwise. If any two edges in $\mathcal{H}$ share at most one vertex, then $\mathcal{H}$ is said to be a *linear hypergraph*.

In a hypergraph $\mathcal{H}$, two vertices are *adjacent* if there is an edge containing both of them and two edges are called *adjacent* if their intersection is not empty. A set $S \subseteq V$ or $S \subseteq E$ is *independent* if no two of its elements are adjacent. A vertex $v$ is said to be *incident* to an edge $e$ if $v \in e$. A *path* from $x$ to $y$ in $\mathcal{H}$ is an alternating sequence of vertices and edges $v_1e_1v_2e_2 \dots v_\ell e_\ell v_{\ell+1}$ such that $x = v_1$, $y = v_{\ell+1}$ and

$v_1, v_2, \dots, v_\ell, v_{\ell+1}$ are distinct vertices with the possibility that $v_1 = v_{\ell+1}$;

$e_1, e_2, \dots, e_\ell$ are distinct edges;

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$v_i$ and $v_{i+1}$ are incident to $e_i$ for $1 \leqslant i \leqslant \ell$.

If $x = v_1 = v_{\ell+1} = y$ the path is called a *cycle*. A hypergraph is called *connected* if for any pair of vertices $u$ and $v$, there is a path from $u$ to $v$. A hypergraph is called *acyclic* if it contains no cycle. A connected and acyclic hypergraph is called a *hypertree*. By definition, every hypertree is linear.

Let $\mathcal{H} = (V, E)$ be an $r$-uniform hypergraph of order $n$ and size $m$. *A matching* of $\mathcal{H}$ is a set of independent edges in $\mathcal{H}$. A *k-matching* is a matching consisting of $k$ edges. We denote by $m(\mathcal{H}, k)$ the number of $k$-matchings of $\mathcal{H}$. The maximum number of edges in a matching of $\mathcal{H}$ is called the *matching number* of $\mathcal{H}$ and denoted by $\nu(\mathcal{H})$.

Zhang et al. [19] introduced the polynomial $\varphi(\mathcal{H}, x)$ of a hypertree $\mathcal{H}$ and obtained some properties between the eigenvalues of $\mathcal{H}$ and this polynomial, where $\varphi(\mathcal{H}, x) = \sum_{k=0}^{\nu(\mathcal{H})}(-1)^k m(\mathcal{H}, k)x^{(\nu(\mathcal{H})-k)r}$ which is called the matching polynomial of $\mathcal{H}$ in [2].

Su et al. [13] redefined the *matching polynomial* of $\mathcal{H}$ as

$$\varphi(\mathcal{H}, x) = \sum_{k\geq 0}(-1)^k m(\mathcal{H}, k)x^{n-kr}.$$

This definition seems more appropriate as it guarantees that matching polynomials of hypergraphs of the same order have the same degree and spectral radius of $\mathcal{H}$ is still the maximum real root of $\varphi(\mathcal{H}, x)$ with algebraic multiplicity one.

Based on the matching polynomial of hypergraph, we proposed a *matching polynomial method* which can be described as follows.

(1) First introduce an ordering on hypertrees by positivity of the difference of matching polynomials; (2) The ordering of hypertrees is compatible with the order of their spectral radii in value; (3) The determination of the ordering of hypertrees is usually easier than comparing their spectral radius directly; (4) Additional tools for comparing spectral radius of hypertrees can be obtained via matching polynomial method, such as edge-grafting theorem; (5) Matching polynomial method, combining with other tools, such as weighted incidence matrix method, edge-moving theorem and so on, turns to be more powerful.

In [7], some transformations on hypergraphs such as "edge-moving" and "edge-releasing" were introduced and the first two largest spectral radii of hypertrees on $n$ vertices were characterized. Yuan et al. [18] further determined the first eight uniform hypertrees on $n$ vertices with the largest spectral radii. Xiao et al. [16] characterized the unique uniform hypertree with the maximum spectral radius among all uniform hypertrees with a given degree sequence. The first two largest spectral radii of uniform hypertrees with given diameter were characterized in [17]. Su et al. [13] defined the matching polynomial of hypergraphs and introduced the matching polynomial method. Applying this method, Su et al. characterized the first $\lfloor \frac{d}{2} \rfloor + 1$ hypertrees among all $r$-uniform hypertrees with given size $m$ and diameter $d$, and also determined the first two minimal hypertrees among all $r$-uniform hypertrees with given size. Further, the first two largest hypertrees among all $r$-uniform hypertrees with given size $m$ and a given matching were characterized in [14, 15].

In this paper, applying the matching polynomial method, we determine the first two maximal hypertrees among all $r$-uniform hypertrees with given size and strong stability number. The structure of the remaining part of the paper is as follows: In Section 2, some definitions and results related to the spectral radius of hypergraph are presented. In Section 3, some transformations on hypertrees are introduced and the effect on the ordering of hypertrees are investigated. In Section 4, some properties on strong stability number of hypertrees are obtained. In the last section, we completely determine the first two largest hypertrees among all $r$-uniform hypertrees with given size and strong stability number.

## 2 Preliminaries

Let $\mathcal{H} = (V, E)$ be an $r$-uniform hypergraph on $n$ vertices. A *subgraph* $\mathcal{H}' = (V', E')$ of $\mathcal{H}$ is a hypergraph with $V' \subseteq V$ and $E' \subseteq E$. A *proper subgraph* $\mathcal{H}'$ of $\mathcal{H}$ is a subgraph of $\mathcal{H}$ with $\mathcal{H}' \neq \mathcal{H}$. For a vertex subset $S \subset V$, let $\mathcal{H} - S$ be the subgraph of $\mathcal{H}$ by deleting all the vertices in $S$ and their incident edges. When $S = \{v\}$, $\mathcal{H} - S$ is simply written as $\mathcal{H} - v$. For an edge $e$ with $V(e) = \{v_1, \ldots, v_r\} \in E(\mathcal{H})$, let $\mathcal{H} \setminus e$ stand for the subgraph of $\mathcal{H}$ obtained by deletion of the edge $e$ from $\mathcal{H}$, i.e. $\mathcal{H} \setminus e = (V, E \setminus \{e\})$, and $\mathcal{H} - V(e)$ stand for the subgraph $\mathcal{H} - \{v_1, \ldots, v_r\}$. A *subgraph* generated by an edge subset $F \subseteq E$ of $\mathcal{H}$ is a hypergraph $\mathcal{H}' = (V', F)$, where $V' = \bigcup_{e \in F} V(e)$. For two $r$-uniform hypergraphs $\mathcal{G}$ and $\mathcal{H}$ with $V(\mathcal{G}) \cap V(\mathcal{H}) = \emptyset$, we use $\mathcal{G} \dot{\cup} \mathcal{H}$ to denote the disjoint union of $\mathcal{G}$ and $\mathcal{H}$. Let $t$ be a positive integer, $t\mathcal{G}$ stand for the disjoint union of $t$ copies of $\mathcal{G}$.

An edge $e$ of $\mathcal{H}$ is called a *pendent edge* if $e$ contains exactly $r - 1$ core vertices. If $e$ is not a pendent edge, it is called a *non-pendent edge*. The intersection vertex of a pendent edge is called *support vertex*. An $r$-uniform hypergraph $\mathcal{H}$ is called a *hyperstar*, denoted by $S_m^r$, if there is a partition of the vertex set $V$ as $V = \{v\} \cup V_1 \cup \cdots \cup V_m$ such that $|V_1| = \cdots = |V_m| = r - 1$, and $E = \{\{v\} \cup V_i \mid i = 1, \ldots, m\}$, and $v$ is the *center* of $S_m^r$. Use $P_m^r$ to denote the $r$-uniform hypertree with $m$ edges which is a path.

For positive integers $r$ and $n$, a real *tensor* $\mathcal{A} = (a_{i_1 i_2 \ldots i_r})$ of order $r$ and dimension $n$ refers to a multidimensional array (also called *hypermatrix*) with entries $a_{i_1 i_2 \ldots i_r}$ such that $a_{i_1 i_2 \ldots i_r} \in \mathbb{R}$ for all $i_1, i_2, \ldots, i_r \in [n]$, where $[n] = \{1, 2, \ldots, n\}$.

Qi [10] and Lim [8] independently introduced the concepts of tensor eigenvalues. Let $\mathcal{A}$ be an order $r$ dimension $n$ tensor, $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{C}^n$ a column vector of dimension $n$. If there exists a number $\lambda \in \mathbb{C}$ and a nonzero vector $x \in \mathbb{C}^n$ such that

$$\mathcal{A}x = \lambda x^{[r-1]},$$

where $x^{[r-1]}$ is a vector with $i$-th entry $x_i^{r-1}$, then $\lambda$ is called an *eigenvalue* of $\mathcal{A}$, $x$ is called an *eigenvector* of $\mathcal{A}$ corresponding to the eigenvalue $\lambda$. The *spectral radius* of $\mathcal{A}$ is the maximum modulus of the eigenvalues of $\mathcal{A}$.

**Definition 2.1** ([3])**.** Let $\mathcal{H} = (V, E)$ be an $r$-uniform hypergraph on $n$ vertices. The adjacency tensor of $\mathcal{H}$ is defined as the order $r$ and dimension $n$ tensor $\mathcal{A}(\mathcal{H}) = (a_{i_1 i_2 \ldots i_r})$, whose $(i_1 i_2 \ldots i_r)$-entry is

$$a_{i_1 i_2 \ldots i_r} = \begin{cases} \frac{1}{(r-1)!}, & \text{if } \{i_1, i_2, \ldots, i_r\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

The *spectral radius of hypergraph* $\mathcal{H}$ is defined as spectral radius of its adjacency tensor, denoted by $\rho(\mathcal{H})$. In [4] the weak irreducibility of nonnegative tensors was defined. It was proved that an $r$-uniform hypergraph $\mathcal{H}$ is connected if and only if its adjacency tensor $\mathcal{A}(\mathcal{H})$ is weakly irreducible (see [4]). Part of the Perron-Frobenius theorem for nonnegative tensors is stated in the following.

**Theorem 2.2** ([11])**.** *Let $\mathcal{A}$ be a nonnegative tensor of order $r$ and dimension $n$, where $r, n \geq 2$. Then $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector corresponding to it. If $\mathcal{A}$ is weakly irreducible, then $\rho(\mathcal{A})$ is a positive eigenvalue of $\mathcal{A}$ with a positve eigenvector $x$. Furthermore, $\rho(\mathcal{A})$ is the unique eigenvalue of $\mathcal{A}$ with a positive eigenvector, and $x$ is the unique positive eigenvector associated with $\rho(\mathcal{A})$, up to a multiplicative constant.*

The unique positive eigenvector $x$ with $\sum_{i=1}^{n} x_i^r = 1$ corresponding to $\rho(\mathcal{H})$ is called the *principal eigenvector* of $\mathcal{H}$.

**Lemma 2.3.** ([3]) *Suppose that $\mathcal{G}$ is a uniform hypergraph, and $\mathcal{G}'$ is a subgraph of $\mathcal{G}$. Then $\rho(\mathcal{G}') \le \rho(\mathcal{G})$. Furthermore, if in addition $\mathcal{G}$ is connected and $\mathcal{G}'$ is a proper subgraph, we have $\rho(\mathcal{G}') < \rho(\mathcal{G})$.*

An *edge-moving operation* on hypergraphs was introduced in [7]. Let $\mathcal{H} = (V, E)$ be a hypergraph with $u \in V$ and $e_1, \dots, e_k \in E$, such that $u \notin e_i$ for $i = 1, \dots, k$. Suppose that $v_i \in e_i$ and write $e_i' = (e_i \setminus \{v_i\}) \cup \{u\}$ $(i = 1, \dots, k)$. Let $\mathcal{H}' = (V, E')$ be the hypergraph with $E' = (E \setminus \{e_1, \dots, e_k\}) \cup \{e_1', \dots, e_k'\}$. Note that $v_1, \dots, v_k$ need not be distinct. Assume that $v_1, \dots, v_r$ are all distinct vertices of them. Then we say that $\mathcal{H}'$ is obtained from $\mathcal{H}$ by *moving edges* $(e_1, \dots, e_k)$ from $(v_1, \dots, v_k)$ (or from $v_1, \dots, v_r$) to $u$.

**Lemma 2.4** ([7])**.** *Let $\mathcal{H}$ be a connected and uniform hypergraph, $\mathcal{H}'$ be the hypergraph obtained from $\mathcal{H}$ by moving edges $(e_1, \dots, e_k)$ from $(v_1, \dots, v_k)$ to $u$, and $\mathcal{H}'$ contains no multiple edges. If $x$ is the principal eigenvector of $\mathcal{H}$ corresponding to $\rho(\mathcal{H})$, and suppose that $x_u \ge \max_{1 \le i \le k}\{x_{v_i}\}$, then $\rho(\mathcal{H}') > \rho(\mathcal{H})$.*

The matching polynomial is fundamental in determining extremal spectral radius of hypertrees, and so some basic properties of matching polynomial of hypergraph are listed as follows.

**Lemma 2.5** ([13])**.** *Let $\mathcal{G}$ and $\mathcal{H}$ be two $r$-uniform hypergraphs. Then the following statements hold.*

(a) $\varphi(\mathcal{G} \dot\cup \mathcal{H}, x) = \varphi(\mathcal{G}, x)\varphi(\mathcal{H}, x)$.

(b) $\varphi(\mathcal{G}, x) = \varphi(\mathcal{G} \setminus e, x) - \varphi(\mathcal{G} - V(e), x)$ *if $e$ is an edge of $\mathcal{G}$.*

(c) *If $u \in V(\mathcal{G})$ and $I = \{i | e_i \in E(u)\}$, for any $J \subseteq I$, we have*

$$\varphi(\mathcal{G}, x) = \varphi(\mathcal{G} \setminus \{e_i | i \in J\}, x) - \sum_{i \in J} \varphi(\mathcal{G} - V(e_i), x)$$

*and*

$$\varphi(\mathcal{G}, x) = x\varphi(\mathcal{G} - u, x) - \sum_{e \in E(u)} \varphi(\mathcal{G} - V(e), x).$$

Let $\mathcal{T}$ and $\mathcal{T}'$ be acyclic uniform hypergraphs of $n$ vertices. We defined $\mathcal{T}' \preceq \mathcal{T}$ iff $\varphi(\mathcal{T}', x) \ge \varphi(\mathcal{T}, x)$ for every $x \ge \rho(\mathcal{T})$; let $\mathcal{T}' \prec \mathcal{T}$ iff $\mathcal{T}' \preceq \mathcal{T}$ and $\varphi(\mathcal{T}', x) - \varphi(\mathcal{T}, x) \neq 0$ at the point $x = \rho(\mathcal{T})$, which implies that $\varphi(\mathcal{T}', x) - \varphi(\mathcal{T}, x) > 0$ for any $x \ge \rho(\mathcal{T})$. Note that $\mathcal{T}' \prec \mathcal{T}$ ($\mathcal{T}' \preceq \mathcal{T}$, resp.) implies $\rho(\mathcal{T}') < \rho(\mathcal{T})$ ($\rho(\mathcal{T}') \le \rho(\mathcal{T})$, resp.).

Now we list some useful results proposed in [13].

**Lemma 2.6** ([13])**.** *If $\mathcal{T}$ is a uniform hypertree, and $\mathcal{T}'$ is a proper subgraph of $\mathcal{T}$ with $V(\mathcal{T}') = V(\mathcal{T})$, then $\mathcal{T}' \prec \mathcal{T}$.*

Let $\mathcal{H}$ be an $r$-uniform linear hypergraph, $e$ be a non-pendent edge of $\mathcal{H}$ and $u \in e$. Let $e_1, e_2, \dots, e_k$ be all edges of $\mathcal{H}$ adjacent to $e$ but not containing $u$, and suppose that $e_i \cap e = \{v_i\}$ for $i = 1, \dots, k$. Let $\mathcal{H}'$ be the hypergraph obtained from $\mathcal{H}$ by moving edges $(e_1, \dots, e_k)$ from $(v_1, \dots, v_k)$ to $u$. Then $\mathcal{H}'$ is said to be obtained by an *edge-releasing operation* on $e$ at $u$. Note that if $\mathcal{H}'$ and $\mathcal{H}''$ are the hypergraphs obtained from a linear hypergraph $\mathcal{H}$ by an edge-releasing operation on some $e$ at two distinct vertices of $e$, respectively. Then $\mathcal{H}'$ and $\mathcal{H}''$ are isomorphic. So we simply say that $\mathcal{H}'$ is obtained from $\mathcal{H}$ by an *edge-releasing operation* on $e$ (or *releasing the edge* $e$).

**Lemma 2.7** ([13])**.** *Let $\mathcal{T}'$ be an $r$-uniform hypertree obtained by releasing a non-pendent edge of $\mathcal{T}$. Then $\mathcal{T}'$ is a uniform hypertree and $\mathcal{T} \prec \mathcal{T}'$.*

**Lemma 2.8** ([13])**.** *Let $\mathcal{H}$ be a nontrivial $r$-uniform hypertree, and $v$ a vertex of $\mathcal{H}$. Let $\mathcal{H}(v)\mathcal{T}$ denote the hypertree with an attached hypertree $\mathcal{T}$ at $v$ of $\mathcal{H}$. Then*

$$\mathcal{H}(v)P_m^r \preceq \mathcal{H}(v)\mathcal{T} \preceq \mathcal{H}(v)S_m^r$$

*where the left–hand side equality holds if and only if $\mathcal{T} \cong P_m^r$ with $v$ as its end vertex whereas the right–hand side equality holds if and only if $\mathcal{T} \cong S_m^r$ with $v$ as its center.*

Let $\mathcal{T}$ be a non-trivial $r$-uniform hypertree, and $u$ be a vertex of $\mathcal{T}$. We consider such edge $e$ incident to $u$ satisfying that edges incident to vertices in $V(e) \setminus \{u\}$ are only pendent edges. Let $a$ be the number of support vertices of $V(e) \setminus \{u\}$. We define the edge $e$ to be

- $L$-type if $a = 0$, equivalently, $e$ is a pendent edge.
- $F$-type if $a > 0$, equivalently, $e$ is not $L$-type.
- $\bar{F}$-type if $e$ is $F$-type and every support vertex is of degree two.
- $W$-type if $e$ is $F$-type and $a = r - 1$.
- $\bar{W}$-type if $e$ is $\bar{F}$-type and $a = r - 1$.
- $\hat{W}$-type if $e$ is $W$-type and just one of $r - 1$ support vertices has degree more than 2.
- $K$-type if $e$ is $F$-type but not $W$-type.
- $\hat{K}$-type if $e$ is $K$-type and $a = 1$.

Let $\mathcal{T}_j$ be an $r$-uniform hypertree with the root $v_j$ for $j = 1, \ldots, l$ with $l \geq 2$. Use $R(\mathcal{T}_1, \ldots, \mathcal{T}_l)$ to denote the $r$-uniform hypertree obtained from $\mathcal{T}_1, \ldots, \mathcal{T}_l$ by identifying roots $v_1, \ldots, v_l$ into a single vertex which is called *center vertex*, see (a) of Figure 1. $R(\mathcal{T}_1, \mathcal{T}_2)$ may be written as $\mathcal{T}_1(v_1, v_2)\mathcal{T}_2$, known as the *coalescence* of $\mathcal{T}_1$ and $\mathcal{T}_2$ at $v_1, v_2$. If some $\mathcal{T}_j$ consists of only one $\bar{F}$-type with $b$ support vertices, then $\mathcal{T}_j$ is simply written as the number $b$ in $R(\mathcal{T}_1, \ldots, \mathcal{T}_l)$. For convenience, $b^x$ in $R(\mathcal{T}_1, \ldots, b^x, \ldots, \mathcal{T}_l)$ means the hypertree represented by the number $b$ appears $x$ times.

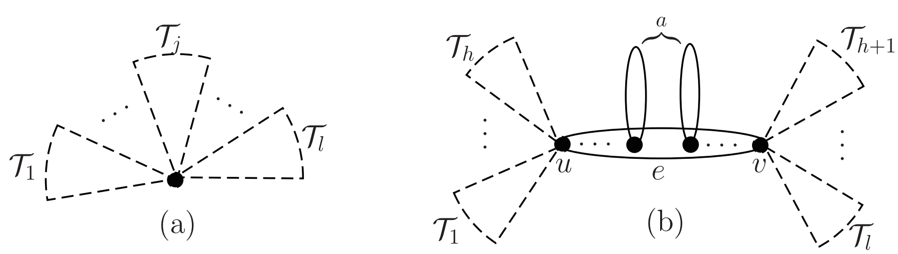


Figure 1: (a) $R(\mathcal{T}_1, \ldots, \mathcal{T}_l)$; (b) $R(\mathcal{T}_1, \ldots, \mathcal{T}_h; a; \mathcal{T}_{h+1}, \ldots, \mathcal{T}_l)$
.

Let $e$ be an $\bar{F}$-type edge with $a$ ($a \leq r - 2$) support vertices, and $u$, $v$ be its two core vertices. Denote by $R(\mathcal{T}_1, \ldots, \mathcal{T}_h; a; \mathcal{T}_{h+1}, \ldots, \mathcal{T}_l)$ an $r$-uniform hypertree obtained by identifying the center vertex of $R(\mathcal{T}_1,\dots,\mathcal{T}_h)$ and that of $R(\mathcal{T}_{h+1},\mathcal{T}_{h+2},\dots,\mathcal{T}_l)$ with $u$ and $v$, respectively. See (b) of Figure 1. Vertices $u$ and $v$ are called *center vertices* and edge $e$ is called *center edge* of $R(\mathcal{T}_1,\dots,\mathcal{T}_h;a;\mathcal{T}_{h+1},\dots,\mathcal{T}_l)$. When $\mathcal{T}_{h+1},\dots,\mathcal{T}_l$ are all trivial, then $R(\mathcal{T}_1,\dots,\mathcal{T}_h;a;\mathcal{T}_{h+1},\dots,\mathcal{T}_l)$ is just $R(\mathcal{T}_1,\dots,\mathcal{T}_h,a)$.

**Lemma 2.9** ([13])**.** *Let $H$ and $G$ be arbitrary $r$-uniform hypertrees with $H$ nontrivial. Let $a,b,c$ be integers.*

(1) *If $0\le b<a\le r-1$, then*
$$R(H,0;a-1;G,b)\succ R(H;b;G,a).$$
(2) *If $0\le b<a\le r-1$, then*
$$R(H,0;a-1;G,b)\succ R(H,0;b-1;G,a).$$
(3) *If $a>c$, then*
$$R(H,a,0;b;0,c)\succ R(H,c,0;b;0,a).$$

## 3 Transformations on Hypertree

To characterize the extremal hypertrees in the class of uniform hypertrees with given size and strong stability number, we need to transform them and then compare them in terms of the ordering "$\succ$". For those with similar structure, the common part of them may be neglected and it suffices to compare part of them, and then the following is very useful in simplifying the procedure of comparing them in terms of the ordering "$\succ$".

**Lemma 3.1.** *Let $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{H}$ be nontrivial $r$-uniform hypertrees, $w_i\in V(\mathcal{G}_i)$ for $i=1,2$ and $w_0\in V(\mathcal{H})$. For $i=1,2$, let $\mathcal{G}_i(w_i,w_0)\mathcal{H}$ be the coalescence of $\mathcal{G}_i$ and $\mathcal{H}$. If $\mathcal{G}_1\succeq\mathcal{G}_2$ and $\mathcal{G}_1-w_1\preceq\mathcal{G}_2-w_2$, then $\mathcal{G}_1(w_1,w_0)\mathcal{H}\succeq\mathcal{G}_2(w_2,w_0)\mathcal{H}$. Furthermore, if at least one of $\mathcal{G}_1\succ\mathcal{G}_2$ and $\mathcal{G}_1-w_1\prec\mathcal{G}_2-w_2$ holds, then $\mathcal{G}_1(w_1,w_0)\mathcal{H}\succ\mathcal{G}_2(w_2,w_0)\mathcal{H}$.*

*Proof.* Assume $E_{\mathcal{H}}(w_0)=\{e_1,\dots,e_l\}$. Applying (c) of Lemma 2.5 to $\mathcal{G}_1(w_1,w_0)\mathcal{H}$ and $e_1,\dots,e_l$, we have
$$\begin{aligned}\varphi(\mathcal{G}_1(w_1,w_0)\mathcal{H},x)&=\varphi(\mathcal{G}_1(w_1,w_0)\mathcal{H}\setminus\{e_1,\dots,e_l\},x)\\&\quad-\sum_{i=1}^{l}\varphi(\mathcal{G}_1(w_1,w_0)\mathcal{H}-V(e_i),x)\\&=\varphi(\mathcal{G}_1,x)\varphi(\mathcal{H}-w_0,x)\\&\quad-\varphi(\mathcal{G}_1-w_1,x)\sum_{i=1}^{l}\varphi(\mathcal{H}-V(e_i),x).\end{aligned}\tag{3.1}$$

Applying (c) of Lemma 2.5 to $\mathcal{G}_2(w_2,w_0)\mathcal{H}$ and $e_1,\dots,e_l$, we have
$$\begin{aligned}\varphi(\mathcal{G}_2(w_2,w_0)\mathcal{H},x)&=\varphi(\mathcal{G}_2(w_2,w_0)\mathcal{H}\setminus\{e_1,\dots,e_l\},x)\\&\quad-\sum_{i=1}^{l}\varphi(\mathcal{G}_2(w_2,w_0)\mathcal{H}-V(e_i),x)\\&=\varphi(\mathcal{G}_2,x)\varphi(\mathcal{H}-w_0,x)\\&\quad-\varphi(\mathcal{G}_2-w_2,x)\sum_{i=1}^{l}\varphi(\mathcal{H}-V(e_i),x).\end{aligned}\tag{3.2}$$

Subtracting the equations (3.1) and (3.2) yields

$$\varphi(\mathcal{G}_1(w_1,w_0)\mathcal{H},x)-\varphi(\mathcal{G}_2(w_2,w_0)\mathcal{H},x)=(\varphi(\mathcal{G}_1,x)-\varphi(\mathcal{G}_2,x))\varphi(\mathcal{H}-w_0,x) + (\varphi(\mathcal{G}_2-w_2,x)-\varphi(\mathcal{G}_1-w_1,x))\sum_{i=1}^{l}\varphi(\mathcal{H}-V(e_i),x) \tag{3.3}$$

By Lemma 2.3, $\mathcal{G}_1, \mathcal{G}_2 - w_2, \mathcal{H} - w_0$ and $\mathcal{H} - V(e_i)$ for $i = 1, \dots, l$ all have less spectral radius than $\mathcal{G}_1(w_1, w_0)\mathcal{H}$. Since the maximum real roots (which actually are their respective spectral radii) of $\varphi(\mathcal{H} - w_0, x)$ and $\varphi(\mathcal{H} - V(e_i), x)$ are less than $\rho(\mathcal{G}_1(w_1, w_0)\mathcal{H})$, we have that $\varphi(\mathcal{H}-w_0, x) > 0$ and $\varphi(\mathcal{H}-V(e_i), x) > 0$ for all $i = 1, \dots, l$ when $x \geq \rho(\mathcal{G}_1(w_1, w_0)\mathcal{H})$. By the definition of "$\succeq$" and "$\succ$", when $x \geq \rho(\mathcal{G}_1(w_1, w_0)\mathcal{H})$, the following statements hold.

(a) $\mathcal{G}_1 \succeq \mathcal{G}_2 \Rightarrow \varphi(\mathcal{G}_1, x) - \varphi(\mathcal{G}_2, x) \leq 0$;

(b) $\mathcal{G}_2 - w_2 \succeq \mathcal{G}_1 - w_1 \Rightarrow \varphi(\mathcal{G}_2 - w_2, x) - \varphi(\mathcal{G}_1 - w_1, x) \leq 0$;

(c) $\mathcal{G}_1 \succ \mathcal{G}_2 \Rightarrow \varphi(\mathcal{G}_1, x) - \varphi(\mathcal{G}_2, x) < 0$;

(d) $\mathcal{G}_2 - w_2 \succ \mathcal{G}_1 - w_1 \Rightarrow \varphi(\mathcal{G}_2 - w_2, x) - \varphi(\mathcal{G}_1 - w_1, x) < 0$.

It follows immediately from Eq. (3.3) that when $x \geq \rho(\mathcal{G}_1(w_1, w_0)\mathcal{H})$, the consequences of (a) and (b) imply $\varphi(\mathcal{G}_1(w_1, w_0)\mathcal{H}, x) - \varphi(\mathcal{G}_2(w_2, w_0)\mathcal{H}, x) \leq 0$, which is equivalent to $\mathcal{G}_1(w_1, w_0)\mathcal{H} \succeq \mathcal{G}_2(w_2, w_0)\mathcal{H}$, and the consequences of (a) and (b) together with (c) or (d) imply $\varphi(\mathcal{G}_1(w_1, w_0)\mathcal{H}, x) - \varphi(\mathcal{G}_2(w_2, w_0)\mathcal{H}, x) < 0$, which is equivalent to $\mathcal{G}_1(w_1, w_0)\mathcal{H} \succ \mathcal{G}_2(w_2, w_0)\mathcal{H}$. We are done. □

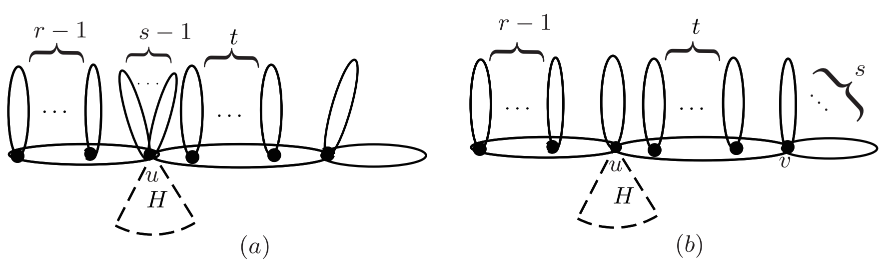


Figure 2: (a) $R(H, r-1, 0^{s-1}; t; 0^2)$; (b) $R(H, r-1, 0; t; 0^s)$

The following two lemmas are direct applications of the above lemma and will play a vital role in comparing four candidates to find out the second maximal hypertree.

**Lemma 3.2.** *Let $H$ be any $r$-uniform hypertree. Then for any nonnegative integers $s \geq 3$ and $t \leq r-2$,*

$$R(H, r-1, 0^{s-1}; t; 0^2) \succ R(H, r-1, 0; t; 0^s).$$

*Proof.* Let $G_1 = R(r-1, 0^{s-1}; t; 0^2)$ and $G_2 = R(r-1, 0; t; 0^s)$. Choose vertices $u \in V(H)$, $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$ such that $R(H, r-1, 0^{s-1}; t; 0^2) = H(u, u_1)G_1$ and $R(H, r-1, 0; t; 0^s) = H(u, u_2)G_2$, as in Figure 2.

Observe that $G_1$ may be viewed as $R(G, r-1, 0; t; 0^2)$ and $G_2$ as $R(G, 0^2; t; 0, r-1)$, where $G = S^r_{s-2}$. By (3) of Lemma 2.9, we have that $G_1 = R(G, r-1, 0; t; 0^2) \succ R(G, 0^2; t; 0, r-1) = G_2$.

It is easy to see that $G_1 - u_1$ is the disjoint union of $r + t - 1$ independent edges, $s(r-1) - t - 1$ isolated vertices and one $S^r_2$, $G_2 - u_2$ is the disjoint union of $r + t - 1$ independent edges, $2(r-1) - t - 1$ isolated vertices and one $S^r_s$. Obviously, $G_1 - u_1$ may be regarded as a proper subgraph of $G_2 - u_2$ obtained by deleting $s - 2$ edges from $S^r_s$ in $G_2 - u_2$. By Lemma 2.6, we have $G_1 - u_1 \prec G_2 - u_2$. Together with $G_1 \succ G_2$ shown earlier, we have $H(u, u_1)G_1 \succ H(u, u_2)G_2$ by Lemma 3.1. We are done. □

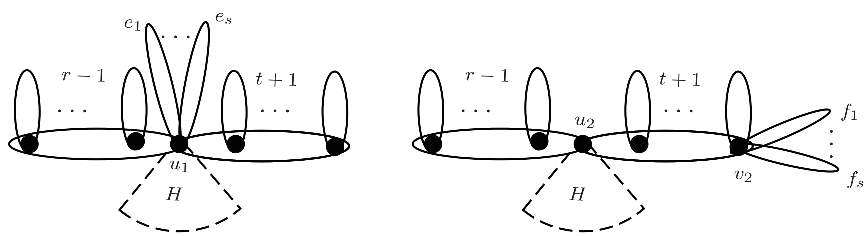


Figure 3: $R(H, r-1, t+1, 0^s)$ and $R(H, r-1; t; 0^{s+1})$
.

**Lemma 3.3.** *Let $H$ be an arbitrary $r$-uniform hypertree. Then for any nonnegative integer $s \geq 1$ and $t \leq r-2$,*

$$R(H, r-1, t+1, 0^s) \succ R(H, r-1; t; 0^{s+1}).$$

*Proof.* Let $G_1 = R(r-1, t+1, 0^s)$ and $G_2 = R(r-1; t; 0^{s+1})$. Choose vertices $u \in V(H)$, $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$ such that $R(H, r-1, t+1, 0^s) = H(u, u_1)G_1$ and $R(H, r-1; t; 0^{s+1}) = H(u, u_2)G_2$, as in Figure 3. Use $v_2$ to denote the unique vertex of degree $s+2$ in $G_2$. Assume that $e_1, \ldots, e_s$ are pendent edges of $G_1$ attached at $u_1$, and $f_1, \ldots, f_s$ (of $s+1$ totally) pendent edges of $G_2$ attached at $v_2$. By applying (c) of Lemma 2.5 to $G_1$ and $e_1, \ldots, e_s$, we have

$$\begin{aligned}\varphi(G_1, x) &= \varphi(G_1 \setminus \{e_1, \ldots, e_s\}, x) - s\varphi(G_1 - V(e_1), x) \\ &= \varphi(R(r-1, t+1), x)x^{(r-1)s} - s(x^r - 1)^{r+t}x^{s(r-1)-t-1}.\end{aligned} \tag{3.4}$$

Applying (c) of Lemma 2.5 to $G_2$ and $f_1, \ldots, f_s$, we have

$$\begin{aligned}\varphi(G_2, x) &= \varphi(G_2 \setminus \{f_1, \ldots, f_s\}, x) - s\varphi(G_2 - V(f_1), x) \\ &= \varphi(R(r-1, t+1), x)x^{(r-1)s} - s(x(x^r - 1)^{r-1} - x^{(r-1)^2})(x^r - 1)^t x^{(r-1)(s+1)-t-1}.\end{aligned} \tag{3.5}$$

Substraction of the equations(3.4) and (3.5) yields

$$\begin{aligned}\varphi(G_1, x) - \varphi(G_2, x) &= s(x^r - 1)^t x^{(r-1)s-t-1}((x^r - 1)^{r-1}x^r - x^{r(r-1)} - (x^r - 1)^r) \\ &= s(x^r - 1)^t x^{(r-1)s-t-1}((x^r - 1)^{r-1} - x^{r(r-1)}).\end{aligned} \tag{3.6}$$

Obviously, the value of (3.6) is always negative for any $x \geq \rho(G_1)$, and this implies that $G_1 \succ G_2$.

Observe that $G_1 - u_1$ is the disjoint union of $r+t$ independent edges and $(s+1)(r-1)-t-1$ isolated vertices, and $G_2 - u_2$ is the disjoint union of $r+t-1$ independent edges and $r-t-2$ isolated vertices and one $S^r_{s+1}$. Obviously, $G_1 - u_1$ may be regarded as a proper subgraph of $G_2 - u_2$ by deleting $s$ edges from $S^r_{s+1}$ in $G_2 - u_2$. By Lemma 2.6, we have $G_1 - u_1 \prec G_2 - u_2$. Together with $G_1 \succ G_2$ shown earlier, we have $H(u, u_1)G_1 \succ H(u, u_2)G_2$ by Lemma 3.1. We are done. □

## 4 Strong Stability Number

Let $\mathcal{H} = (V, E)$ be a hypergraph without isolated vertices. Recall that a subset of vertices in $\mathcal{H}$ is independent(or *strong stable*) if no two of these vertices are adjacent. The *strong stability number* of $\mathcal{H}$, denoted by $\alpha(\mathcal{H})$, is the cardinality of a maximum strong stable set in $\mathcal{H}$. However, another generalization for independent set of graphs is that a subset of vertices in $\mathcal{H}$ is *stable* if it contains no edge of $\mathcal{H}$. In this paper, we are only concerned with strong stable set of hypergraphs. See [1] for more details.

**Lemma 4.1.** *For every hypertree $\mathcal{T}$, if $C$ is the set of some independent core vertices in $\mathcal{T}$, then there exists a maximum strong stable set $S(\mathcal{T})$ of $\mathcal{T}$ such that $C \subseteq S(\mathcal{T})$.*

*Proof.* Assume that independent set $C = \{v_1, v_2, \ldots, v_k\}$ and $v_i \in e_i$ with $d(v_i) = 1$. Note that edges $e_1, e_2, \ldots, e_k$ are distinct. Choose a maximum strong stable set $S(\mathcal{T})$ of $\mathcal{T}$.

If $v_1 \notin S(\mathcal{T})$, then it is easy to verify that $(S(\mathcal{T}) \setminus V(e_1)) \cup \{v_1\}$, denoted by $S_1(\mathcal{T})$, is still a strong stable set of $\mathcal{T}$. Since $|S(\mathcal{T}) \cap V(e_1)| \leq 1$, $|S(\mathcal{T})| \leq |S_1(\mathcal{T})|$. Thus $|S_1(\mathcal{T})| = \alpha(\mathcal{T})$.

If $v_2 \notin S_1(\mathcal{T})$, then we proceed to consider $(S_1(\mathcal{T}) \setminus V(e_2)) \cup \{v_2\}$, denoted by $S_2(\mathcal{T})$. Since $v_1 \notin e_2$, $v_1 \in S_2(\mathcal{T})$ and similarly $S_2(\mathcal{T})$ is a strong stable set of $\mathcal{T}$. By the same way, $|S_1(\mathcal{T})| \leq |S_2(\mathcal{T})|$. Thus $|S_2(\mathcal{T})| = \alpha(\mathcal{T})$.

Continuing in this way, we finally arrive at a maximum strong stable set of $\mathcal{T}$ containing all of $v_1, v_2, \ldots, v_k$. □

Let $\mathcal{H}$ be an $r$-uniform hypertree with a vertex $u \in V(\mathcal{H})$. Suppose that there is a $W$-type edge $e \in E(u)$. Let $C(e)$ denote the vertex subset obtained by taking one core vertex from every pendent edge attached at $r-1$ support vertices of $e$.

Now we investigate the strong stability number of hypertree with respect to these kind of edges incident to a given vertex, which enables us to apply some transformations on hypertree while fixing its strong stability number.

**Lemma 4.2.** *Let $\mathcal{H}$ be an $r$-uniform hypertree with $u \in V(\mathcal{H})$.*

(1) *If $e$ is a $W$-type edge incident to $u$, let $\mathcal{H}'$ denote the component containing $u$ of $\mathcal{H} - e$, then $\alpha(\mathcal{H}) = \alpha(\mathcal{H}') + |C(e)|$.*

(2) *If $e$ is a $K$-type edge incident to $u$, let $\mathcal{H}'$ denote the hypertree obtained from $\mathcal{H}$ by edge-releasing $e$, then $\alpha(\mathcal{H}) = \alpha(\mathcal{H}')$.*

*Proof.* (1) By Lemma 4.1, we can choose a maximum strong stable set $S(\mathcal{H})$ of $\mathcal{H}$ such that $C(e) \subseteq S(\mathcal{H})$. It is easy to verify that $S(\mathcal{H}) \setminus C(e)$ is a strong stable set of $\mathcal{H}'$, and then $|S(\mathcal{H}) \setminus C(e)| \leq \alpha(\mathcal{H}')$. Thus

$$\alpha(\mathcal{H}) = |S(\mathcal{H})| = |S(\mathcal{H}) \setminus C(e)| + |C(e)| \leq \alpha(\mathcal{H}') + |C(e)|. \tag{4.1}$$

Conversely, for any strong stable set $S(\mathcal{H}')$ of $\mathcal{H}'$, vertex $u$ may or may not lie in $S(\mathcal{H}')$. Note that $u$ is adjacent to no one of $C(e)$. In either case, $S(\mathcal{H}') \cup C(e)$ is a strong stable set of $\mathcal{H}$ and so

$$\alpha(\mathcal{H}') + |C(e)| \leq \alpha(\mathcal{H}). \tag{4.2}$$

Combining Eqs. (4.1) and (4.2) yields that $\alpha(\mathcal{H}) = \alpha(\mathcal{H}') + |C(e)|$.

(2) First consider the case that $e$ has only one pendent edge, say $f$, attached at a vertex $v$ (different from $u$) of $e$. Take a core vertex of $e$ different from $u$ and $v$, say $u'$, and a core vertex of $f$ different from $v$, say $v'$.

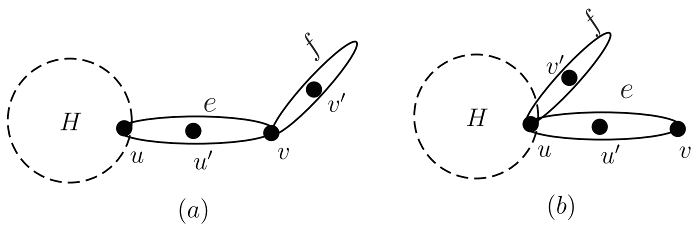


Figure 4: (a) $\mathcal{H}$; (b) $\mathcal{H}'$

By Lemma 4.1, we can choose a maximum strong stable set $S(\mathcal{H})$ of $\mathcal{H}$ such that $u', v' \in S(\mathcal{H})$. Since $u' \in S(\mathcal{H})$, we have $u \notin S(\mathcal{H})$. Thus $S(\mathcal{H})$ is also a strong stable set of $\mathcal{H}'$ and so $\alpha(\mathcal{H}) \leq \alpha(\mathcal{H}')$.

Conversely, by Lemma 4.1, we can choose a maximum strong stable set $S(\mathcal{H}')$ of $\mathcal{H}'$ such that $u', v' \in S(\mathcal{H}')$. Since $u' \in S(\mathcal{H}')$, we have $v \notin S(\mathcal{H}')$. Thus $S(\mathcal{H}')$ is also a strong stable set of $\mathcal{H}$ and so $\alpha(\mathcal{H}') \leq \alpha(\mathcal{H})$. Therefore, $\alpha(\mathcal{H}) = \alpha(\mathcal{H}')$.

By the same way, we can prove that $\alpha(\mathcal{H}) = \alpha(\mathcal{H}')$ for the general case. □

**Remark 4.3.** From Lemma 4.2, it follows immediately that moving $W$-type edge or releasing $K$-type edge keeps the strong stability number of hypergraph fixed.

For a hypertree $\mathcal{T}$, let $N(\mathcal{T})$ denote the maximum distance among support vertices of $\mathcal{T}$, and $CO(\mathcal{T})$ denote the nontrivial component of the hypertree obtained from $\mathcal{T}$ by deleting all pendent edges from $\mathcal{T}$. Denote by $\mathcal{T}(m, r, \alpha)$ the set of all $r$-uniform hypertrees with $m$ edges and strong stability number $\alpha$. We assume from now on that $\alpha < m$. Let $d$ be a nonnegative integer, and $\mathcal{T}_d(m, r, \alpha) = \{\mathcal{T} \in \mathcal{T}(m, r, \alpha) | N(CO(\mathcal{T})) = d\}$.

**Lemma 4.4.** *For any $\mathcal{T} \in \mathcal{T}_d(m, r, \alpha)$ with $d \geq 2$, there exists a hypertree in $\mathcal{T}_j(m, r, \alpha)$ such that it has larger spectral radius than $\mathcal{T}$ and $j \in \{0, 1\}$.*

*Proof.* For $\mathcal{T} \in \mathcal{T}_d(m, r, \alpha)$ with $d \geq 2$, choose two vertices $v_1$ and $v_2$ of $\mathcal{T}$ such that they are support vertices of $CO(\mathcal{T})$ and the distance between them in $CO(\mathcal{T})$ is equal to $d$. Assume that $w_1 e_1 w_2 \ldots w_{d-1} e_{d-1} w_d e_d w_{d+1}$ is the path from $v_1$ to $v_2$ in $\mathcal{T}$, where $v_1 = w_1$ and $v_2 = w_{d+1}$. Since $v_1$ and $v_2$ are support vertices of $CO(\mathcal{T})$ and $d(v_1, v_2) = N(CO(\mathcal{T}))$, any edge other than $e_1$ ( $e_d$, resp.) incident to any vertex in $V(e_1) \setminus \{w_2\}$ ($V(e_d) \setminus \{w_d\}$, resp.) must be $L$-type, $K$-type, or $W$-type.

In $\mathcal{T}$, denote by $W_i$ and $K_i$ the set of $W$-type and $K$-type edges incident to $v_i$ respectively, for $i = 1, 2$. Since $v_1$ and $v_2$ are support vertices of $CO(\mathcal{T})$, $W_i \cup K_i \neq \emptyset$ for $i = 1, 2$. Let $x$ be the principal eigenvector of $\mathcal{T}$. Now we introduce an MR-operation on $\mathcal{T}$ and the pair $(v_1, v_2)$, defined by doing M-operation if $W_1 \neq \emptyset$ and $W_2 \neq \emptyset$, and doing R-operation otherwise, where

1. M-operation is to move all $W$-type edges in $W_{i_2}$ from $v_{i_2}$ to $v_{i_1}$, with the assumption that $\{i_1, i_2\} = \{1, 2\}$ and $x_{v_{i_1}} = \max\{x_{v_1}\, x_{v_2}\}$.

2. R-operation is to release all $K$-type edges in $K_1 \cup K_2$.

After applying MR-operation on $\mathcal{T}$ and a pair $(v_1, v_2)$, the resulting hypertree is denoted by $\mathcal{T}_1$ and the effect by this operation is

(1) $\alpha(\mathcal{T}_1) = \alpha(\mathcal{T})$.

(2) $\rho(\mathcal{T}_1) > \rho(\mathcal{T})$.

(3) $N(CO(\mathcal{T}_1)) \leq N(CO(\mathcal{T}))$.

The statement (1) follows from Lemma 4.2 and the statement (2) holds by Lemmas 2.4 and 2.7. While applying MR-operation on $\mathcal{T}$ and a pair $(v_1, v_2)$, the part of M-operation decreases the number of vertices incident to $W$-type edges in $\mathcal{T}$ and the part of R-operation decreases the number of vertices incident to $K$-type edges in $\mathcal{T}$. Thus the statement (3) holds as well.

If $N(CO(\mathcal{T}_1)) \geq 2$, then continue the process and obtain the sequence $\mathcal{T}_1, \mathcal{T}_2, \ldots$ satisfying that

(1) $\alpha(\mathcal{T}) = \alpha(\mathcal{T}_1) = \alpha(\mathcal{T}_2) = \cdots$

(2) $\rho(\mathcal{T}) < \rho(\mathcal{T}_1) < \rho(\mathcal{T}_2) < \cdots$

(3) $N(CO(\mathcal{T})) \geq N(CO(\mathcal{T}_1)) \geq N(CO(\mathcal{T}_2)) \geq \cdots$

Since each $\rho(\mathcal{T}_i)$ is upper bounded by the spectral radius of hyperstar $S_m^r$, the process will certainly terminate after a finite number of steps, say at some $\mathcal{T}_j$. This means that $N(CO(\mathcal{T}_j)) < 2$ and we are done. □

## 5 Main Results

Now we shall find those hypertrees in $\mathcal{T}(m, r, \alpha)$ with larger spectral radius. First consider the case when $\alpha = m$. By Theorems 19 and 21 in [7], $S_m^r$ and $S(1, 0^{(m-2)})$ are the first two hypertrees with larger spectral radius among all $r$-uniform hypertrees with $m$ edges. Note that $\alpha(S_m^r) = \alpha(S(1, 0^{(m-2)})) = m$. Thus in $\mathcal{T}(m, r, m)$, $S_m^r$ and $S(1, 0^{(m-2)})$ have the first and second largest spectral radius.

Further consider the case when $\alpha = m-1$. Observe that for any $\mathcal{T} \in \mathcal{T}(m, r, m-1)$, there exists at least one edge, say $e_0$, of $\mathcal{T}$ having no core vertex. Assume that $e_0 = \{v_1, \ldots, v_r\}$. Denote by $\mathcal{T}_i$ the hypertree attached at vertex $v_i$ with $m_i$ edges, $i \in \{1, \ldots, r\}$. Without loss of generality we may assume that $m_1 \geq m_2 \geq \ldots \geq m_r > 0$. Use $E(m_1, \ldots, m_r)$ to denote the r-uniform hypertree which is obtained from $e_0$ by attaching $m_i$ pendent edges at vertex $v_i$, for $i = 1, 2, \ldots, r$. By Lemma 2.8, $\rho(\mathcal{T}) \leq \rho(E(m_1, \ldots, m_r))$, with equality holds if and only if $\mathcal{T} = E(m_1, \ldots, m_r)$. Note that $E(m_1, \ldots, m_r)$ has strong stability number equal to $m-1$ if and only if all of $m_1, m_2, \ldots, m_r$ are not zero. If we can find two of $m_1, m_2, \ldots, m_r$ bigger than 1, say $m_1 \geq m_2 > 1$, then after moving only one pendent edge in $E(m_1, \ldots, m_r)$ from $v_2$ to $v_1$ or moving $m_1 - m_2 + 1$ pendent edges in $E(m_1, \ldots, m_r)$ from $v_1$ to $v_2$, the resulting hypertrees are both isomorphic to $E(m_1 + 1, m_2 - 1, m_3 \ldots, m_r)$ and have larger spectral radius by Lemma 2.4. Assume that $m \geq r + 3$. Thus in $\mathcal{T}(m, r, m-1)$, $E(m - r, 1, \ldots, 1) = R(r - 1, 0^{(m-r)})$ uniquely has the largest spectral radius and $E(m -$

$r-1, 2, 1, \dots, 1) = R(0^{(m-r-1)}; (r-2); 0^{(2)})$ uniquely has the second largest spectral radius. From now on we assume that $m \geq r+3$ and $\alpha < m-1$.

For convenience, we introduce the notations $\epsilon, \tau$ defined as

$$\epsilon = m - \alpha, \quad \tau = m - \epsilon r. \tag{5.1}$$

We use $A(m, r, \alpha)$ to denote the hypertree $R((r-1)^\epsilon, 0^\tau)$ with $\varepsilon \geq 1$ and $\tau \geq 1$, and $B(m, r, \alpha)$ to denote the following

$$B(m, r, \alpha) = \begin{cases} R(r-1; 0; r-1), & \text{if } \epsilon = 2 \text{ and } \tau = 1; \\ R((r-1)^{\epsilon-2}, 0^\tau; r-2; r-1, 0), & \text{if } \epsilon \geq 3 \text{ and } \tau = 1; \\ R((r-1)^{\epsilon-1}, 0^{\tau-1}; r-2; 0^2), & \text{if } \epsilon \geq 2 \text{ and } \tau \geq 2. \end{cases}$$

Next we shall show that $A(m, r, \alpha)$ and $B(m, r, \alpha)$ are the first two hypertrees in $\mathcal{T}(m, r, \alpha)$. For our purpose, we first investigate what kind of properties the maximal one in $\mathcal{T}(m, r, \alpha) \setminus \{A(m, r, \alpha)\}$ should have.

**Lemma 5.1.** *Suppose that $\mathcal{T} \in \mathcal{T}(m, r, \alpha) \setminus \{A(m, r, \alpha)\}$ has the largest spectral radius. For any vertex $u$ of $\mathcal{T}$ which is not a support vertex of degree 2, the following statements hold.*

(1) *If $E_{\mathcal{T}}(u)$ contains $W$-type edges, then it must be $\bar{W}$-type or $\hat{W}$-type. If $E_{\mathcal{T}}(u)$ contains $\hat{W}$-type edges, then it contains only one.*

(2) *If $E_{\mathcal{T}}(u)$ contains $K$-type edges, then it contains only one and it is $\hat{K}$-type.*

(3) *$E_{\mathcal{T}}(u)$ cannot contain both $\hat{W}$-type and $\hat{K}$-type edges.*

*Proof.* (1) Suppose to the contrary that $E_{\mathcal{T}}(u)$ contains a $W$-type edge $e$ which is neither $\bar{W}$-type nor $\hat{W}$-type. Then there exists two vertices $w_1$ and $w_2$ in $V(e) \setminus \{u\}$ with degree larger than two. Denote by $\mathcal{T}_0$ the hypertree obtained from $\mathcal{T}$ by moving $d(w_1)-2$ pendent edges incident with $w_1$ from $w_1$ to $w_2$, which is isomorphic to the one obtained by moving $d(w_2)-2$ pendent edges incident with $w_2$ from $w_2$ to $w_1$. It is easy to see that $\mathcal{T}_0 \ncong A(m, r, \alpha)$, $\alpha(\mathcal{T}_0) = \alpha(\mathcal{T})$ and $\rho(\mathcal{T}_0) > \rho(\mathcal{T})$ by Lemma 2.4, a contradiction to the maximality of $\mathcal{T}$ in $\mathcal{T}(m, r, \alpha) \setminus \{A(m, r, \alpha)\}$.

Suppose that there are two $\hat{W}$-type edges, say $e_1$ and $e_2$. Then there exists a vertex $w_i' \in V(e_i) \setminus \{u\}$ such that $d(w_i') > 2$, for $i = 1, 2$. Moving $d(w_1') - 2$ pendent edges from $w_1'$ to $w_2'$, and moving $d(w_2') - 2$ pendent edges from $w_2'$ to $w_1'$, reapectively. The resulting two hypertrees are isomorphic, denoted by $\mathcal{T}_0'$. By the same way, it can be shown that $\mathcal{T}_0' \in \mathcal{T}(m, r, \alpha) \setminus \{A(m, r, \alpha)\}$ and $\rho(\mathcal{T}_0') > \rho(\mathcal{T})$, a contradiction once again.

(2) Suppose to the contrary that $E_{\mathcal{T}}(u)$ contains a $K$-type edge $e$ which is not $\hat{K}$-type. Then there exists two support vertices $w_1, w_2 \in V(e) \setminus \{u\}$. Moving all pendent edges from $w_1$ to $w_2$ or from $w_2$ to $w_1$, the resulting hypertree obviously is not $A(m, r, \alpha)$, and has the same strong stability number as $\mathcal{T}$ but has larger spectral radius than $\mathcal{T}$, contradicting the maximality of $\mathcal{T}$ in $\mathcal{T}(m, r, \alpha) \setminus \{A(m, r, \alpha)\}$.

Suppose there are two $K$-type edges attached at $u$. Releasing one of them, similarly we get a counterexample to the maximality of $\mathcal{T}$ in $\mathcal{T}(m, r, \alpha) \setminus \{A(m, r, \alpha)\}$.

(3) Suppose that $E_{\mathcal{T}}(u)$ contains both $\hat{W}$-type and $\hat{K}$-type edges. Then releasing a $K$-type edge of them, the resulting hypertree cannot be $A(m, r, \alpha)$ as it has $\hat{W}$-type edge, but has the same strong stability number as $\mathcal{T}$ and larger spectral radius than $\mathcal{T}$, a contradiction once again. □

**Lemma 5.2.** (1) *If $\epsilon \geq 2$ and $\tau \geq 2$, then*

(a) $R((r-1)^{\epsilon-1}, 0^{\tau-1}; r-2; 0^2) \succ R((r-1)^{\epsilon}, 0^{\tau-2}, 1)$.

(b) $R((r-1)^{\epsilon-1}, 0^{\tau-1}; r-2; 0^2) \succ R((r-1)^{\epsilon-1}, 0^{\tau-1}; 0; r-1)$.

(2) *If $\epsilon \geq 3$ and $\tau \geq 2$, then*

$$R((r-1)^{\epsilon-1}, 0^{\tau-1}; r-2; 0^2) \succ R((r-1)^{\epsilon-2}, 0^{\tau}; r-2; r-1, 0).$$

(3) *If $\epsilon \geq 3$ and $\tau = 1$, then*

$$R((r-1)^{\epsilon-2}, 0^{\tau}; r-2; r-1, 0) \succ R((r-1)^{\epsilon-1}, 0^{\tau-1}; 0; r-1).$$

*Proof.* (1) Let $G = R((r-1)^{\epsilon-1}, 0^{\tau-2})$, and $H = S_1^r$ consisting of only one edge. Then $R((r-1)^{\epsilon-1}, 0^{\tau-1}; r-2; 0^2)$ and $R((r-1)^{\epsilon}, 0^{\tau-2}, 1)$ may be viewed as $R(H, 0; r-2; G, 0)$ and $R(H; 0; G, r-1)$, respectively. By (1) of Lemma 2.9, $R(H, 0; r-2; G, 0) \succ R(H; 0; G, r-1)$ and we are done.

Let $H = R((r-1)^{\epsilon-1}, 0^{\tau-2})$, and $\Phi$ be the trivial hypergraph consisting of only one vertex. Then $R((r-1)^{\epsilon-1}, 0^{\tau-1}; 0; r-1)$ may be viewed as $R(H, 0; 0; \Phi, r-1)$. By (2) Lemma 2.9, $R(H, 0; 0; \Phi, r-1) \prec R(H, 0; r-2; \Phi, 1)$. Here we take $R(H, 0; r-2; \Phi, 1)$ as an intermediate one and then compare it with $R((r-1)^{\epsilon-1}, 0^{\tau-1}; r-2; 0^2)$. Let $G = R((r-1)^{\epsilon-1}, 0^{\tau-1}, r-2)$, and $u$ be its only core vertex in the edge with $r-2$ pendent edges. Then $R(H, 0; r-2; \Phi, 1)$ may be viewed as $G(u)P_2^r$ and $R((r-1)^{\epsilon-1}, 0^{(\tau-1)}; r-2; 0^2)$ may be viewed as $G(u)S_2^r$. By Lemma 2.8, $G(u)P_2^r \prec G(u)S_2^r$ and we are done.

(2) Let $H = R((r-1)^{\epsilon-2}, 0^{\tau-2})$. Then $R((r-1)^{\epsilon-1}, 0^{\tau-1}; r-2; 0^2)$ and $R((r-1)^{\epsilon-2}, 0^{\tau}; r-2; r-1, 0)$ may be viewed as $R(H, r-1, 0; r-2; 0^2)$ and $R(H, 0^2; r-2; r-1, 0)$, respectively. By (3) of Lemma 2.9, $R(H, r-1, 0; r-2; 0^2) \succ R(H, 0^2; r-2; r-1, 0)$ and we are done.

(3) Let $G = R((r-1)^{\epsilon-2})$ and $H = R((r-1)^1)$. Then $R((r-1)^{\epsilon-2}, 0; r-2; r-1, 0)$ and $R((r-1)^{\epsilon-1}; 0; r-1)$ may be viewed as $R(G, 0; r-2; H, 0)$ and $R(G, r-1; 0; H)$, respectively. By (1) of Lemma 2.9, $R(G, 0; r-2; H, 0) \succ R(G, r-1; 0; H)$ and we are done. □

The remaining case of $\alpha < m-1$ is much harder and the following is the key in determining the extremal hypertrees in $\mathcal{T}(m, r, \alpha)$.

**Theorem 5.3.** *In $\mathcal{T}(m, r, \alpha) \backslash \{A(m, r, \alpha)\}$ with $\alpha < m-1$ and $m \geq r+3$, $B(m, r, \alpha)$ uniquely has the largest spectral radius.*

*Proof.* Suppose that $\mathcal{T} \in \mathcal{T}(m, r, \alpha) \setminus \{A(m, r, \alpha)\}$ has the largest spectral radius. We will show $\mathcal{T} = B(m, r, \alpha)$ case by case. Recall that $CO(\mathcal{T})$ (which is well-defined because $\mathcal{T}$ is not a hyperstar) denotes the unique nontrivial component obtained from $\mathcal{T}$ by deleting all pendent edges in $\mathcal{T}$, and use $N(CO(\mathcal{T}))$ for the maximum distance among support vertices of $CO(\mathcal{T})$. Since $\alpha < m-1$, $CO(\mathcal{T})$ has at least one support vertex. We consider three cases as follows.

**Case 1.** $N(CO(\mathcal{T})) = 0$.

Then $CO(\mathcal{T})$ has just one support vertex, say $u$, and so $CO(\mathcal{T})$ is a hyperstar. Assume that $E_{\mathcal{T}}(u) = W_u \cup K_u \cup L_u$, where $W_u$, $K_u$ and $L_u$ denote the set of $W$-type, $K$-type and $L$-type edges, respectively. Observe that $|W_u| \geq 2$ due to $\alpha < m-1$. By Lemma 5.1, we know that each edge in $W_u$ is either $\bar{W}$-type or $\hat{W}$-type, and if $W_u$ contains $\hat{W}$-type edge, then it contains only one.

If $E_u$ contains no $\hat{W}$-type edge, then $K_u \neq \emptyset$ since otherwise $E_u(\mathcal{T})$ consists of $\bar{W}$-type and $L$-type edges, and then $\mathcal{T} = A(m, r, \alpha)$, a contradiction. By Lemma 5.1, $K_u$ contains exactly one $\hat{K}$-type edge. Thus $\mathcal{T} = R((r-1)^{\epsilon}, 0^{\tau-t-1}; 0; 0^t)$ for some integer $t$, where $\epsilon = m - \alpha = |W_u| \geq 2$, and further can be viewed as $R(H, r-1; 0; 0^t)$, where $H = R((r-1)^{\epsilon-1}, 0^{\tau-t-1})$. Suppose, by way of contradiction, that $t \geq 2$. By Lemma 3.3, $R(H, r-1; 0; 0^t) \prec R(H, r-1, 1, 0^{t-1})$ and $R(H, r-1, 1, 0^{t-1})$ has the same strong stability number $\alpha$, contradicting the maximality of $\mathcal{T}$ in $\mathcal{T}(m, r, \alpha) \setminus \{A(m, r, \alpha)\}$. Thus in this case $\mathcal{T}$ can only be $R((r-1)^{\epsilon}, 0^{\tau-2}, 1)$, with $\epsilon \geq 2$ and $\tau \geq 2$.

If $W_u$ contains $\hat{W}$-type edges, then by Lemma 5.1, $K_u = \emptyset$ and $W_u$ contains only one $\hat{W}$-type edge. Thus we can assume that $\mathcal{T} = R((r-1)^{\epsilon-1}, 0^{\tau-t+1}; r-2; 0^t)$ for some integer $t$. Note that $\epsilon \geq 2$ and $t \geq 2$. We proceed to show that the following statements hold in this case.

(1) $\tau - t + 1 > 0$. Suppose $L_u = \emptyset$. Then after releasing one $\bar{W}$-type edge at $u$ (it exists due to $\epsilon \geq 2$), the resulting hypertree is not isomorphic to $A(m, r, \alpha)$ but has the same strong stability number (in this process if $L_u \neq \emptyset$, then the strong stability number will become bigger) and larger spectral radius, a contradiction. Thus $L_u \neq \emptyset$, equivalently $\tau - t + 1 > 0$.

(2) $t = 2$. Suppose $t \geq 3$. By Lemma 3.2, $R((r-1)^{\epsilon-1}, 0^{\tau-t+1}; r-2; 0^t) \prec R((r-1)^{\epsilon-1}, 0^{\tau-1}; r-2; 0^2)$ and the latter has the same strong stability number $\alpha$, contradicting the maximality of $\mathcal{T}$ in $\mathcal{T}(m, r, \alpha) \setminus \{A(m, r, \alpha)\}$.

Consequently, $\mathcal{T}$ can only be $R((r-1)^{\epsilon-1}, 0^{\tau-1}; r-2; 0^2)$, with $\epsilon \geq 2$ and $\tau \geq 2$.

**Case 2.** $N(CO(\mathcal{T})) = 1$.

Then all support vertices of $CO(\mathcal{T})$ lie in one edge, say $e_0$. Assume that $e_0 = \{v_1, \ldots, v_r\}$. Note that each $E_{\mathcal{T}}(v_i) \setminus \{e_0\}$ only consist of $W$-type, $K$-type and $L$-type edges, say $E_{\mathcal{T}}(v_i) \setminus \{e_0\} = W_i \cup K_i \cup L_i$. If $\cup_{i=1}^r W_i = \emptyset$, then $\alpha(\mathcal{T}) \geq m-1$, contradicting with the assumption of $\alpha < m-1$. Then there exists at least one vertex of $e_0$ which is incident to $W$-type edge. Let $x$ be the principal eigenvector of $\mathcal{T}$. Without loss of generality, assume that $x_{v_1}$ takes the maximum value of all $x_{v_i}$ satisfying $W_i \neq \emptyset$ $(1 \leq i \leq r)$.

Since $N(CO(\mathcal{T})) = 1$, there exists another vertex of $e_0$, say $v_2$, such that $W_2 \cup K_2 \neq \emptyset$. If $K_2 \neq \emptyset$, then after edge-releasing a $K$-type edge in $K_2$, the resulting hypertree cannot be isomorphic to $A(m, r, \alpha)$ and by Lemma 4.2 it has the same strong stability number as $\mathcal{T}$, but has larger spectral radius by Lemma 2.7, contradicting with the maximality of $\mathcal{T}$ in $\mathcal{T}(m, r, \alpha) \setminus \{A(m, r, \alpha)\}$. Thus $K_2 = \emptyset$ and so $W_2 \neq \emptyset$. Then we can move a $W$-type edge in $W_2$ from $v_2$ to $v_1$ in $\mathcal{T}$, and the resulting hypertree, denoted by $\mathcal{T}_1$, has the same strong stability number as $\mathcal{T}$ by Lemma 4.2, but has larger spectral radius by Lemma 2.4. If $\mathcal{T}_1$ is not isomorphic to $A(m, r, \alpha)$, this is a contradiction with the maximality of $\mathcal{T}$ in $\mathcal{T}(m, r, \alpha) \setminus \{A(m, r, \alpha)\}$. Thus $\mathcal{T}_1 \cong A(m, r, \alpha)$. Because $\mathcal{T}_1$ is obtained from $\mathcal{T}$ by moving a $W$-type edge in $W_2$ from $v_2$ to $v_1$, together with the maximality of $\mathcal{T}$ in $\mathcal{T}(m, r, \alpha) \setminus \{A(m, r, \alpha)\}$, $\mathcal{T}$ must satisfy the following conditions:

(1) All $W$-type edges of $\mathcal{T}$ are $\bar{W}$-type, and $|W_1| \geq 1$, $|W_2| = 1$ and $|W_3| = \cdots = |W_r| = 0$.

(2) $K_1 = K_2 = \cdots = K_r = \emptyset$.

(3) $|L_2| = \cdots = |L_r| = 0$ or $|L_2| = \cdots = |L_r| = 1$.

(4) $L_1 \neq \emptyset$ when $L_2 \neq \emptyset$.

The correctness of (1) to (3) is obvious. To prove (4), when $L_2 \neq \emptyset$, which implies that $|L_2| = \cdots = |L_r| = 1$, by contradiction, suppose that $L_1 = \emptyset$ . Since there is at least one $W$-type edge at $v_1$, after releasing this edge, the resulting hypertree $\mathcal{T}'$ has larger spectral radius than $\mathcal{T}$ by Lemma 2.4. It is easy to verify that $\mathcal{T}'$ has the same strong stability number as $\mathcal{T}$. This is a contradiction with the maximality of $\mathcal{T}$ in $\mathcal{T}(m,r,\alpha) \setminus \{A(m,r,\alpha)\}$. Thus $L_1 \neq \emptyset$ when $L_2 \neq \emptyset$.

Therefore, according to whether $L_2 \neq \emptyset$ or not, $\mathcal{T}$ is isomorphic to either $R((r-1)^{\epsilon-2}, 0^{\tau}; r-2; r-1, 0)$ with $\epsilon \geq 3$ and $\tau \geq 1$, or $R((r-1)^{\epsilon-1}, 0^{\tau-1}; 0; r-1)$ with $\epsilon \geq 2$ and $\tau \geq 1$.

**Case 3.** $N(CO(\mathcal{T})) \geq 2$.

By Lemma 4.4, there is a $\mathcal{T}' \in \mathcal{T}_j(m,r,\alpha)$, where $j \in \{0,1\}$, such that $\rho(\mathcal{T}) < \rho(\mathcal{T}')$. If either $N(CO(\mathcal{T}')) = 1$ or $N(CO(\mathcal{T}')) = 0$ and $\mathcal{T}' \neq A(m,r,\alpha)$, then this is a contradiction with the maximality of $\mathcal{T}$ in $\mathcal{T}(m,r,\alpha) \setminus \{A(m,r,\alpha)\}$.

It remains to consider the case of $N(CO(\mathcal{T}')) = 0$ and $\mathcal{T}' = A(m,r,\alpha)$, and that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by applying MR-operation one times on $\mathcal{T}$ and the pair $(v_1, v_2)$. Since $\mathcal{T}' = A(m,r,\alpha)$, $\mathcal{T}'$ cannot be obtained from $\mathcal{T}$ by applying the part of R-operation on the pair $(v_1, v_2)$, and so $\mathcal{T}'$ must be obtained from $\mathcal{T}$ by applying the part of M-operation on the pair $(v_1, v_2)$. Without loss of generality, we assume that M-operation on the pair $(v_1, v_2)$ is to move all $W$-type edges in $W_2$ from $v_2$ to $v_1$. Suppose that $|W_2| \geq 2$. Denote by $\mathcal{T}_1$ the hypertree obtained from $\mathcal{T}$ by moving just one $W$-type edge in $W_2$ from $v_2$ to $v_1$. Then $\mathcal{T}_1$ is not isomorphic to $A(m,r,\alpha)$, but has the same strong stability number $\alpha$ with $\mathcal{T}$ and larger spectral radius than $\mathcal{T}$, a contradiction. Therefore, $|W_2| = 1$ and $\mathcal{T}$ must be one of the forms in Figure 5.

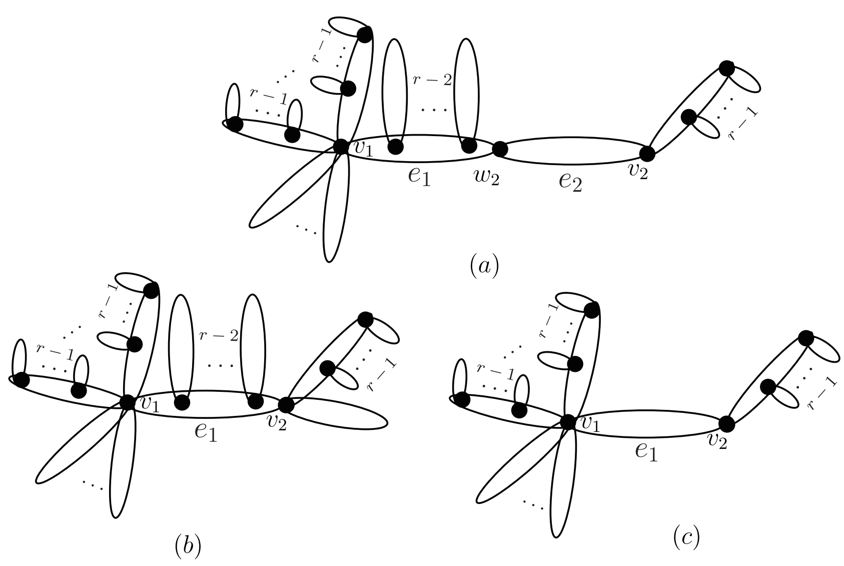


Figure 5: Possible forms of $\mathcal{T}$

Since $N(CO(\mathcal{T})) \geq 2$, $\mathcal{T}$ cannot be of the form as (b) or (c). Now apply edge-moving operation on $\mathcal{T}$ by moving the unique $W$-type edge from $v_2$ to $w_2$, the resulting hypertree is isomorphic to the one obtained by moving edge $e_1$ from $w_2$ to $v_2$, and it has the same strong stability number as $\mathcal{T}$ by Lemma 5.3 and has larger spectral radius than $\mathcal{T}$ by Lemma 2.4.

Thus we come to a contradiction with the maximality of $\mathcal{T}$ in $\mathcal{T}(m,r,\alpha)\setminus\{A(m,r,\alpha)\}$. This means the case of $N(CO(\mathcal{T}))\geq 2$ cannot occur.

From all cases discussed, we conclude that $\mathcal{T}$ must be one of four hypertrees as follows:

1. $R((r-1)^{\epsilon},0^{\tau-2},1)$, with $\epsilon\geq 2$ and $\tau\geq 2$.
2. $R((r-1)^{\epsilon-1},0^{\tau-1};r-2;0^2)$, with $\epsilon\geq 2$ and $\tau\geq 2$.
3. $R((r-1)^{\epsilon-2},0^{\tau};r-2;r-1,0)$ with $\epsilon\geq 3$ and $\tau\geq 1$.
4. $R((r-1)^{\epsilon-1},0^{\tau-1};0;r-1)$ with $\epsilon\geq 2$ and $\tau\geq 1$.

By Lemma 5.2, we have that $\mathcal{T}=R((r-1)^{\epsilon-1},0^{\tau-1};r-2;0^2)$ when $\epsilon\geq 2$ and $\tau\geq 2$, and $\mathcal{T}=R((r-1)^{\epsilon-2},0^{\tau};r-2;r-1,0)$ when $\epsilon\geq 3$ and $\tau=1$, and $\mathcal{T}=R(r-1;0;r-1)$ when $\epsilon=2$ and $\tau=1$. □

As an immediate consequence of Theorem 5.3, we can conclude the section with the following two corollaries. It should be pointed out that the first one has been obtained by Guo and Zhou [5] and now we give a simple proof of it using different methods.

**Corollary 5.4.** *$A(m,r,\alpha)$ uniquely attains the largest spectral radius in $\mathcal{T}(m,r,\alpha)$ with $\alpha<m-1$ and $m\geq r+3$.*

*Proof.* By Theorem 5.3, we know that in $\mathcal{T}(m,r,\alpha)\setminus\{A(m,r,\alpha)\}$, $B(m,r,\alpha)$ uniquely has the largest spectral radius. It is easy to verify that $A(m,r,\alpha)\succ B(m,r,\alpha)$ by Lemma 2.4. □

**Corollary 5.5.** *$B(m,r,\alpha)$ uniquely attains the second largest spectral radius in $\mathcal{T}(m,r,\alpha)$ with $\alpha<m-1$ and $m\geq r+3$.*

*Proof.* It follows directly from the combination of Theorem 5.3 and Corollary 5.4. □

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