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A SIMPLE METHOD FOR COMPUTING THE VANDERMONDE DECOMPOSITION OF A HANKEL TENSOR*

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Abstract: This paper aims to compute the exact Vandermonde decomposition of a Hankel tensor. We first reformulate the Hankel tensor decomposition problem as the system of nonlinear equations. Then we design a new direct method to solve the system of nonlinear equations by a series of matrix transformations. Finally, Some numerical examples illustrate that the new method is feasible and effective.

Key words: *Hankel tensor; Vandermonde decomposition; System of nonlinear equations*

Mathematics Subject Classification: *15A30, 15A69*

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1 Introduction

Throughout this paper, to distinguish scalars, vectors, matrices and higher-order tensors, scalars will be denoted by lower-case Greek letters, e.g. α, β, vectors will be denoted by lower-case bold-faced letters, e.g. $\mathbf{v}, \mathbf{w}$, matrices will be defined by capital bold-faced letters, e.g. $\mathbf{A}, \mathbf{B}$ and higher-order tensors will be denoted by calligraphic letters, e.g. $\mathcal{A}, \mathcal{B}$. Let $\mathbb{R}$ denote the set of real numbers. Let $\mathbb{R}^n$ represents an n-dimensional real vector space. The set of positive integers is represented by the symbol $\mathbb{N}_+$. Denote $[n] := \{1, 2, \cdots, n\}$, for $n \in \mathbb{N}_+$. The set of real mth order tensors is denoted by the symbol $\mathbb{R}^{I_1 \times I_2 \times \cdots \times I_m}$, where $I_1, I_2, \cdots, I_m \in \mathbb{N}_+$ denote index upper bounds. A tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_m}$ is said to be an mth order n-dimensional real tensor if $I_1 = I_2 = \cdots = I_m = n$, and it is represented by $\mathcal{T}_{m,n}$. An mth order n-dimensional real tensor $\mathcal{A} \in \mathcal{T}_{m,n}$ is said to be an mth order n-dimensional real Hankel tensor if $a_{i_1 i_2 \cdots i_m} = a_{j_1 j_2 \cdots j_m}$, where $i_1 + i_2 + \cdots + i_m = j_1 + j_2 + \cdots + j_m$, and it is denoted by $\mathcal{H}_{m,n}$.

In order to develop this paper, we need the following definitions.

Definition 1.1. Suppose that $\mathbf{u} \in \mathbb{R}^n$. If $\mathbf{u} = (1, u, u^2, \cdots, u^{n-1})^\top \in \mathbb{R}^n$, then $\mathbf{u}$ is called a Vandermonde vector. If

$$\mathcal{A} = \sum_{k=1}^{r} \alpha_k (\underbrace{\mathbf{u}_k \circ \mathbf{u}_k \circ \cdots \circ \mathbf{u}_k}_{\mathrm{m}}), \tag{1.1}$$

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where $\alpha_k \in \mathbb{R}, \alpha_k \neq 0, \mathbf{u}_k = (1, u_k, u_k^2, \cdots, u_k^{n-1})^\top \in \mathbb{R}^n$ are Vandermonde vectors for $k = 1, 2, \cdots, r$, r is the CANDECOMP/PARAFAC (CP) rank of the tensor $\mathcal{A}$, and $u_i \neq u_j$ for $i \neq j$, then we say $\mathcal{A}$ has a Vandermonde decompositon.

According to Qi [20], the mth order n-dimensional Hankel tensor $\mathcal{A}$ has the Vandermonde decomposition (1.1), and we will use the CANDECOMP/PARAFAC (CP) rank [11] throughout the paper.

In this paper, we consider an mth order n-dimensional Hankel tensor $\mathcal{A} \in \mathcal{H}_{m,n}$ with $rank(\mathcal{A}) = r$, and its elements are

$$a_{i_1 i_2 \cdots i_m} = h_{i_1+i_2+\cdots+i_m-m}, \text{ for all } i_1, i_2, \cdots, i_m \in [n]. \tag{1.2}$$

Our goal is to obtain the exact Vandermonde decomposition of the Hankel tensor $\mathcal{A}$ by (1.2), that is

$$\begin{aligned}\mathcal{A} = {} & \alpha_1 \underbrace{\begin{bmatrix} 1 \\ x_1 \\ x_1^2 \\ \vdots \\ x_1^{n-1} \end{bmatrix} \circ \begin{bmatrix} 1 \\ x_1 \\ x_1^2 \\ \vdots \\ x_1^{n-1} \end{bmatrix} \circ \cdots \circ \begin{bmatrix} 1 \\ x_1 \\ x_1^2 \\ \vdots \\ x_1^{n-1} \end{bmatrix}}_{m} + \alpha_2 \underbrace{\begin{bmatrix} 1 \\ x_2 \\ x_2^2 \\ \vdots \\ x_2^{n-1} \end{bmatrix} \circ \begin{bmatrix} 1 \\ x_2 \\ x_2^2 \\ \vdots \\ x_2^{n-1} \end{bmatrix} \circ \cdots \circ \begin{bmatrix} 1 \\ x_2 \\ x_2^2 \\ \vdots \\ x_2^{n-1} \end{bmatrix}}_{m} + \cdots \\ & + \alpha_r \underbrace{\begin{bmatrix} 1 \\ x_r \\ x_r^2 \\ \vdots \\ x_r^{n-1} \end{bmatrix} \circ \begin{bmatrix} 1 \\ x_r \\ x_r^2 \\ \vdots \\ x_r^{n-1} \end{bmatrix} \circ \cdots \circ \begin{bmatrix} 1 \\ x_r \\ x_r^2 \\ \vdots \\ x_r^{n-1} \end{bmatrix}}_{m},\end{aligned} \tag{1.3}$$

where $\alpha_i \in \mathbb{R}, \alpha_i \neq 0, x_i \neq 0$, and $x_i \neq x_j$ for $i \neq j, i, j = 1, 2, \cdots, r$.

The Hankel tensor decomposition problem often arises in seismic image [29], seismic signal [21, 16], blind system identification [8], subspace system identification [27], over-the-horizon radar [2, 3], signal processing [12] and some other applications. The tensor decomposition problem and variants thereof have been discussed a number of times in the literature. Most of this work built upon existing work for the special case $m = 2$, also known as Hankel matrix decomposition problem, see [28] for a comprehensive overview. When $m > 2$, Bo et al. [2] constructed the third order Hankel tensor, which is established by echo data of the distance unit containing the ship target. For example, the echo data was written by $\mathbf{s} = [s_1, s_2, \cdots, s_N]$, and its first P elements constructed the first $I_1 \times I_2$ dimensional Hankel matrix of the Hankel tensor, and the second $I_1 \times I_2$ dimensional Hankel matrix of the Hankel tensor was constructed by $[s_2, s_3, \cdots, s_{P+1}]$, by analogy, the other Hankel matrices of the Hankel tensor were constructed until the last element of the one-dimensional array $\mathbf{s}$. These Hankel matrices were arranged in order from front to back to obtain a third-order Hankel tensor $\mathcal{H} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$. Then the sea clutter subspace and target subspace of that tensor were solved by higher order SVD (HOSVD). Finally, in order to suppress sea clutter, Hankel tensor was mapped to the target subspace by orthogonal projection method. For most post-stack seismic datasets, due to the complexity of the underground structure or the influence of noise, the data itself may not be able to satisfy the absolute low rank, thus the recovery accuracy of the data will be affected. In order to solve this problem, Qian et al. [21] designed a Hankel construction method, and the original data tensor was constructed by it to obtain the Hankel tensor, so as to improve the low rank of the data volume. Then a target function of Hankelization was constructed and it was solved by the alternating minimization method. Recovery accuracy was much higher than similar algorithms.

Recently the structured tensor, such as nonnegative tensor and Toeplitz tensor, attracted much attentions, especially, the Hankel tensor got great development as it had wide properties [24] and applications, such as signal processing [7], data mining [1], computer vision [19] and machine learning [22]. For example, Song and Qi [24] discussed some relationships of positive semi-definite tensors and some other structured tensors. Xu [26] introduced Hankel tensors, Vandermonde tensors and their positivities. In many practical problems, the tensors need to be decomposed, such as canonical polyadic (CP) decomposition [25, 10, 11] and Tucker decomposition [18]. Boizard et al. [5] proposed two ways to extend Hankel structure to fourth order tensors. For these two types of tensors, a method to build a reordered mode was proposed, which highlighted the column redundancy and derived a fast algorithm to compute their HOSVD. Nie et al. [17] studied the relations among various ranks of Hankel tensors and gave an algorithm that can compute the Vandermonde ranks and decompositions for all Hankel tensors. The exact decomposition of tensors is a difficult problem. The previous work mainly concentrated on the numerical solutions of the decomposition of tensors. A Multilinear generalization of the singular value decomposition (SVD) was discussed and it was called the higher order SVD by Lieven et al. [14]. Although truncation of the HOSVD of a given tensor may lead to a good rank-$(R_1, R_2, \cdots, R_N)$ approximation ([15] contains an error bound), it turned out that this tensor was in general not the best possible (least-squares) approximation under the given n-mode rank constraints. Tichavsky et al. [25] proposed a numerical method for CP decomposition of small size tensors and he was primarily on decomposition of tensors that correspond to small matrix multiplications. For higher-order tensors, Li et al. [13] proposed a novel, adaptive tensor memorization algorithm (ADATM). This method behaves better as the tensor order grows, making its performance more scalable for higher-order data problems. Smith et al. [23] produced an algorithm to accelerate the Tucker decomposition based on a compressed data structure for sparse tensors and show that many computational redundancies during tensor-matrix multiplications (TTMc) can be identified and pruned without the memory overheads of memorization. For tensor decomposition, some numerical algorithms [6] were described in details, and their numerical complexity was calculated. It was also pointed out that the tensor decomposition was eventually an approximation rather than an exact decomposition. Goulart et al. [9] introduced a CP decomposition model which has structured matrix factors, such as Toeplitz, Hankel or circulant matrices, and studied its associated estimation problem. Structured CP decompositions, i.e. with Toeplitz, circulant, or Hankel matrix factors, were also studied in [4]. However, the previous works mainly concentrated on the approximation tensor decomposition and the study of exact decomposition of Hankel tensor is very rare.

In this paper, we study the problem (1.3) of exact Vandermonde decomposition for the Hankel tensor. Firstly, this problem can be rewritten as the matrix equation, and then a new method is designed to solve the problem. Finally, some examples are given to show that the new method is feasible and effective.

This paper is organized as follows. In Section 2, we propose a new method for solving the problem (1.3). In Section 3, some examples are given to illustrate that the new method is feasible and effective.

$\boxed{2}$ A Simple Method for Computing the Vandermonde Decomposition (1.3) of a Hankel Tensor

By combining the formulas (1.2) and (1.3), we can obtain the system of nonlinear equations

$$\begin{cases} \alpha_1 + \alpha_2 + \cdots + \alpha_r = h_0, \\ \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_r x_r = h_1, \\ \alpha_1 x_1^2 + \alpha_2 x_2^2 + \cdots + \alpha_r x_r^2 = h_2, \\ \alpha_1 x_1^3 + \alpha_2 x_2^3 + \cdots + \alpha_r x_r^3 = h_3, \\ \vdots \\ \alpha_1 x_1^{m(n-1)} + \alpha_2 x_2^{m(n-1)} + \cdots + \alpha_r x_r^{m(n-1)} = h_{m(n-1)}. \end{cases} \tag{2.1}$$

Let

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & \cdots & x_r \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & \cdots & x_r^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 & x_5^3 & \cdots & x_r^3 \\ x_1^4 & x_2^4 & x_3^4 & x_4^4 & x_5^4 & \cdots & x_r^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{r-1} & x_2^{r-1} & x_3^{r-1} & x_4^{r-1} & x_5^{r-1} & \cdots & x_r^{r-1} \\ x_1^r & x_2^r & x_3^r & x_4^r & x_5^r & \cdots & x_r^r \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{m(n-1)} & x_2^{m(n-1)} & x_3^{m(n-1)} & x_4^{m(n-1)} & x_5^{m(n-1)} & \cdots & x_r^{m(n-1)} \end{bmatrix}, \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_r \end{bmatrix}, \mathbf{h} = \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ \vdots \\ h_{m(n-1)} \end{bmatrix},$$

where $\mathbf{X} \in \mathbb{R}^{[m(n-1)+1]\times r}, \alpha \in \mathbb{R}^r, \mathbf{h} \in \mathbb{R}^{m(n-1)+1}$, then (2.1) can be equivalently written as $\mathbf{X}\alpha = \mathbf{h}$, that is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & \cdots & x_r \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & \cdots & x_r^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 & x_5^3 & \cdots & x_r^3 \\ x_1^4 & x_2^4 & x_3^4 & x_4^4 & x_5^4 & \cdots & x_r^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{r-1} & x_2^{r-1} & x_3^{r-1} & x_4^{r-1} & x_5^{r-1} & \cdots & x_r^{r-1} \\ x_1^r & x_2^r & x_3^r & x_4^r & x_5^r & \cdots & x_r^r \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{m(n-1)} & x_2^{m(n-1)} & x_3^{m(n-1)} & x_4^{m(n-1)} & x_5^{m(n-1)} & \cdots & x_r^{m(n-1)} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_r \end{bmatrix} = \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ \vdots \\ h_{m(n-1)} \end{bmatrix}, \tag{2.2}$$

and the augmented matrix of the equations (2.2) is $\overline{\mathbf{X}} = [\mathbf{X} \vdots \mathbf{h}] \in \mathbb{R}^{[m(n-1)+1]\times(r+1)}$, that is

$$\overline{\mathbf{X}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 & h_0 \\ x_1 & x_2 & x_3 & x_4 & x_5 & \cdots & x_r & h_1 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & \cdots & x_r^2 & h_2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 & x_5^3 & \cdots & x_r^3 & h_3 \\ x_1^4 & x_2^4 & x_3^4 & x_4^4 & x_5^4 & \cdots & x_r^4 & h_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_1^{r-1} & x_2^{r-1} & x_3^{r-1} & x_4^{r-1} & x_5^{r-1} & \cdots & x_r^{r-1} & h_{r-1} \\ x_1^r & x_2^r & x_3^r & x_4^r & x_5^r & \cdots & x_r^r & h_r \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_1^{m(n-1)} & x_2^{m(n-1)} & x_3^{m(n-1)} & x_4^{m(n-1)} & x_5^{m(n-1)} & \cdots & x_r^{m(n-1)} & h_{m(n-1)} \end{bmatrix}. \quad (2.3)$$

Then, we make the elementary row operations for the augmented matrix $\overline{\mathbf{X}}$. Let

$$\mathbf{L}_1 = \begin{bmatrix} 1 & & & & & & & & & \\ -x_1 & 1 & & & & & & & & \\ & -x_1 & 1 & & & & & & & \\ & & -x_1 & 1 & & & & & & \\ & & & \ddots & \ddots & & & & & \\ & & & & -x_1 & 1 & & & & \\ & & & & & -x_1 & 1 & & & \\ & & & & & & -x_1 & 1 & & \\ & & & & & & & \ddots & \ddots & \\ & & & & & & & & -x_1 & 1 \end{bmatrix}_{[m(n-1)+1]\times[m(n-1)+1]}, \quad (2.4)$$

then, we gain

$$\mathbf{L}_1\overline{\mathbf{X}} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & h_0 \\ 0 & x_2 - x_1 & x_3 - x_1 & \cdots & x_r - x_1 & \mathbf{h}_1(x_1) \\ 0 & x_2(x_2 - x_1) & x_3(x_3 - x_1) & \cdots & x_r(x_r - x_1) & \mathbf{h}_2(x_1) \\ 0 & x_2^2(x_2 - x_1) & x_3^2(x_3 - x_1) & \cdots & x_r^2(x_r - x_1) & \mathbf{h}_3(x_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & x_2^{r-2}(x_2 - x_1) & x_3^{r-2}(x_3 - x_1) & \cdots & x_r^{r-2}(x_r - x_1) & \mathbf{h}_{r-1}(x_1) \\ 0 & x_2^{r-1}(x_2 - x_1) & x_3^{r-1}(x_3 - x_1) & \cdots & x_r^{r-1}(x_r - x_1) & \mathbf{h}_r(x_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & x_2^{(m(n-1)-1)}(x_2 - x_1) & x_3^{(m(n-1)-1)}(x_3 - x_1) & \cdots & x_r^{(m(n-1)-1)}(x_r - x_1) & \mathbf{h}_{(m(n-1))}(x_1) \end{bmatrix},$$

where $\mathbf{h}_k(x_1) = h_k - h_{k-1}x_1$, for $k = 1, 2, \cdots, m(n-1)$.

Let

$$\mathbf{L}_2 = \begin{bmatrix} 1 & & & & & & & & & \\ 0 & 1 & & & & & & & & \\ & -x_2 & 1 & & & & & & & \\ & & -x_2 & 1 & & & & & & \\ & & & \ddots & \ddots & & & & & \\ & & & & -x_2 & 1 & & & & \\ & & & & & -x_2 & 1 & & & \\ & & & & & & -x_2 & 1 & & \\ & & & & & & & \ddots & \ddots & \\ & & & & & & & & -x_2 & 1 \end{bmatrix}_{[m(n-1)+1]\times[m(n-1)+1]}, \quad (2.5)$$

then

$$\mathbf{L}_2\mathbf{L}_1\overline{\mathbf{X}} =$$

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & h_0 \\ 0 & x_2 - x_1 & x_3 - x_1 & \cdots & x_r - x_1 & \mathbf{h}_1(x_1) \\ 0 & 0 & (x_3 - x_1)(x_3 - x_2) & \cdots & (x_r - x_1)(x_r - x_2) & \mathbf{h}_2(x_1, x_2) \\ 0 & 0 & x_3(x_3 - x_1)(x_3 - x_2) & \cdots & x_r(x_r - x_1)(x_r - x_2) & \mathbf{h}_3(x_1, x_2) \\ 0 & 0 & x_3^2(x_3 - x_1)(x_3 - x_2) & \cdots & x_r^2(x_r - x_1)(x_r - x_2) & \mathbf{h}_4(x_1, x_2) \\ 0 & 0 & x_3^3(x_3 - x_1)(x_3 - x_2) & \cdots & x_r^3(x_r - x_1)(x_r - x_2) & \mathbf{h}_5(x_1, x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & x_3^{r-2}(x_3 - x_1)(x_3 - x_2) & \cdots & x_r^{r-2}(x_r - x_1)(x_r - x_2) & \mathbf{h}_r(x_1, x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & x_3^{m(n-1)-2}(x_3 - x_1)(x_3 - x_2) & \cdots & x_r^{m(n-1)-2}(x_r - x_1)(x_r - x_2) & \mathbf{h}_{m(n-1)}(x_1, x_2) \end{bmatrix},$$

where $\mathbf{h}_1(x_1) = h_1 - h_0 x_1, \mathbf{h}_k(x_1, x_2) = h_k - h_{k-1}(x_1 + x_2) + h_{k-2}x_1x_2$, for $k = 2, 3, \ldots, m(n-1)$.
(2.6)

Similarly, let

$$\mathbf{L}_r = \begin{bmatrix} 1 & & & & & & & & & \\ 0 & 1 & & & & & & & & \\ & \ddots & \ddots & & & & & & & \\ & & 0 & 1 & & & & & & \\ & & & 0 & 1 & & & & & \\ & & & & 0 & 1 & & & & \\ & & & & & -x_r & 1 & & & \\ & & & & & & -x_r & 1 & & \\ & & & & & & & \ddots & \ddots & \\ & & & & & & & & -x_r & 1 \end{bmatrix}_{[m(n-1)+1]\times[m(n-1)+1]}, \quad (2.7)$$

then

$$\mathbf{L}_r \cdots \mathbf{L}_2 \mathbf{L}_1 \overline{\mathbf{X}} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & h_0 \\ 0 & x_2 - x_1 & x_3 - x_1 & x_4 - x_1 & \cdots & x_r - x_1 & \mathbf{h}_1(x_1) \\ 0 & 0 & \prod_{j=1}^{2}(x_3 - x_j) & \prod_{j=1}^{2}(x_4 - x_j) & \cdots & \prod_{j=1}^{2}(x_r - x_j) & \mathbf{h}_2(x_1, x_2) \\ 0 & 0 & 0 & \prod_{j=1}^{3}(x_4 - x_j) & \cdots & \prod_{j=1}^{3}(x_r - x_j) & \mathbf{h}_3(x_1, x_2, x_3) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \prod_{j=1}^{r-1}(x_r - x_j) & \mathbf{h}_{r-1}(x_1, x_2, \cdots, x_{r-1}) \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mathbf{h}_r(x_1, x_2, \cdots, x_r) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mathbf{h}_{m(n-1)}(x_1, x_2, \cdots, x_r) \end{bmatrix}, \tag{2.8}$$

where,

$$\begin{aligned} \mathbf{h}_k(x_1, x_2, \cdots, x_k) &= (-1)^0 h_k + (-1)^1 h_{k-1}(\sum_{i_1=1}^{k} x_{i_1}) + (-1)^2 h_{k-2}(\sum_{1 \le i_1 < i_2 \le k} x_{i_1} x_{i_2}) + \cdots \\ &\quad + (-1)^k h_{k-k}(\sum_{1 \le i_1 < i_2 < \cdots < i_k \le k} x_{i_1} x_{i_2} \cdots x_{i_k}), \text{for } k = 1, 2, \cdots, r-1; \\ \mathbf{h}_k(x_1, x_2, \cdots, x_r) &= (-1)^0 h_k + (-1)^1 h_{k-1}(\sum_{i_1=1}^{r} x_{i_1}) + (-1)^2 h_{k-2}(\sum_{1 \le i_1 < i_2 \le r} x_{i_1} x_{i_2}) + \cdots \\ &\quad + (-1)^r h_{k-r}(\sum_{1 \le i_1 < i_2 < \cdots < i_r \le r} x_{i_1} x_{i_2} \cdots x_{i_r}), \text{for } k = r, r+1, \cdots, m(n-1). \end{aligned} \tag{2.9}$$

Therefore, the system (2.2) has a solution if and only if

$$\begin{cases} \mathbf{h}_r(x_1, x_2, \cdots, x_r) = 0, \\ \mathbf{h}_{r+1}(x_1, x_2, \cdots, x_r) = 0, \\ \mathbf{h}_{r+2}(x_1, x_2, \cdots, x_r) = 0, \\ \mathbf{h}_{r+3}(x_1, x_2, \cdots, x_r) = 0, \\ \mathbf{h}_{r+4}(x_1, x_2, \cdots, x_r) = 0, \\ \vdots \\ \mathbf{h}_{m(n-1)}(x_1, x_2, \cdots, x_r) = 0. \end{cases} \tag{2.10}$$

By (2.9), the system (2.10) can be equivalently written as

$$\begin{bmatrix} (-1)^1 h_{r-1} & (-1)^2 h_{r-2} & \cdots & (-1)^{r-1} h_1 & (-1)^r h_0 \\ (-1)^1 h_r & (-1)^2 h_{r-1} & \cdots & (-1)^{r-1} h_2 & (-1)^r h_1 \\ (-1)^1 h_{r+1} & (-1)^2 h_r & \cdots & (-1)^{r-1} h_3 & (-1)^r h_2 \\ (-1)^1 h_{r+2} & (-1)^2 h_{r+1} & \cdots & (-1)^{r-1} h_4 & (-1)^r h_3 \\ (-1)^1 h_{r+3} & (-1)^2 h_{r+2} & \cdots & (-1)^{r-1} h_5 & (-1)^r h_4 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^1 h_{m(n-1)-1} & (-1)^2 h_{m(n-1)-2} & \cdots & (-1)^{r-1} h_{m(n-1)-r+1} & (-1)^r h_{m(n-1)-r} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ \vdots \\ t_r \end{bmatrix} = \begin{bmatrix} -h_r \\ -h_{r+1} \\ -h_{r+2} \\ -h_{r+3} \\ -h_{r+4} \\ \vdots \\ -h_{m(n-1)} \end{bmatrix}, \tag{2.11}$$

where,

$$\begin{cases} t_1 = \sum_{i_1=1}^{r} x_{i_1} = x_1 + x_2 + \cdots + x_r, \\ t_2 = \sum_{1 \le i_1 < i_2 \le r} x_{i_1} x_{i_2} = x_1 x_2 + \cdots + x_1 x_r + x_2 x_3 + \cdots + x_2 x_r + \cdots + x_{r-1} x_r, \\ \vdots \\ t_r = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le r} x_{i_1} x_{i_2} \cdots x_{i_r} = x_1 x_2 \cdots x_r. \end{cases} \tag{2.12}$$

Next we will solve the system (2.11).

Theorem 2.1. *If $m(n-1)+1 \ge 2r$, set $\mathbf{t} = [t_1, t_2, t_3, t_4, t_5, \cdots, t_r]^\intercal$. Then the nonhomogeneous linear equations (2.11) has a solution*

$$\mathbf{t} = \begin{bmatrix} (-1)^1 h_{r-1} & (-1)^2 h_{r-2} & \cdots & (-1)^{r-1} h_1 & (-1)^r h_0 \\ (-1)^1 h_r & (-1)^2 h_{r-1} & \cdots & (-1)^{r-1} h_2 & (-1)^r h_1 \\ (-1)^1 h_{r+1} & (-1)^2 h_r & \cdots & (-1)^{r-1} h_3 & (-1)^r h_2 \\ (-1)^1 h_{r+2} & (-1)^2 h_{r+1} & \cdots & (-1)^{r-1} h_4 & (-1)^r h_3 \\ (-1)^1 h_{r+3} & (-1)^2 h^3_{r+2} & \cdots & (-1)^{r-1} h_5 & (-1)^r h_4 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^1 h_{2r-2} & (-1)^2 h_{2r-3} & \cdots & (-1)^{r-1} h_r & (-1)^r h_{r-1} \end{bmatrix}_{r \times r}^{-1} \begin{bmatrix} -h_r \\ -h_{r+1} \\ -h_{r+2} \\ -h_{r+3} \\ -h_{r+4} \\ \vdots \\ -h_{2r-1} \end{bmatrix}. \tag{2.13}$$

Proof. If $m(n-1)+1 \ge 2r$, i.e. the number of rows of the coefficient matrix of the system (2.11) is $m(n-1)-r+1 \ge r$, we can get the augmented matrix from the nonhomogeneous linear equations (2.11), i.e,

$$
\overline{\mathbf{H}} = \begin{bmatrix}
(-1)^1 h_{r-1} & (-1)^2 h_{r-2} & \cdots & (-1)^{r-1} h_1 & (-1)^r h_0 & (-1)^1 h_r \\
(-1)^1 h_r & (-1)^2 h_{r-1} & \cdots & (-1)^{r-1} h_2 & (-1)^r h_1 & (-1)^1 h_{r+1} \\
(-1)^1 h_{r+1} & (-1)^2 h_r & \cdots & (-1)^{r-1} h_3 & (-1)^r h_2 & (-1)^1 h_{r+2} \\
(-1)^1 h_{r+2} & (-1)^2 h_{r+1} & \cdots & (-1)^{r-1} h_4 & (-1)^r h_3 & (-1)^1 h_{r+3} \\
(-1)^1 h_{r+3} & (-1)^2 h_{r+2} & \cdots & (-1)^{r-1} h_5 & (-1)^r h_4 & (-1)^1 h_{r+4} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
(-1)^1 h_{m(n-1)-1} & (-1)^2 h_{m(n-1)-2} & \cdots & (-1)^{r-1} h_{m(n-1)-r+1} & (-1)^r h_{m(n-1)-r} & (-1)^1 h_{m(n-1)}
\end{bmatrix}. \tag{2.14}
$$

For convenience, we move the $(r+1)$th column of the matrix $\overline{\mathbf{H}}$ (2.14) to the 1th column by elementary column transformation of matrices, and we get

$$
\widetilde{\mathbf{H}} = \begin{bmatrix}
(-1)^0 h_r & (-1)^1 h_{r-1} & (-1)^2 h_{r-2} & \cdots & (-1)^{r-1} h_1 & (-1)^r h_0 \\
(-1)^0 h_{r+1} & (-1)^1 h_r & (-1)^2 h_{r-1} & \cdots & (-1)^{r-1} h_2 & (-1)^r h_1 \\
(-1)^0 h_{r+2} & (-1)^1 h_{r+1} & (-1)^2 h_r & \cdots & (-1)^{r-1} h_3 & (-1)^r h_2 \\
(-1)^0 h_{r+3} & (-1)^1 h_{r+2} & (-1)^2 h_{r+1} & \cdots & (-1)^{r-1} h_4 & (-1)^r h_3 \\
(-1)^0 h_{r+4} & (-1)^1 h_{r+3} & (-1)^2 h_{r+2} & \cdots & (-1)^{r-1} h_5 & (-1)^r h_4 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(-1)^0 h_{m(n-1)} & (-1)^1 h_{m(n-1)-1} & (-1)^2 h_{m(n-1)-2} & \cdots & (-1)^{r-1} h_{m(n-1)-r+1} & (-1)^r h_{m(n-1)-r}
\end{bmatrix}, \tag{2.15}
$$

and the elements of the augmented matrix are decided by the (2.1), and we can get

$$
\widetilde{\mathbf{H}} = \begin{bmatrix}
(-1)^0 \sum_{i=1}^r \alpha_i x_i^r & (-1)^1 \sum_{i=1}^r \alpha_i x_i^{r-1} & \cdots & (-1)^{r-1} \sum_{i=1}^r \alpha_i x_i^1 & (-1)^r \sum_{i=1}^r \alpha_i x_i^0 \\
(-1)^0 \sum_{i=1}^r \alpha_i x_i^{r+1} & (-1)^1 \sum_{i=1}^r \alpha_i x_i^r & \cdots & (-1)^{r-1} \sum_{i=1}^r \alpha_i x_i^2 & (-1)^r \sum_{i=1}^r \alpha_i x_i^1 \\
(-1)^0 \sum_{i=1}^r \alpha_i x_i^{r+2} & (-1)^1 \sum_{i=1}^r \alpha_i x_i^{r+1} & \cdots & (-1)^{r-1} \sum_{i=1}^r \alpha_i x_i^3 & (-1)^r \sum_{i=1}^r \alpha_i x_i^2 \\
(-1)^0 \sum_{i=1}^r \alpha_i x_i^{r+3} & (-1)^1 \sum_{i=1}^r \alpha_i x_i^{r+2} & \cdots & (-1)^{r-1} \sum_{i=1}^r \alpha_i x_i^4 & (-1)^r \sum_{i=1}^r \alpha_i x_i^3 \\
(-1)^0 \sum_{i=1}^r \alpha_i x_i^{r+4} & (-1)^1 \sum_{i=1}^r \alpha_i x_i^{r+3} & \cdots & (-1)^{r-1} \sum_{i=1}^r \alpha_i x_i^5 & (-1)^r \sum_{i=1}^r \alpha_i x_i^4 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(-1)^0 \sum_{i=1}^r \alpha_i x_i^{2r-1} & (-1)^1 \sum_{i=1}^r \alpha_i x_i^{2r-2} & \cdots & (-1)^{r-1} \sum_{i=1}^r \alpha_i x_i^r & (-1)^r \sum_{i=1}^r \alpha_i x_i^{r-1} \\
(-1)^0 \sum_{i=1}^r \alpha_i x_i^{2r} & (-1)^1 \sum_{i=1}^r \alpha_i x_i^{2r-1} & \cdots & (-1)^{r-1} \sum_{i=1}^r \alpha_i x_i^{r+1} & (-1)^r \sum_{i=1}^r \alpha_i x_i^r \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(-1)^0 \sum_{i=1}^r \alpha_i x_i^{m(n-1)} & (-1)^1 \sum_{i=1}^r \alpha_i x_i^{m(n-1)-1} & \cdots & (-1)^{r-1} \sum_{i=1}^r \alpha_i x_i^{m(n-1)-r+1} & (-1)^r \sum_{i=1}^r \alpha_i x_i^{m(n-1)-r}
\end{bmatrix}. \tag{2.16}
$$

Add $-x_1$ times row i to row $(i+1)$, where $i = m(n-1)-r, m(n-1)-r-1, \cdots, 2, 1$, then we get

$$\widetilde{\mathbf{H}}_1 =$$
$$\begin{bmatrix}
(-1)^0 \sum\limits_{i=1}^{r} \alpha_i x_i^r & (-1)^1 \sum\limits_{i=1}^{r} \alpha_i x_i^{r-1} & \cdots & (-1)^r \sum\limits_{i=1}^{r} \alpha_i x_i^0 \\
(-1)^1 \sum\limits_{i=2}^{r} \alpha_i x_i^r (x_1 - x_i) & (-1)^2 \sum\limits_{i=2}^{r} \alpha_i x_i^{r-1} (x_1 - x_i) & \cdots & (-1)^{r+1} \sum\limits_{i=2}^{r} \alpha_i x_i^0 (x_1 - x_i) \\
(-1)^1 \sum\limits_{i=2}^{r} \alpha_i x_i^{r+1} (x_1 - x_i) & (-1)^2 \sum\limits_{i=2}^{r} \alpha_i x_i^{r} (x_1 - x_i) & \cdots & (-1)^{r+1} \sum\limits_{i=2}^{r} \alpha_i x_i^1 (x_1 - x_i) \\
(-1)^1 \sum\limits_{i=2}^{r} \alpha_i x_i^{r+2} (x_1 - x_i) & (-1)^2 \sum\limits_{i=2}^{r} \alpha_i x_i^{r+1} (x_1 - x_i) & \cdots & (-1)^{r+1} \sum\limits_{i=2}^{r} \alpha_i x_i^2 (x_1 - x_i) \\
(-1)^1 \sum\limits_{i=2}^{r} \alpha_i x_i^{r+3} (x_1 - x_i) & (-1)^2 \sum\limits_{i=2}^{r} \alpha_i x_i^{r+2} (x_1 - x_i) & \cdots & (-1)^{r+1} \sum\limits_{i=2}^{r} \alpha_i x_i^3 (x_1 - x_i) \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^1 \sum\limits_{i=2}^{r} \alpha_i x_i^{2r-2} (x_1 - x_i) & (-1)^2 \sum\limits_{i=2}^{r} \alpha_i x_i^{2r-3} (x_1 - x_i) & \cdots & (-1)^{r+1} \sum\limits_{i=2}^{r} \alpha_i x_i^{r-2} (x_1 - x_i) \\
(-1)^1 \sum\limits_{i=2}^{r} \alpha_i x_i^{2r-1} (x_1 - x_i) & (-1)^2 \sum\limits_{i=2}^{r} \alpha_i x_i^{2r-2} (x_1 - x_i) & \cdots & (-1)^{r+1} \sum\limits_{i=2}^{r} \alpha_i x_i^{r-1} (x_1 - x_i) \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^1 \sum\limits_{i=2}^{r} \alpha_i x_i^{m(n-1)-1} (x_1 - x_i) & (-1)^2 \sum\limits_{i=2}^{r} \alpha_i x_i^{m(n-1)-2} (x_1 - x_i) & \cdots & (-1)^{r+1} \sum\limits_{i=2}^{r} \alpha_i x_i^{m(n-1)-r-1} (x_1 - x_i)
\end{bmatrix}. \tag{2.17}$$

Add $-x_2$ times row i to row $(i+1)$, where $i = m(n-1)-r, m(n-1)-r-1, \cdots, 3, 2$, then we get

$$\widetilde{\mathbf{H}}_2 =$$
$$\begin{bmatrix}
(-1)^0 \sum\limits_{i=1}^{r} \alpha_i x_i^r & (-1)^1 \sum\limits_{i=1}^{r} \alpha_i x_i^{r-1} & \cdots & (-1)^r \sum\limits_{i=1}^{r} \alpha_i x_i^0 \\
(-1)^1 \sum\limits_{i=2}^{r} \alpha_i x_i^r (x_1 - x_i) & (-1)^2 \sum\limits_{i=2}^{r} \alpha_i x_i^{r-1} (x_1 - x_i) & \cdots & (-1)^{r+1} \sum\limits_{i=2}^{r} \alpha_i x_i^0 (x_1 - x_i) \\
(-1)^2 \sum\limits_{i=3}^{r} \alpha_i x_i^r \prod\limits_{j=1}^{2} (x_j - x_i) & (-1)^3 \sum\limits_{i=3}^{r} \alpha_i x_i^{r-1} \prod\limits_{j=1}^{2} (x_j - x_i) & \cdots & (-1)^{r+2} \sum\limits_{i=3}^{r} \alpha_i x_i^0 \prod\limits_{j=1}^{2} (x_j - x_i) \\
(-1)^2 \sum\limits_{i=3}^{r} \alpha_i x_i^{r+1} \prod\limits_{j=1}^{2} (x_j - x_i) & (-1)^3 \sum\limits_{i=3}^{r} \alpha_i x_i^{r} \prod\limits_{j=1}^{2} (x_j - x_i) & \cdots & (-1)^{r+2} \sum\limits_{i=3}^{r} \alpha_i x_i^1 \prod\limits_{j=1}^{2} (x_j - x_i) \\
(-1)^2 \sum\limits_{i=3}^{r} \alpha_i x_i^{r+2} \prod\limits_{j=1}^{2} (x_j - x_i) & (-1)^3 \sum\limits_{i=3}^{r} \alpha_i x_i^{r+1} \prod\limits_{j=1}^{2} (x_j - x_i) & \cdots & (-1)^{r+2} \sum\limits_{i=3}^{r} \alpha_i x_i^2 \prod\limits_{j=1}^{2} (x_j - x_i) \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^2 \sum\limits_{i=3}^{r} \alpha_i x_i^{2r-3} \prod\limits_{j=1}^{2} (x_j - x_i) & (-1)^3 \sum\limits_{i=3}^{r} \alpha_i x_i^{2r-4} \prod\limits_{j=1}^{2} (x_j - x_i) & \cdots & (-1)^{r+2} \sum\limits_{i=3}^{r} \alpha_i x_i^{r-3} \prod\limits_{j=1}^{2} (x_j - x_i) \\
(-1)^2 \sum\limits_{i=3}^{r} \alpha_i x_i^{2r-2} \prod\limits_{j=1}^{2} (x_j - x_i) & (-1)^3 \sum\limits_{i=3}^{r} \alpha_i x_i^{2r-3} \prod\limits_{j=1}^{2} (x_j - x_i) & \cdots & (-1)^{r+2} \sum\limits_{i=3}^{r} \alpha_i x_i^{r-2} \prod\limits_{j=1}^{2} (x_j - x_i) \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^2 \sum\limits_{i=3}^{r} \alpha_i x_i^{m(n-1)-2} \prod\limits_{j=1}^{2} (x_j - x_i) & (-1)^3 \sum\limits_{i=3}^{r} \alpha_i x_i^{m(n-1)-3} \prod\limits_{j=1}^{2} (x_j - x_i) & \cdots & (-1)^{r+2} \sum\limits_{i=3}^{r} \alpha_i x_i^{m(n-1)-r-2} \prod\limits_{j=1}^{2} (x_j - x_i)
\end{bmatrix}. \tag{2.18}$$

Similar to above and the rest may be deduced by analogy, and add $-x_{r-1}$ times row i

to row $(i+1)$, $i = m(n-1)-r, m(n-1)-r-1, \cdots, r, r-1$, then we get

$$\widetilde{\mathbf{H}}_{r-1} =$$
$$\begin{bmatrix}
(-1)^0 \sum\limits_{i=1}^{r} \alpha_i x_i^r & (-1)^1 \sum\limits_{i=1}^{r} \alpha_i x_i^{r-1} & \cdots & (-1)^r \sum\limits_{i=1}^{r} \alpha_i x_i^0 \\
(-1)^1 \sum\limits_{i=2}^{r} \alpha_i x_i^r (x_1 - x_i) & (-1)^2 \sum\limits_{i=2}^{r} \alpha_i x_i^{r-1} (x_1 - x_i) & \cdots & (-1)^{r+1} \sum\limits_{i=2}^{r} \alpha_i x_i^0 (x_1 - x_i) \\
(-1)^2 \sum\limits_{i=3}^{r} \alpha_i x_i^r \prod\limits_{j=1}^{2} (x_j - x_i) & (-1)^3 \sum\limits_{i=3}^{r} \alpha_i x_i^{r-1} \prod\limits_{j=1}^{2} (x_j - x_i) & \cdots & (-1)^{r+2} \sum\limits_{i=3}^{r} \alpha_i x_i^0 \prod\limits_{j=1}^{2} (x_j - x_i) \\
(-1)^3 \sum\limits_{i=4}^{r} \alpha_i x_i^r \prod\limits_{j=1}^{3} (x_j - x_i) & (-1)^4 \sum\limits_{i=4}^{r} \alpha_i x_i^{r-1} \prod\limits_{j=1}^{3} (x_j - x_i) & \cdots & (-1)^{r+3} \sum\limits_{i=4}^{r} \alpha_i x_i^0 \prod\limits_{j=1}^{3} (x_j - x_i) \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^{r-2} \sum\limits_{i=r-1}^{r} \alpha_i x_i^r \prod\limits_{j=1}^{r-2} (x_j - x_i) & (-1)^{r-1} \sum\limits_{i=r-1}^{r} \alpha_i x_i^{r-1} \prod\limits_{j=1}^{r-2} (x_j - x_i) & \cdots & (-1)^{2r-2} \sum\limits_{i=r-1}^{r} \alpha_i x_i^0 \prod\limits_{j=1}^{r-2} (x_j - x_i) \\
(-1)^{r-1} \sum\limits_{i=r}^{r} \alpha_i x_i^r \prod\limits_{j=1}^{r-1} (x_j - x_i) & (-1)^{r} \sum\limits_{i=r}^{r} \alpha_i x_i^{r-1} \prod\limits_{j=1}^{r-1} (x_j - x_i) & \cdots & (-1)^{2r-1} \sum\limits_{i=r}^{r} \alpha_i x_i^0 \prod\limits_{j=1}^{r-1} (x_j - x_i) \\
(-1)^{r-1} \sum\limits_{i=r}^{r} \alpha_i x_i^{r+1} \prod\limits_{j=1}^{r-1} (x_j - x_i) & (-1)^{r} \sum\limits_{i=r}^{r} \alpha_i x_i^{r} \prod\limits_{j=1}^{r-1} (x_j - x_i) & \cdots & (-1)^{2r-1} \sum\limits_{i=r}^{r} \alpha_i x_i^1 \prod\limits_{j=1}^{r-1} (x_j - x_i) \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^{r-1} \sum\limits_{i=r}^{r} \alpha_i x_i^{m(n-1)-r+1} \prod\limits_{j=1}^{r-1} (x_j - x_i) & (-1)^{r} \sum\limits_{i=r}^{r} \alpha_i x_i^{m(n-1)-r} \prod\limits_{j=1}^{r-1} (x_j - x_i) & \cdots & (-1)^{2r-1} \sum\limits_{i=r}^{r} \alpha_i x_i^{m(n-1)-2r+1} \prod\limits_{j=1}^{r-1} (x_j - x_i)
\end{bmatrix}. \tag{2.19}$$

Add $-x_r^p$ times row r to row $(r+p)$, $p = 1, 2, \cdots, m(n-1)-2r+1$, we get

$$\widetilde{\mathbf{H}}_{r} =$$
$$\begin{bmatrix}
(-1)^0 \sum\limits_{i=1}^{r} \alpha_i x_i^r & (-1)^1 \sum\limits_{i=1}^{r} \alpha_i x_i^{r-1} & \cdots & (-1)^r \sum\limits_{i=1}^{r} \alpha_i x_i^0 \\
(-1)^1 \sum\limits_{i=2}^{r} \alpha_i x_i^r (x_1 - x_i) & (-1)^2 \sum\limits_{i=2}^{r} \alpha_i x_i^{r-1} (x_1 - x_i) & \cdots & (-1)^{r+1} \sum\limits_{i=2}^{r} \alpha_i x_i^0 (x_1 - x_i) \\
(-1)^2 \sum\limits_{i=3}^{r} \alpha_i x_i^r \prod\limits_{j=1}^{2} (x_j - x_i) & (-1)^3 \sum\limits_{i=3}^{r} \alpha_i x_i^{r-1} \prod\limits_{j=1}^{2} (x_j - x_i) & \cdots & (-1)^{r+2} \sum\limits_{i=3}^{r} \alpha_i x_i^0 \prod\limits_{j=1}^{2} (x_j - x_i) \\
(-1)^3 \sum\limits_{i=4}^{r} \alpha_i x_i^r \prod\limits_{j=1}^{3} (x_j - x_i) & (-1)^4 \sum\limits_{i=4}^{r} \alpha_i x_i^{r-1} \prod\limits_{j=1}^{3} (x_j - x_i) & \cdots & (-1)^{r+3} \sum\limits_{i=4}^{r} \alpha_i x_i^0 \prod\limits_{j=1}^{3} (x_j - x_i) \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^{r-2} \sum\limits_{i=r-1}^{r} \alpha_i x_i^r \prod\limits_{j=1}^{r-2} (x_j - x_i) & (-1)^{r-1} \sum\limits_{i=r-1}^{r} \alpha_i x_i^{r-1} \prod\limits_{j=1}^{r-2} (x_j - x_i) & \cdots & (-1)^{2r-2} \sum\limits_{i=r-1}^{r} \alpha_i x_i^0 \prod\limits_{j=1}^{r-2} (x_j - x_i) \\
(-1)^{r-1} \sum\limits_{i=r}^{r} \alpha_i x_i^r \prod\limits_{j=1}^{r-1} (x_j - x_i) & (-1)^{r} \sum\limits_{i=r}^{r} \alpha_i x_i^{r-1} \prod\limits_{j=1}^{r-1} (x_j - x_i) & \cdots & (-1)^{2r-1} \sum\limits_{i=r}^{r} \alpha_i x_i^0 \prod\limits_{j=1}^{r-1} (x_j - x_i) \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}. \tag{2.20}$$

Add x_r times column j to column $(j-1)$, $j = 2, 3, \cdots, r+1$, then we get

$$\widehat{\mathbf{H}}_1 =$$

$$\begin{bmatrix}
(-1)^0 \sum\limits_{i=1}^{r-1} \alpha_i x_i^{r-1}(x_i - x_r) & \cdots & (-1)^{r-1} \sum\limits_{i=1}^{r-1} \alpha_i x_i^0 (x_i - x_r) & (-1)^r \sum\limits_{i=1}^{r} \alpha_i x_i^0 \\
(-1)^1 \sum\limits_{i=2}^{r-1} \alpha_i x_i^{r-1} \prod\limits_{j=1}^{1}(x_j - x_i) \prod\limits_{k=r}^{r}(x_i - x_k) & \cdots & (-1)^r \sum\limits_{i=2}^{r-1} \alpha_i x_i^0 \prod\limits_{j=1}^{1}(x_j - x_i) \prod\limits_{k=r}^{r}(x_i - x_k) & (-1)^{r+1} \sum\limits_{i=2}^{r} \alpha_i x_i^0 (x_1 - x_i) \\
(-1)^2 \sum\limits_{i=3}^{r-1} \alpha_i x_i^{r-1} \prod\limits_{j=1}^{2}(x_j - x_i) \prod\limits_{k=r}^{r}(x_i - x_k) & \cdots & (-1)^{r+1} \sum\limits_{i=3}^{r-1} \alpha_i x_i^0 \prod\limits_{j=1}^{2}(x_j - x_i) \prod\limits_{k=r}^{r}(x_i - x_k) & (-1)^{r+2} \sum\limits_{i=3}^{r} \alpha_i x_i^0 \prod\limits_{j=1}^{2}(x_j - x_i) \\
(-1)^3 \sum\limits_{i=4}^{r-1} \alpha_i x_i^{r-1} \prod\limits_{j=1}^{3}(x_j - x_i) \prod\limits_{k=r}^{r}(x_i - x_k) & \cdots & (-1)^{r+2} \sum\limits_{i=4}^{r-1} \alpha_i x_i^0 \prod\limits_{j=1}^{3}(x_j - x_i) \prod\limits_{k=r}^{r}(x_i - x_k) & (-1)^{r+3} \sum\limits_{i=4}^{r} \alpha_i x_i^0 \prod\limits_{j=1}^{3}(x_j - x_i) \\
(-1)^4 \sum\limits_{i=5}^{r-1} \alpha_i x_i^{r-1} \prod\limits_{j=1}^{4}(x_j - x_i) \prod\limits_{k=r}^{r}(x_i - x_k) & \cdots & (-1)^{r+3} \sum\limits_{i=5}^{r-1} \alpha_i x_i^0 \prod\limits_{j=1}^{4}(x_j - x_i) \prod\limits_{k=r}^{r}(x_i - x_k) & (-1)^{r+4} \sum\limits_{i=5}^{r} \alpha_i x_i^0 \prod\limits_{j=1}^{4}(x_j - x_i) \\
\vdots & \ddots & \vdots & \vdots \\
(-1)^{r-2} \sum\limits_{i=r-1}^{r-1} \alpha_i x_i^{r-1} \prod\limits_{j=1}^{r-2}(x_j - x_i) \prod\limits_{k=r}^{r}(x_i - x_k) & \cdots & (-1)^{2r-3} \sum\limits_{i=r-1}^{r-1} \alpha_i x_i^0 \prod\limits_{j=1}^{r-2}(x_j - x_i) \prod\limits_{k=r}^{r}(x_i - x_k) & (-1)^{2r-2} \sum\limits_{i=r-1}^{r} \alpha_i x_i^0 \prod\limits_{j=1}^{r-2}(x_j - x_i) \\
0 & \cdots & 0 & (-1)^{2r-1} \sum\limits_{i=r}^{r} \alpha_i x_i^0 \prod\limits_{j=1}^{r-1}(x_j - x_i) \\
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0
\end{bmatrix}. \tag{2.21}$$

Similar to above, and the rest may be deduced by analogy, and add x_1 times column 2 to column 1, we get

$$\widehat{\mathbf{H}}_r =$$

$$\begin{bmatrix}
0 & [\widehat{\mathbf{H}}_r]_{12} & [\widehat{\mathbf{H}}_r]_{13} & \cdots & (-1)^{r-1} \sum\limits_{i=1}^{r-1} \alpha_i \prod\limits_{k=r}^{r}(x_i - x_k) & (-1)^r \sum\limits_{i=1}^{r} \alpha_i \\
0 & 0 & [\widehat{\mathbf{H}}_r]_{23} & \cdots & (-1)^r \sum\limits_{i=2}^{r-1} \alpha_i \prod\limits_{j=1}^{1}(x_j - x_i) \prod\limits_{k=r}^{r}(x_i - x_k) & (-1)^{r+1} \sum\limits_{i=2}^{r} \alpha_i \prod\limits_{j=1}^{1}(x_j - x_i) \\
0 & 0 & 0 & \cdots & (-1)^{r+1} \sum\limits_{i=3}^{r-1} \alpha_i \prod\limits_{j=1}^{2}(x_j - x_i) \prod\limits_{k=r}^{r}(x_i - x_k) & (-1)^{r+2} \sum\limits_{i=3}^{r} \alpha_i \prod\limits_{j=1}^{2}(x_j - x_i) \\
0 & 0 & 0 & \cdots & (-1)^{r+2} \sum\limits_{i=4}^{r-1} \alpha_i \prod\limits_{j=1}^{3}(x_j - x_i) \prod\limits_{k=r}^{r}(x_i - x_k) & (-1)^{r+3} \sum\limits_{i=4}^{r} \alpha_i \prod\limits_{j=1}^{3}(x_j - x_i) \\
0 & 0 & 0 & \cdots & (-1)^{r+3} \sum\limits_{i=5}^{r-1} \alpha_i \prod\limits_{j=1}^{4}(x_j - x_i) \prod\limits_{k=r}^{r}(x_i - x_k) & (-1)^{r+4} \sum\limits_{i=5}^{r} \alpha_i \prod\limits_{j=1}^{4}(x_j - x_i) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & (-1)^{2r-3} \sum\limits_{i=r-1}^{r-1} \alpha_i \prod\limits_{j=1}^{r-2}(x_j - x_i) \prod\limits_{k=r}^{r}(x_i - x_k) & (-1)^{2r-2} \sum\limits_{i=r-1}^{r} \alpha_i \prod\limits_{j=1}^{r-2}(x_j - x_i) \\
0 & 0 & 0 & \cdots & 0 & (-1)^{2r-1} \sum\limits_{i=r}^{r} \alpha_i \prod\limits_{j=1}^{r-1}(x_j - x_i) \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}. \tag{2.22}$$

where, $[\widehat{\mathbf{H}}_r]_{12} = (-1)^1 \sum\limits_{i=1}^{1} \alpha_i \prod\limits_{k=2}^{r}(x_i - x_k)$, $[\widehat{\mathbf{H}}_r]_{13} = (-1)^2 \sum\limits_{i=1}^{2} \alpha_i \prod\limits_{k=3}^{r}(x_i - x_k)$,
$[\widehat{\mathbf{H}}_r]_{23} = (-1)^3 \sum\limits_{i=2}^{2} \alpha_i \prod\limits_{j=1}^{1}(x_j - x_i) \prod\limits_{k=3}^{r}(x_i - x_k)$.

Due to $\alpha_i \neq 0, x_i \neq 0, x_j \neq 0, x_k \neq 0$, and $x_i \neq x_j \neq x_k$ for $i \neq j \neq k, i, j, k = 1, 2, \cdots, r$, we can get that the rank of the matrix $\widehat{\mathbf{H}}_r$ is $rank(\widehat{\mathbf{H}}_r) = r$. From the elementary transformations above, we can also see that the nonhomogeneous linear equations (2.11) is equivalent to the preceding r rows of the equations (2.11). Therefore the nonhomogeneous linear equations (2.11) has the solutions, that is,

$$\mathbf{t} = \begin{bmatrix} (-1)^1 h_{r-1} & (-1)^2 h_{r-2} & \cdots & (-1)^{r-1} h_1 & (-1)^r h_0 \\ (-1)^1 h_r & (-1)^2 h_{r-1} & \cdots & (-1)^{r-1} h_2 & (-1)^r h_1 \\ (-1)^1 h_{r+1} & (-1)^2 h_r & \cdots & (-1)^{r-1} h_3 & (-1)^r h_2 \\ (-1)^1 h_{r+2} & (-1)^2 h_{r+1} & \cdots & (-1)^{r-1} h_4 & (-1)^r h_3 \\ (-1)^1 h_{r+3} & (-1)^2 h_{r+2}^3 & \cdots & (-1)^{r-1} h_5 & (-1)^r h_4 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^1 h_{2r-2} & (-1)^2 h_{2r-3} & \cdots & (-1)^{r-1} h_r & (-1)^r h_{r-1} \end{bmatrix}^{-1} \begin{bmatrix} -h_r \\ -h_{r+1} \\ -h_{r+2} \\ -h_{r+3} \\ -h_{r+4} \\ \vdots \\ -h_{2r-1} \end{bmatrix}. \tag{2.23}$$

□

Moreover, we can transform the system (2.12) into the following equations

$$\begin{cases} (-1)^r x_1^r + (-1)^{r-1} t_1 x_1^{r-1} + (-1)^{r-2} t_2 x_1^{r-2} + \cdots + (-1)^1 t_{r-1} x_1^1 + (-1)^0 t_r = 0, \\ (-1)^r x_2^r + (-1)^{r-1} t_1 x_2^{r-1} + (-1)^{r-2} t_2 x_2^{r-2} + \cdots + (-1)^1 t_{r-1} x_2^1 + (-1)^0 t_r = 0, \\ \vdots \\ (-1)^r x_r^r + (-1)^{r-1} t_1 x_r^{r-1} + (-1)^{r-2} t_2 x_r^{r-2} + \cdots + (-1)^1 t_{r-1} x_r^1 + (-1)^0 t_r = 0. \end{cases} \tag{2.24}$$

From the equations (2.24), we know that each equation is a one dimensional equation of the r-degree with the same coefficients, hence the system of equations (2.24) is equivalent to the following equation

$$(-1)^r x^r + (-1)^{r-1} t_1 x^{r-1} + (-1)^{r-2} t_2 x^{r-2} + \cdots + (-1)^1 t_{r-1} x^1 + (-1)^0 t_r = 0. \tag{2.25}$$

Thus, the solutions of the decomposition (1.3) can be obtained by the equation (2.13) and the formula (2.25), i.e.,

$$\mathbf{x} = [x_1, x_2, x_3, \cdots, x_r]^\intercal. \tag{2.26}$$

Now we can input the solution (2.26) to equation (2.2), and according to the formulas (2.8) and (2.10), we can obtain

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & \cdots & x_r \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & \cdots & x_r^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 & x_5^3 & \cdots & x_r^3 \\ x_1^4 & x_2^4 & x_3^4 & x_4^4 & x_5^4 & \cdots & x_r^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{r-1} & x_2^{r-1} & x_3^{r-1} & x_4^{r-1} & x_5^{r-1} & \cdots & x_r^{r-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \vdots \\ \alpha_r \end{bmatrix} = \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ \vdots \\ h_{r-1} \end{bmatrix}. \tag{2.27}$$

and the solution of the equation (2.27) is

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \vdots \\ \alpha_r \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & \cdots & x_r \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & \cdots & x_r^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 & x_5^3 & \cdots & x_r^3 \\ x_1^4 & x_2^4 & x_3^4 & x_4^4 & x_5^4 & \cdots & x_r^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{r-1} & x_2^{r-1} & x_3^{r-1} & x_4^{r-1} & x_5^{r-1} & \cdots & x_r^{r-1} \end{bmatrix}^{-1} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ \vdots \\ h_{r-1} \end{bmatrix}. \tag{2.28}$$

Therefore, the new method to solve problem (1.3) can be stated as follows.

Algorithm 2.2 (This algorithm attempts to solve problem (1.3))**.**
1. Given initial values $h_0, h_1, h_2, \cdots, h_{m(n-1)}$ of the Hankel tensor $\mathcal{A}$ and $rank(\mathcal{A}) = r$.
2. Compute $\mathbf{t} = [t_1, t_2, t_3, t_4, t_5, \cdots, t_r]^\intercal$ by (2.13).
3. Compute $\mathbf{x} = [x_1, x_2, x_3, \cdots, x_r]^\intercal$ by (2.25).
4. Compute $\alpha = [\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_r]^\intercal$ by (2.28).
5. Combining $\mathbf{x}$ and α, the exact Vandermonde decomposition (1.3) of Hankel tensor $\mathcal{A}$ is obtained.

3 Numerical Experiments

In this section, we use some examples to show that the new method is feasible to solve problem (1.3). All experiments are performed in $MATLABR_{2012b}$ on a PC with an Intel Core i7 processor at $2.4GH_z$.

Example 3.1. We consider the 3th order 4-dimensional Hankel tensor $\mathcal{A}$ as below

$$\mathcal{A}(:,:,1) = \begin{pmatrix} 23/10 & 353/50 & 6317/250 & 51095/498 \\ 353/50 & 6317/250 & 51095/498 & 24091/53 \\ 6317/250 & 51095/498 & 24091/53 & 55313/26 \\ 51095/498 & 24091/53 & 55313/26 & 82401/8 \end{pmatrix},$$

$$\mathcal{A}(:,:,2) = \begin{pmatrix} 353/50 & 6317/250 & 51095/498 & 24091/53 \\ 6317/250 & 51095/498 & 24091/53 & 55313/26 \\ 51095/498 & 24091/53 & 55313/26 & 82401/8 \\ 24091/53 & 55313/26 & 82401/8 & 152837/3 \end{pmatrix},$$

$$\mathcal{A}(:,:,3) = \begin{pmatrix} 6317/250 & 51095/498 & 24091/53 & 55313/26 \\ 51095/498 & 24091/53 & 55313/26 & 82401/8 \\ 24091/53 & 55313/26 & 82401/8 & 152837/3 \\ 55313/26 & 82401/8 & 152837/3 & 766576/3 \end{pmatrix},$$

$$\mathcal{A}(:,:,4) = \begin{pmatrix} 51095/498 & 24091/53 & 55313/26 & 82401/8 \\ 24091/53 & 55313/26 & 82401/8 & 152837/3 \\ 55313/26 & 82401/8 & 152837/3 & 766576/3 \\ 82401/8 & 152837/3 & 766576/3 & 1293794 \end{pmatrix},$$

where, $h_0 = 23/10, h_1 = 353/50, h_2 = 6317/250, h_3 = 51095/498, h_4 = 24091/53, h_5 = 55313/26, h_6 = 82401/8, h_7 = 152837/3, h_8 = 766576/3, h_9 = 1293794$, and $r(\mathcal{A}) = 4$.

We use Algorithm 2.1 to solve this problem. Firstly, we get the $\mathbf{t} = [t_1, t_2, t_3, t_4]^\intercal = [79/5, 1803/20, 4357/20, 371/2]^\intercal$ by the equation (2.13), input them to equation (2.25), so we can get the solutions $\mathbf{x} = [x_1, x_2, x_3, x_4]^\intercal = [5.3, 5, 3.5, 2]^\intercal$ by the equation (2.25), then input them to equation (2.28), and we can get $\alpha = [\alpha_1, \alpha_2, \alpha_3, \alpha_4]^\intercal = [0.2, 0.3, 0.6, 1.2]^\intercal$. Therefore, we get the Vandermonde decomposition of the Hankel tensor $\mathcal{A}$, i.e.,

$$\mathcal{A} = \sum_{k=1}^{4} \alpha_k (\mathbf{x}_k \circ \mathbf{x}_k \circ \mathbf{x}_k), \tag{3.1}$$

where $\alpha_1 = 0.2, \alpha_2 = 0.3, \alpha_3 = 0.6, \alpha_4 = 1.2$,
$\mathbf{x}_1 = [1, 5.3^1, 5.3^2, 5.3^3]^\intercal \in \mathbb{R}^4, \mathbf{x}_2 = [1, 5^1, 5^2, 5^3]^\intercal \in \mathbb{R}^4$,
$\mathbf{x}_3 = [1, 3.5^1, 3.5^2, 3.5^3]^\intercal \in \mathbb{R}^4, \mathbf{x}_4 = [1, 2^1, 2^2, 2^3]^\intercal \in \mathbb{R}^4$.

Example 3.2. We consider the 3th order 10-dimensional Hankel tensor $\mathcal{A}$, where $h_0 = 1/2, h_1 = -171/50, h_2 = 42/125, h_3 = -1057/267, h_4 = 1749/118, h_5 = 7405/239, h_6 = 6998/39, h_7 = 28594/57, h_8 = 124270/69, h_9 = 90465/17, h_{10} = 17003, h_{11} = 560011/11, h_{12} = 625735/4, h_{13} = 469667, h_{14} = 1422385, h_{15} = 4271122, h_{16} = 12862307, h_{17} = 38610953, h_{18} = 116013806, h_{19} = 348164064, h_{20} = 1045168064, h_{21} = 3136080811, h_{22} = 9410790796, h_{23} = 28234957768, h_{24} = 84714564350, h_{25} = 254154938458, h_{26} = 762501952016, h_{27} = 2287553766609$, and $r(\mathcal{A}) = 5$.

We use Algorithm 2.2 to solve this problem. Firstly, we get the $\mathbf{t} = [t_1, t_2, t_3, t_4, t_5]^\intercal = [39/10, -34/25, -741/50, -144/25, 162/25]^\intercal$ by the equation (2.13), input them to equation (2.25), so we can get the solutions $\mathbf{x} = [x_1, x_2, x_3, x_4, x_5]^\intercal = [3, -9/5, 2, 6/5, -1/2]^\intercal$ by the equation (2.25), then input them to equation (2.28), and we can get $\alpha = [\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5]^\intercal = [3/10, 3/5, -9/10, -7/10, 6/5]^\intercal$. Therefore, we get the Vandermonde decomposition of the Hankel tensor $\mathcal{A}$, i.e.,

$$\mathcal{A} = \sum_{k=1}^{5} \alpha_k (\mathbf{x}_k \circ \mathbf{x}_k \circ \mathbf{x}_k), \tag{3.2}$$

where $\alpha_1 = 3/10, \alpha_2 = 3/5, \alpha_3 = -9/10, \alpha_4 = -7/10, \alpha_5 = 6/5$,

$$\begin{aligned}
\mathbf{x}_1 &= [1, 3^1, 3^2, 3^3, 3^4, 3^5, 3^6, 3^7, 3^8, 3^9]^\intercal \in \mathbb{R}^{10},\\
\mathbf{x}_2 &= [1, (-1.8)^1, (-1.8)^2, (-1.8)^3, (-1.8)^4, (-1.8)^5, (-1.8)^6, (-1.8)^7, (-1.8)^8, (-1.8)^9]^\intercal \in \mathbb{R}^{10},\\
\mathbf{x}_3 &= [1, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^9]^\intercal \in \mathbb{R}^{10},\\
\mathbf{x}_4 &= [1, 1.2^1, 1.2^2, 1.2^3, 1.2^4, 1.2^5, 1.2^6, 1.2^7, 1.2^8, 1.2^9]^\intercal \in \mathbb{R}^{10},\\
\mathbf{x}_5 &= [1, (-0.5)^1, (-0.5)^2, (-0.5)^3, (-0.5)^4, (-0.5)^5, (-0.5)^6, (-0.5)^7, (-0.5)^8, (-0.5)^9]^\intercal \in \mathbb{R}^{10}.
\end{aligned}$$

Example 3.3. We consider the 10th order 20-dimensional Hankel tensor $\mathcal{A}$ (There are a large number of elements in the Hankel tensor $\mathcal{A}$, and its element values are very large. For simplicity, it is not shown here.), which is generated by MATLAB and $r(\mathcal{A}) = 7$.

We use Algorithm 2.2 to solve this problem. Firstly, we get the $\mathbf{t} = [t_1, t_2, t_3, t_4, t_5, t_6, t_7]^\intercal = [18, -3102/25, -13709/33, -8551/12, 7883/13, -27737/123, 648/25]^\intercal$ by the equation (2.13), input them to equation (2.25), so we can get the solutions $\mathbf{x} = [x_1, x_2, x_3, x_4, x_5, x_6, x_7]^\intercal = [6, 5, 3, 2, 6/5, 3/5, 1/5]^\intercal$ by the equation (2.25), then input them to equation (2.28), and we can get $\alpha = [\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7]^\intercal = [27/10, 3/2, 8/25, 6, 9, 5, 7/2]^\intercal$. Therefore, we get the Vandermonde decomposition of the Hankel tensor $\mathcal{A}$, i.e.,

$$\mathcal{A} = \sum_{k=1}^{7} \alpha_k (\underbrace{\mathbf{x}_k \circ \mathbf{x}_k \circ \cdots \circ \mathbf{x}_k}_{10}), \tag{3.3}$$

where $\alpha_1 = 27/10, \alpha_2 = 3/2, \alpha_3 = 8/25, \alpha_4 = 6, \alpha_5 = 9, \alpha_6 = 5, \alpha_7 = 7/2$,
$\mathbf{x}_1 = [1, 6^1, 6^2, 6^3, 6^4, 6^5, \cdots, 6^{18}, 6^{19}]^\intercal \in \mathbb{R}^{20}$,
$\mathbf{x}_2 = [1, 5^1, 5^2, 5^3, 5^4, 5^5, 5^6, \cdots, 5^{18}, 5^{19}]^\intercal \in \mathbb{R}^{20}$,
$\mathbf{x}_3 = [1, 3^1, 3^2, 3^3, 3^4, 3^5, 3^6, \cdots, 3^{18}, 3^{19}]^\intercal \in \mathbb{R}^{20}$,
$\mathbf{x}_4 = [1, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, \cdots, 2^{18}, 2^{19}]^\intercal \in \mathbb{R}^{20}$,
$\mathbf{x}_5 = [1, 1.2^1, 1.2^2, 1.2^3, 1.2^4, 1.2^5, 1.2^6, \cdots, 1.2^{18}, 1.2^{19}]^\intercal \in \mathbb{R}^{20}$,
$\mathbf{x}_6 = [1, 0.6^1, 0.6^2, 0.6^3, 0.6^4, 0.6^5, 0.6^6, \cdots, 0.6^{18}, 0.6^{19}]^\intercal \in \mathbb{R}^{20}$,
$\mathbf{x}_7 = [1, 0.2^1, 0.2^2, 0.2^3, 0.2^4, 0.2^5, 0.2^6, \cdots, 0.2^{18}, 0.2^{19}]^\intercal \in \mathbb{R}^{20}$.

4 Conclusion

The exact Vandermonde decomposition problem of Hankel tensor is studied in this paper. We first reformulate this problem as the systems of nonlinear equations, then design a new method to solve this problem. Finally, we use some numerical examples to illustrate that the new method is feasible and effective to solve the problem. In future works, we plan to get the exact decomposition for other tensors, such as general tensor, symmetric tensor, and Toeplitz tensor.

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