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A CONVERGENCE ANALYSIS FOR AN ALGORITHM COMPUTING A SYMMETRIC LOW RANK ORTHOGONAL APPROXIMATION OF A SYMMETRIC TENSOR*

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Abstract: In this paper, we present an algorithm for solving the problem of the symmetric low rank orthogonal tensor approximation for a given symmetric tensor. Proximality technique and shifted power technique are tailored into this algorithm. Interestingly, we can show that this algorithm converges globally without any assumption once the parameters are chosen appropriately, and moreover the convergence rate is sublinear with an explicitly given rate and it is better than the usual $O(\frac{1}{p})$ of first order methods in optimization.

Key words: *symmetric tensor, low rank orthogonal approximation, polar decomposition, global convergence, sublinear rate*

Mathematics Subject Classification: *15A18, 15A69*

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1 Introduction

Tensors are the higher order equivalents of vectors and matrices, and they play a fundamental role in data analysis, signal processing, machine learning and scientific computing [10,23,36]. In computations of tensors, low rank approximations of tensors are basic research topics and are the cornerstone for further study as well as applications of tensor techniques [23,26]. In this paper, we will focus on approximating a given symmetric tensor by a symmetric low rank tensor which is also orthogonal.

The problem of approximating a given tensor by low rank tensors has a rich and long history, dating back to the research of invariants in algebraic geometry [26]. Some earlier works can be found in [8,17,38], with both deep theoretical investigations as well as fantastic real applications. The research on orthogonal tensor approximations also has a rich history, partly from a formal generalization of the celebrated Echart-Young theorem for the matrix singular value decomposition, and partly from applications arising from diverse areas where orthogonality is a necessity, we refer to [9,14,16,20,21,30,37,39,40] and references therein for more details. While, we can see from the literature that most works are designed for tensors in the multilinear case, not for symmetric tensors. Works on symmetric orthogonal

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decomposable tensors and symmetric low rank orthogonal approximations (SLROA) of given tensors are much fewer than the general nonsymmetric case. Nevertheless, symmetric tensors have sources from a very broad areas with important applications [26,29]. Moreover, stable progress has been carried out in recent years, such as algebraic variety characterization in [7], a numerical identifiability method [22], approximation method for structured cases [30], and Jacobi-type methods [27], see references therein. The mostly investigated case is the best rank one approximation, see [18,23,24,33] and references therein. In [32], a method, together with a global convergence analysis, is proposed for the symmetric low rank orthogonal approximation problem, and applications are demonstrated in image processing. Under mild conditions, the algorithm is shown to be converged for a variant of the symmetric low rank orthogonal approximation problem. Convergence study in the general case with application to this problem is also studied in [28]. In [12], the hypothesis guaranteeing global convergence in [32] is vastly waived to only on a condition for the input parameter of the algorithm. In the same paper, a problem is raised on establishing an algorithm with nice convergence properties for the SLROA problem directly, other than a variant of it. A main purpose of this paper is presenting a complete answer to this question.

In this paper, the proximality technique introduced in [20] along the polar decomposition in tensor low rank approximations and the shifted power method technique introduced in [24] will be combined and tailored to the SLROA problem. Then, with advanced techniques developed recently in the literature, we will present the global convergence of the proposed algorithm for SLROA without any assumption, other than appropriately chosen parameters. Sublinear convergence with explicit rate will also be presented.

The rest paper is organized as follows. Some preliminaries on the symmetric best low rank orthogonal tensor approximation problem is described in Section 2. Section 3 presents some technical lemmas that are necessary for the subsequent analysis. The global convergence is established in Sections 4 and 5, corresponding respectively to the global convergence analysis with proximality and without proximality in the implementation. Section 6 gives a sublinear convergence rate analysis of the studied algorithm. Some final remarks are given in the last section to conclude this paper.

2 Preliminaries and Algorithmic Description

2.1 Notation

In this subsection, we review some basic notions of tensors and give some notations. Let $m \geq 3$ and n be given positive integers and $\mathbb{R}$ the field of real numbers. We denote $\otimes^m\mathbb{R}^n$ as the space of real tensors of order m and dimension n. The subspace of symmetric tensors inside $\otimes^m\mathbb{R}^n$ is denoted as $\mathrm{S}(\otimes^m\mathbb{R}^n)$. We refer to [23,26] for more details on tensors.

Let $\mathcal{A} \in \mathrm{S}(\otimes^m\mathbb{R}^n)$ be a given symmetric tensor with entries $a_{i_1,\dots,i_m}$, $k \leq m$ and $\mathbf{x} = (x_1, x_2, \cdots, x_n)^\top \in \mathbb{R}^n$. We define the tensor-vector product $\mathcal{A}\mathbf{x}^k \in \mathrm{S}(\otimes^{m-k}\mathbb{R}^n)$ via

$$(\mathcal{A}\mathbf{x}^k)_{i_{k+1},\cdots,i_m} := \sum_{i_1,\cdots,i_k=1}^{n} a_{i_1,\cdots,i_m} x_{i_1} \cdots x_{i_k}, \text{ for all } i_{k+1}, \dots, i_m \in \{1, \dots, n\}.$$

It is easy to see that this definition is well-defined by the symmetry of $\mathcal{A}$.

Let $\mathbf{x}^m$ be the symmetric *decomposable tensor* in $\mathrm{S}(\otimes^m\mathbb{R}^n)$ defined via

$$(\mathbf{x}^m)_{i_1,\cdots,i_m} := x_{i_1} \cdots x_{i_m}, \quad 1 \leq i_1, \cdots, i_m \leq n.$$

Define the *Hilbert-Schmidt inner product* of two given tensors $\mathcal{A}, \mathcal{B} \in \otimes^m \mathbb{R}^n$, as follows:

$$\langle \mathcal{A}, \mathcal{B} \rangle := \sum_{i_1=1}^{n_1} \cdots \sum_{i_m=1}^{n_m} a_{i_1,\cdots,i_m} b_{i_1,\cdots,i_m}.$$

Accordingly, the *Hilbert-Schmidt norm* $\|\mathcal{A}\|$ of $\mathcal{A}$ is defined by

$$\|\mathcal{A}\| := \sqrt{\langle \mathcal{A}, \mathcal{A} \rangle}.$$

In this paper, the *spectral radius* $\rho(\mathcal{A})$ of $\mathcal{A}$ is involved, which is defined as

$$\rho(\mathcal{A}) := \max\{|\langle \mathcal{A}, \mathbf{x}^m \rangle| \colon \mathbf{x}^\top \mathbf{x} = 1\}, \tag{2.1}$$

which is equal to

$$\max\{|\langle \mathcal{A}\mathbf{x}^{m-2}, \mathbf{y}^2 \rangle| \colon \mathbf{x}^\top \mathbf{x} = 1 \text{ and } \mathbf{y}^\top \mathbf{y} = 1\} = \max\{\rho(\mathcal{A}\mathbf{x}^{m-2}) \colon \mathbf{x}^\top \mathbf{x} = 1\}$$

by Banach's theorem of [3]. It can be shown that

$$\rho(\mathcal{A}) \leq \|\mathcal{A}\|.$$

Here if $m = 2$, then $\rho(\mathcal{A})$ becomes the spectral radius of the underlying matrix. For more discussions on the spectral radius and its consequence on the convergence analysis for tensor approximation problems, we refer to [33] and references herein.

Finally, for a given matrix X, the *Frobenius norm* of X is denoted by $\|X\|_F$ and the spectral norm of X is denoted by $\|X\|$, which is equal to $\rho(X)$.

2.2 The symmetric low rank orthogonal approximation problem

Next, we illustrate the *symmetric low rank orthogonal approximation* (SLROA) problem for a given symmetric tensor.

Let $r \leq n$ be a given positive integer and $\mathcal{A} \in \mathrm{S}(\otimes^m \mathbb{R}^n)$ be a given nonzero symmetric tensor. Let $X = [\mathbf{x}_1, \cdots, \mathbf{x}_r] \in \mathbb{R}^{n \times r}$ be a matrix with its i-th column being $\mathbf{x}_i$ and $\lambda = (\lambda_1, \ldots, \lambda_r)^\top \in \mathbb{R}^r$. The SLROA problem is characterized by the following optimization problem:

$$\begin{array}{ll} \min\limits_{X,\lambda} & F(X, \lambda) := \|\mathcal{A} - \sum\limits_{i=1}^{r} \lambda_i (\mathbf{x}_i)^m\|^2 \\ \text{s.t.} & X^\top X = I, \end{array} \tag{2.2}$$

where I is the identity matrix of appropriate size. The constraint set is the *Stiefel manifold* $\mathrm{St}(r, n) := \{X \in \mathbb{R}^{n \times r} \colon X^\top X = I\}$.

Problem (2.2) is a nonlinear least square problem with orthogonality constraint. It is very difficult to solve it, even when $r = 1$ [18]. Moreover, even if an algorithm is designed to solve (2.2), the convergence analysis starting from (2.2) is subtle. Actually, utilizing the Lagrange multiplier theory [5], the optimization problem (2.2) can be reformulated equivalently as the following maximization problem:

$$\begin{array}{ll} \max & \sum\limits_{i=1}^{r} (\mathcal{A}(\mathbf{x}_i)^m)^2 \\ \text{s.t.} & X^\top X = I. \end{array} \tag{2.3}$$

Let the objective function of (2.3) be defined as

$$f(X) := \sum_{i=1}^{r} (\mathcal{A}(\mathbf{x}_i)^m)^2. \tag{2.4}$$

In this paper, we will propose an algorithm for solving the maximization problem (2.3) and present a global convergence of the algorithm.

Before that, we give the optimality condition of (2.3), which is crucial for the tolerance of the algorithm and the convergence analysis. It follows from [1, 35] that at an optimizer X^* of (2.3) we have

$$\nabla f(X^*) = 2X^* S^* \tag{2.5}$$

for a unique symmetric matrix S^*. Actually, (2.5) characterizes all the KKT points of problem (2.3).

Algorithm A An algorithm for SLROA

Input: Given a symmetric tensor $\mathcal{A} \in \mathrm{S}(\otimes^m \mathbb{R}^n)$, an orthonormal matrix $X^{(0)}$, a parameter γ, a proximal parameter ε, a criterion tolerance η.

Output: $\lambda = (\lambda_1, \cdots, \lambda_r)^\top, X = [\mathbf{x}_1, \cdots, \mathbf{x}_r] \in \mathrm{St}(r, n)$.

for $i = 1, \cdots, r$, **do**
  $\lambda_i^{(0)} := \mathcal{A}(\mathbf{x}_i^{(0)})^m$
end for
for $p = 0, 1, 2, \ldots,$ **do**
  for $i = 1, \cdots, r$, **do**
    $\mathbf{v}_i^{(p+1)} := \lambda_i^{(p)} \mathcal{A}(\mathbf{x}_i^{(p)})^{m-1} + \frac{\gamma+\epsilon}{m} \mathbf{x}_i^{(p)}$
  end for
  $V^{(p+1)} := \left[\mathbf{v}_1^{(p+1)}, \cdots, \mathbf{v}_r^{(p+1)}\right]$
  $X^{(p+1)} :=$ Polar orthogonal factor of $V^{(p+1)}$
  for $i = 1, \cdots, r$, **do**
    $\lambda_i^{(p+1)} := \mathcal{A}(\mathbf{x}_i^{(p+1)})^m$
  end for
  if $\left\| X^{(p+1)} (\nabla f(X^{(p+1)}))^\top X^{(p+1)} - \nabla f(X^{(p+1)}) \right\|_F < \eta$**, then**
    break
  end if
end for

The termination criterion is chosen as the more standard KKT condition for optimization problem (2.3).

In this paper, we will show the following result.

Theorem 2.1 (Global Convergence with Sublinear Rate)**.** *Under either of the following conditions*

(a) $\gamma > m(m-1)\rho(\mathcal{A})^2$ *and* $\epsilon > 0$*,*

(b) $\gamma > m \max\{2\sqrt{r}, m-1\}\rho(\mathcal{A})^2$ *and* $\epsilon \geq 0$*,*

the iterative sequence $\{X^{(p)}\}$ *converges for all tensors in* $\mathrm{S}(\otimes^m \mathbb{R}^n)$ *to a KKT point of* (2.3) *with convergence rate at least* $O(p^{-1-\kappa})$ *for some constant* $\kappa \in (0,1)$ *depending only on* m *and* n*.*

Proof. The result follows from Theorems 4.1, 5.1 and 6.1. $\square$

3 Technical Lemmas

Let $\mathrm{St}(r,n) := \{X \in \mathbb{R}^{n\times r} : X^\top X = I\}$. The indicator function $\delta_{\mathrm{St}(r,n)}$ of $\mathrm{St}(r,n)$ is defined as

$$\delta_{\mathrm{St}(r,n)}(X) := \begin{cases} 0 & \text{if } X \in \mathrm{St}(r,n), \\ +\infty & \text{otherwise.} \end{cases}$$

The *subdifferential* of the indicator function $\delta_{\mathrm{St}(r,n)}$ at $X \in \mathrm{St}(r,n)$ is (cf. [35])

$$\partial\delta_{\mathrm{St}(r,n)}(X) = N_{\mathrm{St}(r,n)}(X),$$

where $N_{\mathrm{St}(r,n)}(X)$ is the *normal cone* of $\mathrm{St}(r,n)$ at $X \in \mathrm{St}(r,n)$, and from [1,13] that

$$N_{\mathrm{St}(r,n)}(X) = \{XS \mid S \in \mathrm{S}^{r\times r}\},$$

where $\mathrm{S}^{r\times r} \subset \mathbb{R}^{r\times r}$ is the subspace of $r \times r$ real symmetric matrices.

The Kurdyka-Łojasiewicz property is needed in our proof, we refer to [6,25] for more details. Let p be an extended real-valued function and $\partial p(\mathbf{x})$ be the set of subdifferentials of p at $\mathbf{x}$. Let $\mathbf{x}^* \in dom(\partial p)$, where $dom(\partial p) := \{\mathbf{x} : \partial p(\mathbf{x}) \neq \varnothing\}$. If there exist some $\eta \in (0,+\infty]$, a neighborhood U of $\mathbf{x}^*$, and a continuous concave function $\varphi : [0,\eta) \to \mathbb{R}_+$, such that

1. $\varphi(0) = 0$,
2. φ is continuously differentiable on $(0,\eta)$,
3. for all $s \in (0,\eta)$, $\varphi'(s) > 0$, and
4. for all $\mathbf{x} \in U \cap \{\mathbf{x} : p(\mathbf{x}^*) < p(\mathbf{x}) < p(\mathbf{x}^*) + \eta\}$, the Kurdyka-Łojasiewicz inequality holds
$$\varphi'(p(\mathbf{x}) - p(\mathbf{x}^*))\,\mathrm{dist}(0, \partial p(\mathbf{x})) \geq 1,$$

then we say that p has the Kurdyka-Łojasiewicz property (abbreviated as KL) at $\mathbf{x}^*$. If p has the KL property at every point, then we say that p is a KL function.

A *critical point* of a proper lower semicontinuous function p is a point $\mathbf{x}$ such that $\mathbf{0} \in \partial p(\mathbf{x})$.

The following abstract convergence result is classic [2].

Lemma 3.1 (Abstract Convergence)**.** *Let $p : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a proper lower semicontinuous function and $\{\mathbf{x}^{(k)}\} \subset \mathbb{R}^n$ be a sequence satisfying the following properties*

(a) *there is a constant $\alpha > 0$ such that*
$$p(\mathbf{x}^{(k)}) - p(\mathbf{x}^{(k+1)}) \geq \alpha\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\|^2,$$

(b) *there is a constant $\beta > 0$ and a $\mathbf{w}^{(k+1)} \in \partial p(\mathbf{x}^{(k+1)})$ such that*
$$\|\mathbf{w}^{(k+1)}\| \leq \beta\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\|,$$

3. *there is a subsequence $\{\mathbf{x}^{(k_i)}\}$ of $\{\mathbf{x}^{(k)}\}$ and $\mathbf{x}^* \in \mathbb{R}^n$ such that*
$$\mathbf{x}^{(k_i)} \to \mathbf{x}^* \text{ and } p(\mathbf{x}^{(k_i)}) \to p(\mathbf{x}^*) \text{ as } i \to \infty.$$

If p has the Kurdyka-Łojasiewicz property at the point $\mathbf{x}^*$, then the whole sequence $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}^*$, and $\mathbf{x}^*$ is a critical point of p.

Lemma 3.2. *Let $\mathcal{A} \in \mathrm{S}(\otimes^m \mathbb{R}^n)$ be a symmetric tensor and $X = \left[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_r\right] \in \mathrm{St}(r,n)$ be an orthonormal matrix. Define $K := \left[\mathcal{A}(\mathbf{x}_1)^{m-1}, \cdots, \mathcal{A}(\mathbf{x}_r)^{m-1}\right] \Lambda \in \mathbb{R}^{n\times r}$ with $\Lambda = diag(\lambda_1, \cdots, \lambda_r)$ and $\lambda_i := \mathcal{A}(\mathbf{x}_i)^m$, then we have $\rho(X^\top K + K^\top X) \leq 2\sqrt{r}\rho(\mathcal{A})^2$.*

Proof. Let $B := \left[\mathcal{A}(\mathbf{x}_1)^{m-1}, \cdots, \mathcal{A}(\mathbf{x}_r)^{m-1}\right] \in \mathbb{R}^{n\times r}$, so we have $K = B\Lambda$. We know that $\rho(X^\top K + K^\top X) \leq \left\|X^T K + K^\top X\right\|_F$, where $\|\cdot\|_F$ represents the standard Frobenius norm of a given matrix. Since X is an orthonormal matrix, it follows that

$$\begin{aligned}
\rho(X^\top K + K^\top X) &\leq \left\|X^\top K + K^\top X\right\|_F \\
&\leq \left\|X^\top K\right\|_F + \left\|K^\top X\right\|_F \\
&\leq 2\left\|K\right\|_F \\
&= 2\left\|B\Lambda\right\|_F \\
&\leq 2\left\|B\right\|_F \left\|\Lambda\right\|.
\end{aligned} \tag{3.1}$$

It also follows from (2.1) that $\left\|\mathcal{A}(\mathbf{x}_i)^{m-1}\right\| \leq \rho(\mathcal{A})$. Consequently, we have

$$\|B\|_F^2 = \left\|\left[\mathcal{A}(\mathbf{x}_1)^{m-1}, \cdots, \mathcal{A}(\mathbf{x}_r)^{m-1}\right]\right\|_F^2 = \sum_{i=1}^{r} \left\|\mathcal{A}(\mathbf{x}_i)^{m-1}\right\|^2 \leq r\rho(\mathcal{A})^2. \tag{3.2}$$

We also have

$$\|\Lambda\|^2 = \max_{1\leq i\leq r} \lambda_i^2 \leq \rho(\mathcal{A})^2. \tag{3.3}$$

Combining (3.1), (3.2)and (3.3), we obtain the desired conclusion

$$\rho(X^\top K + K^\top X) \leq 2\sqrt{r}\rho(\mathcal{A})^2.$$

The conclusion then follows. □

The spectral radius of a tensor is not easy to calculate, while $\|\mathcal{A}\|$ can be computed out. By $\rho(\mathcal{A}) \leq \|\mathcal{A}\|$, we have $\rho(X^\top K + K^\top X) \leq 2\sqrt{r}\left\|\mathcal{A}\right\|^2$. Let $\|X\|_{2,\infty}$ be the $(2,\infty)$-norm of a given matrix $X = [\mathbf{x}_1, \ldots, \mathbf{x}_r] \in \mathbb{R}^{n\times r}$, defined as

$$\|X\|_{2,\infty} := \max\{\|\mathbf{x}_i\| \colon i = 1, \ldots, r\}.$$

Lemma 3.3. *Let $\mathcal{A} \in \mathrm{S}(\otimes^m \mathbb{R}^n)$ be a symmetric tensor and $X = \left[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_r\right] \in \mathrm{St}(r,n)$ be an orthonormal matrix. Define $g(X) := \sum_{i=1}^{r} (\mathcal{A}(\mathbf{x}_i)^m)^2 + \gamma\|X\|_F^2$. If $\gamma > m(m-1)\rho(\mathcal{A})^2$, then $g(X)$ is convex over an open neighborhood of the unit disc $\Omega = \{X \mid X \in \mathbb{R}^{n\times r}, \|X\|_{2,\infty} \leq 1\}$.*

Proof. As $g(X) = \sum_{i=1}^{r} (\mathcal{A}(\mathbf{x}_i)^m)^2 + \gamma\|X\|_F^2$, we have

$$\nabla g(X) = 2m\left[\lambda_1\mathcal{A}(\mathbf{x}_1)^{m-1}, \ldots, \lambda_r\mathcal{A}(\mathbf{x}_r)^{m-1}\right] + 2\gamma X \in \mathbb{R}^{n\times r}.$$

Therefore, $\nabla^2 g(X)$ is a linear operator such that

$$\langle\nabla^2 g(X)Y, Y\rangle = 2\gamma\|Y\|_F^2 + 2m(m-1)\sum_{i=1}^{r} \lambda_i\mathcal{A}(\mathbf{x}_i)^{m-2}\mathbf{y}_i^2 + 2m^2\sum_{i=1}^{r}(\mathcal{A}(\mathbf{x}_i)^{m-1}\mathbf{y}_i)^2$$

for all $Y \in \mathbb{R}^{n\times r}$. Therefore, whenever

$$\rho(m(m-1)\lambda_i \mathcal{A}(\mathbf{x}_i)^{m-2}) < \gamma,$$

we can conclude that g is a convex function over an open neighborhood of Ω [34].

On the other hand, we have that

$$\rho(\lambda_i \mathcal{A}(\mathbf{x}_i)^{m-2}) \le \rho(\mathcal{A})^2.$$

Consequently, whenever

$$\gamma > m(m-1)\rho(\mathcal{A})^2,$$

the convexity conclusion follows. □

The following polar decomposition is classic [15].

Lemma 3.4 (Polar Decomposition)**.** *Let $A \in \mathbb{R}^{n\times r}$ with $n \ge r$. Then there exist an orthonormal matrix $X \in \mathrm{St}(r,n)$ and a unique symmetric positive semidefinite matrix $P \in \mathrm{S}^{r\times r}$ such that $A = XP$ and*

$$X \in \mathsf{argmax}\{\langle Q, A\rangle : Q \in \mathrm{St}(r,n)\}.$$

Moreover, if A is of full rank , then the matrix X is uniquely determined and P is positive definite.

The next result can be found in [20].

Lemma 3.5 (Error Bound)**.** *If $A \in \mathbb{R}^{n\times r}$ is of full rank, and the polar decomposition of A is $A = XP$ with $X \in \mathrm{St}(r,n)$. Let $\sigma_{\min}$ be the minimum positive singular value of A, then for all $Y \in \mathrm{St}(r,n)$, we have*

$$\langle A, X - Y\rangle \ \ge \frac{\sigma_{\min}}{2}\left\|X - Y\right\|_F^2 .$$

Lemma 3.6 ([20])**.** *For any orthonormal matrices $U, W \in V(m,n)$, we have*

$$\|U^\top W - I\|_F^2 \le \|U - W\|_F^2.$$

The next result is a basic fact in matrix analysis, which is recorded for subsequent analysis.

Lemma 3.7. *Let $r \le n$ and $A \in \mathbb{R}^{n\times r}$, if $A^\top A$ is a positive definite matrix, then A is of full rank. If $\lambda_{\min}$ is the minimum eigenvalue of $A^\top A$, then $\sigma_{\min} = \sqrt{\lambda_{\min}}$ is the minimum singular value of A.*

The next result shows the Lipschitz property of the Veronese mapping over the sphere.

Lemma 3.8. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be unit vectors, then we have $\|\mathbf{x}^m - \mathbf{y}^m\| \le m\,\|\mathbf{x} - \mathbf{y}\|$.*

Proof. We have

$$\begin{aligned}
\|\mathbf{x}^m - \mathbf{y}^m\| &= \left\|\mathbf{x}^{m-1}\otimes\mathbf{x} - \mathbf{x}^{m-1}\otimes\mathbf{y} + \mathbf{x}^{m-1}\otimes\mathbf{y} - \mathbf{y}^{m-1}\otimes\mathbf{y}\right\| \\
&= \left\|\mathbf{x}^{m-1}\otimes(\mathbf{x}-\mathbf{y}) + (\mathbf{x}^{m-1}-\mathbf{y}^{m-1})\otimes\mathbf{y}\right\| \\
&\le \left\|\mathbf{x}^{m-1}\otimes(\mathbf{x}-\mathbf{y})\right\| + \left\|(\mathbf{x}^{m-1}-\mathbf{y}^{m-1})\otimes\mathbf{y}\right\| \\
&\le \left\|\mathbf{x}^{m-1}\right\|\|\mathbf{x}-\mathbf{y}\| + \left\|\mathbf{x}^{m-1}-\mathbf{y}^{m-1}\right\|\|\mathbf{y}\| .
\end{aligned}$$

We know $\mathbf{x}$ and $\mathbf{y}$ are unit vectors, it is easy to see $\left\|\mathbf{x}^{m-1}\right\| = 1$ and $\|\mathbf{y}\| = 1$. It follows that

$$\|\mathbf{x}^m - \mathbf{y}^m\| \le \|\mathbf{x} - \mathbf{y}\| + \left\|\mathbf{x}^{m-1} - \mathbf{y}^{m-1}\right\|. \tag{3.4}$$

Inductively, we have

$$\left\|\mathbf{x}^{m-1} - \mathbf{y}^{m-1}\right\| \le \|\mathbf{x} - \mathbf{y}\| + \left\|\mathbf{x}^{m-2} - \mathbf{y}^{m-2}\right\|.$$

By putting the above inequality into (3.4) and using an induction, we get that $\|\mathbf{x}^m - \mathbf{y}^m\| \le m\|\mathbf{x} - \mathbf{y}\|$. $\square$

Below is the Łojasiewicz's gradient inequality for polynomials (cf. [11]) which will play a key role in our sublinear convergence rate analysis.

Lemma 3.9 (Łojasiewicz's Gradient Inequality for Polynomials)**.** *Let f be a real polynomial on $\mathbb{R}^n$ with degree $d \in \mathbb{N}$. Suppose that $f(\mathbf{0}) = 0$ and $\nabla f(\mathbf{0}) = \mathbf{0}$. Then there exist constants $c, \epsilon > 0$ such that for all $\|\mathbf{x}\| \le \epsilon$, we have*

$$\|\nabla f(\mathbf{x})\| \ge c|f(\mathbf{x})|^\kappa \quad \textit{with} \quad \kappa = 1 - \frac{1}{d(3d-3)^{n-1}}.$$

4 The Global Convergence with Proximality

We are now in the position to present the global convergence of Algorithm A with proximality, one of our main results.

Theorem 4.1 (Convergence under Proximality)**.** *Let $\mathcal{A} \in \mathrm{S}(\otimes^m \mathbb{R}^n)$ be a symmetric tensor. Suppose that $\epsilon > 0$ and $\gamma > m(m-1)\rho(\mathcal{A})^2$. Then any sequence $\{X^{(p)}\}$ generated by Algorithm A converges to a KKT point X^* of the problem* (2.3).

Proof. Recall that $f(X) = \sum_{i=1}^{r} (\mathcal{A}(\mathbf{x}_i)^m)^2$, and let $h(X) := -f(X) + \delta_{\mathrm{St}(r,n)}(X)$. Then we have

$$\nabla f(X) = 2m\left[\mathcal{A}(\mathbf{x}_1)^{m-1}, \ldots, \mathcal{A}(\mathbf{x}_r)^{m-1}\right]\Lambda, \tag{4.1}$$

where

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_r)$$

with $\lambda_i = \mathcal{A}(\mathbf{x}_i)^m$.

With the function h, we can see that the problem (2.3) is equivalent to the following unconstrained optimization problem

$$-\min_{X \in \mathbb{R}^{n\times r}} h(X). \tag{4.2}$$

In the following, we will apply Lemma 3.1 to problem (4.2). Thus, the rest proof is divided into three parts for clarity accordingly.

Part I. Let $X \in \mathbb{R}^{n\times r}$ and $g(X) = \sum_{i=1}^{r} (\mathcal{A}(\mathbf{x}_i)^m)^2 + \gamma\|X\|_F^2$ as before. We will restrict the function g over a suitable open neighborhood of the unit disc $\Omega = \{X \mid X \in \mathbb{R}^{n\times r}, \|X\|_{2,\infty} \le 1\}$. It can be shown that g is convex over such a neighborhood of Ω when $\gamma > m(m-1)\rho(\mathcal{A})^2$ by Lemma 3.3. We also have that

$$\begin{aligned}\nabla g(X^{(p)}) &= 2m\left[\mathcal{A}(\mathbf{x}_1^{(p)})^{m-1}, \ldots, \mathcal{A}(\mathbf{x}_r^{(p)})^{m-1}\right]\Lambda^{(p)} + 2\gamma X^{(p)} \\ &= 2mK^{(p)} + 2\gamma X^{(p)},\end{aligned} \tag{4.3}$$

where $\Lambda^{(p)} = \mathrm{diag}(\lambda_1^{(p)}, \cdots, \lambda_r^{(p)})$ and $K^{(p)} = \left[\mathcal{A}(\mathbf{x}_1^{(p)})^{m-1}, \ldots, \mathcal{A}(\mathbf{x}_r^{(p)})^{m-1}\right] \Lambda^{(p)}$.

On the one hand, we have

$$
\begin{aligned}
& h(X^{(p)}) - h(X^{(p+1)}) \\
&= f(X^{(p+1)}) - f(X^{(p)}) \\
&= \sum_{i=1}^{r} \left(\mathcal{A}(\mathbf{x}_i^{(p+1)})^m\right)^2 - \sum_{i=1}^{r} \left(\mathcal{A}(\mathbf{x}_i^{(p)})^m\right)^2 \\
&= \left(\sum_{i=1}^{r} \left(\mathcal{A}(\mathbf{x}_i^{(p+1)})^m\right)^2 + \gamma \|X^{(p+1)}\|_F^2\right) - \left(\sum_{i=1}^{r} \left(\mathcal{A}(\mathbf{x}_i^{(p)})^m\right)^2 + \gamma \|X^{(p)}\|_F^2\right) \\
&= g(X^{(p+1)}) - g(X^{(p)}),
\end{aligned} \tag{4.4}
$$

where we used the fact that both $X^{(p+1)} \in \mathrm{St}(r,n)$ and $X^{(p)} \in \mathrm{St}(r,n)$.

On the other hand, we know that $X^{(p+1)}$ is the polar orthonormal matrix of $V^{(p+1)}$ by Algorithm A, which is equal to the $K^{(p)} + \frac{\gamma+\epsilon}{m} X^{(p)}$, so by Lemma 3.4 we have that

$$
\left\langle X^{(p+1)}, K^{(p)} + \frac{\gamma+\epsilon}{m} X^{(p)} \right\rangle \geq \left\langle X^{(p)}, K^{(p)} + \frac{\gamma+\epsilon}{m} X^{(p)} \right\rangle,
$$

i.e.,

$$
\left\langle \ X^{(p+1)} - X^{(p)}, K^{(p)} + \frac{\gamma+\epsilon}{m} X^{(p)} \right\rangle \ \geq 0.
$$

So we could obtain that

$$
\begin{aligned}
\left\langle \ X^{(p+1)} - X^{(p)}, K^{(p)} + \frac{\gamma}{m} X^{(p)} \right\rangle \ &\geq \left\langle \ X^{(p)} - X^{(p+1)}, \frac{\epsilon}{m} X^{(p)} \right\rangle \\
&= \frac{\epsilon}{2m} \|X^{(p+1)} - X^{(p)}\|_F^2,
\end{aligned} \tag{4.5}
$$

where the equality follows from the fact that both $X^{(p)} \in \mathrm{St}(r,n)$ and $X^{(p+1)} \in \mathrm{St}(r,n)$.

By using the convexity of g and combining (4.3),(4.4) and (4.5), we have

$$
\begin{aligned}
h(X^{(p)}) - h(X^{(p+1)}) &= g(X^{(p+1)}) - g(X^{(p)}) \\
&\geq \left\langle \nabla g(X^{(p)}), X^{(p+1)} - X^{(p)} \right\rangle \\
&= 2m \left\langle K^{(p)} + \frac{\gamma}{m} X^{(p)}, X^{(p+1)} - X^{(p)} \right\rangle \\
&\geq \epsilon \|X^{(p+1)} - X^{(p)}\|_F^2.
\end{aligned} \tag{4.6}
$$

Hence, the first condition of Lemma 3.1 is established.

Part II. Recall that $h(X) = -f(X) + \delta_{\mathrm{St}(r,n)}(X)$, $\partial \delta_{\mathrm{St}(r,n)} = N_{\mathrm{St}(r,n)}$ and

$$
\partial \delta_{\mathrm{St}(r,n)}(X^{(p+1)}) = N_{\mathrm{St}(r,n)}(X^{(p+1)}) = \{X^{(p+1)} S | S \in \mathrm{S}^{r \times r}\}.
$$

It follows that (cf. [35])

$$
\partial h(X^{(p+1)}) = -\nabla f(X^{(p+1)}) + N_{\mathrm{St}(r,n)}(X^{(p+1)}). \tag{4.7}
$$

It follows from (4.1) and Algorithm A that the polar decomposition of $\frac{1}{2m} \nabla f(X^{(p)}) + \frac{\gamma+\epsilon}{m} X^{(p)}$ is

$$
\frac{1}{2m} \nabla f(X^{(p)}) + \frac{\gamma+\epsilon}{m} X^{(p)} = X^{(p+1)} S^{(p+1)}. \tag{4.8}
$$

Let $\bar{S}^{(p+1)} := S^{(p+1)} - \frac{\gamma+\epsilon}{m} I$, then $\bar{S}^{(p+1)}$ is a symmetric matrix and hence $X^{(p+1)}\bar{S}^{(p+1)} \in N_{\mathrm{St}(r,n)}(X^{(p+1)})$. Therefore, by (4.7), we have

$$2mW^{(p+1)} \in \partial h(X^{(p+1)}) \text{ with } W^{(p+1)} := -\frac{1}{2m}\nabla f(X^{(p+1)}) + X^{(p+1)}\bar{S}^{(p+1)}. \tag{4.9}$$

On the other hand, by (4.8), we have that

$$\begin{aligned} X^{(p+1)}\bar{S}^{(p+1)} &= X^{(p+1)}(S^{(p+1)} - \frac{\gamma+\epsilon}{m} I) \\ &= X^{(p+1)}S^{(p+1)} - \frac{\gamma+\epsilon}{m} X^{(p+1)} \\ &= \frac{1}{2m}\nabla f(X^{(p)}) + \frac{\gamma+\epsilon}{m}(X^{(p)} - X^{(p+1)}). \end{aligned} \tag{4.10}$$

By (4.9) and (4.10), we have

$$\begin{aligned} \left\|W^{(p+1)}\right\|_F &= \left\|-\frac{1}{2m}\nabla f(X^{(p+1)}) + X^{(p+1)}\bar{S}^{(p+1)}\right\|_F \\ &= \left\|-\frac{1}{2m}\nabla f(X^{(p+1)}) + \frac{1}{2m}\nabla f(X^{(p)}) + \frac{\gamma+\epsilon}{m}(X^{(p)} - X^{(p+1)})\right\|_F \\ &\le \frac{1}{2m}\left\|\nabla f(X^{(p+1)}) - \nabla f(X^{(p)})\right\|_F + \frac{\gamma+\epsilon}{m}\left\|X^{(p+1)} - X^{(p)}\right\|_F. \end{aligned} \tag{4.11}$$

We also have that

$$\begin{aligned} &\frac{1}{2m}\left\|\nabla f(X^{(p+1)}) - \nabla f(X^{(p)})\right\|_F \\ &\le \sum_{i=1}^{r}\left\|\lambda_i^{(p+1)}\mathcal{A}(\mathbf{x}_i^{(p+1)})^{m-1} - \lambda_i^{(p)}\mathcal{A}(\mathbf{x}_i^{(p)})^{m-1}\right\| \\ &\le \sum_{i=1}^{r}\left\|\lambda_i^{(p+1)}\mathcal{A}(\mathbf{x}_i^{(p+1)})^{m-1} - \lambda_i^{(p+1)}\mathcal{A}(\mathbf{x}_i^{(p)})^{m-1}\right\| \\ &\quad + \sum_{i=1}^{r}\left\|\lambda_i^{(p+1)}\mathcal{A}(\mathbf{x}_i^{(p)})^{m-1} - \lambda_i^{(p)}\mathcal{A}(\mathbf{x}_i^{(p)})^{m-1}\right\| \\ &\le \|\mathcal{A}\|\sum_{i=1}^{r}|\lambda_i^{(p+1)}|\left\|(\mathbf{x}_i^{(p+1)})^{m-1} - (\mathbf{x}_i^{(p)})^{m-1}\right\| \\ &\quad + \sum_{i=1}^{r}\left(|\lambda_i^{(p+1)} - \lambda_i^{(p)}|\left\|\mathcal{A}(\mathbf{x}_i^{(p)})^{m-1}\right\|\right) \\ &\le \|\mathcal{A}\|^2\sum_{i=1}^{r}\left\|(\mathbf{x}_i^{(p+1)})^{m-1} - (\mathbf{x}_i^{(p)})^{m-1}\right\| + \|\mathcal{A}\|\sum_{i=1}^{r}\left\|\mathcal{A}(\mathbf{x}_i^{(p+1)})^{m} - \mathcal{A}(\mathbf{x}_i^{(p)})^{m}\right\| \\ &\le (m-1)\|\mathcal{A}\|^2\sum_{i=1}^{r}\left\|\mathbf{x}_i^{(p+1)} - \mathbf{x}_i^{(p)}\right\| + m\|\mathcal{A}\|^2\sum_{i=1}^{r}\left\|\mathbf{x}_i^{(p+1)} - \mathbf{x}_i^{(p)}\right\| \\ &\le (m-1)r\|\mathcal{A}\|^2\left\|X^{(p+1)} - X^{(p)}\right\|_F + mr\|\mathcal{A}\|^2\left\|X^{(p+1)} - X^{(p)}\right\|_F \\ &= (2mr - r)\|\mathcal{A}\|^2\left\|X^{(p+1)} - X^{(p)}\right\|_F, \end{aligned} \tag{4.12}$$

where the first inequality follows from (4.1) and the fifth follows from Lemma 3.8.

Combining (4.11) and (4.12), we have

$$\begin{aligned}\left\|W^{(p+1)}\right\|_F &\le \frac{1}{2m}\left\|\nabla f(X^{(p+1)})-\nabla f(X^{(p)})\right\|_F+\frac{\gamma+\epsilon}{m}\left\|X^{(p+1)}-X^{(p)}\right\|_F\\ &\le (2mr-r)\left\|\mathcal{A}\right\|^2\left\|X^{(p+1)}-X^{(p)}\right\|_F+\frac{\gamma+\epsilon}{m}\left\|X^{(p+1)}-X^{(p)}\right\|_F\\ &=\left((2m-1)r\left\|\mathcal{A}\right\|^2+\frac{\gamma+\epsilon}{m}\right)\left\|X^{(p+1)}-X^{(p)}\right\|_F.\end{aligned}\tag{4.13}$$

Let $\beta := 2m(2m-1)r\left\|\mathcal{A}\right\|^2+2(\gamma+\epsilon)>0$. Then, we see that

$$\left\|2mW^{(p+1)}\right\|_F\le\beta\left\|X^{(p+1)}-X^{(p)}\right\|_F.$$

Then, the second condition of Lemma 3.1 is established.

Part III. Recall from Algorithm A that $\{X^{(p)}\}\subset \mathrm{St}(r,n)$ and the fact that $\mathrm{St}(r,n)$ is a compact set. So there is a subsequence $\{X^{(p_i)}\}$ of $\{X^{(p)}\}$ and $X^*\in\mathrm{St}(r,n)$ such that

$$X^{(p_i)}\to X^* \text{ as } i\to\infty.$$

Obviously, h is continuous over $\mathrm{St}(r,n)$. Hence when $X^{(p_i)}\to X^*$ as $i\to\infty$, we have $h(X^{(p_i)})\to h(X^*)$. The third condition of Lemma 3.1 follows.

Finally, the fact that h is a KL function is also known (cf. [2]). Therefore, the whole sequence $\{X^{(p)}\}$ converges to X^*, and X^* is a critical point of $h(X)$ by Lemma 3.1. □

5 The Global Convergence without Proximality

In this section, we present a global convergence proof for Algorithm A without the proximality. Avoiding proximality is preferable in numerical computations [20].

Theorem 5.1 (Convergence without Proximality)**.** *Let $\mathcal{A}\in \mathrm{S}(\otimes^m\mathbb{R}^n)$ be a symmetric tensor. Suppose that $\epsilon\ge 0$ and $\gamma>m\max\{2\sqrt{r},(m-1)\}\rho(\mathcal{A})^2$. Then any sequence $\{X^{(p)}\}$ generated by Algorithm A converges to a KKT point X^* of the problem* (2.3).

Proof. The notations in the proof of Theorem 4.1 will be adopted and Lemma 3.1 will be applied again. The proof is divided into three parts for clarity accordingly, and similar proof as that of Theorem 4.1 will be omitted for simplicity.

Part I. From Algorithm A, we know that

$$\begin{aligned}V^{(p+1)}&=\left[\mathbf{v}_1^{(p+1)},\cdots,\mathbf{v}_r^{(p+1)}\right]\\ &=\left[\lambda_1^{(p)}\mathcal{A}(\mathbf{x}_1^{(p)})^{m-1}+\tfrac{\gamma+\epsilon}{m}\mathbf{x}_1^{(p)},\cdots,\lambda_r^{(p)}\mathcal{A}(\mathbf{x}_r^{(p)})^{m-1}+\tfrac{\gamma+\epsilon}{m}\mathbf{x}_r^{(p)}\right]\\ &=\left[\mathcal{A}(\mathbf{x}_1^{(p)})^{m-1},\cdots,\mathcal{A}(\mathbf{x}_r^{(p)})^{m-1}\right]\mathrm{diag}(\lambda_1^{(p)},\cdots,\lambda_r^{(p)})+\frac{\gamma+\epsilon}{m}X^{(p)}\\ &=K^{(p)}+\frac{\gamma+\epsilon}{m}X^{(p)},\end{aligned}$$

where

$$K^{(p)}:=\left[\mathcal{A}(\mathbf{x}_1^{(p)})^{m-1},\cdots,\mathcal{A}(\mathbf{x}_r^{(p)})^{m-1}\right]\mathrm{diag}(\lambda_1^{(p)},\cdots,\lambda_r^{(p)}).$$

Thus, we have

$$\begin{aligned}(V^{(p+1)})^\top V^{(p+1)} &= (K^{(p)} + \frac{\gamma+\epsilon}{m}X^{(p)})^\top (K^{(p)} + \frac{\gamma+\epsilon}{m}X^{(p)})\\ &= \frac{(\gamma+\epsilon)^2}{m^2}I + \frac{\gamma+\epsilon}{m}(X^{(p)})^\top K^{(p)} + \frac{\gamma+\epsilon}{m}(K^{(p)})^\top X^{(p)} + (K^{(p)})^\top K^{(p)},\end{aligned}$$

where the second equality follows from the fact that $X^{(p)} \in \mathrm{St}(r,n)$. By Lemma 3.2, we know

$$\rho((X^{(p)})^\top K^{(p)} + (K^{(p)})^\top X^{(p)}) \le 2\sqrt{r}\rho(\mathcal{A})^2.$$

On the other hand, $(K^{(p)})^\top K^{(p)}$ is a positive semidefinite symmetric matrix. Therefore, when $\frac{(\gamma+\epsilon)^2}{m^2} > 2\frac{\gamma+\epsilon}{m}\sqrt{r}\rho(\mathcal{A})^2$, i.e., $\gamma+\epsilon > 2m\sqrt{r}\rho(\mathcal{A})^2$, $(V^{(p+1)})^\top V^{(p+1)}$ is a positive definite matrix. By Lemma 3.7, we get that $V^{(p+1)}$ is of full rank. Consequently, by Lemma 3.4, we know $V^{(p+1)}$ has a unique polar decomposition. Moreover, by Lemma 3.7, we have

$$\begin{aligned}\sigma_{\min} &\ge \sqrt{\frac{(\gamma+\epsilon)^2}{m^2} - 2\frac{\gamma+\epsilon}{m}\sqrt{r}\rho(\mathcal{A})^2}\\ &= \sqrt{\gamma+\epsilon}\sqrt{\frac{\gamma+\epsilon}{m^2} - \frac{2}{m}\sqrt{r}\rho(\mathcal{A})^2}\\ &\ge \sqrt{\frac{\gamma^2}{m^2} - \frac{2\gamma}{m}\sqrt{r}\rho(\mathcal{A})^2} > 0,\end{aligned}$$

where $\sigma_{\min}$ is the minimum singular value of $V^{(p+1)}$.

Let

$$\widetilde{g}(X) := \sum_{i=1}^{r}(\mathcal{A}(\mathbf{x}_i)^m)^2 + (\gamma+\epsilon)\|X\|_F^2.$$

By Lemma 3.3 and the fact that $\gamma+\epsilon > m(m-1)\rho(\mathcal{A})^2$, we see that $\widetilde{g}$ is convex over an open neighborhood of the unit disc $\Omega = \{X \mid X \in \mathbb{R}^{n\times r}, \|X\|_{2,\infty} \le 1\}$. Therefore, from (4.4) and by using the convexity of $\widetilde{g}$ and Lemma 3.5, we have

$$\begin{aligned}h(X^{(p)}) - h(X^{(p+1)}) &= g(X^{(p+1)}) - g(X^{(p)})\\ &= \widetilde{g}(X^{(p+1)}) - \widetilde{g}(X^{(p)})\\ &\ge \left\langle \nabla\widetilde{g}(X^{(p)}), X^{(p+1)} - X^{(p)}\right\rangle\\ &= 2m\left\langle V^{(p+1)}, X^{(p+1)} - X^{(p)}\right\rangle\\ &\ge m\sigma_{\min}\left\|X^{(p+1)} - X^{(p)}\right\|_F^2\\ &\ge \sqrt{\gamma^2 - 2m\gamma\sqrt{r}\rho(\mathcal{A})^2}\left\|X^{(p+1)} - X^{(p)}\right\|_F^2,\end{aligned}$$

where the second equality follows from the fact that both $X^{(p)}$ and $X^{(p+1)}$ are orthonormal matrices by Algorithm A.

Thus, the first condition in Lemma 3.1 is satisfied.

Parts II. and III. The proofs are the same as those for Theorem 4.1.

In summary, the conclusion follows. □

Theorem 5.1 presents a possibility of Algorithm A without the proximality. This is in particular meaningful whenever $m-1 \ge 2\sqrt{r}$. We see that this is the case for low rank approximations for higher order tensors.

6 The Sublinear Convergence Rate

In this section, we will give a sublinear convergence rate analysis for Algorithm A. The proof follows from a similar framework as that in [19], with the global convergence results established in the above sections. We want to remark that the general result Lemma 3.1 and the Łojasiwicz property with explicit exponent in Lemma 3.9 cannot directly imply the conclusions in this section. An apparent difference is the second condition in Lemma 3.1 and a key inequality (6.8) to be used in the proof for the next Theorem 6.1.

Recall from (2.4) that

$$f(X) = \sum_{i=1}^{r} (\mathcal{A}(\mathbf{x}_i)^m)^2.$$

Theorem 6.1 (Sublinear Convergence Rate)**.** *Let $\{X^{(p)}\}$ be a sequence generated by Algorithm A for a given nonzero tensor $\mathcal{A} \in \mathrm{S}(\otimes^m \mathbb{R}^n)$. Suppose that the sequence $\{X^{(p)}\}$ converges globally to a KKT point of problem (2.3). Let $N := \frac{r}{2}(2n + r + 1)$ and*

$$\tau := 1 - \frac{1}{2m(6m-3)^{N-1}}. \tag{6.1}$$

The following statements hold:

(a) *the sequence $\{f(X^{(p)})\}$ converges to f^*, with sublinear convergence rate at least $O(p^{\frac{1}{1-2\tau}})$, that is, there exist $M_1 > 0$ and $p_1 \in \mathbb{N}$ such that for all $p \geq p_1$,*

$$f^* - f(X^{(p)}) \leq M_1 p^{\frac{1}{1-2\tau}};$$

(b) *the sequence $\{X^{(p)}\}$ converges to X^* globally with the sublinear convergence rate at least $O(p^{\frac{1-\tau}{2\tau-1}})$, that is, there exist $M_2 > 0$ and $p_1 \in \mathbb{N}$ such that for all $p \geq p_1$,*

$$\|X^{(p)} - X^*\|_F \leq M_2 p^{\frac{\tau-1}{2\tau-1}}.$$

Proof. By the hypothesis, the sequence $\{X^{(p)}\}$ converges to a KKT point X^* of (2.3) with the corresponding unique multiplier P^*. Note that P^* is symmetric and by (2.5)

$$P^* = \frac{1}{2}(X^*)^\top \nabla f(X^*).$$

Let

$$q(X,P) := f(X) - \langle P, X^\top X - I \rangle$$

for $X \in \mathbb{R}^{n\times r}$ and $P \in \mathrm{S}^r$, and

$$\hat{q}(X,P) := q(X,P) - q(X^*,P^*).$$

Thus, we have $\hat{q}(X^*,P^*) = 0$ and $\nabla\hat{q}(X^*,P^*) = 0$. Note that $\hat{q}$ is a polynomial in $X \in \mathbb{R}^{n\times r}$ and $P \in \mathrm{S}^r$ of degree $2m$. Thus, the number of variables is $N = nr + \frac{r(r+1)}{2} = \frac{r}{2}(2n+r+1)$. Consequently, by Lemma 3.9, there exist constants $\delta, c > 0$ such that

$$\|\nabla\hat{q}(X,P)\|_F \geq c|\hat{q}(X,P)|^\tau \text{ for all } \|(X,P) - (X^*,P^*)\|_F \leq \delta,$$

where τ is given by (6.1).

Therefore,

$$\|\nabla f(X) - 2XP\|_F^2 \geq c^2\big(f(X) - f(X^*)\big)^{2\tau} \tag{6.2}$$

for any feasible point X of (2.3) and $P \in \mathrm{S}^r$ satisfying $\|(X, P) - (X^*, P^*)\|_F \leq \delta$, since in this case

$$\nabla_P \hat{q}(X, P) = X^\top X - I = 0 \text{ and } \nabla_X \hat{q}(X, P) = \nabla f(X) - 2XP.$$

The rest proof will be divided into two parts, for respectively (a) and (b).

Part I. Proof of (a).

Let

$$P^{(p)} := mS^{(p)} - (\gamma + \epsilon)I = m(X^{(p)})^\top V^{(p)} - (\gamma + \epsilon)I,$$

where $S^{(p)}$ is the positive semidefinite factor matrix of the matrix $V^{(p)}$ from Algorithm A. By (4.11) and (4.13), we have

$$\begin{aligned} &\| - \nabla f(X^{(p+1)}) + 2X^{(p+1)}P^{(p+1)}\|_F \\ =&\| - \nabla f(X^{(p+1)}) + 2mX^{(p+1)}S^{(p+1)} - 2(\gamma + \epsilon)X^{(p+1)}\|_F \\ \leq&\big(2m(2m - 1)r\|\mathcal{A}\|^2 + 2(\gamma + \epsilon)\big)\|X^{(p+1)} - X^{(p)}\|_F. \end{aligned} \tag{6.3}$$

Let

$$\hat{P}^{(p)} := \frac{1}{4}\Big((X^{(p)})^\top \nabla f(X^{(p)}) + \big(\nabla f(X^{(p)})\big)^\top X^{(p)}\Big). \tag{6.4}$$

Let

$$M^{(p)} := \Big[\lambda_1^{(p)}\mathcal{A}(\mathbf{x}_1^{(p)})^{m-1} \quad \ldots \quad \lambda_r^{(p)}\mathcal{A}(\mathbf{x}_r^{(p)})^{m-1}\Big]. \tag{6.5}$$

Then

$$V^{(p+1)} = M^{(p)} + \frac{\gamma + \epsilon}{m}X^{(p)}.$$

Note that

$$\nabla f(X^{(p)}) = 2mM^{(p)}.$$

We have

$$\begin{aligned} &\|P^{(p+1)} - \hat{P}^{(p+1)}\|_F \\ =&\|m(X^{(p+1)})^\top V^{(p+1)} - (\gamma + \epsilon)I - \frac{1}{4}\Big((X^{(p+1)})^\top \nabla f(X^{(p+1)}) + \big(\nabla f(X^{(p+1)})\big)^\top X^{(p+1)}\Big)\|_F \\ =&\|\frac{m}{2}(X^{(p+1)})^\top M^{(p)} + \frac{m}{2}(M^{(p)})^\top X^{(p+1)} + \frac{(\gamma + \epsilon)}{2}((X^{(p+1)})^\top X^{(p)} + (X^{(p)})^\top X^{(p+1)} - 2I) \\ &- \frac{m}{2}\Big((X^{(p+1)})^\top M^{(p+1)} + (M^{(p+1)})^\top X^{(p+1)}\Big)\|_F \\ \leq&\frac{m}{2}\|(X^{(p+1)})^\top M^{(p)} - (X^{(p+1)})^\top M^{(p+1)} + (M^{(p)})^\top X^{(p+1)} - (M^{(p+1)})^\top X^{(p+1)}\|_F \\ &+ (\gamma + \epsilon)\|(X^{(p+1)})^\top X^{(p)} - I\|_F \\ \leq&m\|(X^{(p+1)})^\top M^{(p)} - (X^{(p+1)})^\top M^{(p+1)}\|_F + (\gamma + \epsilon)\|X^{(p+1)} - X^{(p)}\|_F \\ \leq&m\sqrt{r}\|M^{(p)} - M^{(p+1)}\|_F + (\gamma + \epsilon)\|X^{(p+1)} - X^{(p)}\|_F \\ \leq&C_0\|X^{(p+1)} - X^{(p)}\|_F, \end{aligned} \tag{6.6}$$

where $C_0 > 0$ is a constant, in the second equality we used the fact that the matrix $(X^{(p+1)})^\top V^{(p+1)}$ is symmetric by Lemma 3.4 and Algorithm A, the second inequality follows from Lemma 3.6, and the last one follows from the fact that the matrix M defined in (6.5) is Lipschitz continuous.

Therefore, we have

$$\begin{aligned}
&\|\nabla f(X^{(p+1)}) - 2X^{(p+1)}\hat{P}^{(p+1)}\|_F \\
\leq&\|\nabla f(X^{(p+1)}) - 2X^{(p+1)}P^{(p+1)}\|_F + \|2X^{(p+1)}P^{(p+1)} - 2X^{(p+1)}\hat{P}^{(p+1)}\|_F \\
\leq&C\|X^{(p+1)} - X^{(p)}\|_F
\end{aligned} \tag{6.7}$$

for some constant $C > 0$ by (6.3) and (6.6).

It follows from the definition of $\hat{P}^{(p)}$ (cf. (6.4)) that $\hat{P}^{(p)} \in \mathrm{S}^r$ and it converges as $X^{(p)}$ converges. We have

$$\lim_{p\to\infty} \hat{P}^{(p)} = \frac{m}{2}((X^*)^\top M^* + (M^*)^\top X^*) = \frac{1}{2}(X^*)^\top \nabla f(X^*) = P^*.$$

Hence for sufficiently large p (saying $p \geq p_0$ for some positive p_0), we may conclude that

$$\|(X^{(p)}, \hat{P}^{(p)}) - (X^*, P^*)\|_F \leq \delta.$$

Therefore,

$$\begin{aligned}
&c^2\big(f(X^{(p)}) - f(X^*)\big)^{2\tau} \\
&\leq \|\nabla f(X^{(p)}) - 2X^{(p)}\hat{P}^{(p)}\|_F^2 \\
&\leq \|\nabla f(X^{(p+1)}) - 2X^{(p+1)}\hat{P}^{(p+1)}\|_F^2 \\
&\quad + \|\nabla f(X^{(p+1)}) - 2X^{(p+1)}\hat{P}^{(p+1)} - \big(\nabla f(X^{(p)}) - 2X^{(p)}\hat{P}^{(p)}\big)\|_F^2 \\
&\leq (C^2 + L^2)\|X^{(p+1)} - X^{(p)}\|_F^2 \\
&\leq \frac{C^2 + L^2}{\epsilon}\big(f(X^{(p+1)}) - f(X^{(p)})\big),
\end{aligned} \tag{6.8}$$

where the first inequality follows from (6.2), the third from (6.7) and the fact that the funciton $\nabla f(X^{(p)}) - 2X^{(p)}\hat{P}^{(p)}$ is Lipschitz continuous with respect to $X^{(p)}$, and the last from (4.6). Here L is the Lipschitz constant of the function $\nabla f(X^{(p)}) - 2X^{(p)}\hat{P}^{(p)}$ on the Stiefel manifold.

We let $\beta_p := f(X^*) - f(X^{(p)})$, from (6.8), there exists a constant $D > 0$, we have

$$\beta_p - \beta_{p+1} = f(X^{(p+1)}) - f(X^{(p)}) \geq D\big(f(X^p) - f(X^{(*)})\big)^{2\tau} = D\beta_p^{2\tau}. \tag{6.9}$$

Note that the sequence $\{\beta_p\}$ is a sequence of positive numbers, since otherwise the algorithm terminates in finitely many steps.

Define a function $h(x) := x^{-2\tau}$, it follows that

$$\beta_p - \beta_{p+1} \geq D\beta_p^{2\tau} = Dh(\beta_p)^{-1}.$$

Define another function $t(x) := \frac{x^{1-2\tau}}{1-2\tau}$, then we have $h(x) = t'(x)$ and it is easy to verify that h is non-increasing on $\mathbb{R}_{++}$, the set of positive real numbers. Thus, we have

$$\begin{aligned}
D \leq h(\beta_p)(\beta_p - \beta_{p+1}) &\leq \int_{\beta_{p+1}}^{\beta_p} h(x)\,dx \\
&= t(\beta_p) - t(\beta_{p+1}) = \frac{1}{1-2\tau}(\beta_p^{1-2\tau} - \beta_{p+1}^{1-2\tau}) \\
&= \frac{1}{2\tau - 1}(\beta_{p+1}^{1-2\tau} - \beta_p^{1-2\tau}).
\end{aligned}$$

Consequently, we have by induction that

$$\beta_p^{1-2\tau} \geq D(2\tau-1) + \beta_{p-1}^{1-2\tau} \geq \cdots \geq D(2\tau-1)(p-p_0) + \beta_{p_0}^{1-2\tau} \geq D(2\tau-1)(p-p_0).$$

By the definition of τ (cf. (6.1)), we have that $1 < 2\tau$. Let $p_1 = 2p_0$, so for all $p \geq p_1$, we have

$$\beta_p \leq [D(2\tau-1)(p-p_0)]^{\frac{1}{1-2\tau}} = [D(2\tau-1)\frac{(p-p_0)}{p}]^{\frac{1}{1-2\tau}} \cdot p^{\frac{1}{1-2\tau}} \leq [D(2\tau-1)\frac{1}{2}]^{\frac{1}{1-2\tau}} \cdot p^{\frac{1}{1-2\tau}}.$$

Therefore, there exists $M_1 > 0$ such that for all $p \geq p_1$,

$$0 \leq \beta_p \leq M_1 p^{\frac{1}{1-2\tau}}.$$

The conclusion (a) then follows.

Part II. Proof of (b).

From (6.8), we have

$$c^2\big(f(X^{(p)}) - f(X^*)\big)^{2\tau} \leq (C^2 + L^2)\|X^{(p+1)} - X^{(p)}\|_F^2.$$

Let $d := C^2 + L^2$, it follows that

$$\big(f(X^*) - f(X^{(p)})\big)^{\tau} \leq \frac{\sqrt{d}}{c}\|X^{(p+1)} - X^{(p)}\|_F.$$

Define $s_p := \|X^{(p+1)} - X^{(p)}\|_F$, so we could have

$$\beta_p^{\tau} \leq \frac{\sqrt{d}}{c} s_p. \tag{6.10}$$

Define a function $\phi(s) := -s^{1-\tau}$. It is easy to verify that ϕ is convex on $\mathbb{R}_{++}$, and hence we have

$$\phi(\beta_{p+1}) - \phi(\beta_p) \geq \phi'(\beta_p)(\beta_{p+1} - \beta_p).$$

Therefore, it follows that

$$\begin{aligned}
\beta_p^{1-\tau} - \beta_{p+1}^{1-\tau} &\geq (1-\tau)\beta_p^{-\tau}(\beta_p - \beta_{p+1}) \\
&= (1-\tau)\beta_p^{-\tau}\big(f(X^{(p+1)}) - f(X^{(p)})\big) \\
&\geq (1-\tau)\epsilon\beta_p^{-\tau}\|X^{(p+1)} - X^{(p)}\|_F^2 \\
&= (1-\tau)\epsilon\beta_p^{-\tau}s_p^2 \\
&\geq (1-\tau)\epsilon\frac{c}{\sqrt{d}}s_p,
\end{aligned}$$

where the first equality follows from (6.9), the second inequality from (4.6) and the last from (6.10). Let $C_d := \frac{\sqrt{d}}{(1-\tau)\epsilon c}$, hence, we obtain

$$s_p \leq C_d(\beta_p^{1-\tau} - \beta_{p+1}^{1-\tau}). \tag{6.11}$$

For any $N > N_0 > p_0$, summing the inequality (6.11) from $p = N_0$ to $p = N$ it follows that

$$\sum_{p=N_0}^{N} s_p \leq C_d(\beta_{N_0}^{1-\tau} - \beta_{N+1}^{1-\tau}).$$

Note that we have $\beta_N \to 0$ as $N \to \infty$. It then follows that

$$\sum_{p=N_0}^{\infty} s_p \le C_d \beta_{N_0}^{1-\tau}. \tag{6.12}$$

This shows that $\sum_{p=1}^{\infty} s_p < +\infty$. For $p \ge p_0$, define $\Delta_p := \sum_{i=p}^{\infty} s_i$. Then, combining (6.10) and (6.12), we have

$$\Delta_p \le C_d \left(\frac{\sqrt{d}}{c} s_p \right)^{\frac{1-\tau}{\tau}}.$$

As $0 < \frac{1-\tau}{\tau} < 1$, there exists constant $K > 0$ such that

$$\Delta_p^{\frac{\tau}{1-\tau}} \le K s_p = K(\Delta_p - \Delta_{p+1}).$$

Consequently, we have a relation

$$\Delta_p \ge \Delta_{p+1} + \frac{1}{K} \Delta_p^{\frac{\tau}{1-\tau}}.$$

Note that this is a relation for the sequence $\{\Delta_p\}$ in the same formula as $\{\beta_p\}$ in (6.9). We also have that $\frac{\tau}{1-\tau} > 1$ since $2\tau > 1$. Therefore, a similar argument as that in Part I after the relation (6.9) will give the conclusion that there exists constant $M_2 > 0$ such that

$$\Delta_p \le M_2 p^{\frac{\tau-1}{2\tau-1}}.$$

With this, finally we have

$$\|X^{(p)} - X^*\|_F \le \sum_{i=p}^{\infty} \|X^{(p)} - X^{(p+1)}\|_F = \Delta_p \le M_2 p^{\frac{\tau-1}{2\tau-1}}.$$

The conclusion (b) then follows.

The proof is then complete. □

7 Conclusions

In this paper, we studied the problem of low rank symmetric orthogonal approximations for given symmetric tensors and proposed an algorithm for solving this problem. The main conclusion is that under only a condition on the parameters of the proposed algorithm, we can show that this algorithm converges globally with an explicit sublinear convergence rate without any further assumption. Furthermore, this sublinear rate is better than the usual $O(\frac{1}{p})$ rate for first order methods in optimization [4], and it is of order $O(\frac{1}{p^{1+\kappa}})$ for some $\kappa \in (0,1)$. Since the best possible rate is $O(\frac{1}{p^2})$ in the convexity case under some additional assumptions [31], the derived sublinear rate is sharp in this sense.

Further investigations on the linear convergence rate for the generic case may be carried out as [20], which is left as our next work.

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