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PRECONDITIONED SOR-TYPE ITERATIVE METHODS FOR SOLVING MULITI-LINEAR SYSTEMS WITH $\mathcal{L}$-TENSORS*

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Abstract: Recently, many algorithms are presented for solving the multi-linear systems, and most of them focus on the case when the coefficient tensor is an $\mathcal{M}$-tensor. In this paper, we propose successive over-relaxation (SOR) iterative methods to solve the multi-linear systems when the coefficient tensor is an $\mathcal{L}$-tensor. We present four iterative tensors with two new preconditioners. The corresponding comparison theorems for spectral radius of iterative tensors are given. Numerical experiments are tested to show the efficiency of the proposed SOR methods.

Key words: *multi-linear systems, tensor splitting, SOR method, preconditioned method, tensor splitting algorithms*

Mathematics Subject Classification: *15A48, 15A69, 65F10, 65H10*

1 Introduction

Consider the following multi-liner systems:

$$\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}, \tag{1.1}$$

where $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ is an m-th order n-dimensional tensor, $\mathbf{b}$, $\mathbf{x}$ are vectors in $\mathbb{R}^n$, and the n dimensional vector $\mathcal{A}\mathbf{x}^{m-1}$ is defined as [24]:

$$(\mathcal{A}\mathbf{x}^{m-1})_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{i i_2 \ldots i_m} x_{i_2} \ldots x_{i_m}, \quad i = 1, 2, \ldots, n, \tag{1.2}$$

where x_i denotes the i-th component of $\mathbf{x}$.

The multi-liner systems (1.1) appear in many practical fields including data mining and numerical partial differential equations [1, 5, 10, 11, 12, 15, 16, 18, 26, 27]. There have been many theoretical analysis and algorithms for solving the systems (1.1). Most of the existing methods for solving (1.1) focus on the case when $\mathcal{A}$ is an $\mathcal{M}$-tensor. Ding, Wei [10] extended the classical iterative methods and the Newton method for solving system of linear equations to solve the systems (1.1) with $\mathcal{A}$ being a strong $\mathcal{M}$-tensor. Han [13] proposed a homotopy method for finding the unique positive solution of (1.1) with a strong $\mathcal{M}$-tensor

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and a positive right side vector $\mathbf{b}$. Liu, Li and Vong [20] proposed some tensor splitting algorithms for solving (1.1) with strong $\mathcal{M}$-tensors. He, Ling, Qi and Zhou [14] proposed a Newton-type method to solve (1.1) with $\mathcal{M}$-tensors. Li, Xie and Xu [16] extended the classic splitting methods for solving the system of linear equations to solve symmetric tensor equations. Liu, Li and Vong [21] proposed a preconditioned SOR method for solving (1.1) with $\mathcal{M}$-tensors. There are also some preconditioned tensor splitting iterative methods for solving (1.1) with $\mathcal{M}$-tensors, see [4, 6, 30].

A matrix is called an L-matrix if it has positive diagonal entries and non-positive off-diagonal elements. L-matrices are also an important class of matrices and have been well studied for solving linear equations [8, 19, 28]. In this paper, we will deal with the systems (1.1) with $\mathcal{A}$ being an $\mathcal{L}$-tensor. A tensor $\mathcal{A}$ is called an $\mathcal{L}$-tensor if it has positive diagonal entries and non-positive off-diagonal elements, which is a natural extension of the L-matrix. We can see that it is very easy to verify a tensor is an $\mathcal{L}$-tensor or not, and also it is easy to see that an $\mathcal{L}$-tensor is not necessary an $\mathcal{M}$-tensor. In this paper, we propose new preconditioned SOR methods to solve the systems (1.1) with $\mathcal{A}$ being an $\mathcal{L}$-tensor. In the whole paper, it is assumed that

$$\mathcal{A} = \mathcal{I} - \mathcal{L} - \mathcal{F}, \tag{1.3}$$

where $\mathcal{I}$ is the unit tensor, $\mathcal{L} = L\mathcal{I}$(that is the product of a matrix and a tensor, which is defined in (2.1)), and $-L$ is the strictly lower triangular part of the majorization matrix of $\mathcal{A}$. The majorization matrix of $\mathcal{A}$, denoted by $M(\mathcal{A})$, is a $n \times n$ matrix with the entries

$$M(\mathcal{A})_{ij} = a_{ij\ldots j}, i, j = 1, \ldots, n.$$

The iterative tensor of the classical SOR method is represented by

$$\mathcal{T}(\omega) = (I - \omega L)^{-1}((1-\omega)\mathcal{I} + \omega\mathcal{F}), \tag{1.4}$$

where ω is a real parameter with $\omega \neq 0$. The spectral radius of the iterative tensor is conclusive for the convergence and stability of the method, and the smaller it is, the faster the method converges when the spectral radius is smaller than 1. The effective method to decrease the spectral radius is to precondition the multi-liner systems (1.1), namely,

$$P\mathcal{A}\mathbf{x}^{m-1} = P\mathbf{b},$$

where P is a nonsingular matrix and is called a preconditioner for solving the multi-liner systems (1.1). In this paper, we give two new preconditioners: $\overline{P} = I + \overline{S}$ and $\widetilde{P} = I + \widetilde{S}$ with

$$\overline{S} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ -(a_{21\ldots 1} + \gamma_2) & 0 & 0 & \ldots & 0 \\ -(a_{31\ldots 1} + \gamma_3) & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -(a_{n1\ldots 1} + \gamma_n) & 0 & 0 & \ldots & 0 \end{pmatrix}, \tag{1.5}$$

$$\widetilde{S} = \begin{pmatrix} 0 & \ldots & 0 & 0 & -(a_{1n\ldots n} + \delta_1) \\ 0 & \ldots & 0 & 0 & -(a_{2n\ldots n} + \delta_2) \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & -(a_{(n-1)n\ldots n} + \delta_{n-1}) \\ 0 & \ldots & 0 & 0 & 0 \end{pmatrix}, \tag{1.6}$$

where $\gamma_2, \gamma_3, \ldots, \gamma_n$ and $\delta_1, \delta_2, \ldots, \delta_{n-1}$ are real parameters. The preconditioners have been applied in [8] for solving linear equations. We consider the two preconditioned multi-liner systems as follows:

$$\overline{\mathcal{A}}\mathbf{x}^{m-1} = \overline{\mathbf{b}} \quad \text{where} \quad \overline{\mathcal{A}} = \overline{P}\mathcal{A} \quad \text{and} \quad \overline{\mathbf{b}} = \overline{P}\mathbf{b};$$
$$\widetilde{\mathcal{A}}\mathbf{x}^{m-1} = \widetilde{\mathbf{b}} \quad \text{where} \quad \widetilde{\mathcal{A}} = \widetilde{P}\mathcal{A} \quad \text{and} \quad \widetilde{\mathbf{b}} = \widetilde{P}\mathbf{b}.$$

We propose four iterative tensors based on the preconditioned equations. The corresponding comparison theorems for the spectral radius of the proposed iterative tensors are shown when $\mathcal{A}$ is an irreducible $\mathcal{L}$-tensor.

The rest of this paper is organized as follows. In Sect.2, we introduce some related definitions and lemmas. Also, we discuss the relations between $\mathcal{M}$-tensors and irreducible $\mathcal{L}$-tensors. In Sect.3, four iterative tensors are presented with the two new precondtioners, and the corresponding theoretical analysis is given. In Sect.4, numerical examples are given to show the efficiency of the proposed SOR methods. The final section is the concluding remark.

2 Preliminaries

In this section, we introduce some definitions, lemmas, and some related properties which will be used in the sequel.

Let $\langle n \rangle = \{1, \ldots, n\}$ for a positive integer n. A tensor $\mathcal{A}$ consists of $n_1 \times \cdots \times n_m$ elements in the complex number field $\mathbb{C}$:

$$\mathcal{A} = (a_{i_1 \ldots i_m}), a_{i_1 \ldots i_m} \in \mathbb{C}, i_j \in \langle n_j \rangle, j = 1, \ldots, m.$$

We sometimes denote $a_{i_1 i_2 \ldots i_m}$ as $a_{i_1 \alpha}$, where $\alpha = i_2 \ldots i_m$. When $m = 2$, $\mathcal{A}$ is an $n_1 \times n_2$ matrix. If $n_1 = \cdots = n_m = n$, $\mathcal{A}$ is called an m-th order n-dimensional tensor. We denote all m-th order tensors consisting of $n_1 \times \cdots \times n_m$ entries by $\mathbb{C}^{n_1 \times \cdots \times n_m}$ and the set of all m-th order n-dimensional tensors by $\mathbb{C}^{[m,n]}$. Similarly, the above notions can be used to the real number field $\mathbb{R}$. Let $\mathcal{I} = (\delta_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]}$ be a unit tensor with its entries given by

$$\delta_{i_1 \ldots i_m} = \begin{cases} 1, & i_1 = \cdots = i_m, \\ 0, & else. \end{cases}$$

Let $\mathbf{0}, O$ and $\mathcal{O}$ denote a zero vector, a zero matrix and a zero tensor, respectively. Let $\mathcal{A}$ and $\mathcal{B}$ be two tensors (vectors or matrices) with the same size, the order $\mathcal{A} \geq \mathcal{B} (> \mathcal{B})$ means that each element of $\mathcal{A}$ is no less than (larger than) corresponding one of $\mathcal{B}$.

Definition 2.1 ([2]). Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_2}$ and $\mathcal{B} \in \mathbb{C}^{n_2 \times \cdots \times n_{k+1}}$ be two tensors of order $m (\geq 2)$ and $k (\geq 1)$, respectively. The product $\mathcal{AB}$ is the tensor of order $(m-1)(k-1)+1$ with entries:

$$(\mathcal{AB})_{j\alpha_2 \ldots \alpha_m} = \sum_{j_2, \ldots, j_m = 1}^{n_2} (a_{jj_2 \ldots j_m} \prod_{i=2}^{m} b_{j_i \alpha_i}),$$

where $j \in \langle n_1 \rangle$, $\alpha_2, \ldots, \alpha_m \in \langle n_3 \rangle \times \cdots \times \langle n_{k+1} \rangle$.

In this paper, we will use one special case of the above definition: If $A \in \mathbb{R}^{[2,n]}$(i.e., A is an n-dimensional square matrix) and $\mathcal{B} \in \mathbb{R}^{[k,n]}$, then the tensor $\mathcal{C} = A\mathcal{B} \in \mathbb{R}^{[k,n]}$ is defined by

$$c_{ji_2 \ldots i_k} = \sum_{j_2=1}^{n} a_{jj_2} b_{j_2 i_2 \ldots i_k}, \tag{2.1}$$

where $j, i_2, \ldots, i_k \in \langle n \rangle$.

Definition 2.2 ([23])**.** Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[m,n]}$. Then the majorization matrix $M(\mathcal{A})$ of $\mathcal{A}$ is the $n \times n$ matrix with the entries

$$M(\mathcal{A})_{ij} = a_{ij\ldots j}, i, j \in \langle n \rangle.$$

Definition 2.3 ([25])**.** Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ and $R_i(\mathcal{A}) = (r_{ii_2 \ldots i_m})_{i_2,\ldots,i_m}^n \in \mathbb{R}^{[m-1,n]}$ with $r_{ii_2\ldots i_m} = a_{ii_2\ldots i_m}$. Then $\mathcal{A}$ is called row-subtensor diagonal, or simply row diagonal, if all its row-subtensors $R_1(\mathcal{A}), \ldots, R_n(\mathcal{A})$ are diagonal tensors, namely, if $a_{ii_2\ldots i_m}$ can take nonzero value only when $i_2 = \cdots = i_m$.

Lemma 2.4 ([25])**.** *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. Then $\mathcal{A}$ is row diagonal if and only if $\mathcal{A} = M(\mathcal{A})\mathcal{I}$.*

Now, we give some properties about the majorization matrix of the product of a matrix and a tensor.

Lemma 2.5. *Let $\mathcal{C} = A\mathcal{B} \in \mathbb{R}^{[k,n]}$ with $A \in \mathbb{R}^{[2,n]}$ and $\mathcal{B} \in \mathbb{R}^{[k,n]}$, then we have $M(\mathcal{C}) = AM(\mathcal{B})$, that is, $M(A\mathcal{B}) = AM(\mathcal{B})$.*

Proof. By the definition of the product in (2.1), we have

$$c_{ji\ldots i} = \sum_{j_2=1}^{n} a_{jj_2} b_{j_2 i \ldots i},$$

that is, $M(\mathcal{C}) = M(A\mathcal{B}) = AM(\mathcal{B})$. □

Lemma 2.6. *Let $A \in \mathbb{R}^{[2,n]}$ and $\mathcal{B}$ is a row diagonal tensor in $\mathbb{R}^{[k,n]}$, then $\mathcal{C} = A\mathcal{B} = AM(\mathcal{B})\mathcal{I}$.*

Proof. By Lemma 2.5, we have $M(\mathcal{C}) = AM(\mathcal{B})$. Next, we only need show $\mathcal{C}$ is row diagonal, that is, $c_{ii_2\ldots i_m} = 0$ when $i_2, \ldots, i_m$ are not all equal. In fact, as $\mathcal{B}$ is row diagonal, $b_{ii_2\ldots i_m} = 0$ when $i_2, \ldots, i_m$ are not all equal, then $c_{ji_2\ldots i_m} = \sum_{j_2=1}^{n} a_{jj_2} b_{j_2 i_2 \ldots i_m} = 0$. □

Next, we give some definitions of structured tensors.

Definition 2.7 ([9, 29])**.** Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. A tensor $\mathcal{A}$ is called a $\mathcal{Z}$-tensor if its off-diagonal entries are non-positive. A tensor $\mathcal{A}$ is called an $\mathcal{M}$-tensor if there exist a nonnegative tensor $\mathcal{B}$ and a positive real number $s \geq \rho(\mathcal{B})$ such that

$$\mathcal{A} = s\mathcal{I} - \mathcal{B},$$

where $\rho(\mathcal{B})$ is the spectral radius of tensor $\mathcal{B}$, that is

$$\rho(\mathcal{B}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{B}\}.$$

If $s > \rho(\mathcal{B})$, $\mathcal{A}$ is called a strong or nonsingular $\mathcal{M}$-tensor.

Definition 2.8 ([7])**.** Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. We say that $\mathcal{A}$ is an $\mathcal{L}$-tensor if it is a $\mathcal{Z}$-tensor with positive diagonal entries.

Definition 2.9 ([17])**.** An $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called reducible if there exists a nonempty proper index subset $\mathbb{I} \subseteq \langle n \rangle$ such that

$$a_{i_1 i_2 \ldots i_m} = 0, \forall i_1 \in \mathbb{I}, \forall i_2, \ldots, i_m \notin \mathbb{I}.$$

If $\mathcal{A}$ is not reducible, we call $\mathcal{A}$ is irreducible.

Remark 2.10. In this paper, we will solve the systems (1.1) with $\mathcal{A}$ being an irreducible $\mathcal{L}$-tensor. As most of the algorithms proposed in the references for solving (1.1) focus on $\mathcal{A}$ being an $\mathcal{M}$-tensor, an interesting question is what are the relations between the irreducible $\mathcal{L}$-tensors and $\mathcal{M}$-tensors. Now we discuss this problem in the following examples. The two examples show that an irreducible $\mathcal{L}$-tensor may not be a strong $\mathcal{M}$-tensor, and a strong $\mathcal{M}$-tensor may not be an irreducible $\mathcal{L}$-tensor.

Example 2.11 ([20])**.** Let $\mathcal{A} = (a_{i_1 i_2 i_3}) \in \mathbb{R}^{[3,2]}$ be defined as follows:

$$a_{111} = 2, a_{121} = -3, a_{112} = -3, a_{122} = -1,$$

$$a_{211} = -1, a_{221} = -3, a_{212} = -3, a_{222} = 2.$$

It is obvious that $\mathcal{A}$ is an irreducible $\mathcal{L}$-tensor.

For arbitrary $t \geq 0$, let $s = 2 + t$, then $\mathcal{A} = s\mathcal{I} - \mathcal{B}$ and $\mathcal{B} = (b_{i_1 i_2 i_3})$ is given by

$$b_{111} = t, b_{121} = 3, b_{112} = 3, b_{122} = 1,$$

$$b_{211} = 1, b_{221} = 3, b_{212} = 3, b_{222} = t,$$

Notice that $\rho(\mathcal{B}) = 7 + t > s = 2 + t$. Therefore $\mathcal{A}$ is an irreducible $\mathcal{L}$-tensor and not a strong $\mathcal{M}$-tensor.

Example 2.12. It can be easily verified that the unit tensor $\mathcal{I}$ is a strong $\mathcal{M}$-tensor. Obviously, $\mathcal{I}$ is an $\mathcal{L}$-tensor, however it is reducible, which means that $\mathcal{I}$ is not an irreducible $\mathcal{L}$-tensor.

Remark 2.13. It is known that for any positive vector $\mathbf{b}$, i.e. $\mathbf{b} > \mathbf{0}$, the systems (1.1) have a unique positive solution when $\mathcal{A}$ is a strong $\mathcal{M}$-tensor [10]. However, we do not know much about the properties of the solution for the systems (1.1) with $\mathcal{A}$ being an $\mathcal{L}$-tensor. Now we survey the solution of (1.1) when $\mathcal{A}$ is the tensor in Example 2.11 with different $\mathbf{b} = (b_1, b_2)^T$ in three cases. We rewrite (1.1) as

$$\begin{cases} 2x_1^2 - 6x_1x_2 - x_2^2 = b_1, \\ -x_1^2 - 6x_1x_2 + 2x_2^2 = b_2. \end{cases} \tag{2.2}$$

(i) $\mathbf{b} = (0, 0)^T$. In this case, the systems (2.2) have a unique solution $\mathbf{x} = (0, 0)^T$.

(ii) $\mathbf{b} \geq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$. In this case, the systems (2.2) have two solutions, and for every solution, the components have different signs. Specifically, when $b_1 \geq b_2 > 0$, the solutions are

$$\left(-\sqrt{\frac{32b_1 - 38b_2 + 3\sqrt{\Delta}}{210}}, \sqrt{\frac{32b_2 - 38b_1 + 3\sqrt{\Delta}}{210}}\right)^T; \quad \left(\sqrt{\frac{32b_1 - 38b_2 + 3\sqrt{\Delta}}{210}}, -\sqrt{\frac{32b_2 - 38b_1 + 3\sqrt{\Delta}}{210}}\right)^T,$$

where

$$\Delta = (\frac{38}{3}b_1 - \frac{32}{3}b_2)^2 + 140(\frac{1}{3}b_1 + \frac{2}{3}b_2)^2 > 0.$$

And when $0 \leq b_1 \leq b_2$, the solutions can be similarly given.

(iii) $\mathbf{b} \leq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$. In this case, the systems (2.2) have two solutions, and for every solution, the components have the same signs. Specifically, when $0 > b_1 \geq b_2$, the solutions are

$$\left(\sqrt{\tfrac{32b_1-38b_2+3\sqrt{\Delta}}{210}}, \sqrt{\tfrac{32b_2-38b_1+3\sqrt{\Delta}}{210}}\right)^T; \quad \left(-\sqrt{\tfrac{32b_1-38b_2+3\sqrt{\Delta}}{210}}, -\sqrt{\tfrac{32b_2-38b_1+3\sqrt{\Delta}}{210}}\right)^T,$$

where

$$\Delta = (\frac{38}{3}b_1 - \frac{32}{3}b_2)^2 + 140(\frac{1}{3}b_1 + \frac{2}{3}b_2)^2 > 0.$$

And when $b_1 \leq b_2 < 0$, the solutions can be similarly given.

Consequently, the systems (2.2) have a unique positive solution for every $\mathbf{b} < \mathbf{0}$.

We also need the following definitions and lemmas which will be used in the sequel.

Definition 2.14 ([2])**.** Let $\mathcal{A} \in \mathbb{C}^{[m,n]}$, $\mathcal{B} \in \mathbb{C}^{[k,n]}$. If $\mathcal{AB} = \mathcal{I}$, then $\mathcal{A}$ is called an order m left inverse of $\mathcal{B}$, and $\mathcal{B}$ is called an order k right inverse of $\mathcal{A}$.

Definition 2.15 ([22])**.** Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. If $M(\mathcal{A})$ is a nonsingular matrix and $\mathcal{A} = M(\mathcal{A})\mathcal{I}$, we call $M(\mathcal{A})^{-1}$ the order 2 left-inverse of $\mathcal{A}$.

Definition 2.16 ([22])**.** Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. If $\mathcal{A}$ has an order k left(right) inverse, $\mathcal{A}$ is called a left(right)-invertible or left(right)-nonsingular tensor, where $k \geq 2$.

Definition 2.17 ([20])**.** Let $\mathcal{A}, \mathcal{E}, \mathcal{F} \in \mathbb{R}^{[m,n]}$. Then $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is called a splitting of $\mathcal{A}$ if $\mathcal{E}$ is a left-invertible tensor. The splitting is called:

(1) regular if $M(\mathcal{E})^{-1} \geq O$ and $\mathcal{F} \geq \mathcal{O}$;

(2) convergent if $\rho(M(\mathcal{E})^{-1}\mathcal{F}) < 1$;

(3) weak regular if $M(\mathcal{E})^{-1} \geq O$ and $M(\mathcal{E})^{-1}\mathcal{F} \geq \mathcal{O}$.

By Theorem 1.3 and Theorem 1.4 in [3], we have the following Lemma 2.18.

Lemma 2.18. *Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ be an irreducible nonnegative tensor. Then $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a positive eigenvector $\mathbf{x} \in \mathbb{R}^n$.*

Lemma 2.19 ([3])**.** *Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ be an irreducible nonnegative tensor. If $(\lambda, \mathbf{x})$ and $(\mu, \mathbf{y}) \in \mathbb{R}_+ \times (\mathbb{R}^n \backslash \mathbf{0})$ satisfy $\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}$ and $\mathcal{A}\mathbf{y}^{m-1} \geq \mu \mathbf{y}^{[m-1]}$ (or, respectively, $\mathcal{A}\mathbf{y}^{m-1} \leq \mu \mathbf{y}^{[m-1]}$), then $\lambda = \rho(\mathcal{A})$ and $\mu \leq \lambda$ (or, respectively, $\lambda \leq \mu$).*

3 SOR Method

3.1 The tensor splitting with two preconditioners

In the subsection, we give four iterative tensors for SOR methods with two preconditioners, and then prove the iterative tensors are nonnegative and irreducible under the conditions that $\mathcal{A}$ is an irreducible $\mathcal{L}$-tensor and some other assumptions are defined on $\mathcal{A}$.

Two iteration tensors associated with $\overline{P}$. As $\overline{S}$, which is defined in (1.5), is a strictly lower triangular matrix and $\mathcal{L} = L\mathcal{I}$ in (1.3) is a row diagonal tensor with $L = M(\mathcal{L})$ being a strictly lower triangular matrix, we know that $\overline{S}\mathcal{L} = \overline{S}L\mathcal{I} = \mathcal{O}$ by Lemma 2.6. Thus,

$$\begin{aligned}
\overline{\mathcal{A}} &= \overline{P}\mathcal{A} = (I + \overline{S})(\mathcal{I} - \mathcal{L} - \mathcal{F}) \\
&= \mathcal{I} - \mathcal{L} - \mathcal{F} + \overline{S}\mathcal{I} - \overline{S}\mathcal{L} - \overline{S}\mathcal{F} \\
&= \mathcal{I} - \mathcal{L} - \mathcal{F} + \overline{S}\mathcal{I} - \overline{S}\mathcal{F} \\
&= (\mathcal{I} + \overline{\mathcal{D}}_1) - (\mathcal{L} - \overline{S}\mathcal{I} + \overline{\mathcal{L}}_1) - (\mathcal{F} + \overline{\mathcal{F}}_1) \\
&= \overline{\mathcal{D}} - \overline{\mathcal{L}} - \overline{\mathcal{F}},
\end{aligned}$$

where $\overline{\mathcal{D}} = \mathcal{I} + \overline{\mathcal{D}}_1, \overline{\mathcal{L}} = \mathcal{L} - \overline{S}\mathcal{I} + \overline{\mathcal{L}}_1, \overline{\mathcal{F}} = \mathcal{F} + \overline{\mathcal{F}}_1, \overline{\mathcal{D}}_1 = \overline{D}_1\mathcal{I}, \overline{\mathcal{L}}_1 = \overline{L}_1\mathcal{I}, \overline{\mathcal{F}}_1 = \overline{S}\mathcal{F} + \overline{\mathcal{D}}_1 - \overline{\mathcal{L}}_1$, and $-\overline{D}_1, \overline{L}_1$ are the diagonal, strictly lower triangular parts of $M(\overline{S}\mathcal{F})$. It is easy to see that $\overline{\mathcal{D}}$ and $\overline{\mathcal{L}}$ are row diagonal tensors. Denote $\overline{D} = M(\overline{\mathcal{D}})$ and $\overline{L} = M(\overline{\mathcal{L}})$. Then $\overline{D} = I + \overline{D}_1$ and $\overline{L} = L - \overline{S} + \overline{L}_1$. For $0 < \omega \leq 1$,

$$\begin{aligned}
\omega\overline{\mathcal{A}} &= \omega(\overline{\mathcal{D}} - \overline{\mathcal{L}} - \overline{\mathcal{F}}) \\
&= \overline{\mathcal{D}} - \omega\overline{\mathcal{L}} - [(1-\omega)\overline{\mathcal{D}} + \omega\overline{\mathcal{F}}] \\
&= (\mathcal{I} - \omega\overline{\mathcal{L}}) - [(1-\omega)\mathcal{I} + \omega(\overline{\mathcal{F}} - \overline{\mathcal{D}}_1)].
\end{aligned}$$

Denote $\mathcal{E}_1 = \overline{\mathcal{D}} - \omega\overline{\mathcal{L}}, \mathcal{F}_1 = (1-\omega)\overline{\mathcal{D}} + \omega\overline{\mathcal{F}}$ and $\mathcal{E}_2 = \mathcal{I} - \omega\overline{\mathcal{L}}, \mathcal{F}_2 = (1-\omega)\mathcal{I} + \omega(\overline{\mathcal{F}} - \overline{\mathcal{D}}_1)$. We have

$$\omega\overline{\mathcal{A}} = \mathcal{E}_1 - \mathcal{F}_1 = \mathcal{E}_2 - \mathcal{F}_2,$$

and $M(\mathcal{E}_1) = \overline{D} - \omega\overline{L}$, $M(\mathcal{E}_2) = I - \omega\overline{L}$. Two different forms of SOR iteration tensor associated with $\overline{\mathcal{A}}$ can be represented by

$$\mathcal{T}_1(\omega) = M(\mathcal{E}_1)^{-1}\mathcal{F}_1 = (\overline{D} - \omega\overline{L})^{-1}[(1-\omega)\overline{\mathcal{D}} + \omega\overline{\mathcal{F}}], \tag{3.1}$$

$$\mathcal{T}_2(\omega) = M(\mathcal{E}_2)^{-1}\mathcal{F}_2 = (I - \omega\overline{L})^{-1}[(1-\omega)\mathcal{I} + \omega(\overline{\mathcal{F}} - \overline{\mathcal{D}}_1)]. \tag{3.2}$$

Two iteration tensors associated with $\widetilde{P}$. For a similar discussion with the preconditioner $\widetilde{P} = I + \widetilde{S}$, we have

$$\begin{aligned}
\widetilde{\mathcal{A}} &= \widetilde{P}\mathcal{A} = (I + \widetilde{S})\mathcal{A} = \mathcal{A} + \widetilde{S}\mathcal{A} \\
&= \mathcal{I} - \mathcal{L} - \mathcal{F} + (\widetilde{\mathcal{D}}_1 - \widetilde{\mathcal{L}}_1 - \widetilde{\mathcal{F}}_1) \\
&= \mathcal{I} + \widetilde{\mathcal{D}}_1 - (\mathcal{L} + \widetilde{\mathcal{L}}_1) - (\mathcal{F} + \widetilde{\mathcal{F}}_1) \\
&= \widetilde{\mathcal{D}} - \widetilde{\mathcal{L}} - \widetilde{\mathcal{F}},
\end{aligned}$$

where $\widetilde{\mathcal{D}} = \mathcal{I} + \widetilde{\mathcal{D}}_1, \widetilde{\mathcal{L}} = \mathcal{L} + \widetilde{\mathcal{L}}_1, \widetilde{\mathcal{F}} = \mathcal{F} + \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{D}}_1 = \widetilde{D}_1\mathcal{I}, \widetilde{\mathcal{L}}_1 = \widetilde{L}_1\mathcal{I}, \widetilde{\mathcal{F}}_1 = -\widetilde{S}\mathcal{A} + \widetilde{\mathcal{D}}_1 - \widetilde{\mathcal{L}}_1$, and $-\widetilde{D}_1, \widetilde{L}_1$ are the diagonal, strictly lower triangular parts of $M(\widetilde{S}\mathcal{A})$. Also $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{L}}$ are row diagonal tensors. Denote $\widetilde{D} = M(\widetilde{\mathcal{D}})$ and $\widetilde{L} = M(\widetilde{\mathcal{L}})$. Then $\widetilde{D} = I + \widetilde{D}_1$ and $\widetilde{L} = L - \widetilde{S} + \widetilde{L}_1$. For $0 < \omega \leq 1$,

$$\begin{aligned}
\omega\widetilde{\mathcal{A}} &= \omega(\widetilde{\mathcal{D}} - \widetilde{\mathcal{L}} - \widetilde{\mathcal{F}}) \\
&= \widetilde{\mathcal{D}} - \omega\widetilde{\mathcal{L}} - [(1-\omega)\widetilde{\mathcal{D}} + \omega\widetilde{\mathcal{F}}] \\
&= (\mathcal{I} - \omega\widetilde{\mathcal{L}}) - [(1-\omega)\mathcal{I} + \omega(\widetilde{\mathcal{F}} - \widetilde{\mathcal{D}}_1)].
\end{aligned}$$

Denote $\mathcal{E}_3 = \widetilde{\mathcal{D}} - \omega\widetilde{\mathcal{L}}, \mathcal{F}_3 = (1-\omega)\widetilde{\mathcal{D}} + \omega\widetilde{\mathcal{F}}$ and $\mathcal{E}_4 = \mathcal{I} - \omega\widetilde{\mathcal{L}}, \mathcal{F}_4 = \mathcal{I} - \omega\widetilde{\mathcal{D}} + \omega\widetilde{\mathcal{F}}$. Then

$$\omega\widetilde{\mathcal{A}} = \mathcal{E}_3 - \mathcal{F}_3 = \mathcal{E}_4 - \mathcal{F}_4,$$

and $M(\mathcal{E}_3) = \widetilde{D} - \omega\widetilde{L}$, $M(\mathcal{E}_4) = I - \omega\widetilde{L}$. Two different forms of SOR iteration tensor associated with $\widetilde{\mathcal{A}}$ can be represented by

$$\mathcal{T}_3(\omega) = M(\mathcal{E}_3)^{-1}\mathcal{F}_3 = (\widetilde{D} - \omega\widetilde{L})^{-1}[(1-\omega)\widetilde{\mathcal{D}} + \omega\widetilde{\mathcal{F}}], \tag{3.3}$$

$$\mathcal{T}_4(\omega) = M(\mathcal{E}_4)^{-1}\mathcal{F}_4 = (I - \omega\widetilde{L})^{-1}[(1-\omega)\mathcal{I} + \omega(\widetilde{\mathcal{F}} - \widetilde{\mathcal{D}}_1)]. \tag{3.4}$$

Theorem 3.1. *Let $\mathcal{T}(\omega), \mathcal{T}_1(\omega), \mathcal{T}_2(\omega), \mathcal{T}_3(\omega)$ and $\mathcal{T}_4(\omega)$ be defined by* (1.4), (3.1)-(3.4), *respectively. If $\mathcal{A}$ is an irreducible $\mathcal{L}$-tensor, then for $0<\omega<1$,*

(i) *$\mathcal{T}(\omega)$ is a nonnegative and irreducible tensor;*

(ii) *if $a_{1q\dots q}a_{q1\dots 1}>0$ and $\gamma_q \in ((1-a_{1q\dots q}a_{q1\dots 1})/a_{1q\dots q}, -a_{q1\dots 1}) \cap (0, -a_{q1\dots 1})$ for $q=2,3,\dots n$, $\mathcal{T}_1(\omega)$ and $\mathcal{T}_2(\omega)$ are nonnegative and irreducible tensors;*

(iii) *if $a_{ns\dots s}a_{sn\dots n}>0$ and $\delta_s \in ((1-a_{ns\dots s}a_{sn\dots n})/a_{ns\dots s}, -a_{sn\dots n}) \cap (0, -a_{sn\dots n})$ for $s=1,2,\dots,n-1$, $\mathcal{T}_3(\omega)$ and $\mathcal{T}_4(\omega)$ are nonnegative and irreducible tensors.*

Proof. (i) Since $\mathcal{A}$ is an $\mathcal{L}$-tensor, by the splitting of $\mathcal{A}$ in (1.3), we know that $\mathcal{L} \geq \mathcal{O}$ and $\mathcal{F} \geq \mathcal{O}$. Then

$$\begin{aligned}\mathcal{T}(\omega) &= (I-\omega L)^{-1}[(1-\omega)\mathcal{I}+\omega\mathcal{F}]\\ &= (I+\omega L+\omega^2L^2+\dots\omega^{n-1}L^{n-1})[(1-\omega)\mathcal{I}+\omega\mathcal{F}]\\ &= [(1-\omega)\mathcal{I}+\omega(1-\omega)\mathcal{L}+\omega\mathcal{F}] + \text{nonnegative terms.}\end{aligned}$$

Consequently, $T(\omega)$ is nonnegative, and if $\mathcal{A}$ is irreducible, $(1-\omega)\mathcal{I}+\omega(1-\omega)\mathcal{L}+\omega\mathcal{F}$ is irreducible. Therefore $\mathcal{T}(\omega)$ is a nonnegative and irreducible tensor.

(ii) As $\overline{D}=I+\overline{D}_1$, it is easy to get

$$\overline{D} = \mathrm{diag}(1, 1-a_{12\dots 2}(a_{21\dots 1}+\gamma_2), 1-a_{13\dots 3}(a_{31\dots 1}+\gamma_3), \dots, 1-a_{1n\dots n}(a_{n1\dots 1}+\gamma_n)), \tag{3.5}$$

where

$$1-a_{1q\dots q}(a_{q1\dots 1}+\gamma_q) > \begin{cases} 1-a_{1q\dots q}(a_{q1\dots 1}+\frac{1-a_{1q\dots q}a_{q1\dots 1}}{a_{1q\dots q}})=0, & if\, \frac{1-a_{1q\dots q}a_{q1\dots 1}}{a_{1q\dots q}}>0,\\ 1-a_{1q\dots q}a_{q1\dots 1}>0, & if\, \frac{1-a_{1q\dots q}a_{q1\dots 1}}{a_{1q\dots q}}\leq 0,\end{cases}$$

for $\gamma_q \in ((1-a_{1q\dots q}a_{q1\dots 1})/a_{1q\dots q}, -a_{q1\dots 1}) \cap (0, -a_{q1\dots 1})$, $q=2,3,\dots,n$. That is, $\overline{D}$ is a diagonal matrix with positive diagonal elements. So $\overline{D}$ is invertible and $\overline{D}^{-1} \geq O$.

Since $\overline{L}$ is a strictly lower triangular matrix, i.e.,

$$\overline{L} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0\\ \gamma_2 & 0 & 0 & \dots & 0\\ \gamma_3 & -a_{32\dots 2}+a_{12\dots 2}(a_{31\dots 1}+\gamma_3) & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \gamma_n & -a_{n2\dots 2}+a_{12\dots 2}(a_{n1\dots 1}+\gamma_n) & \dots & -a_{n(n-1)\dots(n-1)}+a_{1(n-1)\dots(n-1)}(a_{n1\dots 1}+\gamma_n) & 0\end{pmatrix}.$$

The element $-a_{ij\dots j}+a_{1j\dots j}(a_{i1\dots 1}+\gamma_i) \geq -a_{ij\dots j}+a_{1j\dots j}(a_{i1\dots 1}-a_{i1\dots 1}) = -a_{ij\dots j} \geq 0$, $i=3,\dots,n, j=2,\dots,i-1$, together with $\gamma_q \geq 0$, $q=2,3,\dots,n$, implies that $\overline{L} \geq O$. By $\overline{S} \geq O$ and $\mathcal{F} \geq \mathcal{O}$, then $\overline{S}\mathcal{F} \geq \mathcal{O}$, $\overline{\mathcal{F}}_1 \geq \mathcal{O}$, furthermore, $\overline{\mathcal{F}} = \mathcal{F}+\overline{\mathcal{F}}_1 \geq \mathcal{O}$.

$$\begin{aligned}\mathcal{T}_1(\omega) &= M(\mathcal{E}_1)^{-1}\mathcal{F}_1\\ &= (\overline{D}-\omega\overline{L})^{-1}[(1-\omega)\overline{\mathcal{D}}+\omega\overline{\mathcal{F}}]\\ &= (I-\omega\overline{D}^{-1}\overline{L})^{-1}[(1-\omega)\mathcal{I}+\omega\overline{D}^{-1}\overline{\mathcal{F}}]\\ &= [I+\omega\overline{D}^{-1}\overline{L}+\omega^2(\overline{D}^{-1}\overline{L})^2+\dots+\omega^{n-1}(\overline{D}^{-1}\overline{L})^{n-1}][(1-\omega)\mathcal{I}+\omega\overline{D}^{-1}\overline{\mathcal{F}}]\\ &= (1-\omega)\mathcal{I}+\omega\overline{D}^{-1}\overline{\mathcal{F}}+(1-\omega)\omega\overline{D}^{-1}\overline{\mathcal{L}} + \text{nonnegative terms.}\end{aligned}$$

From the above results, we easily know that $\mathcal{T}_1(\omega)$ is a nonnegative tensor for any $0 < \omega < 1$. Also as $\overline{D}$ is a diagonal matrix with positive diagonal elements, $(1-\omega)\mathcal{I} + \omega\overline{D}^{-1}\overline{\mathcal{F}} + (1-\omega)\omega\overline{D}^{-1}\overline{\mathcal{L}}$ is irreducible when $\mathcal{A}$ is irreducible. So $\mathcal{T}_1(\omega)$ is a nonnegative and irreducible tensor. Similarly, we can prove that $\mathcal{T}_2(\omega)$ is a nonnegative and irreducible tensor.

(iii) By a similar computation, we have

$$\begin{aligned}\widetilde{D} = \operatorname{diag}(&1 - a_{n1\ldots1}(a_{1n\ldots n} + \delta_1), 1 - a_{n2\ldots2}(a_{2n\ldots n} + \delta_2), \ldots, \\ &1 - a_{n(n-1)\ldots(n-1)}(a_{(n-1)n\ldots n} + \delta_{n-1}), 1),\end{aligned}$$

$$\widetilde{L} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -a_{21\ldots1} + a_{n1\ldots1}(a_{2n\ldots n} + \delta_2) & 0 & 0 & 0 & 0 \\ -a_{31\ldots1} + a_{n1\ldots1}(a_{3n\ldots n} + \delta_3) & -a_{32\ldots2} + a_{n2\ldots2}(a_{3n\ldots n} + \delta_3) & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & 0 \\ -a_{n1\ldots1} & -a_{n2\ldots2} & \cdots & -a_{n(n-1)\ldots(n-1)} & 0 \end{pmatrix}.$$

Similarly, it can be proved that $\mathcal{T}_3(\omega)$ and $\mathcal{T}_4(\omega)$ are nonnegative and irreducible tensors. $\square$

3.2 The comparison theorem

In this section, we will discuss the properties for the spectral radius of the four iterative tensors.

Theorem 3.2. *Let $\mathcal{T}(\omega)$ and $\mathcal{T}_1(\omega)$ be defined by* (1.4) *and* (3.1)*, respectively. If $\mathcal{A}$ is an irreducible $\mathcal{L}$-tensor with $a_{1q\ldots q}a_{q1\ldots1} > 0$, and $\gamma_q \in ((1 - a_{1q\ldots q}a_{q1\ldots1})/a_{1q\ldots q}, -a_{q1\ldots1}) \cap (0, -a_{q1\ldots1})$, $q = 2, 3, \ldots n$, then for $0 < \omega < 1$, one of the following statements holds:*

(1) $\rho(\mathcal{T}_1(\omega)) \le \rho(\mathcal{T}(\omega)) < 1$;

(2) $\rho(\mathcal{T}_1(\omega)) = \rho(\mathcal{T}(\omega)) = 1$;

(3) $\rho(\mathcal{T}_1(\omega)) \ge \rho(\mathcal{T}(\omega)) > 1$.

Proof. From Theorem 3.1 (i), we know that $\mathcal{T}(\omega)$ is an irreducible and nonnegative tensor. Thus, by Lemma 2.18, there is a positive vector $\mathbf{z} \in \mathbb{R}^n$, such that

$$\mathcal{T}(\omega)\mathbf{z}^{m-1} = \chi\mathbf{z}^{[m-1]},$$

where $\chi = \rho(\mathcal{T}(\omega))$. By the definition of $\mathcal{T}(\omega)$, we get

$$((1-\omega)\mathcal{I} + \omega\mathcal{F})\mathbf{z}^{m-1} = \chi(\mathcal{I} - \omega\mathcal{L})\mathbf{z}^{m-1}. \tag{3.6}$$

Then

$$
\begin{aligned}
&\mathcal{T}_1(\omega)\mathbf{z}^{m-1}-\chi\mathbf{z}^{[m-1]}\\
=&(\overline{D}-\omega\overline{L})^{-1}[(1-\omega)\overline{\mathcal{D}}+\omega\overline{\mathcal{F}}]\mathbf{z}^{m-1}-\chi\mathcal{I}\mathbf{z}^{m-1}\\
=&(\overline{D}-\omega\overline{L})^{-1}[(1-\omega)\overline{\mathcal{D}}+\omega\overline{\mathcal{F}}-\chi(\overline{\mathcal{D}}-\omega\overline{\mathcal{L}})]\mathbf{z}^{m-1}\\
=&(\overline{D}-\omega\overline{L})^{-1}[(1-\omega)(\mathcal{I}+\overline{\mathcal{D}}_1)+\omega(\mathcal{F}+\overline{\mathcal{F}}_1)-\chi(\mathcal{I}+\overline{\mathcal{D}}_1)+\chi\omega(\mathcal{L}-\overline{S}\mathcal{I}+\overline{\mathcal{L}}_1)]\mathbf{z}^{m-1}\\
=&(\overline{D}-\omega\overline{L})^{-1}[(1-\omega-\chi)\overline{\mathcal{D}}_1+\chi\omega(\overline{\mathcal{L}}_1-\overline{S}\mathcal{I})+\omega\overline{\mathcal{F}}_1]\mathbf{z}^{m-1}\\
=&(\overline{D}-\omega\overline{L})^{-1}[(1-\chi)\overline{\mathcal{D}}_1+\chi\omega(\overline{\mathcal{L}}_1-\overline{S}\mathcal{I})-\omega\overline{\mathcal{L}}_1+\omega\overline{S}\mathcal{F}]\mathbf{z}^{m-1}\\
=&(\overline{D}-\omega\overline{L})^{-1}[(1-\chi)\overline{\mathcal{D}}_1+(\chi-1)\omega\overline{\mathcal{L}}_1-\chi\omega\overline{S}\mathcal{I}+(\chi+\omega-1)\overline{S}\mathcal{I}]\mathbf{z}^{m-1}\\
=&(\overline{D}-\omega\overline{L})^{-1}[(\chi-1)(-\overline{\mathcal{D}}_1+\omega\overline{\mathcal{L}}_1)+(\chi-1)(1-\omega)\overline{S}\mathcal{I}]\mathbf{z}^{m-1}\\
=&(\chi-1)(\overline{D}-\omega\overline{L})^{-1}[-\overline{\mathcal{D}}_1+\omega\overline{\mathcal{L}}_1+(1-\omega)\overline{S}\mathcal{I}]\mathbf{z}^{m-1},
\end{aligned}
$$

where the third equation is because $\overline{\mathcal{D}}=\mathcal{I}+\overline{\mathcal{D}}_1, \overline{\mathcal{L}}=\mathcal{L}-\overline{S}\mathcal{I}+\overline{\mathcal{L}}_1, \overline{\mathcal{F}}=\mathcal{F}+\overline{\mathcal{F}}_1$; the fourth equation is by the equation (3.6); the fifth equation is by $\overline{\mathcal{F}}_1=\overline{S}\mathcal{F}+\overline{\mathcal{D}}_1-\overline{\mathcal{L}}_1$; the sixth equation is by (3.6) and $\overline{S}\mathcal{L}=\mathcal{O}$, that is, $\omega\overline{S}\mathcal{F}\mathbf{z}^{m-1}=(\chi\overline{S}\mathcal{I}-\chi\omega\overline{S}\mathcal{L})\mathbf{z}^{m-1}-(1-\omega)\overline{S}\mathcal{I}\mathbf{z}^{m-1}=(\chi+\omega-1)\overline{S}\mathcal{I}\mathbf{z}^{m-1}$, and the other equations are by simple computations. Since $\mathbf{z}>\mathbf{0}$, $-\overline{D}_1\geq O$, $\overline{L}_1\geq O$ and $\overline{S}\geq O$, $[-\overline{\mathcal{D}}_1+\omega\overline{\mathcal{L}}_1+(1-\omega)\overline{S}\mathcal{I}]\mathbf{z}^{m-1}\geq\mathbf{0}$. As $\overline{D}\geq O$ and is invertible, $(\overline{D}-\omega\overline{L})^{-1}=(I-\omega\overline{D}^{-1}\overline{L})^{-1}\overline{D}^{-1}=[I+\omega\overline{D}^{-1}\overline{L}+\omega^2(\overline{D}^{-1}\overline{L})^2+\cdots+\omega^{n-1}(\overline{D}^{-1}\overline{L})^{n-1}]\overline{D}^{-1}\geq O$. Together with the above two results, we have $(\overline{D}-\omega\overline{L})^{-1}[-\overline{\mathcal{D}}_1+\omega\overline{\mathcal{L}}_1+(1-\omega)\overline{S}\mathcal{I}]\mathbf{z}^{m-1}\geq\mathbf{0}$. Thus,

(1) If $\chi<1$, then $\mathcal{T}_1(\omega)\mathbf{z}^{m-1}-\chi\mathbf{z}^{[m-1]}\leq\mathbf{0}$, i.e., $\mathcal{T}_1(\omega)\mathbf{z}^{m-1}\leq\chi\mathbf{z}^{[m-1]}$. By Lemma 2.19, we have $\rho(\mathcal{T}_1(\omega))\leq\chi=\rho(\mathcal{T}(\omega))$;

(2) If $\chi=1$, then $\mathcal{T}_1(\omega)\mathbf{z}^{m-1}-\chi\mathbf{z}^{[m-1]}=\mathbf{0}$, i.e., $\mathcal{T}_1(\omega)\mathbf{z}^{m-1}=\chi\mathbf{z}^{[m-1]}$. By Lemma 2.19, we have $\rho(\mathcal{T}_1(\omega))=\chi=\rho(\mathcal{T}(\omega))$;

(3) If $\chi>1$, then $\mathcal{T}_1(\omega)\mathbf{z}^{m-1}-\chi\mathbf{z}^{[m-1]}\geq\mathbf{0}$, i.e., $\mathcal{T}_1(\omega)\mathbf{z}^{m-1}\geq\chi\mathbf{z}^{[m-1]}$. By Lemma 2.19, we have $\rho(\mathcal{T}_1(\omega))\geq\chi=\rho(\mathcal{T}(\omega))$.

The proof is finished. □

Theorem 3.3. *Let $\mathcal{T}(\omega)$ and $\mathcal{T}_2(\omega)$ be defined by* (1.4) *and* (3.2)*, respectively. If $\mathcal{A}$ is an irreducible $\mathcal{L}$-tensor with $a_{1q\ldots q}a_{q1\ldots 1}>0$, and $\gamma_q\in((1-a_{1q\ldots q}a_{q1\ldots 1})/a_{1q\ldots q},-a_{q1\ldots 1})\cap(0,-a_{q1\ldots 1})$, $q=2,3,\ldots n$, then for $0<\omega<1$, one of the following statements holds:*

(1) $\rho(\mathcal{T}_2(\omega))\leq\rho(\mathcal{T}(\omega))<1$*;*

(2) $\rho(\mathcal{T}_2(\omega))=\rho(\mathcal{T}(\omega))=1$*;*

(3) $\rho(\mathcal{T}_2(\omega))\geq\rho(\mathcal{T}(\omega))>1$*.*

Proof. By a similar discussion as the proof of Theorem 3.2, we can get

$$\mathcal{T}_2(\omega)\mathbf{z}^{m-1}-\chi\mathbf{z}^{[m-1]}=(\chi-1)(I-\omega\overline{L})^{-1}[\omega\overline{\mathcal{L}}_1+(1-\omega)\overline{S}\mathcal{I}]\mathbf{z}^{m-1},$$

and it also can be shown that $(I-\omega\overline{L})^{-1}[\omega\overline{\mathcal{L}}_1+(1-\omega)\overline{S}\mathcal{I}]\mathbf{z}^{m-1}\geq 0$. Therefore,

(1) If $\chi < 1$, then $\mathcal{T}_2(\omega)\mathbf{z}^{m-1} - \chi\mathbf{z}^{[m-1]} \leq \mathbf{0}$, i.e., $\mathcal{T}_2(\omega)\mathbf{z}^{m-1} \leq \chi\mathbf{z}^{[m-1]}$. By Lemma 2.19, we have $\rho(\mathcal{T}_2(\omega)) \leq \chi = \rho(\mathcal{T}(\omega))$;

(2) If $\chi = 1$, then $\mathcal{T}_2(\omega)\mathbf{z}^{m-1} - \chi\mathbf{z}^{[m-1]} = \mathbf{0}$, i.e., $\mathcal{T}_2(\omega)\mathbf{z}^{m-1} = \chi\mathbf{z}^{[m-1]}$. By Lemma 2.19, we have $\rho(\mathcal{T}_2(\omega)) = \chi = \rho(\mathcal{T}(\omega))$;

(3) If $\chi > 1$, then $\mathcal{T}_2(\omega)\mathbf{z}^{m-1} - \chi\mathbf{z}^{[m-1]} \geq \mathbf{0}$, i.e., $\mathcal{T}_2(\omega)\mathbf{z}^{m-1} \geq \chi\mathbf{z}^{[m-1]}$. By Lemma 2.19, we have $\rho(\mathcal{T}_2(\omega)) \geq \chi = \rho(\mathcal{T}(\omega))$.

$\square$

Theorem 3.4. *Let $\mathcal{T}(\omega)$ and $\mathcal{T}_3(\omega)$ be defined by* (1.4) *and* (3.3)*, respectively. If $\mathcal{A}$ is an irreducible $\mathcal{L}$-tensor with $a_{ns\ldots s}a_{sn\ldots n} > 0$, and $\delta_s \in ((1 - a_{ns\ldots s}a_{sn\ldots n})/a_{ns\ldots s}, -a_{sn\ldots n}) \cap (0, -a_{sn\ldots n})$, $s = 1, 2, \ldots, n-1$, then for $0 < \omega < 1$, one of the following statements holds:*

(1) $\rho(\mathcal{T}_3(\omega)) \leq \rho(\mathcal{T}(\omega)) < 1$;

(2) $\rho(\mathcal{T}_3(\omega)) = \rho(\mathcal{T}(\omega)) = 1$;

(3) $\rho(\mathcal{T}_3(\omega)) \geq \rho(\mathcal{T}(\omega)) > 1$.

Proof. By the definition of $\mathcal{T}_3(\omega)$, we can write

$$
\begin{aligned}
&\mathcal{T}_3(\omega)\mathbf{z}^{m-1} - \chi\mathbf{z}^{[m-1]}\\
=&(\widetilde{D} - \omega\widetilde{L})^{-1}[(1-\omega)\widetilde{\mathcal{D}} + \omega\widetilde{\mathcal{F}}]\mathbf{z}^{m-1} - \chi\mathbf{z}^{[m-1]}\\
=&(\widetilde{D} - \omega\widetilde{L})^{-1}[(1-\omega)\widetilde{\mathcal{D}} + \omega\widetilde{\mathcal{F}} - \chi(\widetilde{\mathcal{D}} - \omega\widetilde{\mathcal{L}})]\mathbf{z}^{m-1}\\
=&(\widetilde{D} - \omega\widetilde{L})^{-1}[(1-\omega)(\mathcal{I} + \widetilde{\mathcal{D}}_1) + \omega(\mathcal{F} + \widetilde{\mathcal{F}}_1) - \chi(\mathcal{I} + \widetilde{\mathcal{D}}_1) + \chi\omega(\mathcal{L} + \widetilde{\mathcal{L}}_1)]\mathbf{z}^{m-1}\\
=&(\widetilde{D} - \omega\widetilde{L})^{-1}[(1-\omega)\mathcal{I} + \omega\mathcal{F} + (1-\omega-\chi)\widetilde{\mathcal{D}}_1 + \omega\widetilde{\mathcal{F}}_1 - \chi(\mathcal{I} - \omega\mathcal{L}) + \chi\omega\widetilde{\mathcal{L}}_1]\mathbf{z}^{m-1}\\
=&(\widetilde{D} - \omega\widetilde{L})^{-1}[(1-\omega-\chi)\widetilde{\mathcal{D}}_1 + \omega\widetilde{\mathcal{F}}_1 + \chi\omega\widetilde{\mathcal{L}}_1]\mathbf{z}^{m-1}\\
=&(\widetilde{D} - \omega\widetilde{L})^{-1}[\omega(-\widetilde{\mathcal{D}}_1 + \widetilde{\mathcal{L}}_1 + \widetilde{\mathcal{F}}_1) - (\chi-1)\widetilde{\mathcal{D}}_1 + \omega(\chi-1)\widetilde{\mathcal{L}}_1]\mathbf{z}^{m-1}\\
=&(\widetilde{D} - \omega\widetilde{L})^{-1}[-\omega\widetilde{S}\mathcal{A} - (\chi-1)\widetilde{\mathcal{D}}_1 + \omega(\chi-1)\widetilde{\mathcal{L}}_1]\mathbf{z}^{m-1}\\
=&\frac{\chi-1}{\chi}(\widetilde{D} - \omega\widetilde{L})^{-1}[(1-\omega)\widetilde{S}\mathcal{I} + \omega\widetilde{S}\mathcal{F} - \chi\widetilde{\mathcal{D}}_1 + \omega\chi\widetilde{\mathcal{L}}_1]\mathbf{z}^{m-1},
\end{aligned}
$$

where the second equation is because $\widetilde{\mathcal{D}} = \widetilde{D}\mathcal{I}$, $\widetilde{\mathcal{L}} = \widetilde{L}\mathcal{I}$; the third equation is by $\widetilde{\mathcal{D}} = \mathcal{I} + \widetilde{\mathcal{D}}_1$, $\widetilde{\mathcal{L}} = \mathcal{L} + \widetilde{\mathcal{L}}_1$ and $\widetilde{\mathcal{F}} = \mathcal{F} + \widetilde{\mathcal{F}}_1$; the fifth equation is by the equation (3.6); the seventh equation is by $\widetilde{S}\mathcal{A} = \widetilde{\mathcal{D}}_1 - \widetilde{\mathcal{L}}_1 - \widetilde{\mathcal{F}}_1$; and the eighth equation is because $-\omega\widetilde{S}\mathcal{A}\mathbf{z}^{m-1} = \widetilde{S}(\omega\mathcal{I} - \omega\mathcal{L} - \omega\mathcal{F})\mathbf{z}^{m-1} = -\widetilde{S}[(\mathcal{I} - \omega\mathcal{L})\mathbf{z}^{m-1} - ((1-\omega)\mathcal{I} + \omega\mathcal{F})\mathbf{z}^{m-1}] = -(\frac{1}{\chi} - 1)\widetilde{S}[(1-\omega)\mathcal{I} + \omega\mathcal{F}]\mathbf{z}^{m-1} = \frac{\chi-1}{\chi}[(1-\omega)\widetilde{S}\mathcal{I} + \omega\widetilde{S}\mathcal{F}]\mathbf{z}^{m-1}$.

By a similar proof as Theorem 3.2, we can prove that $(\widetilde{D} - \omega\widetilde{L})^{-1}[(1-\omega)\widetilde{S}\mathcal{I} + \omega\widetilde{S}\mathcal{F} - \chi\widetilde{\mathcal{D}}_1 + \omega\chi\widetilde{\mathcal{L}}_1]\mathbf{z}^{m-1} \geq \mathbf{0}$. Hence

(1) If $\chi < 1$, then $\mathcal{T}_3(\omega)\mathbf{z}^{m-1} - \chi\mathbf{z}^{[m-1]} \leq \mathbf{0}$, i.e., $\mathcal{T}_3(\omega)\mathbf{z}^{m-1} \leq \chi\mathbf{z}^{[m-1]}$. By Lemma 2.19, we have $\rho(\mathcal{T}_3(\omega)) \leq \chi = \rho(\mathcal{T}(\omega))$;

(2) If $\chi = 1$, then $\mathcal{T}_3(\omega)\mathbf{z}^{m-1} - \chi\mathbf{z}^{[m-1]} = \mathbf{0}$, i.e., $\mathcal{T}_3(\omega)\mathbf{z}^{m-1} = \chi\mathbf{z}^{[m-1]}$. By Lemma 2.19, we have $\rho(\mathcal{T}_3(\omega)) = \chi = \rho(\mathcal{T}(\omega))$;

(3) If $\chi > 1$, then $\mathcal{T}_3(\omega)\mathbf{z}^{m-1} - \chi\mathbf{z}^{[m-1]} \geq \mathbf{0}$, i.e., $\mathcal{T}_3(\omega)\mathbf{z}^{m-1} \geq \chi\mathbf{z}^{[m-1]}$. By Lemma 2.19, we have $\rho(\mathcal{T}_3(\omega)) \geq \chi = \rho(\mathcal{T}(\omega))$.

□

Theorem 3.5. *Let $\mathcal{T}(\omega)$ and $\mathcal{T}_4(\omega)$ be defined by (1.4) and (3.4), respectively. If $\mathcal{A}$ is an irreducible $\mathcal{L}$-tensor with $a_{ns\ldots s}a_{sn\ldots n} > 0$, and $\delta_s \in ((1 - a_{ns\ldots s}a_{sn\ldots n})/a_{ns\ldots s}, -a_{sn\ldots n}) \cap (0, -a_{sn\ldots n})$, $s = 1, 2, \ldots, n-1$, then for $0 < \omega < 1$, one of the following statements holds:*

(1) $\rho(\mathcal{T}_4(\omega)) \leq \rho(\mathcal{T}(\omega)) < 1$;

(2) $\rho(\mathcal{T}_4(\omega)) = \rho(\mathcal{T}(\omega)) = 1$;

(3) $\rho(\mathcal{T}_4(\omega)) \geq \rho(\mathcal{T}(\omega)) > 1$.

Proof. By a similar proof as Theorem 3.4, we can get

$$\mathcal{T}_4(\omega)\mathbf{z}^{m-1} - \chi\mathbf{z}^{[m-1]} = \frac{(\chi-1)}{\chi}(\widetilde{D} - \omega\widetilde{L})^{-1}[(1-\omega)\widetilde{S}\mathcal{I} + \omega\widetilde{S}\mathcal{F} + \omega\chi\widetilde{\mathcal{L}}_1]\mathbf{z}^{m-1},$$

and $(\widetilde{D} - \omega\widetilde{L})^{-1}[(1-\omega)\widetilde{S}\mathcal{I} + \omega\widetilde{S}\mathcal{F} - \chi\widetilde{\mathcal{D}}_1 + \omega\chi\widetilde{\mathcal{L}}_1]\mathbf{z}^{m-1} \geq \mathbf{0}$. Thus,

(1) If $\chi < 1$, then $\mathcal{T}_4(\omega)\mathbf{z}^{m-1} - \chi\mathbf{z}^{[m-1]} \leq \mathbf{0}$, i.e., $\mathcal{T}_4(\omega)\mathbf{z}^{m-1} \leq \chi\mathbf{z}^{[m-1]}$. By Lemma 2.19, we have $\rho(\mathcal{T}_4(\omega)) \leq \chi = \rho(\mathcal{T}(\omega))$;

(2) If $\chi = 1$, then $\mathcal{T}_4(\omega)\mathbf{z}^{m-1} - \chi\mathbf{z}^{[m-1]} = \mathbf{0}$, i.e., $\mathcal{T}_4(\omega)\mathbf{z}^{m-1} = \chi\mathbf{z}^{[m-1]}$. By Lemma 2.19, we have $\rho(\mathcal{T}_4(\omega)) = \chi = \rho(\mathcal{T}(\omega))$;

(3) If $\chi > 1$, then $\mathcal{T}_4(\omega)\mathbf{z}^{m-1} - \chi\mathbf{z}^{[m-1]} \geq \mathbf{0}$, i.e., $\mathcal{T}_4(\omega)\mathbf{z}^{m-1} \geq \chi\mathbf{z}^{[m-1]}$. By Lemma 2.19, we have $\rho(\mathcal{T}_4(\omega)) \geq \chi = \rho(\mathcal{T}(\omega))$.

□

3.3 The comparison of the convergence rate of the SOR methods

In this section, we mainly discuss the comparison of the convergence rate of the SOR iterative methods, specifically, the comparison of the spectral radius of $\mathcal{T}_1(\omega)$ and $\mathcal{T}_2(\omega)$, $\mathcal{T}_3(\omega)$ and $\mathcal{T}_4(\omega)$.

Lemma 3.6 ([17])**.** *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $\mathcal{A} = \mathcal{E}_1 - \mathcal{F}_1 = \mathcal{E}_2 - \mathcal{F}_2$ be a regular splitting and a weak regular splitting respectively, and $\mathcal{F}_1 \leq \mathcal{F}_2, \mathcal{F}_2 \neq \mathcal{O}$. One of the following statements holds:*

(1) $\rho(M(\mathcal{E}_2)^{-1}\mathcal{F}_2) \leq \rho(M(\mathcal{E}_1)^{-1}\mathcal{F}_1) < 1$.

(2) $\rho(M(\mathcal{E}_2)^{-1}\mathcal{F}_2) \geq \rho(M(\mathcal{E}_1)^{-1}\mathcal{F}_1) \geq 1$. *If $\mathcal{F}_1 < \mathcal{F}_2, \mathcal{F}_1 \neq \mathcal{O}$ and $\rho(M(\mathcal{E}_1)^{-1}\mathcal{F}_1) > 1$, the first inequality is strict.*

Theorem 3.7. *Let $\mathcal{T}_1(\omega)$ and $\mathcal{T}_2(\omega)$ be defined by (3.1) and (3.2), respectively. If $\mathcal{A}$ is an irreducible $\mathcal{L}$-tensor with $a_{1q\ldots q}a_{q1\ldots 1} > 0$, and $\gamma_q \in ((1 - a_{1q\ldots q}a_{q1\ldots 1})/a_{1q\ldots q}, -a_{q1\ldots 1}) \cap (0, -a_{q1\ldots 1})$ for $q = 2, 3, \ldots n$, then for $0 < \omega < 1$, one of the following statements holds:*

(1) $\rho(\mathcal{T}_1(\omega)) \leq \rho(\mathcal{T}_2(\omega)) < 1$;

(2) $\rho(\mathcal{T}_1(\omega)) \geq \rho(\mathcal{T}_2(\omega)) \geq 1$.

Proof. By the proof of Theorem 3.1, we know that the two splittings $\omega\overline{\mathcal{A}} = \mathcal{E}_1 - \mathcal{F}_1 = \mathcal{E}_2 - \mathcal{F}_2$ are both weak regular splittings. Next, we need show that $\mathcal{F}_1 \leq \mathcal{F}_2$, and $\mathcal{F}_2 \neq \mathcal{O}$. We see that $\mathcal{F}_2 - \mathcal{F}_1 = -\overline{\mathcal{D}}_1 = -\overline{D}_1\mathcal{I}$, where $\overline{D}_1 = \overline{D} - I$. By (3.5), we have

$$\overline{D}_1 = \mathrm{diag}(0, -a_{12\ldots2}(a_{21\ldots1} + \gamma_2), -a_{13\ldots3}(a_{31\ldots1} + \gamma_3), \ldots, -a_{1n\ldots n}(a_{n1\ldots1} + \gamma_n)).$$

Under the conditions $a_{1q\ldots q}a_{q1\ldots1} > 0, \gamma_q \in (0, -a_{q1\ldots1})$, we get $-a_{1q\ldots q}(a_{q1\ldots1} + \gamma_q) > 0$ for $q = 2, 3, \ldots n$, that is, $\overline{D}_1 \leq O$ and $\overline{D}_1 \neq O$. Therefore $\mathcal{F}_1 \leq \mathcal{F}_2$, together with $\mathcal{F}_1 \geq \mathcal{O}$, then $\mathcal{F}_2 \neq \mathcal{O}$. The proof is completed. $\square$

By a similar analysis, we can get the following theorem.

Theorem 3.8. *Let $\mathcal{T}_3(\omega), \mathcal{T}_4(\omega)$ be defined by* (3.3) *and* (3.4)*. If $\mathcal{A}$ is an irreducible $\mathcal{L}$-tensor with $a_{ns\ldots s}a_{sn\ldots n} > 0$, and $\delta_s \in ((1 - a_{ns\ldots s}a_{sn\ldots n})/a_{ns\ldots s}, -a_{sn\ldots n}) \cap (0, -a_{sn\ldots n})$, for $s = 1, 2, \ldots, n-1$, then for $0 < \omega < 1$, one of the following statements holds:*

(1) $\rho(\mathcal{T}_3(\omega)) \leq \rho(\mathcal{T}_4(\omega)) < 1$;

(2) $\rho(\mathcal{T}_3(\omega)) \geq \rho(\mathcal{T}_4(\omega)) \geq 1$.

4 Numerical Examples

In this section, we do some numerical experiments to illustrate the theory results. By the iteration tensors $\mathcal{T}_1(\omega)$ and $\mathcal{T}_2(\omega)$, which are defined by (3.1) and (3.2), respectively, solving the systems (1.1) is equivalent to solving

$$\mathbf{x} = [\mathcal{T}_i(\omega)\mathbf{x}^{m-1} + \omega M(\mathcal{E}_i)^{-1}\overline{\mathbf{b}}]^{[\frac{1}{m-1}]}, i = 1, 2.$$

In this section, we use the following iterative method for solving (1.1): for a given initial vector $\mathbf{x}_0$,

$$\mathbf{x}_k = [\mathcal{T}_i(\omega)\mathbf{x}_{k-1}^{m-1} + \omega M(\mathcal{E}_i)^{-1}\overline{\mathbf{b}}]^{[\frac{1}{m-1}]}, k = 1, 2, \ldots, i = 1, 2.$$

For the iteration tensors $\mathcal{T}_3(\omega)$ and $\mathcal{T}_4(\omega)$, which are defined by (3.3) and (3.4), respectively, the iterative method for solving (1.1) is: for a given initial vector $\mathbf{x}_0$,

$$\mathbf{x}_k = [\mathcal{T}_i(\omega)\mathbf{x}_{k-1}^{m-1} + \omega M(\mathcal{E}_i)^{-1}\widetilde{\mathbf{b}}]^{[\frac{1}{m-1}]}, k = 1, 2, \ldots, i = 3, 4.$$

In the section, all tests of the examples were done in Matlab R2014b and Tensor Toolbox 2.6. The codes were done on a DELL desktop with Inter(R) Core(TM) i5-5200U CPU 2.20 GHz and 4GB RAM running on Windows 7.

Example 4.1 ([4])**.** Let $\mathcal{A} \in \mathbb{R}^{[3,3]}$ be as follows,

$$\mathcal{A} = \left(\begin{array}{ccc|ccc|ccc} 1 & -0.12 & -0.13 & -0.04 & -0.02 & -0.03 & -0.03 & -0.02 & -0.04 \\ -0.12 & -0.03 & -0.06 & -0.01 & 1 & -0.02 & -0.02 & -0.06 & -0.03 \\ -0.13 & -0.02 & -0.10 & -0.03 & -0.04 & -0.02 & -0.02 & -0.10 & 1 \end{array}\right).$$

It is obvious that $\mathcal{A}$ in Example 4.1 is an irreducible $\mathcal{L}$-tensor with $a_{122}a_{211} > 0$, $a_{133}a_{311} > 0$, $a_{311}a_{133} > 0$ and $a_{322}a_{233} > 0$. For this example, we choose $\gamma_i = 0.005$, $\gamma_i = 0.0005$, $\delta_j = 0.005$ and $\delta_j = 0.0005$ ($i = 2, 3$ and $j = 1, 2$). It is easy to see that parameters γ_i, δ_j satisfy the initial conditions in Theorems 3.2-3.5, respectively. We set the parameter $\omega = 0.2$, 0.4, 0.6, 0.8, 1.0. The numerical results are shown in the Tab.1. The results show that

(i) when $\rho(\mathcal{T}(\omega)) < 1$, $\rho(\mathcal{T}_i(\omega)) \leq \rho(\mathcal{T}(\omega)), i = 1, 2, 3, 4$, which certify the result in case (1) of Theorems 3.2-3.5.

(ii) when $\rho(\mathcal{T}_2(\omega)) < 1$, $\rho(\mathcal{T}_1(\omega)) \leq \rho(\mathcal{T}_2(\omega))$, which certifies the result in case (1) of Theorem 3.7; when $\rho(\mathcal{T}_4(\omega)) < 1$, $\rho(\mathcal{T}_3(\omega)) \leq \rho(\mathcal{T}_4(\omega))$, which certifies the result in case (1) of Theorem 3.8.

Also, we plot $\mathcal{T}(\omega)$, $\mathcal{T}_3(\omega)$ and $\mathcal{T}_4(\omega)$ when $\delta_j = 0.005$ in Figure 1, which shows the relations between them.

Table 1: Numerical results for Example 4.1 with different preconditioners.

		$\gamma_i = 0.005$		$\gamma_i = 0.0005$		$\delta_j = 0.005$		$\delta_j = 0.0005$	
ω	$\rho(\mathcal{T}(\omega))$	$\rho(\mathcal{T}_1(\omega))$	$\rho(\mathcal{T}_2(\omega))$	$\rho(\mathcal{T}_1(\omega))$	$\rho(\mathcal{T}_2(\omega))$	$\rho(\mathcal{T}_3(\omega))$	$\rho(\mathcal{T}_4(\omega))$	$\rho(\mathcal{T}_3(\omega))$	$\rho(\mathcal{T}_4(\omega))$
0.2	0.8815	0.8725	0.8728	0.8721	0.8724	0.8783	0.8785	0.8777	0.8781
0.4	0.7586	0.7441	0.7448	0.7435	0.7442	0.7523	0.7529	0.7513	0.7520
0.6	0.6306	0.6149	0.6159	0.6142	0.6153	0.6214	0.6223	0.6199	0.6210
0.8	0.4968	0.4847	0.4860	0.4842	0.4856	0.4848	0.4861	0.4829	0.4843
1.0	0.3558	0.3536	0.3553	0.3535	0.3552	0.3412	0.3428	0.3388	0.3407

Table 2: Numerical results for Example 4.2 with different preconditioners.

		$\gamma_2 = 0.02$		$\gamma_2 = 0.002$		$\delta_1 = 0.02$		$\delta_1 = 0.002$	
ω	$\rho(\mathcal{T}(\omega))$	$\rho(\mathcal{T}_1(\omega))$	$\rho(\mathcal{T}_2(\omega))$	$\rho(\mathcal{T}_1(\omega))$	$\rho(\mathcal{T}_2(\omega))$	$\rho(\mathcal{T}_3(\omega))$	$\rho(\mathcal{T}_4(\omega))$	$\rho(\mathcal{T}_3(\omega))$	$\rho(\mathcal{T}_4(\omega))$
0.1	1.2563	1.5432	1.3701	1.5521	1.3721	1.5878	1.3894	1.5986	1.3919
0.3	1.8068	2.5685	2.0728	2.5922	2.0777	2.9607	2.2448	3.0004	2.2531
0.5	2.4085	3.5121	2.7257	3.5463	2.7316	4.5763	3.1967	4.6534	3.2118
0.7	3.0620	4.3736	3.3287	4.4142	3.3336	6.4122	4.2393	6.5335	4.2620
0.9	3.7675	5.1531	3.8819	5.1959	3.8840	8.4489	5.3668	8.6204	5.3977

Example 4.2. In this example, we choose the system tensor $\mathcal{A} \in \mathbb{R}^{[3,2]}$ shown in Example 2.11, whose elements are given by,

$$a_{111} = 2, a_{121} = -3, a_{112} = -3, a_{122} = -1,$$

$$a_{211} = -1, a_{221} = -3, a_{212} = -3, a_{222} = 2.$$

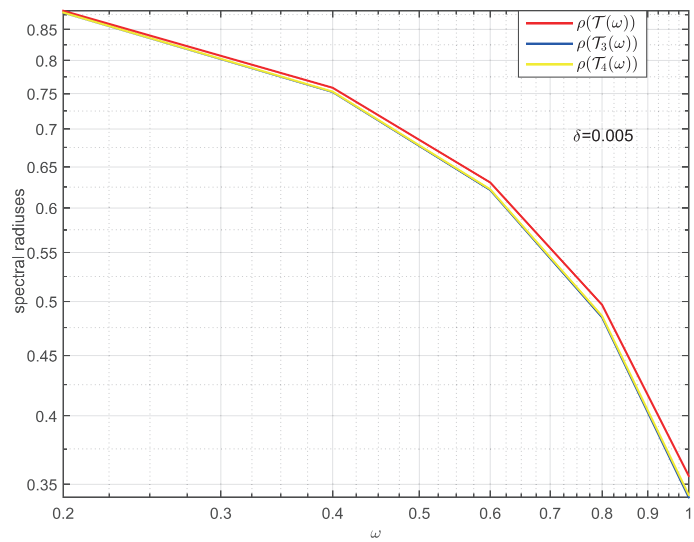

Figure 1: Plot of the different spectral radiuses $\rho(\mathcal{T}(\omega))$, $\rho(\mathcal{T}_3(\omega))$, $\rho(\mathcal{T}_4(\omega))$ against ω with $\delta = 0.005$.

We already know the tensor $\mathcal{A}$ is an irreducible $\mathcal{L}$-tensor, and not a strong $\mathcal{M}$-tensor. We choose $\gamma_2 = 0.02$, $\gamma_2 = 0.002$, $\delta_1 = 0.02$ and $\delta_1 = 0.002$, which satisfy the conditions in Theorems 3.2-3.5, and the parameter $\omega = 0.1, 0.3, 0.5, 0.7, 0.9$. Then the calculation results are shown in the Tab.2 and the relations between $\mathcal{T}(\omega)$, $\mathcal{T}_1(\omega)$ and $\mathcal{T}_2(\omega)$ with $\gamma_2 = 0.02$ are shown in Figure 2. From the results in Tab.2, we can see that

(i) when $\rho(\mathcal{T}(\omega)) > 1$, $\rho(\mathcal{T}_i(\omega)) \geq \rho(\mathcal{T}(\omega)), i = 1, 2, 3, 4$, which certify the results in case (3) of Theorems 3.2-3.5.

(ii) when $\rho(\mathcal{T}_2(\omega)) > 1$, $\rho(\mathcal{T}_1(\omega)) \geq \rho(\mathcal{T}_2(\omega))$, which certifies the result in case (3) of Theorem 3.7; when $\rho(\mathcal{T}_4(\omega)) > 1$, $\rho(\mathcal{T}_3(\omega)) \geq \rho(\mathcal{T}_4(\omega))$, which certifies the result in case (3) of Theorem 3.8.

5 Conclusions

In this paper, we proposed new SOR methods with four iterative tensors engendered by two new preconditioners for solving the multi-linear systems (1.1) with $\mathcal{A}$ being an irreducible $\mathcal{L}$-tensor. We also compared the spectral radius of the four iterative tensors. The numerical experiments show the efficiency of the proposed methods. There are many tensor splitting methods with preconditioners already proposed to solve (1.1) with $\mathcal{A}$ being an $\mathcal{M}$-tensor. Can the methods be applied to solve (1.1) with $\mathcal{A}$ being an irreducible $\mathcal{L}$-tensor, or just an $\mathcal{L}$-tensor? These are the topics we will consider in the future.

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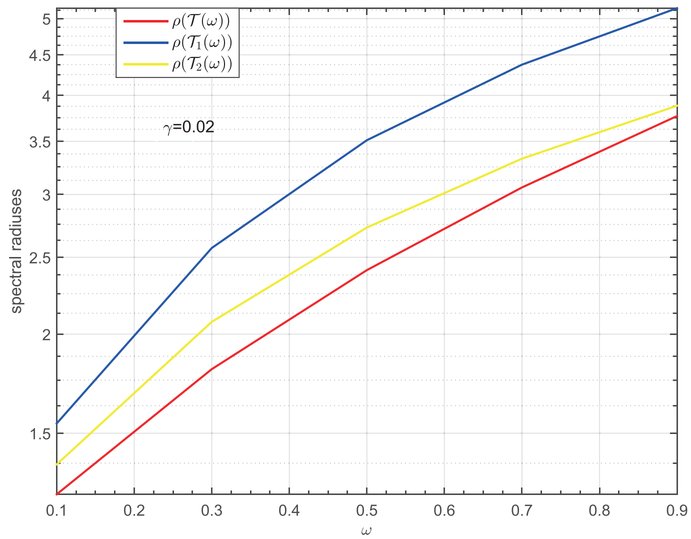

Figure 2: Plot of the different spectral radiuses $\rho(\mathcal{T}(\omega))$, $\rho(\overline{\mathcal{T}}_1(\omega))$, $\rho(\overline{\mathcal{T}}_2(\omega))$ against ω with $\gamma = 0.02$.

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